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178 001

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A SELF-SHIELDING FACTOR FOR URANIUM IN A TEFLON LATTICE

Joel Bengston*

ABSTRACT

A thermal self-shielding factor for the uranium foil in the second assembly of the two-region reflector-moderated reactor has been obtained, using Yvon's double spherical harmonic method in a plane infinite lattice calculation. The calculation has been carried out in three successive orders of approximation, and is believed to be accurate to about 3%. The result is significantly different from those used in multigroup calculations, but its use would apparently decrease agreement between calculation and experiment. It is pointed out that the result of an accurate infinite lattice calculation is actually a lower bound on the correct self-shielding factor for the experimental finite lattice, but is the most accurate method yet available if the fuel region is several mean free paths in thickness. It is indicated that the much simpler formula proposed by R. R. Bate gives an upper bound on the self-shielding factor.

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178 002

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I. INTRODUCTION

The fuel region in the reflector-moderated reactor critical experiments¹ consists of alternate layers of uranium foil (93.17% enriched) and teflon, $(CF_2)_n$. In the multigroup criticality calculations, this lattice was replaced by a homogeneous region with cross sections obtained by averaging over a unit cell of this lattice. In order to obtain the average, it is assumed that the lattice is essentially infinite in extent, so that only one unit cell need be considered. If we let V_i , Σ_i , and ϕ_i be the volume, macroscopic cross section, and average flux in the i^{th} region of the lattice ($i = 1$ for the fuel, 2 for the moderator), then the properly averaged cross section for the lattice is

$$\bar{\Sigma} = \frac{\Sigma_1 V_1 \bar{\phi}_1 + \Sigma_2 V_2 \bar{\phi}_2}{V_1 \bar{\phi}_1 + V_2 \bar{\phi}_2}.$$

Now put $f \equiv \phi_1 / \phi_2$, and let $v_i \equiv V_i / (V_1 + V_2)$ be the volume fraction of the i^{th} ingredient, so that

$$\bar{\Sigma} = \frac{f \Sigma_1 v_1 + \Sigma_2 v_2}{1 - (1 - f) v_1}. \quad (1)$$

In the critical experiments, the uranium foil was ~ 0.004 in. thick, while the teflon was packed in sheets whose nominal thickness was in multiples of $1/32$ -in. The minimum teflon thickness used to date is 0.173 in., for

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1. See ORNL CF-54-4-53 for a preliminary description of the program. Subsequent work in: CF-54-7-159, CF-54-8-180, CF-54-9-185, CF-54-10-119 and CF-54-11-150.

the second assembly of the two-region reactor. In this case, if we assume $f \geq 1/2$, the error in neglecting the second term in the denominator is $\sim 1\%$. The method of calculation customarily used involves making this approximation and finding some simple formula for f , which is called the self-shielding factor for the uranium, and is applied to both fission and absorption cross sections. In the criticality calculations for the reflector moderated critical assemblies, two quite different approximations for F were made at different times. In an effort to appraise the validity of the simple approximations used, an accurate calculation of F for thermal energy was undertaken for a specified thickness of teflon (0.173 in.) by the use of Yvon's new double spherical harmonic approximation.^{2,3}

The uranium foils were manufactured in sheets 2.860 by 8.610 in., and of mean weight 29.2 g/sheet. Their thickness is then 0.183800 g/cm². The 93.2% U²³⁵ was assumed to have the following microscopic properties:

$$\left. \begin{array}{ll} A &= 235.32 \\ \sigma_a &= 623.2 \text{ b} \\ \sigma_{sc} &= 8.2 \text{ b} \\ \sigma_{tot} &= 631.4 \text{ b} \end{array} \right\} \text{ for thermal neutrons}$$

Using these numbers, the foil thickness turns out to be 0.2971 mean free paths. However, since the scattering cross section is $\sim 1\%$ of the total,

2. J. Yvon, unpublished notes on Milne problem, translated by W. K. Ergen and given limited distribution at ORNL.
3. J. Bengston, Neutron Self-Shielding in a Plane Absorbing Foil, ORNL CP-56-3-170 (Mar. 1, 1956).

the calculations were simplified by assuming that the foil is a pure absorber of thickness 0.295 mean free paths.

The teflon was assumed to be pure $(CF_2)_n$, of density 2.1 g/cm^3 . Then $\Sigma_{\text{tot}} = 0.3423 \text{ cm}^{-1}$ and $\Sigma_a / \Sigma = 0.00146$. The 0.173 in. thickness of teflon then corresponds to 0.1504 mean free paths. The lattice calculations were considerably simplified by the assumption that $\Sigma_a = 0$ in the teflon.

II. METHODS OF CALCULATION

The transport equation in plane geometry for one velocity group with isotropic scattering may be written:

$$\mu \frac{\partial \psi(x, \mu)}{\partial x} + \frac{1}{\lambda} \psi(x, \mu) = \frac{c}{2\lambda} \int_{-1}^1 \psi(x, \mu) d\mu + \frac{S(x)}{2} \quad (2)$$

where

c = neutrons emergent per collision

λ = neutron mean free path

S = source strength in neutrons/cm³ sec ,

and $\psi(x, \mu)d\mu$ is the total path length/cm³ sec of neutrons whose direction relative to the x axis is between arc $\cos \mu$ and arc $\cos(\mu + d\mu)$. The double spherical harmonic method of Yvon consists of expanding this angular distribution of path length in two orthogonal expansions, each valid for half the angular range:

$$\begin{aligned} \psi(x, \mu) &= \sum (2\ell + 1) P_\ell(2\mu - 1) \psi_\ell(x), \quad \mu > 0 \\ &= \sum (2\ell + 1) P_\ell(2\mu + 1) \chi_\ell(x), \quad \mu < 0. \end{aligned} \quad (3)$$

The general development of this method for $0 < c < 1$ has been reported elsewhere,³ but the particular case in which $c = 1$ (pure scatterer), requires special treatment and that will be developed here. Measuring distance in units of mean free path and letting $c = 1$, substitute Eqs. (3) in Eq. (2) and multiply the resulting two equations by $P_m(2\mu \pm 1)$; then integrate over the appropriate half of the angular range, using the orthogonality properties of the Legendre functions to obtain:

$$m \psi'_{m-1} + (2m+1) \psi'_m + (m+1) \psi'_{m+1} + 2(2m+1) \psi_m(x) = \left[\psi_0 + \chi_0 + \lambda S(x) \right] \delta_{0,m} \quad (4)$$

$$m \chi'_{m-1} - (2m+1) \chi'_m + (m+1) \chi'_{m+1} + 2(2m+1) \chi_m(x) = \left[\psi_0 + \chi_0 + \lambda S(x) \right] \delta_{0,m}.$$

where $\lambda \equiv \frac{d}{dx}$. The solutions to the homogeneous part of the above equations can be obtained by putting

$$\begin{aligned} \psi_m &= A_m e^{\nu x} \\ \chi_m &= B_m e^{\nu x} \end{aligned} \quad (5)$$

In this way we obtain:

$$\left. \begin{aligned} m \nu A_{m-1} + (2m+1)(2+\nu) A_m + (m+1) \nu A_{m+1} &= (A_0 + B_0) \delta_{m,0} \\ m \nu B_{m-1} + (2m+1)(2-\nu) B_m + (m+1) \nu B_{m+1} &= (A_0 + B_0) \delta_{m,0} \end{aligned} \right\} \quad (6)$$

In the "double P_L approximation", we let A_m and B_m be zero for all $m > L$. Then Eqs. (6) consist of $2(L+1)$ homogeneous equations for the deter-

mination of a like number of constants. Then the condition for the existence of a non-vanishing solution is the vanishing of the determinant of the coefficients of the A_m and B_m . This condition then gives a polynomial of order $(L + 1)$ in ν^2 . One pair of roots corresponds to $\nu^2 = 0$, and we are then left with L pairs of real roots, $\pm \nu_i$, such that $|\nu_i| > 1$. Using these L values of $\pm \nu_i$, we can determine all the other coefficients $A_{m\pm}$, $B_{m\pm}$ in terms of the $2L$ coefficients $A_{0i\pm}$. The $\nu^2 = 0$ solutions, in the absence of sources, would correspond to a solution $\psi''_0 = \chi''_0 = 0$, and thus a linear term in addition to the exponential ones. In the presence of a constant source, however, the second derivatives equal non-vanishing constants, and thus the inhomogeneous solution must be of the form of a quadratic function of x . The form of this function depends upon both the source strength and the boundary conditions, so that the specific lattice problem will be introduced before the solution is explicitly written down.

We will consider a two-region periodic slab lattice consisting of regions of thickness $2a$ in which $c = 0$ (pure absorber) alternating with regions of thickness $2(X - a)$ in which $c = 1$. We will take $x = 0$ at the center of one of the absorbing layers, so that the right boundary of that layer is at $x = a$, and the center of the adjacent scattering layer is at $x = X$.

The solutions in the scatterer will then have a symmetry about X such that

$$\psi(X, \mu) = \psi(X, -\mu) \quad (7a)$$

and Eq. (2) then also requires

$$\frac{\partial \psi(X, \mu)}{\partial x} = - \frac{\partial \psi(X, -\mu)}{\partial x} \quad (7b)$$

In terms of the expansions in Eqs. (3), these conditions evidently become:

$$\psi_\ell(X) = (-)^{\ell} X_\ell(X) \quad (8a)$$

and

$$\psi'_\ell(X) = (-)^{\ell+1} X'_\ell(X) \quad (8b)$$

Without the use of the symmetry relationships, the solution in the moderator due to the L finite values of $|\nu|$ may be written:

$$\psi_{\ell, \text{partial}}(a < x < 2X - a) = \sum_{i=2}^{L+1} \left\{ A_{\ell i-} e^{-|\nu_i| x'} + A_{\ell i+} e^{|\nu_i| x'} \right\} \quad (9)$$

$$X_{\ell, \text{partial}}(a < x < 2X - a) = \sum_{i=2}^{L+1} \left\{ B_{\ell i-} e^{-|\nu_i| x'} + B_{\ell i+} e^{|\nu_i| x'} \right\}$$

where $x' \equiv x - X$.

In the above, the plus and minus subscripts denote the two choices of sign for ν_i ; in subsequent formulas here the absolute value sign will be dropped on the ν . The symmetry requirements of Eqs. (8) will be satisfied if we put

$$B_{mi-} = (-)^m A_{mi+}$$

and

$$B_{mi+} = (-)^m A_{mi-}$$

(10)

It can be verified easily that Eqs. (10) are also consistent with the recurrence relations given by Eqs. (6). In the work reported here, it was found convenient to express all formulas in terms of the $A_{\ell i-}$ and $B_{\ell i-}$, and then to express all of these numerically, via recurrence relations derivable from Eqs. (6), in terms of the L coefficients A_{oi-} which were then evaluated by the boundary conditions at $x = a$. The equations used for determining the γ_i and the recurrence relations are listed in the Appendix for the three lowest orders of approximation.

The solution of the homogeneous equations can be found by explicit substitution of arbitrary quadratics in Eqs. (4), and evaluating the coefficients in terms of the source term and using the boundary conditions in Eqs. (8). In this way one obtains for the complete solution in the moderator,

$$\begin{aligned}\psi_0 &= \frac{3\lambda s_0}{4} [1 + 2X - x] + C + \sum_{i=2}^{L+1} \left[A_{oi-} e^{-\gamma_i x'} + B_{oi-} e^{\gamma_i x'} \right] \\ \chi_0 &= \frac{3\lambda s_0}{4} (2X - x)(x + 1) + C + \sum_{i=2}^{L+1} \left[B_{oi-} e^{-\gamma_i x'} + A_{oi-} e^{\gamma_i x'} \right] \\ \psi_1 &= -\frac{\lambda s_0}{4} [1 - x + X] + \sum_{i=2}^{L+1} \left[A_{li-} e^{-\gamma_i x'} - B_{li-} e^{\gamma_i x'} \right] \\ \chi_1 &= \frac{\lambda s_0}{4} [1 + x - X] + \sum_{i=2}^{L+1} \left[B_{li-} e^{-\gamma_i x'} - A_{li-} e^{\gamma_i x'} \right]\end{aligned}\tag{11}$$

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$$\psi_2 = -\frac{\lambda S_0}{20} + \sum_{i=2}^{L+1} \left[A_{2i-} e^{-\nu_i x'} + B_{2i-} e^{\nu_i x'} \right] \quad (11)$$

$$\chi_2 = -\frac{\lambda S_0}{20} + \sum_{i=2}^{L+1} \left[B_{2i-} e^{-\nu_i x'} + A_{2i-} e^{\nu_i x'} \right]$$

$$\psi_{\ell \geq 3} = \sum_{i=2}^{L+1} \left[A_{\ell i-} e^{-\nu_i x'} + (-)^{\ell} B_{\ell i-} e^{\nu_i x'} \right]$$

$$\chi_{\ell \geq 3} = \sum_{i=2}^{L+1} \left[B_{\ell i-} e^{-\nu_i x'} + (-)^{\ell} A_{\ell i-} e^{\nu_i x'} \right]$$

The $L + 1$ constants (C and the L coefficients A_{0i-}) are then evaluated by the use of $L + 1$ equations obtained by a method derived in Ref. 3, Eqs. (38) and (39). The method consists of using the exact equation valid for a non-scattering foil,

$$\psi(a, \mu) = \psi(-a, \mu) e^{-2a/\mu}, \quad (12)$$

substituting the expansion (3a) in both sides, multiplying by μ^{ℓ} , $\ell = 0, 1, \dots, L$, integrating over $0 \leq \mu \leq 1$, and using the symmetry relations

$$\psi_{\ell}(-a) = (-)^{\ell} \chi_{\ell}(a), \quad (13)$$

which are justified in a manner similar to that used for Eq. (8a) above.

The resulting equations contain the $\psi_{\ell}(a)$ and $\chi_{\ell}(a)$, which can be evaluated numerically from Eqs. (11) in terms of the A_{0i-} and C .

For the double P_3 approximation, which is the highest used in this report, the equations are:

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178 010

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$$\psi_0(a) = U \cdot E_2(2a) - 6V E_3(2a) + 30W E_4(2a) - 140 X_3(a) E_5(2a) \quad (14)$$

$$\frac{1}{2} (\psi_0 + \psi_1)_a = U E_3(2a) - 6V E_4(2a) + 30W E_5(2a) - 140 X_3(a) E_6(2a)$$

$$\left[\frac{1}{3} \psi_0 + \frac{1}{2} \psi_1 + \frac{1}{6} \psi_2 \right]_a = U E_4(2a) - 6V E_5(2a) + 30W E_6(2a) - 140 X_3(a) E_7(2a)$$

$$\begin{aligned} \left[\frac{1}{4} \psi_0 + \frac{9}{20} \psi_1 + \frac{1}{4} \psi_2 + \frac{1}{20} \psi_3 \right]_a \\ = U E_5(2a) - 6V E_6(2a) + 30W E_7(2a) - 140 X_3(a) E_8(2a), \end{aligned}$$

where

$$U = \sum_{\ell=0}^3 (2\ell + 1) X_\ell(a)$$

$$V = \left[X_1 + 5 X_2 + 14 X_3 \right]_a \quad (15)$$

$$W = \left[X_2 + 7 X_3 \right]_a$$

The A's having been obtained, the flux at any point $a \leq x \leq X$ in the moderator can be obtained from its definition,

$$\begin{aligned} \rho(x) &= \int_{-1}^1 \psi(x, \mu) d\mu \\ &= \psi_0(x) + X_0(x) \end{aligned} \quad (16)$$

and Eqs. (11). The desired quantities for Eq. (1), however, are the average values of flux in foil and moderator. For the first, we use the

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178 011

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the method derived in Ref. 3, Eqs. (41) and (42), to give, in the double P_3 approximation,

$$\begin{aligned} \langle \rho \rangle_{\text{Foil}} = \frac{1}{a} \left\{ U \left[\frac{1}{2} - E_3(2a) \right] - 6V \left[\frac{1}{3} - E_4(2a) \right] \right. \\ \left. + 30W \left[\frac{1}{4} - E_5(2a) \right] - 140X_3(a) \left[\frac{1}{5} - E_6(2a) \right] \right\} . \end{aligned} \quad (17)$$

For the average value in the moderator, use Eqs. (11) with an integration,

$$\begin{aligned} \langle \rho \rangle_{\text{Mod}} &= \frac{1}{X-a} \int_a^X dx \left\{ \psi_0 + \chi_0 \right\} \\ &= 2C + 2 \sum_{i=2}^{L+1} \frac{(A_{oi} + B_{oi})}{\gamma_i (X-a)} \sinh \gamma_i (X-a) \\ &\quad + \frac{\lambda S_0}{2} \left[2X^2 + 2aX - a^2 + 3X \right] . \end{aligned} \quad (18)$$

III. RESULTS AND DISCUSSION

The two region uranium-teflon lattice specified in the introduction was investigated in the double P_1 , double P_2 and double P_3 approximations by use of the methods outlined above. The results may be summarized in the following table:

SECRET

178 012

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Quantity \ Method →	Double P ₁	Double P ₂	Double P ₃
$\langle \phi \rangle_U$	.5098 λs_0	.5098 λs_0	.5098 λs_0
$\phi(a)$	.5725 λs_0	.5887 λs_0	.5953 λs_0
$\langle \phi \rangle_{CF_2}$	.5947	.6264	.6447
$\phi(x)$	.6058 λs_0	.6450 λs_0	.6686 λs_0
$\frac{\langle \phi \rangle_U}{\phi(a)}$	.8906	.8660	.8563
$f = \frac{\langle \phi \rangle_U}{\langle \phi \rangle_{CF_2}}$	.8573	.8139	.7906

It then seems that the double P₃ value of f is good to 3% or better and may be compared to the more approximate values used in the multi-group calculations. The procedure which has been used in the past in the ANP Physics group is to use values of f based on KAPL-336. Those results, as has been observed elsewhere,³ really give the ratio of the average value of the flux in a foil of the same thickness to the flux at infinity in the infinite medium in which the foil is assumed to be embedded, if the infinite medium is a pure absorber containing a spatially uniform source of neutrons. For that dissimilar situation, the method of KAPL-336 gives

$$f \text{ (KAPL-336)} = 0.670,$$

which is about 15% lower than the double P₃ result, and probably at least 10% lower than the exact result for the lattice spacing of the second

assembly of the two-region reflector-moderated reactor. In the relevant criticality calculations, the KAPL-336 values of f were used, and it would therefore seem that the critical mass prediction should be too high, since the calculated flux level in the uranium is too low. However, the calculated critical masses actually tended to be low. (There is a certain amount of ambiguity in the proper choice of spherical radii to approximate the actual rhombicuboctahedra used in the experiments, but by no logical criterion was it possible to make a choice such that the calculated mass was greater than that obtained experimentally). This may be due in part to unknown errors in cross sections, etc., but it should also be emphasized that even the lattice picture used here is not sufficiently refined, since it is assumed here that the neutrons are thermalized in the teflon, while in the experimental arrangement they are thermalized for the most part in the beryllium reflector, and then diffuse into the region consisting of U - CF₂ lattices.

Another approximation, due to R. R. Bate, has been used in recent calculations for the three-region reactors. This consists of using the ratio of the average value of the flux in the foil to that at the edge, assuming an isotropic flux incident on the foil. An isotropic incident flux incident on the foil corresponds to a situation in which the foil is embedded in an infinite un-scattering medium in which there is a spatially uniform source of neutrons. The value of this ratio for the 4 mil foil is⁴

4. The relevant formula is given in Ref. 3, Eq. (50).

$$f(\text{Bate}) = 0.909,$$

which is about 15% higher than the double P_3 result for this lattice.

No complete multigroup calculations have as yet been performed using this ratio for the reflector-moderated reactions, but some three-group calculations with constants determined from multigroup calculations have been performed for a three-region reactor. In this case the calculated critical mass seems to be too high, which is again in a direction opposite to that expected from comparison of the result of the Bate formula and the double P_3 calculation. However, the uncertainty of approximating the rhombicuboctahedra by spheres again enters here, as well as the problems of cross sections and group constants, so this again may not be a good test of the validity of the lattice assumption used here. It should be mentioned, however, that the three region reactors assembled to date have been considerably thicker layers of teflon than the 0.173 in. used in the two-region assembly, so that a lattice calculation of the sort undertaken here would give an even smaller value of f , decreasing the accuracy of the Bate formula even further.

It should be noticed, however, that the Bate formula gives a fairly good (to ~6%) approximation to the quantity which it more nearly represents, the ratio of the average value of the flux in the foil to that at the edge. In a closely packed lattice, it is evident that more neutrons will impinge on the foil at oblique angles than perpendicularly, so this ratio should be lower than the Bate formula predicts. Even in this very tight lattice, however, this ratio falls only as far as to 0.856, and the depression in the

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178 015

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teflon accounts for more than half of the discrepancy between the Bate formula and the double P_3 result.

As noted above, however, the infinite lattice assumption made here is an oversimplification, especially inasmuch as the neutrons are actually predominantly thermalized in the beryllium reflector and then return to the region occupied by the lattice. It would seem that the neutron flux incident on the uranium foil nearest the beryllium reflector would be more nearly isotropic than is the case in the postulated infinite periodic lattice. However, after a very short distance of penetration into the fuel region (the unit cell of the lattice is ~ 0.45 mean free paths thick), the angular distribution of neutrons incident on a foil should be very nearly that found here, and our lattice calculation should be reasonably accurate for the central portion of the fuel region. From this reasoning, it would appear that the Bate formula and the method given here are both valuable as giving upper and lower bounds respectively to the correct value of f , but that the lattice calculation is more nearly correct.

A. Double P₃ Approximation for c = 1

The Eqs. (6) are for the double P₃ case:

$$(\gamma+1)A_0 - B_0 + \gamma A_1 = 0$$

$$-A_0 + (1-\gamma)B_0 + \gamma B_1 = 0$$

$$\gamma A_0 + 3(\gamma+2)A_1 + 2\gamma A_2 = 0$$

$$\gamma B_0 + 3(2-\gamma)B_1 + 2\gamma B_2 = 0$$

$$2\gamma A_1 + 5(2+\gamma)A_2 + 3\gamma A_3 = 0$$

$$2\gamma B_1 + 5(2-\gamma)B_2 + 3\gamma B_3 = 0$$

$$3\gamma A_2 + 7(2+\gamma)A_3 = 0$$

$$3\gamma B_2 + 7(2-\gamma)B_3 = 0$$

Evaluating the determinant of the coefficients gives the equation for γ^2 :

$$\gamma^2 (3\gamma^6 - 540\gamma^4 + 4060\gamma^2 - 4900) = 0.$$

Solving for the higher A's in terms of A₀:

$$A_1 = - \frac{\gamma(70 + 70\gamma + 13\gamma^2)}{(2+\gamma)(210 + 210\gamma + 25\gamma^2)} A_0$$

$$A_2 = \frac{7\gamma^2}{(210 + 210\gamma + 25\gamma^2)} A_0$$

$$A_3 = - \frac{3\gamma^3}{(2+\gamma)(210 + 210\gamma + 25\gamma^2)} A_0$$

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178 017

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Solving for the higher B's in terms of B_0 :

$$B_1 = - \frac{\nu(70 - 70\nu + 13\nu^2)}{(2-\nu)(210 - 210\nu + 25\nu^2)} B_0$$

$$B_2 = \frac{7\nu^2}{(210 - 210\nu + 25\nu^2)} B_0$$

$$B_3 = - \frac{3\nu^3}{(2-\nu)(210 - 210\nu + 25\nu^2)} B_0,$$

and expressing A_0 in terms of B_0

$$A_0 = \left\{ (1-\nu) - \frac{\nu^2(70 - 70\nu + 13\nu^2)}{(2-\nu)(210 - 210\nu + 25\nu^2)} \right\} B_0.$$

The formulas above have been evaluated numerically by Ann Forbes of Oak Ridge National Laboratory for the ν_i and the A_{mi} and B_{mi} in terms of A_{oi} , and the results are listed in Table A-1 below (note that $\nu_1 = 0$, so that higher coefficients vanish).

B. Double P_2 Approximation for $c = 1$

In this case we need only the first six of the Eqs. (6), with $A_3 = B_3 = 0$.

The determinant of these yields:

$$\nu^2 (\nu^4 - 60\nu^2 + \frac{400}{3}) = 0.$$

The relations for the A_i and B_i are:

Table A-1DOUBLE P_3 COEFFICIENTS

	i = 2	i = 3	i = 4
v_i	1.2272139	2.5096078	13.119957
A_{1i}/A_{oi}	- 0.57971702	- 0.73443467	- 0.93235818
A_{2i}/A_{oi}	- 1.0475784	- 0.27629553	0.68534494
A_{3i}/A_{oi}	- 0.71296914	0.58313197	- 0.34654667
B_{oi}/A_{oi}	0.48422255	0.33353538	0.11029067
B_{1i}/A_{oi}	0.063939317	0.067970122	0.042477239
B_{2i}/A_{oi}	0.010101302	0.016439380	0.018283338
B_{3i}/A_{oi}	0.0016462396	0.0039208093	0.0067992429

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178 019

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$$A_1 = - \frac{5\nu(2+\nu)}{60 + 60\nu + 11\nu^2} A_0$$

$$A_2 = \frac{2\nu^2}{60 + 60\nu + 11\nu^2} A_0$$

$$A_0 = B_0 \left\{ 1 - \nu - \frac{5\nu^2(2-\nu)}{60 - 60\nu + 11\nu^2} \right\}.$$

$$B_1 = - \frac{5\nu(2-\nu)}{60 - 60\nu + 11\nu^2} B_0$$

$$B_2 = \frac{2\nu^2}{60 - 60\nu + 11\nu^2} B_0$$

The formulas above have also been evaluated numerically by Ann Forbes, and are listed in Table A-2 below.

C. Double P₁ Approximation

The formulas for this case are given in Ref. 3, but the numerical values will be repeated here for completeness:

$$\nu_1 = 0$$

$$\nu_2 = \sqrt{12} = 3.464102$$

$$A_{12-} = -0.7886751 A_{02-}$$

$$B_{02-} = 0.2679492 A_{02-}$$

$$B_{12-} = 0.05662432 A_{02-}.$$

Table A-2DOUBLE P_2 COEFFICIENTS

	$i = 2$	$i = 3$
γ_i	1.5202808	7.5953106
A_{1i}/A_{oi}	- 0.62946730	- 0.88961221
A_{2i}/A_{oi}	- 0.79793937	- 0.48303887
B_{oi}/A_{oi}	0.43668629	0.16157075
B_{1i}/A_{oi}	0.066153078	0.051183286
B_{2i}/A_{oi}	0.011427640	0.016205957

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