1 of 1

Some Remarks About Pseudo-Hamiltonian

N. Malitsky and G. Bourianoff

Superconducting Super Collider Laboratory*
2550 Beckleymeade Avenue
Dallas, Texas 75237

and

Yu Severgin

D. V. Efremov Scientific Research Institute
189631, St. Petersburg, Russia

November 1993

*Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

MASTER

1.0 INTRODUCTION

For the many applied tasks of accelerator physics, the 4D single particle pseudo-Hamiltonian may be presented as the Hamiltonian of the near-integrable system consisting of integrable and perturbed terms. The KAM theorem states that for sufficiently small perturbation the invariant surfaces continue to exist and, for the systems with two degrees of freedom, completely isolate the thin stochastic layers.¹ As the perturbation strength increases, a transition can occur in which these surfaces disappear and the stochastic layers connect, resulting in globally stochastic motion. One of the important problems is to determine this “boundary” invariant surface. There are several approaches that may be used to describe the regular trajectories in the small limited region. The most powerful method is the perturbation theory which allows us to study the combined influence of the different multipoles. The inclusion of Lie operators improved this method and developed it up to high order perturbation. But the perturbation theory failed to describe the change in topology and since the regular trajectories depend discontinuously on choice of initial coordinates, it cannot be used in the whole region of the stable motion. We suggest to limit the attention to the study of the “boundary” invariant and implement the additional “local” transformation. In Section 2, we briefly review the well known theories, their advantages and imperfections, and the necessity of the “local” transformation. In Section 3, we present the comparison of the map tracking with the invariants determined by the perturbation methods.

2.0 PSEUDO-HAMILTONIAN

The pseudo-Hamiltonian of single-particle 4-dimensional motion in modern circular accelerators can be presented as the sum of linear H_0 and perturbed V_0 terms:²

$$H^{(0)} = H_0 + V_0(\theta), \quad (1)$$

where in the usual action-angle variables $(J_{0x}, J_{0y}, \phi_{0x}, \phi_{0y})$

$$\begin{aligned} H_0 &= \mu_{0x} J_{0x} + \mu_{0y} J_{0y}, \\ V_0(\theta) &= \sum_{N=3}^{\infty} V_0^{(N)}(\vec{J}_0, \vec{\phi}_0, \theta), \\ V_0^{(N)}(\vec{J}_0, \vec{\phi}_0, \theta) &= \sum_{m=0}^N J_{0x}^{m/2} J_{0y}^{(N-m)/2} v_m^{(N)}(\vec{\phi}_0, \theta). \end{aligned}$$

These perturbed terms V_0 arise from nonlinear multipole fields distributed around the lattice. Unlike the linear terms, they keep their θ dependence even after the usual canonical transformation and $H^{(0)}$ cannot be used as the invariant of the motion.

Now there are two basic approaches to remove the θ dependence of the pseudo-Hamiltonian and to derive this invariant:

1. Perturbation theory.
2. Single-resonance theory.

At the center of canonical perturbation theory is a generating function that should transport the θ dependence of the pseudo-Hamiltonian to higher order terms:

$$F_n(\vec{J}_n, \vec{\phi}_{n-1}, \theta) = \vec{J}_n \vec{\phi}_{n-1} + \chi_n(\vec{J}_n, \vec{\phi}_{n-1}, \theta), \quad (2)$$

where χ_n is small and satisfies the condition:

$$\vec{\mu}_n(\vec{J}_n) \frac{\delta \chi_n}{\delta \vec{\phi}_{n-1}} + \frac{\delta \chi_n}{\delta \theta} + V_{n-1}^{(N_0)}(\vec{J}_n, \vec{\phi}_{n-1}, \theta) = 0, \quad (3)$$

where n is the order of the canonical transformation, $(\vec{J}_n, \vec{\phi}_n)$ and $(\vec{J}_{n-1}, \vec{\phi}_{n-1})$ are new and old action-angle variables, and N_0 is the lowest order of θ -dependent perturbation terms. Now tune depends on the action variable and may be presented like:

$$\vec{\mu}_n(\vec{J}_n) = \frac{\delta H_n(\vec{J}_n)}{\delta \vec{J}_n}, \quad (4)$$

and

$$H_n(\vec{J}_n) = H_0(\vec{J}_n) + \sum_{N=4,6,\dots}^{N_0} V_n^{(N)}(\vec{J}_n)$$

$$V_n^{(N)} = \sum_{m=0}^N v_{N,m} J_{nx}^{m/2} J_{ny}^{(N-m)/2}.$$

To find a solution of Eq. (3), χ_n and $V_{n-1}^{(N_0)}(\theta)$ should be expanded into a Fourier series:

$$V_{n-1}^{(N_0)}(\vec{J}_n, \vec{\phi}_{n-1}, \theta) = \sum_{l_x, l_y, p} V_{n-1; l_x l_y p}^{(N_0)}(\vec{J}_n) \exp(i[l_x \phi_{n-1; x} + l_y \phi_{n-1; y} - p(\theta - \theta_0)]) \quad (5)$$

and by substituting them in Eq. (3) we can derive the expression for Fourier component χ_n :

$$\chi_{n; l_x l_y p} = i \frac{V_{n-1; l_x l_y p}^{(N_0)}(\vec{J}_n)}{l_x \mu_{nx}(\vec{J}_n) + l_y \mu_{ny}(\vec{J}_n) - p}. \quad (6)$$

After canonical transformation, new pseudo-Hamiltonian $H^{(n)}$ becomes an invariant of motion up to N_0 order:

$$H^{(n)} = H^{(n-1)} + \frac{\delta \chi_n}{\delta \theta} = H_0(\vec{J}_n) + V_n^{N \leq N_0}(\vec{J}_n) + V_n^{N > N_0}(\vec{J}_n, \vec{\phi}_n, \theta), \quad (7)$$

where

$$\vec{J}_{n-1} = \vec{J}_n + \frac{\delta}{\delta \vec{\phi}_{n-1}} \chi_n(\vec{J}_n, \vec{\phi}_{n-1}, \theta).$$

This approach has a very important restriction. Obviously it can be used only far from the resonant condition $l_x \mu_{nx} + l_y \mu_{ny} = p$, when the dominator in Eq. (6) is not small. Near resonances, the regular solutions are strongly perturbed and undergo a change in topology. For this case we must use a different generating function from single-resonance theory:

1. For 2-dimensional motion:

$$F_1(J_1, \phi_0, \theta) = (\phi_0 - \frac{p}{l} \theta) J_1 \quad (8)$$

to find the Hamiltonian

$$H^{(1)} = (\mu_0 - \frac{p}{l}) J_1 + V_1^{(N < N_0)}(J_1) + V_{1;l p}^{(N = N_0)}(J_1) \exp(il\phi_1) + \text{small high-frequency terms.} \quad (9)$$

2. For 4-dimensional motion:

$$F_1(\vec{J}_1, \vec{\phi}_0, \theta) = (l_x \phi_{0x} + l_y \phi_{0y} - p(\theta - \theta_0)) J_{1x} + \phi_{0z} J_{1y} \quad (10)$$

to define two invariants $H^{(1)}$ and J_{1y}

$$H^{(1)} = (l_x \mu_{0x} + l_y \mu_{0y} - p) J_{1x} + \mu_{0y} J_{1y} + V_1^{(N < N_0)}(\vec{J}_1) + V_{1;l_x l_y p}^{(N = N_0)}(\vec{J}_1) \exp(i\phi_{1x}) + \text{small high-frequency terms.} \quad (11)$$

Unfortunately, this approach describes the topology of single resonance for the multipole of the same order and does not determine the influences of lower order multipoles. Then the perturbation and single-resonance theories should be used together. The standard Hamiltonian formalism is very cumbersome and usually is used only up to the second order.

This restriction is removed by the symplectic maps and Lie transformations.^{3,4} The symplectic one-turn map $\mathcal{M}$ for the Hamiltonian which does not explicitly depend on the time can be written in the form:

$$\mathcal{M} = \exp(-2 * \pi : H :) \quad (12)$$

where the pseudo-Hamiltonian H is invariant and satisfies the following condition:

$$H(\mathcal{M} \vec{z}_0) = H(\vec{z}_0), \quad (13)$$

for arbitrary initial coordinates $\vec{z}_0 = (x_0, p_{x0}, y_0, p_{y0})$. Here and farther we will use only the normalized phase space coordinates. For the concrete map $\mathcal{M}$ it is possible to derive this pseudo-Hamiltonian H using the special canonical generator $\mathcal{G}_n(\vec{z})$:

$$\exp(: -2 * \pi * H_n(\vec{J}_0) :) = \mathcal{G}_n(\vec{z}) \mathcal{M} \mathcal{G}_n^{-1}(\vec{z}) + \sigma(n+2), \quad (14)$$

where:

$$\mathcal{G}_n^{-1}(\vec{z}) = \prod_{i=3}^{n+2} \exp(: -F_i(\vec{z}) :),$$

$$\vec{J}_n = \mathcal{G}_n^{-1}(\vec{z}) \vec{J}_0,$$

F_i may be defined from the next equation:

$$F_i(\vec{z}) = \sum_{l+m+n+o=i} \frac{c_{lmno}}{1 - \exp(i[(l-m)\mu_{0x} + (n-o)\mu_{0y}])} \hat{x}_+^l \hat{x}_-^m \hat{y}_+^n \hat{y}_-^o \quad (15)$$

where

$$\hat{x}_{\pm} = \frac{1}{\sqrt{2}}(x \pm ip_x),$$

$$\hat{y}_{\pm} = \frac{1}{\sqrt{2}}(y \pm ip_y).$$

In comparison with Eq. (6), the expression, Eq. (15), for F_i does not include the nonlinear tunes shift directly in the dominator. It means that this canonical transformation may be used only in the limited vicinity of the working point (μ_{0x}, μ_{0y}) when this nonlinear tunes shift is less than the distance to the nearest dominant resonance. In reality, the region of stable motion may be larger and more complicated. We suggest to study the behavior of the pseudo-Hamiltonian in the vicinity of the definite pair of tunes $(\tilde{\mu}_x, \tilde{\mu}_y)$. Then we apply the additional "local" transformation $\mathcal{L}_n$ and use the standard approach for new object $\tilde{\mathcal{M}}$:

$$\tilde{\mathcal{G}}_n^{-1} \mathcal{L}_n \tilde{\mathcal{G}}_n = \exp(: (-\tilde{\mu} \vec{J}_n + 2 * \pi * \tilde{H}_n(\vec{J}_n) + \text{const})/2. :), \quad (16)$$

$$\tilde{\mathcal{M}} = \tilde{\mathcal{G}}_n^{-1} \mathcal{L}_n \tilde{\mathcal{G}}_n \mathcal{M} \tilde{\mathcal{G}}_n^{-1} \mathcal{L}_n \tilde{\mathcal{G}}_n,$$

$$\mathcal{M} = \exp(-2 * \pi : \tilde{H}(\vec{J}_n) :). \quad (17)$$

It allows us to linearize the tunes shift and include it in Eq. (15). In the next section we present the analytical calculations of pseudo-Hamiltonians H_6 , Eq. (14) and $\tilde{H}_6$, Eq. (17), for the simple nonlinear object and compare them with the map tracking.

3.0 EXAMPLES

For the numerical experiment we chose the standard nonlinear object which consists only of the linear uncoupling matrix and the thin sextupole. Zlib library was used for the 2nd map tracking and order-by-order normalization.⁵

3.1 One Degree of Freedom

Our attention was localized on the study of the influence of the nonlinear tunes shift on regular trajectories in the vicinity of 4th and 5th order resonances. Next, we considered two corresponding pairs of tunes (0.245, 0.259) and (0.203, 0.22).

The results of the 2nd-order map tracking for work tunes (0.245, 0.259) are shown on the Figure 1. The particle with the initial coordinate (1.8, 0., 0., 0.) is unstable. Using the order-by-order normalization we constructed the pseudo-Hamiltonian H_n , Eq. (14), and defined its levels (Figure 2a). In this case the incorrect inclusion of the nonlinear tunes shift causes the stable region to decrease. We suggest to determine it from the map tracking using the standard Fast Fourier transform method and then include in the "local" transformation, Eq. (17), as $\tilde{\mu}$. It allows us to study the behavior of the regular trajectories near this tune with good accuracy. The pseudo-Hamiltonian $\tilde{H}_n$ constructed by the 8th-order canonical generator and "local" transformation describes the motion of particle with initial coordinate (1.7, 0., 0., 0.) with the accuracy of 2.6% (Figure 2b).

Figure 1. Horizontal normalized phase space diagram for 2nd-order map tracking with the initial coordinates (1.4, 0., 0., 0.) and (1.7, 0., 0., 0.). Work tunes (0.245, 0.259).

Figure 2. Levels of pseudo-Hamiltonian constructed by the 8th-order canonical generator (a) and the 8th-order additional "local" transformation (b). Work tunes (0.245, 0.259).

A similar result may be obtained for 5th-order resonance. Figure 3 shows the 2nd-order map tracking for working point (0.203, 0.22). The islands restrict the region where the standard approach may be applied (Figure 4a). In reality, the stable region is larger and the tune for the particle with the initial coordinate (1.8, 0., 0., 0.) equals to 0.183. The pseudo-Hamiltonian $\tilde{H}_n$, Eq. (16), describes this regular trajectory with an accuracy of 1.8% (Figure 4b).

Figure 3. Horizontal normalized phase space diagram for 2nd-order map tracking. Work tunes (0.203, 0.23).

Figure 4. Levels of pseudo-Hamiltonian constructed by the 8th-order canonical generator (a) and the 8th-order additional "local" transformation (b). Work tunes (0.203, 0.22).

3.2 Two Degrees of Freedom

To include the coupled nonlinear motion we used the same work tunes but changed only the vertical coordinate. Follow the previous technique for the fixed particles we built the invariants $\vec{J}_n$ and $\vec{\tilde{J}}_n$ and compared them with the 1000-turns tracking (Figure 5 and Figure 6). Here it is very important to notice that the series this perturbation approaches are not convergent. Since the resonances are dense in the frequency space, new resonances are continually encountered as the new orders of perturbation are included, leading to series expansions that are at best asymptotic. To check the convergence we calculated the average values $\bar{J}$ and the relative mean absolute deviation $\Delta\bar{J}$ for the several order transformations:

$$\bar{J}_z = \frac{1}{1000} \sum_{i=1}^{1000} J_z(x_i, px_i, y_i, py_i), \quad (18)$$

$$\Delta J_z = \frac{\sum_{i=1}^{1000} |J_{iz} - \bar{J}_z|}{1000 * \bar{J}_z}.$$

The tuneshift was calculated as:

$$tuneshift_z = \frac{\delta H}{\delta J_z}. \quad (19)$$

Figure 5. Normalized phase space diagram for 2nd-order map tracking with the initial coordinates (0.5, 0., 0.4, 0.); (a) horizontal plane, (b) vertical plane. Work tunes (0.245, 0.259).

Figure 6. Normalized phase space diagram for 2nd-order map tracking with the initial coordinates $(-0.7, 0., 0.4, 0.)$; (a) horizontal plane, (b) vertical plane. Work tunes $(0.203, 0.22)$.

These results are presented in Tables 1 and 2. The numerical label near the invariant means the number of the canonical transformation, Eq. (14). The tables show the good agreement between the analytic calculations and the map tracking for the “local” transformation. The fast expansion of the usual method in the 6th order (Table 1) is caused by the influence of 8th order resonance $5\mu_x + 3\mu_y = 2$ which really is farther for this initial coordinate.

4.0 Conclusion

Results from Section 3 show the restrictions of the standard method. The formal reason is the incorrect inclusion of the nonlinear tunes. In reality, this is a more difficult problem and reflects the complicated relationship between the topology of regular trajectories and the initial conditions. We suggest focusing on the behavior of the “local” invariants in the vicinity of the fixed tune, which allows us to improve the series convergence and describe regular trajectories with good accuracy.

Table 1. Numerical comparison of the invariants with the 1000-turn tracking for the particle with the initial coordinate (0.5, 0., 0.4, 0.) (Figure 5). Tunes shift was calculated by 10 000-turn FFT equals (-0.0001, 0.0026). Work tunes (0.245, 0.259).

	Nonlinear Normal Forms	Nonlinear Normal Forms Plus "Local" Transformation
$\bar{J}_{2x}$	0.152	0.154
ΔJ_{2x}	0.13	0.085
$\bar{J}_{2y}$	0.062	0.065
ΔJ_{2y}	0.23	0.15
hor. tunes shift	-2.4E-04	-1.6E-04
ver. tunes shift	2.8E-03	2.9E-03
$\bar{J}_{4x}$	0.171	0.159
ΔJ_{4x}	0.096	0.035
$\bar{J}_{4y}$	0.079	0.070
ΔJ_{4y}	0.17	0.044
hor. tunes shift	-6.9E-04	-4.4E-04
ver. tunes shift	2.0E-03	2.3E-03
$\bar{J}_{5x}$	0.167	-
ΔJ_{5x}	0.078	-
$\bar{J}_{5y}$	0.076	-
ΔJ_{5y}	0.13	-
hor. tunes shift	-6.7E-04	-
ver. tunes shift	2.0E-03	-
$\bar{J}_{6x}$	0.133	0.159
ΔJ_{6x}	0.22	0.027
$\bar{J}_{6y}$	0.042	0.067
ΔJ_{6y}	0.65	0.030
hor. tunes shift	-3.3E-04	-9.4E-05
ver. tunes shift	2.5E-03	2.6E-03

Table 2. Numerical comparison invariants with the 1000-turn tracking for the particle with the initial coordinate $(-0.7, 0., 0.4, 0.)$ (Figure 6).
Tuneshift calculated by 10 000-turn FFT equals $(-0.0057, 0.0056)$.
Work tunes $(0.203, 0.22)$.

	Nonlinear Normal Forms	Nonlinear Normal Forms Plus "Local" Transformation
$\bar{J}_{4x}$	0.337	0.338
ΔJ_{4x}	0.11	0.033
$\bar{J}_{4y}$	0.045	0.065
ΔJ_{2y}	0.23	0.15
hor. tuneshift	$-6.1\text{E}-03$	$-5.7\text{E}-03$
ver. tuneshift	$4.6\text{E}-03$	$5.5\text{E}-03$
$\bar{J}_{6x}$	0.350	0.340
ΔJ_{6x}	0.20	0.009
$\bar{J}_{6y}$	0.038	0.042
ΔJ_{6y}	0.22	0.016
hor. tuneshift	$-5.6\text{E}-03$	$-5.4\text{E}-03$
ver. tuneshift	$6.4\text{E}-03$	$5.8\text{E}-03$

ACKNOWLEDGEMENTS

We are grateful to Dr. Yiton Yan for providing us with the code ZLIB without which this paper could not have been made. We would also like to thank Prof. Richard Talman and Dr. Yuriy Senechev for many useful discussions.

REFERENCES

1. A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion*, Springer-Verlag, New York Inc. (1983).
2. E. J. N. Wilson, "Nonlinear Resonances," CERN 87-03 (1987).
3. E. Forest, M. Berz, and J. Irwin, "Normal Form Methods for Complicated Periodic System," *Particle Accelerators*, **24**, **91** (1989).
4. Y. Yan, "Applications of Differential Algebra to Single-Particle Dynamics in Storage Rings," SSC Laboratory Report SSCL-500 (1991).
5. Y. Yan and Chiung-Ying Yan, "ZLIB - A Numerical Library for Differential Algebra," SSC Laboratory Report SSCL-300 (1990).

**DATE
FILMED**

1 / 7 / 94

END