THE PROSAIC LAPLACE AND FOURIER TRANSFORM*

G. A. Smith
Brookhaven National Laboratory, Upton NY

ABSTRACT

Integral Transform methods play an extremely important role in many branches of science and engineering. The ease with which many problems may be solved using these techniques is well known. In Electrical Engineering especially, Laplace and Fourier Transforms have been used for a long time as a way to change the solution of differential equations into trivial algebraic manipulations or to provide alternate representations of signals and data. These techniques, while seemingly overshadowed by today's emphasis on digital analysis, still form an invaluable basis in the understanding of systems and circuits. A firm grasp of the practical aspects of these subjects provides valuable conceptual tools.

This tutorial paper is a review of Laplace and Fourier Transforms from an applied perspective with an emphasis on engineering applications. The interrelationship of the time and frequency domains will be stressed, in an attempt to comfort those who, after living so much of their lives in the time domain, find thinking in the frequency domain disquieting.

INTRODUCTION

Laplace and Fourier Transforms are, as Nahin [1] suggests, the mathematical signature of the modern electrical engineer. Even before the development of a rigorous mathematical structure, and in the face of heavy criticism from pure mathematicians, Oliver Heaviside developed his Operational Analysis, a true predecessor of the Laplace Transform applied to electric circuits. In the preface to his book, Network Analysis [2], Van Valkenberg states that;

"The modern EE must be bilingual when speaking of network response. He must speak the language of the time domain and must also be trained in the language of the frequency domain. He may think in either language, but must be able to translate from one to the other at a moment's notice."

In a recent conversation with a vendor of wide bandwidth RF power amplifiers I asked about the pulse response of one of their units. After a long

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pause they said that they simply did not know. No such data had ever been taken. Since this amplifier was being considered for our wide band damper system in the AGS, I felt that it was important to know that pulses could be amplified by the unit with reasonable fidelity so I asked them if they might supply with some Bode plots of the unit thinking that I might infer a pulse response from that. They said they had no such thing. They could talk about gain flatness and maximum phase deviation in the pass band and 3dB points and such but why anyone would even be interested in pulse response seemed to be a mystery to them. In the end we borrowed one of their units and looked at the pulse response ourselves.

In another conversation with an engineer who was designing a high frequency amplifier, I asked how the circuit he was building responded to very fast pulses. "I really don't know", he said, "I've done all my testing using the network analyzer. Never looked at that!" On the other hand, one RF engineer I met recently said something like, "Yeah, every time I say db you can see their eyes get glassy".

In no way do I mean these comments as criticisms of the people involved. But perhaps it is clearer what the motivation for this tutorial paper was. It has begun to become apparent to me that within this world of specialists, myself included, stuck in one domain or the other, we are perhaps overdue for a review of the ways in which we are related. Its a big subject, and I know that what follows does not completely fill in every detail, in fact it only scratches the surface, but I hope that this review does stimulate the curious to further thought.

INTEGRAL TRANSFORMS

An Integral Transform $f^T(q)$, of a function $f(q)$ is defined by the integral;

$$f^T(p) = \int_{\Omega} K(p,q) f(q) dq \quad (1)$$

Where $K(p,q)$ is a chosen function of p and q , and is called the *kernel* of the transform, and $f^T(q)$ is called the *transform* of $f(q)$. The symbol Ω indicates the domain over which the integral is to be evaluated. This domain can be finite or, as in the cases we will study here, one or both of the limits of integration can be infinite.

Now it is implied in mathematical treatments of transforms that there exists an inversion formula, that has the form

$$f(q) = \int_0 K_1(p, q) f^T(p) dp \quad (2)$$

This inverse "undoes" the first operation in the sense that it gives us back the original function. There is a mapping of one function space onto the transform function space that preserves "almost" everything about the original function yet puts it into a different, sometimes hardly recognizable form. In much the same way we can expand a periodic function $f(t + nT)$ into a Fourier series. The series expansion does not particularly resemble the original function, yet they are the same.

Common examples of integral transforms are [4,5],

The Laplace transform (where p is complex, q is real)

$$f^T(p) = \int_{-\infty}^{\infty} e^{-pq} f(q) dq \quad (3)$$

The Hankel transform (of order n , where p and q are both real)

$$f^T(p) = \int_0^{\infty} q J_n(pq) f(q) dq \quad (4)$$

The Mellin transform

$$f^T(p) = \int_{-\infty}^{\infty} q^{p-1} f(q) dq \quad (5)$$

The Fourier sine transform

$$f^T(p) = \int_0^{\infty} \sin(pq) f(q) dq \quad (6)$$

The complex Fourier transform

$$f^T(p) = \int_{-\infty}^{\infty} e^{-ipq} f(q) dq \quad (7)$$

The transforms shown change the initial function $f(q)$, which may be real or complex, into a new function in another real or complex variable. The result of

this "transformation" is to simplify the original function with respect to some operation. In cases involving partial differentiation for example one variable can be removed. In the case of the Laplace and Fourier Transforms, which we will focus on here, a differential equation of order n reverts to a polynomial of the same order making the problem of finding a solution an algebraic one.

Of course implicit in these transform approaches is the fact that once the "solution" is found in the transform space, it must be possible to revert to the original problem space by "inverting" the result. This may not be trivial.

THE LAPLACE TRANSFORM

In most texts on this subject, the Fourier Transform is introduced first. In reality the Fourier Transform is younger than the Laplace Transform by about 30 years. Perhaps the reason for introducing it first is because to truly appreciate the Laplace Theory requires a greater understanding of complex analysis than does Fourier Analysis. In addition the development of Fourier Analysis from Fourier Series may give one a better sense of connection to the idea of a spectrum and hence a feeling of still being tethered to reality. As an EE student however, I was mercilessly thrown the equations below, with no initial knowledge of how they might relate to Fourier Series (Fourier Transforms had never been discussed) and it was not until much later that I learned that there was a connection. We will return to this.

The Laplace Transform of a function of time $f(t)$ is defined as,

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt \quad (8)$$

This is the *bilateral* Laplace Transform. In practice, unlike the Fourier Transform which we will consider later, the Laplace Transform is most often used with time dependant signal functions which are defined for time $t = 0^+$ often written as $f(t) = u(t) f(t)$ where $u(t)$ is the unit step function. This allows us to write equation (8) as,

$$F(s) = \int_{0^+}^{\infty} f(t) e^{-st} dt \quad (9)$$

and this is called the *unilateral* transform.

The inversion formula is

$$f(t) = \frac{1}{2\pi i} \oint_{-\infty}^{\infty} F(s) e^{st} ds \quad (10)$$

where $s = \sigma + i\omega$, the general complex variable. As is often the case with a terse mathematical formula as shown above, much is implied.

Now it is not obvious under what the conditions the integral in (8) exists. The question is one of whether the integral converges as the limits approach infinity. This will surely depend on the choice of $f(t)$. It can be shown that, in general, convergence of the upper limit is assured so long as $f(t)$ is piecewise continuous and if for some complex number s_0 ,

$$\int_0^{\infty} f(t) e^{-s_0 t} dt \quad (11)$$

converges, the integral then converges for all $\text{Re } s > \text{Re } s_0$ [4]. In addition convergence of the lower limit is assured if

$$\int_0^{\infty} f(-t) e^{s_1 t} dt \quad (12)$$

converges, then convergence is assured for all $\text{Re } s < \text{Re } s_1$. Provided $\text{Re } s_1 < \text{Re } s_0$. So for our function $f(t)$, a vertical strip is defined in the s -plane within which (8) exists. This is called the strip of convergence of $F(s)$.

Now the next step is to realize that the function, $F(s)$, found by performing the integration in (8) will be *regular* in its strip of convergence. By the principal of *analytic continuation* then, we know that this is sufficient for the function to exist everywhere outside of the strip of convergence. This may seem like black magic, but it is the result of some powerful theorems from complex function theory.[4,8,9]

Of course we still have to face the problem of finding the inverse, that is, we must solve (10) to recover $f(t)$ from our $F(s)$ if we are truly to have something worthy of being called a transform.

Now it might be more enlightening to write the inversion formula as.

$$f(t) = \frac{1}{2\pi i} \oint_{c-i\infty}^{c+i\infty} F(s) e^{st} ds \quad (13)$$

We need to remember that integration of a function of a complex variable is defined for contours in the s -plane and that the integral of an *analytic* function is independent of the contour we choose. The limits of integration shown above highlight the fact that in evaluating the integral we need to pay attention to the contour in the complex plane (sometimes referred to as a Bromwich contour) if the inverse integral is going to exist. What we must insure here is that the contour we

choose to integrate over in the s -plane is inside the region of convergence of the Laplace integral of $f(t)$. This information is not contained in $F(s)$ and must be known by some independent means.

To evaluate the integral, we first do a partial fraction expansion and then invoke Cauchy's theorem of residues. We can use any contour we wish, but since the regions of convergence are all vertical strips a vertical line inside this strip makes the most sense. Using contour 1 in figure (1) and letting R become infinite, will give us $f(t)$ for $t > 0$ and using contour 2 gives us $t < 0$. This concept shows why the bilateral Laplace Transform is not too useful. Since we must have independent knowledge of the strip of convergence to uniquely accomplish the inversion, and since it is conceivable that after some algebraic manipulations in the Laplace domain this information is obscured, then the bilateral case may not give us a clearly unique result. On the other hand the unilateral case always has all the singularities to the left of some real number and so the strip of convergence always occupies the entire right half plane above some $\text{Re } s > 0$. Moving our contour over to the right sufficiently far (so that all the singularities of $F(s)$ are to the left) gives us a unique result.

Table 1 - Some Laplace Transform Pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{-at} u(t)$	$\frac{1}{s+a}$
$\sin(bt) u(t)$	$\frac{b}{s^2 + b^2}$
$\cos(bt) u(t)$	$\frac{s}{s^2 + b^2}$
$e^{-at} \sin(bt) u(t)$	$\frac{b}{(s+a)^2 + b^2}$

But we are drawn into a strange world with this definition. So far this is strictly a mathematical statement, and the fact that $F(s)$ is a complex function of a complex variable leads to the question of whether or not any real physical meaning can be attached to the operations here. As we continue we will try to understand what this all means in applies to physical reality.

Engineers or system analysts rarely calculate a Laplace Transform from (9). For all practical purposes tables of transforms exist that contain almost all the results anyone faced with a practical problem would ever need. Listed in Table 1 are a few important ones. We will need to refer to these later. In addition to the transform pairs in Table 1, it is helpful to know how certain operations in the time domain are reflected in the complex frequency domain. Some are shown in Table 3, and these are easily proved.

THE FOURIER TRANSFORM

The Fourier Transform of a function of time, $f(t)$ may be defined as

$$F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt \quad (14)$$

And its inversion formula as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dw \quad (15)$$

This is the bilateral Laplace Transform with $s = iw$, that is, $\sigma = 0$, so long as the imaginary axis lies within the region of convergence of the Laplace integral. The function $F(w)$ will, in general, be complex, that is it will have the form

$$F(w) = R(w) + iX(w) \quad (16)$$

where $R(w)$ and $X(w)$ are *real* functions of the *real* variable w . This may appear at first to make it less general than the Laplace Transform, but when it comes to the inversion integral the requirement that (13) possesses a known strip of convergence is actually more restrictive than that (15) exists for all real w .

Now as in the Laplace Transform we are faced with the problem of the existence of an integral, in this case (14). In order that the integral in (14) converge, one set of conditions on $f(t)$, known as the Dirichlet conditions: (1) the function is continuous almost everywhere; (2) at any point of discontinuity t_0 , that

$f(t_0 +)$ and $f(t_0 -)$ exist; (3) The function $f(t)$ is absolutely integrable. that is;

$$\int_{-\infty}^{\infty} |f(t)| dt \quad (17)$$

exists. These requirements are, simply stated, that physically realistic signals always possess a Fourier Transform. It must also be pointed out that they are sufficient, not necessary. The function

$$\frac{\sin(wt)}{t} \quad (18)$$

is not absolutely integrable but has a Fourier Transform. On the other hand, $\sin(t)$ and the unit step, $u(t)$ do not, strictly speaking, have Fourier Transforms. Neither does DC. Dirac delta function is a question mark.

Table 2 - Some Fourier Transform Pairs

$f(t)$	$F(w)$
$\delta(t)$	1
$u(t)$	$\pi \delta(w) + \frac{1}{iw}$
$\cos w_0 t$	$\pi [\delta(w - w_0) + \delta(w + w_0)]$
$u(t+1) - u(t-1)$	$2 \frac{\sin(w)}{w}$
$e^{-a t }$	$\frac{2a}{w^2 + a^2}$

I will not deal in detail with the delta function here, only say that it has always seemed a shame to me that a such an important piece of applied mathematics is still on such thin ice. I believe that by now the mathematical world has managed to embrace this entity under the theory of distributions or in linear operational calculus where it is treated nicely as an ideal point added to a space of ordinary functions. The problem is, however, as it seems always to have been, that the "function" as treated in these mathematical formalisms, seems to be bereft

of a real physical meaning. Anyway, we will go on to use it as usual, and to assume that a large, short, pulse is a good approximation to the mathematical entity. Those who are interested can read further [6,7,8].

Table 2 shows some typical and interesting Fourier Transform pairs. Note that these pairs are for the bilateral case. For the case where we have an $f(t)$ which is limited to $t > 0$, that is, $f(t)u(t)$ the pairs become those in Table 1 substituting $s = iw$.

OPERATOR - TRANSFORM PAIRS

No discussion of Transforms would be complete without at least a look at how some fundamental operations, for example differentiation, integration, etc., are reflected in each domain. These relationships form the basis for understanding the way some behavior or characteristic in one domain is mirrored in the other. They are the result of many of the theorems about the transform, form the basis for deriving new transform pairs, and provide the justification for other conclusions.

Table 3 Operator - Transform Pairs for Laplace and Fourier Transforms

Operation	$f(t)$	$F(s)$	$F(w)$
Linearity	$af_1(t) \pm bf_2(t)$	$aF_1(s) \pm bF_2(s)$	$aF_1(w) \pm bF_2(w)$
Differentiation	$\frac{d^n f(t)}{dt^n}$	$s^n F(s)$	$(iw)^n F(w)$
Integration	$\int f(t) dt$	$\frac{F(s)}{s} + \frac{f^{(-1)}(0+)}{s}$	$\frac{F(w)}{iw}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) F_2(s)$	$F_1(w) F_2(w)$

LINEAR SYSTEMS

Laplace and Fourier Transforms are particularly powerful when applied to the analysis of time invariant linear systems. *Linear systems* are those in which, given a system response described by the operator T , two inputs $i_1(t)$ and $i_2(t)$, and

arbitrary constants a and b , then,

$$T[a i_1(t) + b i_2(t)] = a T[i_1(t)] + b T[i_2(t)] \quad (19)$$

Time invariance is a characteristic of a system with constant, parameters and states that the same input produces the same output no matter when, in time, it is applied. This is a fundamental feature of most analog circuits with fixed components, for example.

Now the following shows the impact of these two conditions. Let $h(t)$ be the response of the system above to a unit impulse $\delta(t)$. (In fact we will reserve the notation $h(t)$ and its corresponding transforms, $H(s)$ and $H(w)$ as call them the system response, system function or impulse response interchangeably.) Time invariance assures that,

$$h(t - t_0) = T[\delta(t - t_0)] \quad (20)$$

for any value of t_0 . An arbitrary input $f(t)$ can always be expressed as a sum of unit impulses in the following way [8].

$$f(t) = \int_{-\infty}^{\infty} f(t_0) \delta(t - t_0) dt_0 \quad (21)$$

Invoking linearity, which should allow us to take the operator under the integral in the above, this gives us,

$$T[f(t)] = \int_{-\infty}^{\infty} f(t_0) T[\delta(t - t_0)] dt_0 = \int_{-\infty}^{\infty} f(t_0) h(t - t_0) dt_0 \quad (22)$$

and,

$$T[f(t)] = \int_{-\infty}^{\infty} f(t_0) h(t - t_0) dt_0 = \int_{-\infty}^{\infty} f(t - t_0) h(t_0) dt_0 \quad (23)$$

This result is called the *convolution theorem* and is a direct result of linearity and time invariance. It is also a statement of superposition. In any event, it can be interpreted to be a direct statement of the principal that once we know the response of a linear, time invariant system to an impulse, we can find the response to any other input.

The next step is to see how this convolution integral behaves under both transformations. This will lead us to a result that is as important and as powerful in the analysis of our systems as the transforms and their inversion formulas. It is

hinted at in Table 3. For the case of the Fourier Transform we consider the transform of (23).

$$O(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \left[\int_{-\infty}^{\infty} f(t_0) h(t - t_0) dt_0 \right] dt = \int_{-\infty}^{\infty} f(t_0) \left[\int_{-\infty}^{\infty} h(t - t_0) e^{-i\omega t} dt \right] dt_0$$

Now as a result of the shifting theorem (not proved here but its easy), see Table 3 as well,

$$O(\omega) = \int_{-\infty}^{\infty} f(t_0) [H(\omega) e^{-i\omega t_0}] dt_0 = F(\omega) H(\omega) \quad (24)$$

So convolution in the time domain corresponds to multiplication in the Fourier domain (and vice versa). Similarly, with the bilateral and unilateral Laplace Transform.

As an additional bit of information, it is worthy of note that most physical systems are *causal*. This simply means that an output can only occur in response to an input. Causality allows us to assume that $h(t) = 0$ for $t < 0$ and, also that if $o(t_1)$ is the output of a system for an input $f(t_1)$, and if $f(t_1) = 0$ for $t < t_1$, then $o(t_1) = 0$ for $t < t_1$.

Causality also allows us to rewrite (22) as,

$$T[f(t)] = \int_{-\infty}^t f(t_0) h(t - t_0) dt_0 \quad (25)$$

because $h(t - t_0) = 0$ when $t_0 > t$. This statement says that only the past counts in finding $o(t)$. In general, when we find the transform of an input we find it for all time. It might happen, although it is rare in practice, that an input does not have a Laplace Transform that is defined for all time, e^{t^2} is one. We can still find solutions to this input by cutting off the function (with a clever use of time shifted step functions) at some time greater than one we would be interested in. This create no loss of generality because response of the system up to that time is only a function of history.

RELATING FOURIER TO LAPLACE

We are now in a position to look briefly at the relationship that exists between the two transform types. We have already begun this by noting that the defining integral definitions for the Fourier Transform and the Laplace Transform

differ in the substitution of the complex variable s for $i\omega$. This of course makes a more than subtle difference in the outcome chiefly because, in the Laplace case, we end up dealing with complex variables. With Fourier we still remain in the domain of the real numbers. But the fact is that, given $H(s)$ for some system, so long as the region of convergence includes the $i\omega$ axis, $H(i\omega) = H(\omega)$. That is, we know the Fourier Transform given the Laplace Transform. This is what allows us to put the Fourier Transform of a signal into a system designed and characterized by Laplace techniques and come out with meaningful result. This fact is also used, in network synthesis for example, together with the principle of analytic continuation in inferring the Laplace Transform from the Fourier Transform. This is a fancy way of saying, for example, that we can design a filter around a desired frequency and phase characteristic.

One thing we have not addressed, although it has been referred to several times is how we arrive at the notion that the variable ω , in either case refers to something we call frequency, or how it comes about that $H(\omega)$ is "frequency" response as we might understand the term as the response of a physical system to a purely sinusoidal excitation. This notion, at least in the Fourier Transform case, can be easily verified if we accept the transforms shown in Table 2. From this it can easily seem that;

$$o(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) H(\omega) e^{i\omega t} d\omega \quad (26)$$

substituting

$$o(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] d\omega \quad (27)$$

and

$$o(t) = \frac{1}{2} [H(\omega_0) e^{-i\omega_0 t} + H(-\omega_0) e^{i\omega_0 t}] \quad (28)$$

This shows, even though we have no idea what the values of $H(\pm\omega_0)$ will be, that the *time* dependence will always be a sine or cosine at the frequency of the input, ω_0 . This is of course what is commonly meant by frequency response. In addition, the fact that the Fourier Transform of sines and cosines in time result in impulses in the frequency domain, forms the final link to the sense of the frequency domain as we commonly speak of it.

The same result can be demonstrated for the Laplace Transform although the situation is different. The unilateral Laplace transform will give us a result as if the sinusoidal input was impressed at $t = 0+$, so the response will include the transients generated by the application of the input signal. to proceed, we will make the following observations about $H(s)$. For a typical system made up of models that are described by integral and differential equations with constant coefficients, $H(s)$ will always be a ratio of two polynomials in the complex variable s . The coefficients, a_n and b_n , will be real constants. That is, it will have the form,

$$H(s) = \frac{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m} = \frac{a_0 (s-z_1)(s-z_2)\dots(s-z_n)}{b_0 (s-p_1)(s-p_2)\dots(s-p_m)} \quad (29)$$

where the last term in the equation shows the polynomials in their generalized factored form.

Now when a sinusoidal input (see Table 1) is applied to such a system we get the following response (still in the Laplace domain),

$$O(s) = H(s) \frac{w_0}{s^2 + w_0^2} \quad (30)$$

We would normally find the inverse of this response by a expanding the result in using a partial fractions expansion of (30). Terms associated with the roots of the denominator of $H(s)$ will represent the transient response of the system. On some scale, however long that may be, these will eventually die out leaving only a term associated only with the sinusoidal signal. This is usually called the steady state response. Since it is this part we are looking for we can say that,

$$O_{ss}(s) = \frac{K_a}{(s - iw_0)} + \frac{K_b}{(s + iw_0)} \quad (31)$$

where

$$K_a = H(s) \frac{w_0}{s - iw_0} \Big|_{s=iw_0} = \frac{H(-iw_0)}{2i} ; \quad K_b = H(s) \frac{w_0}{s + iw_0} \Big|_{s=-iw_0} = \frac{H(iw_0)}{2i}$$

now we invoke another property of real system functions. Since $H(s)$ it is a *real function* of a complex variable (i.e. $H(x)$ is real number if x is real) then

$$H(s^*) = H^*(s) \quad \text{and so} \quad H(-iw_0) = H^*(iw_0)$$

(13)

writing the above in terms of the real and imaginary parts of $H(i\omega)$,

$$H(i\omega_0) = R(\omega_0) + iX(\omega_0) \quad \text{and} \quad H(-i\omega_0) = R(\omega_0) - iX(\omega_0)$$

and with some perseverance, we find that,

$$O(s) = \frac{X(\omega_0)s + \omega_0 R(\omega_0)}{s^2 + \omega_0^2} = X(\omega_0) \frac{s}{s^2 + \omega_0^2} + R(\omega_0) \frac{\omega_0}{s^2 + \omega_0^2} \quad (32)$$

using Table 1 in the reverse we see that,

$$o(t) = X(\omega_0) \cos(\omega_0 t) + R(\omega_0) \sin(\omega_0 t) \quad (33)$$

and so we have the result that the steady state response to a sinusoidal input is a sinusoid of frequency ω_0 with amplitude and phase equal to

$$|H(\omega_0)| = [R(\omega_0)^2 + X(\omega_0)^2]^{\frac{1}{2}} \quad \varphi = \arctan \frac{X(\omega_0)}{R(\omega_0)} \quad (34)$$

In summary we note that, in fact, the Laplace Transform gives, in the steady state, the same result as the Fourier Transform.

APPLYING THE LAPLACE TRANSFORM

When I first studied the Laplace Transform, I remember that the instructor drew an analogy with the action of logarithms and the operations of multiplication and division of real numbers. At the time I dismissed the analogy as an oversimplification of what seemed to me to be a considerably more complicated operation, and thought very little more about it. In my recent revisit to the subject I realized how closely the two do resemble each other. Under the transformation of the real numbers to logarithms, the operations of multiplication and division become addition and subtraction, which, when all you have is a pencil and a piece of paper, can make those operations much easier, especially when the numbers are large and there is a need for accuracy. Of course what you lose in this transformation is the ease of doing addition and subtraction on the numbers. In order to do those operations you are forced to return to the original number space by inverting the original process. In fact without some "feel" for the inversion process, it is very difficult to appreciate the meaning of the result one has obtained.

Well, this is analogous to the situation encountered with the Laplace

transform. We simplify the problem by transforming into a "complex frequency" space. Here the operations of convolution become multiplication of polynomials in our new domain. But we have lost the explicit sense of the time dependence, trading it in for a variable space, in a sense, a mathematical artifact. In order to fully understand the result of our analysis we must be able to return to the time domain or at least to get a "feel" for how the solution we see relates to that original domain.

In the analysis of networks, system functions analogous to the generic $h(t)$, often called a transfer impedance or admittance functions are summations under the rules of Kirchoffs Laws from terms such as,

$$R i(t) \quad , \quad L \frac{di(t)}{dt} \quad , \quad \frac{1}{C} \int i(t) dt \quad , \quad G v(t) \quad , \quad C \frac{dv(t)}{dt} \quad , \quad \frac{1}{L} \int v(t) dt$$

under the Laplace Transform, and invoking linearity, these can be replaced at the component level with terms like,

$$R I(s) \quad , \quad L s I(s) \quad , \quad \frac{1}{Cs} I(s) \quad , \quad G V(s) \quad , \quad Cs V(s) \quad , \quad \frac{1}{Ls} V(s)$$

From this, under whatever technique is used, transfer functions are generated in composite systems that are the ratios of polynomials in the complex variable s just like equation (29).

Now in the last term in equation (29), the polynomials have been factored into their roots. The roots of the denominator (Hurwitz) polynomial are known as poles and are the singular points of the $H(s)$, the numerator roots are called the zeros of the function. Depending on the relative magnitude of m and n , poles or zeros are also said to exist at infinity, the order of the pole or zero being equal to the difference of m and n . In this sense it is always true that the total number of poles always equals the total number of zeros.

Some things are known about these roots when they result from ordinary passive circuit components. For a system to be stable its poles must be in the open left half of the complex plane. Clearly this avoids terms which would produce growing exponentials in the time domain (see Table 1). If they occur off the real axis they must occur in complex conjugate pairs. This is because the coefficients of the polynomials must always be real numbers. Poles must also be simple if they are located on the imaginary axis. There are restrictions on how the numerator and denominator can differ. For driving point functions (impedances and admittances) m and n can differ by no more than one. For transfer functions, numerator and denominator may differ by more than unity but the order of the denominator is always greater. Driving point function zeros must also be in the left half-plane.

Now, one very powerful tool for analysis of is a simple plot in two dimensions of the pole and zero locations. This is known as a *pole-zero* diagram. Illustrated in figure (2), it is a graphical representation of the network function. If we include the scale factor it is a complete description of the function it represents. For a transfer function, it is a representation in the Laplace domain of the impulse response of the network. Interestingly enough, these diagrams are far more often referred to than drawn, still the usefulness of the concept in relating time to frequency is unequaled. For example, as illustrated in the figure, the magnitude and phase response at an arbitrary point on the $j\omega$ axis is easily seen. If one draws a vector from each finite pole and zero, the value of $|H(\omega)|$ is the ratio of the magnitudes of the zero vectors to the pole vectors and the phase angle is the sum of the angles defined to the same point by the vectors associated with the zeros minus the pole angles. This is much easier to visualize than it is to say.

Now a plot of $20 \log_{10} |H(\omega)|$ and $\arg H(\omega)$ constitute what is called a Bode diagram of the system function. It is a graphical representation of the Fourier Transform of the function which, if it is known in analytical form gives the Laplace Transform by substituting $\omega = s/j$ (remember analytic continuation). If as often happens, the magnitude and phase response is known only by measurement, or only over part of the $j\omega$ axis, then it must be fit to some algebraic approximation by some means. For systems with small numbers of simple poles and zeros this is not difficult since these roots cause certain specific behavior in the Bode plot that is easy to see. Because the magnitude plot is done on a log scale it makes behavior associated with both the pole and zeros additive, hence easier to spot and interpret.

A TIME FOR UNDERSTANDING

Figure (3) shows an illustration of how various simple pole-zero patterns reflect in the time domain. The thing to remember here is that it is the poles that determine the time response, the zeros effect only the relative magnitude and sign (remember the final form of the partial fraction expansion). A zero at the origin in a transfer function means no DC.

Figure (4) shows a simple RC circuit that is a good model of one of our accelerator pick-up electrodes. Capacitor C1 represents the coupling of the electrode to the beam, capacitor C2 is the capacitance of the electrode to the beam pipe. The resistor R1 represents the input resistance to any electronics that might be connected to the output of the PUE. The equation on the figure represents the voltage transfer function from the beam to the resistor R1. Notice that it has a zero at the origin and one simple pole on the real axis. The pole position is a function of R1, and it moves away from the origin as the resistance

gets lower. This alters the low frequency response of the circuit, causing the characteristic "walking" of the signal seeking its average value (remember no DC) to take less time. We can see the base line shift of the upper waveform in figure (4). This would not be troublesome if the beam pulses were always the same size, but when one is looking for beam position or betatron tune information, which normally shows up as modulating pulse amplitudes over many beam pulses, a pole in the circuit response near the frequency of interest can cause havoc in attempting to properly detect those effects.

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Figure 1 - Complex integration contours used to find the inverse Laplace transform. The shaded areas represent regions in which the integrals do not converge.

Figure 2 - Using the pole-zero diagram of $H(i\omega)$ to find $|H(\omega)|$

Figure 3 - This illustrates the time domain response corresponding to several simple pole - zero configurations. If this were a network function then they would be the impulse response of that system.

$$\frac{V_o}{V_b} = \frac{C1}{C1+C2} \frac{s}{s+s_0}$$

$$s_1 = \frac{1}{(C1+C2)RI}$$

Figure 4 - Model of the PUE output circuit

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