Ellipsoidal Distributions of Charge or Mass*

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Abstract

The Coulomb (or gravitational) energy is calculated for a distribution of charge (or mass) in which the surfaces of constant density are a family of similar concentric ellipsoids. The density can vary in any manner from one surface to another, and the ellipsoids need not have an axis of symmetry. Two examples are discussed: the charge distribution of a deformed atomic nucleus having a diffuse surface, and the mass distribution of a stellar galaxy. The energy is shown to be a product of two factors. One is the energy of the spherical distribution from which the ellipsoidal distribution can be obtained by a volume-preserving deformation. The other is an anisotropy factor that has a simple geometrical significance and depends only on the two eccentricities of the ellipsoids. Its values range from unity to zero and are tabulated numerically.

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I. INTRODUCTION

Among the largest as well as the smallest physical systems, there occur distributions of matter that have an ellipsoidal shape and are composed of elements that exert inverse-square-law forces on one another. Approximately a quarter of the brightest galaxies belong to a type known as elliptical galaxies, in which the density of stars decreases smoothly in all directions from the center. They generally contain very little interstellar gas or dust, show almost no internal structure, and have nearly ellipsoidal shapes with axial symmetry and various degrees of flattening.¹ Another quarter consists of lenticular

¹ G. de Vaucouleurs, "Classification and Morphology of External Galaxies," Handbuch der Physik (Springer, Berlin, 1959), Vol. LIII, p. 305.

galaxies, many of which are similar to the ellipticals but show a little differentiation between internal regions. The remaining lenticulars and most of the spiral galaxies show too much internal structure to be approximated, except in a crude fashion, by a mass distribution with ellipsoidal surfaces of constant density.

Near the other extreme of size, the atomic nuclei of the rare-earth and actinide regions, as well as some of the light nuclei, have equilibrium deformations of ellipsoidal shape. The departures from spherical symmetry are never as large as they sometimes are in the elliptical galaxies, but both prolate and oblate deformations occur and there is substantial evidence for shapes without axial symmetry (triaxial ellipsoids).²

² The asymmetric rotor theory of Davydov and Filippov is compared with experimental data by D. M. Van Patter, Nuclear Phys. 14, 42 (1959).

Unlike the gravitational attractions between the stars of a galaxy, the

repulsive Coulomb forces between the protons in a nucleus do not control the dynamics of the system. Nevertheless, the Coulomb energy is an important term in the semi-empirical mass formula; it supplies the energy released in fission and plays an important part in the energetics of any disintegration into charged fragments.

To a first approximation the charge distribution of a deformed nucleus can be treated as a homogeneous ellipsoid (a uniform charge density bounded by a sharp ellipsoidal surface). Since the potential in the interior of a homogeneous ellipsoid is a quadratic function of the Cartesian position coordinates,³ it is easy to obtain the Coulomb energy by integrating the

³ O. D. Kellogg, Foundations of Potential Theory (Springer, Berlin, 1929), p. 194 .

product of the potential and the constant charge density over the interior of the ellipsoid. (Alternatively, for small deformations, one can expand the equation of the surface in spherical harmonics of degree two and calculate the Coulomb energy to lowest order in the expansion coefficients.) For a better approximation one would like to take into account the diffuseness of the nuclear surface; furthermore, the mass distribution in a galaxy tapers off gradually and does not resemble a homogeneous ellipsoid.

In order to discuss the energy of such distributions, which are not known in exact detail but are neither spherical nor homogeneous, we shall assume that the surfaces of constant density are a family of similar concentric ellipsoids.⁴ This assumption seems reasonable both

⁴ A series expansion of the potential of such a distribution will be given in a later paper.

for nuclei and for elliptical galaxies, although some reservations about its accuracy will be mentioned in Sec. V. (It also applies moderately well to the mass distribution of the Earth, a slightly flattened ellipsoid with a sharp surface but non-uniform density, although the high-density contours near the Earth's center are thought to be more nearly spherical than the low-density contours near the surface.⁵⁾)

⁵ K. Jung, "Figur der Erde," Handbuch der Physik (Springer, Berlin, 1956), Vol. XLVII, p. 604.

The Coulomb or gravitational energy of a distribution that satisfies this assumption but is otherwise arbitrary will be shown in Sec.II to reduce exactly to a product of two factors. One of these, to be called the anisotropy factor, depends only on the eccentricities of the ellipsoids and has a simple geometrical significance; the other depends only on the variation of density from one surface to another and has a simple physical significance.

Since the energy is not changed by permuting the three axes, it seems appropriate to replace the eccentricities by two parameters that are elementary symmetric functions of the axis lengths. In Sec.IV the anisotropy factor, which is essentially an incomplete elliptic integral of the first kind, is tabulated as a function of two such symmetric parameters. As in the unified model of the nucleus, one parameter is a measure of the departure from spherical symmetry while the other specifies the shape of the deformation (prolate, oblate, or triaxial). The anisotropy factor is insensitive to the shape parameter for small and even for moderately large deformations.

Sec. V contains a discussion of two examples, the Coulomb energy of a Gd^{160} nucleus and the gravitational energy of the Large Magellanic Cloud. An Appendix gives the recursion relations that were used to calculate the incomplete elliptic integrals of the first kind.

II. FACTORIZATION OF THE ENERGY

In a family of similar concentric ellipsoids, each member corresponds to a constant value of

$$R^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \quad (2.1)$$

where x , y , and z are Cartesian coordinates and the corresponding semi-axes have lengths aR , bR , and cR . It is convenient to regard R as a length and to let a , b , and c be dimensionless numbers that are normalized according to

$$abc = 1 \quad (2.2)$$

and are common to all ellipsoids in the family. (Similar ellipsoids, unlike confocal ellipsoids, are simple enlargements of one another.) When a , b , and c are separately equal to unity, the ellipsoids degenerate to spheres and R reduces to the radius r .

We shall want to compare a spherically symmetric distribution having density $\rho(r)$ with a corresponding ellipsoidal distribution having the same total charge but density $\rho(R)$, the functional dependence of ρ on its argument being exactly the same in the two cases. The ellipsoidal distribution can be obtained from the spherical one by a volume-preserving deformation that moves each charge element from its original location P to a final location P' :

$$P = (x, y, z) \longrightarrow P' = (ax, by, cz) \quad (2.3)$$

Each surface of constant density is deformed from a sphere into an ellipsoid of the same volume, the initial value of r and the final value of R being numerically equal.

The Coulomb energy of the ellipsoidal distribution is given (in rationalized mks units) by

$$U(a, b, c) = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \iint d\underline{r} d\underline{r}' \frac{\rho(\underline{R})\rho(\underline{R}')}{|\underline{r} - \underline{r}'|} \quad (2.4)$$

In the gravitational case $(4\pi\epsilon_0)^{-1}$ is replaced by ^{negative of the} the gravitational constant G . The energy $U(1,1,1)$ of the corresponding spherical distribution differs from (2.4) only by the occurrence of r and r' in place of R and R' .

The result to be proved in this section is that the right side of Eq. (2.4) can be factored:

$$U(a, b, c) = \langle r_1^2 \rangle U(1, 1, 1) \quad (2.5)$$

The second factor is the energy of the spherical distribution, which depends only on the functional form of ρ . The first factor is an anisotropy factor that depends only on a , b , and c and has a simple geometrical meaning. The deformation (2.3) changes the unit sphere into a "unit ellipsoid" that has the same volume. The radius vector from its center to a point on its surface is $\underline{r}_1 = (x_1, y_1, z_1)$, where the components are dimensionless numbers satisfying

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \quad (2.6)$$

In spherical polar coordinates the radius vector has components

(r_1, θ, ϕ) and its square is found from Eq.(2.6) to be

$$r_1^2 = \frac{1}{S(\theta, \phi)} = \left(\frac{\sin^2 \theta \cos^2 \phi}{a^2} + \frac{\sin^2 \theta \sin^2 \phi}{b^2} + \frac{\cos^2 \theta}{c^2} \right)^{-1} \quad (2.7)$$

The anisotropy factor is the mean square radius of the unit ellipsoid,

where the mean is an average over angles:

$$\langle r_1^2 \rangle = \int r_1^2 \frac{d\Omega}{4\pi} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{\sin \theta d\theta d\phi}{S(\theta, \phi)} \quad (2.8)$$

We shall give two proofs of Eq.(2.5). The first is formal and self-contained; the second is more physical but makes use of a theorem from potential theory. The formal proof begins by inserting a Fourier integral for the reciprocal distance in Eq.(2.4):

$$U(a,b,c) = \frac{1}{2\epsilon_0} \frac{1}{(2\pi)^3} \int d\underline{r} d\underline{r}' d\underline{k} \rho(\underline{r}) \frac{e^{i\underline{k} \cdot (\underline{r} - \underline{r}')}}{k^2} \rho(\underline{r}'). \quad (2.9)$$

If we define $\underline{R} = (x/a, y/b, z/c)$ and $\underline{K} = (ak_x, bk_y, ck_z)$, then the Fourier transform of $\rho(\underline{r})$ can be written in the form

$$\begin{aligned} \int d\underline{r} e^{i\underline{k} \cdot \underline{r}} \rho(\underline{r}) &= \int d\underline{R} e^{i\underline{K} \cdot \underline{R}} \rho(\underline{R}) \\ &= \frac{4\pi}{K} \int_0^\infty \sin KR \rho(R) R dR \\ &= 4\pi \epsilon_0 \int_0^\infty \cos KR \tau(R) dR, \end{aligned} \quad (2.10)$$

where

$$\tau(R) = \frac{1}{\epsilon_0} \int_0^\infty \rho(R') R' dR' \quad (2.11)$$

The last step in Eq.(2.10) follows from an integration by parts in which the boundary terms vanish.⁶ Eq. (2.9) now becomes

$$U(a,b,c) = \frac{\epsilon_0}{\pi} \int \frac{d\underline{K}}{K^2} \left(\int_0^\infty \cos KR \tau(R) dR \right)^2. \quad (2.12)$$

⁶ Both the total charge and the energy of the spherical distribution are assumed to exist. Then $r \tau(r)$ necessarily tends to zero at the origin and also at large distances.

If we choose spherical polar coordinates (K, θ, ϕ) in K-space, then $k^2 = K^2 S(\theta, \phi)$, where $S(\theta, \phi)$ is defined by Eq.(2.7). The integral

over angles is just the one in Eq.(2.8):

$$\begin{aligned} U(a,b,c) &= 4\epsilon_0 \langle r_i^2 \rangle \int_0^\infty dK \left(\int_0^\infty \cos KR \tau(R) dR \right)^2 \\ &= \langle r_i^2 \rangle 2\pi\epsilon_0 \int_0^\infty [\tau(R)]^2 dR, \end{aligned} \quad (2.13)$$

where Parseval's formula has been used in the last step. For the particular case of spherical symmetry, Eq.(2.13) reduces to

$$U(1,1,1) = 2\pi\epsilon_0 \int_0^\infty [\tau(r)]^2 dr, \quad (2.14)$$

an expression for the energy of a spherical distribution that has been derived more directly elsewhere.⁷ Since the variable of integration

⁷B. C. Carlson, Iowa State Journal of Science, 35, 000 (1961).

is immaterial, Eq.(2.14) can be substituted back into Eq.(2.13) to obtain Eq.(2.5).

The second proof starts from Newton's theorem: if a thin shell between two similar concentric ellipsoids is filled with a uniform charge density, the potential is constant in the region enclosed by the shell.⁸ In order

⁸ W. D. MacMillan, The Theory of the Potential (Dover Publications, New York, 1958), p. 10.

to evaluate the constant, it suffices to find the potential at the center. An element of solid angle $d\Omega$ about the center cuts from the shell a small cylinder with slant height dr and projected base area $r^2 d\Omega$, where r is determined as a function of angle by $r^2 = R^2/S(\theta, \phi)$ in accordance with Eqs.(2.1) and (2.7). The charge contained in this cylinder contributes

to the potential at the center an amount

$$\rho r^2 dr d\Omega / 4\pi \epsilon_0 r = \rho R dR d\Omega / 4\pi \epsilon_0 S(\theta, \phi) .$$

Integration over angles shows that the total potential at the center is

$$\langle r_1^2 \rangle \rho R dR / \epsilon_0 .$$

Hence, when a spherical shell is subjected to the deformation (2.3), the constant potential inside the shell is changed only by a factor $\langle r_1^2 \rangle$.

We now assemble the complete ellipsoidal distribution by bringing up shells of successively smaller size. When all shells larger than the one labeled by R have been assembled, the potential in the empty space enclosed by them has the constant value

$$\frac{\langle r_1^2 \rangle}{\epsilon_0} \int_R^\infty \rho(R') R' dR' = \langle r_1^2 \rangle \tau(R) .$$

(We note in passing that the potential at the center of an ellipsoidal distribution is $\langle r_1^2 \rangle \tau(0)$.) The work required to bring up the next shell of thickness dR and volume $4\pi R^2 dR$ is

$$\rho(R) 4\pi R^2 dR \langle r_1^2 \rangle \tau(R) .$$

The total work required to assemble the distribution is

$$U(a, b, c) = \langle r_1^2 \rangle 4\pi \int_0^\infty \tau(R) \rho(R) R^2 dR . \quad (2.15)$$

As in the case of Eq.(2.13), the coefficient of $\langle r_1^2 \rangle$ on the right-hand side must be the energy of a spherical distribution; this particular expression for it is again obtained more directly in Reference 7.

III. THE ANISOTROPY FACTOR AS A FUNCTION OF THE ECCENTRICITIES

The mean square radius of the unit ellipsoid, given by Eq. (2.8), can be written as

$$\langle r_1^2 \rangle = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{\sin^2 \theta \cos^2 \phi}{a^2} + \frac{\sin^2 \theta \sin^2 \phi}{b^2} + \frac{\cos^2 \theta}{c^2} \right)^{-1} \sin \theta d\theta d\phi, \quad (3.1)$$

where the integration now extends over one octant of the ellipsoid.

By taking $\tan \phi$ as a new variable, the integration over ϕ can be carried out:⁹

$$\langle r_1^2 \rangle = \int_0^{\pi/2} \left(\frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{c^2} \right)^{-1/2} \left(\frac{\sin^2 \theta}{b^2} + \frac{\cos^2 \theta}{c^2} \right)^{-1/2} \sin \theta d\theta. \quad (3.2)$$

⁹ Most of the changes of variable used in Sec. III and the first paragraph of Sec. IV can be found in texts that discuss the potential of a homogeneous ellipsoid. Several well-known relations will be essentially reproduced here because of minor differences arising from our normalization of the semi-axes.

Because of its geometrical meaning, the integral is a symmetric function of a , b , and c ; therefore, no generality is lost by assuming

$$a \leq b \leq c. \quad (3.3)$$

We shall retain this assumption throughout Sec. III and define the eccentricities

$$\begin{aligned} \epsilon &= \left(1 - \frac{a^2}{c^2} \right)^{1/2}, \\ \epsilon' &= \left(1 - \frac{b^2}{c^2} \right)^{1/2} \leq \epsilon. \end{aligned} \quad (3.4)$$

The inverse relations are found from Eq. (2.2) to be

$$\begin{aligned} a &= (1 - \epsilon^2)^{1/3} (1 - \epsilon'^2)^{-1/6}, \\ b &= (1 - \epsilon^2)^{-1/6} (1 - \epsilon'^2)^{1/3}, \\ c &= (1 - \epsilon^2)^{-1/6} (1 - \epsilon'^2)^{-1/6}. \end{aligned} \quad (3.5)$$

Substitution of $\cos \theta = x$ in Eq.(3.2) gives

$$\langle r_1^2 \rangle = \frac{1}{c} \int_0^1 (1 - \epsilon^2 x^2)^{-1/2} (1 - \epsilon'^2 x^2)^{-1/2} dx \quad (3.6)$$

The further substitution of $\epsilon x = \sin \psi$ puts the integral in Legendre's standard form for the incomplete elliptic integral of the first kind:

$$\langle r_1^2 \rangle = \frac{1}{c\epsilon} \int_0^{\phi} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} = \frac{1}{c\epsilon} F(\phi, k), \quad (3.7)$$

where

$$\begin{aligned} \sin \phi &= \epsilon, \\ k &= \epsilon'/\epsilon. \end{aligned} \quad (3.8)$$

When the unit ellipsoid has an axis of symmetry, the elliptic integral reduces to an elementary integral. In the prolate case inverse hyperbolic functions occur:

$$\begin{aligned} a &= b \leq c, \\ \epsilon' &= \epsilon, \quad k = 1, \\ \langle r_1^2 \rangle &= \frac{1}{2c\epsilon} \ln \frac{1+\epsilon}{1-\epsilon} = \frac{1}{c\epsilon} \tanh^{-1} \epsilon \\ &= \frac{(1-\epsilon^2)^{1/3}}{\epsilon} \tanh^{-1} \epsilon \\ &= 1 - \frac{1}{45} \epsilon^4 - \frac{64}{2835} \epsilon^6 - \frac{58}{2835} \epsilon^8 - \dots \end{aligned} \quad (3.9)$$

In the oblate case inverse circular functions occur:

$$\begin{aligned} a &\leq b = c, \\ \epsilon' &= 0 = k, \\ \langle r_1^2 \rangle &= \frac{1}{c\epsilon} \sin^{-1} \epsilon = \frac{(1-\epsilon^2)^{1/6}}{\epsilon} \sin^{-1} \epsilon \\ &= 1 - \frac{1}{45} \epsilon^4 - \frac{62}{2835} \epsilon^6 - \frac{55}{2835} \epsilon^8 - \dots \end{aligned} \quad (3.10)$$

The leading coefficients of the power series in the prolate and oblate cases are nearly equal. However, this does not imply that $\langle r_1^2 \rangle$ is nearly independent of ϵ' for a general ellipsoid with small eccentricities; on the contrary, the right-hand side of Eq.(3.6) is formally symmetric in ϵ and ϵ' . Change of variable to $u = x^2$ gives the integral form of Appell's first hypergeometric function of two variables:¹⁰

$$\begin{aligned} \langle r_1^2 \rangle &= \frac{1}{2c} \int_0^1 u^{-\frac{1}{2}} (1 - \epsilon^2 u)^{-\frac{1}{2}} (1 - \epsilon'^2 u)^{-\frac{1}{2}} du \\ &= \frac{1}{c} F_1 \left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \epsilon^2, \epsilon'^2 \right) \\ &= \frac{1}{c} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{m+n} \left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_{m+n} m! n!} \epsilon^{2m} \epsilon'^{2n}, \end{aligned} \quad (3.11)$$

where

$$(\alpha)_m = \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+m-1).$$

¹⁰ W. N. Bailey, Generalized Hypergeometric Series (Cambridge University Press, 1935), pp. 73, 77.

By making a binomial expansion of $1/c$ from Eq. (3.5) and multiplying the two series, we obtain the leading terms

$$\begin{aligned} \langle r_1^2 \rangle &= 1 - \frac{1}{45} (\epsilon^4 - \epsilon^2 \epsilon'^2 + \epsilon'^4) \\ &\quad - \frac{2}{2835} (\epsilon^2 + \epsilon'^2) (31 \epsilon^4 - 46 \epsilon^2 \epsilon'^2 + 31 \epsilon'^4) + \dots \end{aligned} \quad (3.12)$$

IV. THE ANISOTROPY FACTOR AS A FUNCTION OF SYMMETRIC PARAMETERS

If it should be inconvenient to make the assumption (3.3), the definitions of the eccentricities in terms of a , b , and c could be made to depend on the relative sizes of these numbers. For instance $(1 - \epsilon^2)^{1/2}$ would be defined as the ratio of the smallest to the largest. The smallest of three numbers can be regarded as a symmetric function of the three numbers; similarly, the eccentricities could then be regarded as symmetric functions of a , b , and c but not at the same time as elementary functions in the usual sense of the word. To find a more satisfactory choice of symmetric parameters, we consider an integral representation of $\langle r_1^2 \rangle$ which makes its symmetry conspicuous: substitution of $\lambda = c^2 \tan^2 \theta$ in Eq.(3.2) gives

$$\langle r_1^2 \rangle = \frac{1}{2} \int_0^\infty [(\lambda + a^2)(\lambda + b^2)(\lambda + c^2)]^{-\frac{1}{2}} d\lambda. \quad (4.1)$$

We shall now introduce symmetric parameters ξ and η in the course of a proof that $\langle r_1^2 \rangle$ cannot exceed unity. The expression in square brackets can be rewritten as

$$(\lambda + a^2)(\lambda + b^2)(\lambda + c^2) = (\lambda + 1)^3 + \xi \lambda^2 + \eta \lambda, \quad (4.2)$$

where

$$\begin{aligned} \xi &= a^2 + b^2 + c^2 - 3, \\ \eta &= a^{-2} + b^{-2} + c^{-2} - 3. \end{aligned} \quad (4.3)$$

The quantities $\xi + 3$ and $\eta + 3$ are both of the form $u + v + w$, where u , v , and w are positive and their product is $uvw = 1$ by Eq.(2.2).

Under these conditions we have

$$(\mu + \nu + \omega)^2 = \sum_{\mu, \nu, \omega} (\mu^2 + \frac{1}{\mu}) \geq 9,$$

$$\mu + \nu + \omega \geq 3,$$

where the equality holds only when μ , ν , and ω are all unity. This result implies

$$\xi \geq 0, \quad \eta \geq 0,$$

and

$$\langle r_1^2 \rangle \leq \frac{1}{2} \int_0^\infty (\lambda+1)^{-3/2} d\lambda = 1, \quad (4.4)$$

where the equalities hold only when the unit ellipsoid is a sphere.

Any non-spherical ellipsoidal distribution has a lower self-energy than the corresponding spherical distribution. We note in passing that Eqs.(4.4) and (3.7) give an upper bound for the incomplete elliptic integral of the first kind:

$$F(\phi, k) \leq \frac{\sin \phi}{(\cos \phi)^{1/2} (1 - k^2 \sin^2 \phi)^{1/2}} \quad (\phi < \frac{\pi}{2}). \quad (4.5)$$

The inequality is a close one for small ϕ .

If ξ and η are sufficiently small, we can substitute Eq.(4.2) in Eq.(4.1) and make a binomial expansion of the integrand:

$$\begin{aligned} \langle r_1^2 \rangle &= \frac{1}{2} \int_0^\infty [(\lambda+1)^3 + \xi \lambda^2 + \eta \lambda]^{-1/2} d\lambda \\ &= \sum_{s=0}^\infty (-1)^s (2s-1)!! K_s(\xi, \eta), \end{aligned} \quad (4.6)$$

where

$$K_s(\xi, \eta) = \frac{1}{2^{s+1} s!} \int_0^\infty \frac{(\xi \lambda + \eta)^s \lambda^s}{(\lambda+1)^{3s + \frac{3}{2}}} d\lambda \quad (4.7)$$

is a homogeneous polynomial in ξ and η of degree s . The polynomial

can be written¹¹ as a terminating hypergeometric series:

$$K_s(\xi, \eta) = \frac{(2s-1)!!}{(4s+1)!!} \eta^s {}_2F_1\left(-s, s+1; 2s+\frac{3}{2}; 1-\frac{\xi}{\eta}\right) \quad (4.8)$$

$$= \frac{(4s-1)!!}{(6s+1)!!} \eta^s {}_2F_1\left(-s, s+1; \frac{1}{2}-2s; \frac{\xi}{\eta}\right). \quad (4.9)$$

¹¹ Erdélyi, Magnus, Oberhettinger, and Tricomi, Higher Transcendental Functions (McGraw-Hill Book Company, New York, 1953), Vol.I, pp.60, 108.

Eq.(4.8) is useful in relation to the special case $\xi = \eta$ because the ${}_2F_1$ function then reduces to unity. From Eqs. (4.3) and (2.2) one finds that the condition $\xi = \eta$ means that one semi-axis has unit length while the other two are reciprocals. In the notation of Sec.III we have

$$\begin{aligned} b &= 1, & c &= \frac{1}{a}, \\ e^2 &= e'^2 (2 - e'^2), \\ 1/k^2 &= 1 + \cos \phi, \\ \eta = \xi &= a^2 + c^2 - 2 = \cos \phi + \sec \phi - 2. \end{aligned} \quad (4.10)$$

in this case
The mean square radius turns out to be a generalized hypergeometric series:

$$\begin{aligned} \langle r_1^2 \rangle &= \sum_{s=0}^{\infty} (-1)^s \frac{[(2s-1)!!]^2}{(4s+1)!!} \xi^s \\ &= 1 - \frac{1}{3 \cdot 5} \xi + \frac{1 \cdot 3}{5 \cdot 7 \cdot 9} \xi^2 - \frac{1 \cdot 3 \cdot 5}{7 \cdot 9 \cdot 11 \cdot 13} \xi^3 + \dots \\ &= {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{3}{4}, \frac{5}{4}; -\frac{\xi}{4}\right). \end{aligned} \quad (4.11)$$

In the general case Eq.(4.9) is more convenient for determining $\langle r_1^2 \rangle$ as a power series in ξ and η :

$$\langle r_1^2 \rangle = 1 - \frac{1}{105} (4\xi + 3\eta) + \frac{1}{15,015} (48\xi^2 + 60\xi\eta + 35\eta^2) + \dots \quad (4.12)$$

Although the parameters ξ and η have the advantage of being symmetric polynomials in the three semi-axis lengths, they have some drawbacks too. One of the foremost is their awkwardness for describing ellipsoids of revolution. We shall turn now to a pair of parameters used by Hill and Wheeler¹² in the collective model of the nucleus:

$$\begin{aligned} a &= e^{\sigma \cos(\gamma - \frac{2\pi}{3})}, & \frac{b}{c} &= e^{\sqrt{3} \sigma \sin(\gamma - \frac{2\pi}{3})}, \\ b &= e^{\sigma \cos(\gamma + \frac{2\pi}{3})}, & \frac{c}{a} &= e^{\sqrt{3} \sigma \sin(\gamma + \frac{2\pi}{3})}, \\ c &= e^{\sigma \cos \gamma}, & \frac{a}{b} &= e^{\sqrt{3} \sigma \sin \gamma}. \end{aligned} \quad (4.13)$$

¹² D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953), Fig. 13.

The condition $a \leq b \leq c$ of Sec. III corresponds to $-60^\circ \leq \gamma \leq 0^\circ$.

It would be natural in the present context to reverse the sign of γ in Eqs.(4.13) and thereby to interchange the definitions of a and b , but we shall retain the choice of sign that has become conventional in the unified model of the nucleus.

There is a one-to-one correspondence between the ordered triples of positive numbers (a, b, c) satisfying $abc = 1$ and the points of the $\sigma\gamma$ plane ($0 \leq \sigma$, $0 \leq \gamma < 2\pi$). The inverse relations are

$$\begin{aligned} \sigma^2 &= \frac{2}{3} [(\ln a)^2 + (\ln b)^2 + (\ln c)^2], \\ \cos \gamma &= \frac{1}{\sigma} \ln c, \\ \sin \gamma &= \frac{1}{\sqrt{3} \sigma} \ln \frac{a}{b}, \\ \cos 3\gamma &= \frac{4}{\sigma^3} (\ln a)(\ln b)(\ln c). \end{aligned} \quad (4.14)$$

Permutations of a , b , and c carry the point (σ, γ) into the six points $(\sigma, \pm \gamma)$, $(\sigma, 120^\circ \pm \gamma)$, and $(\sigma, 240^\circ \pm \gamma)$ without

changing the values of the symmetric functions σ and $\cos 3\gamma$. The prolate ellipsoids of revolution are described by $\cos 3\gamma = 1$, the oblate ones by $\cos 3\gamma = -1$, and the unit ellipsoids with one semi-axis of unit length by $\cos 3\gamma = 0$.

A nearly spherical ellipsoid has a radius $r(\theta, \phi)$ that can be written, to first order in σ , as a constant plus spherical harmonics of degree two. More briefly, a small ellipsoidal deformation is a quadrupole deformation. To establish the connection **this case between the** notation of Eqs. (4.13) for ellipsoids and the now widespread notation introduced by A. Bohr¹³ to describe nuclear quadrupole deformations, it suffices to compare the expressions for the semi-axis lengths;

¹³ A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 26, No. 14 (1952).

one finds that Bohr's shape parameter γ is the same as the one used here and that his measure β of deformation magnitude is a constant times σ . To second order in σ , however, these statements are no longer true. By Eq. (2.7) the quadrupole part of

$$r_1(\theta, \phi) = [S(\theta, \phi)]^{-1/2} = \sum_{l,m} a_{lm} Y_l^m(\theta, \phi) \quad (4.15)$$

is given to second order in σ by

$$\begin{aligned} a_{00} &= (4\pi)^{1/2} \left(1 - \frac{1}{5} \sigma^2\right), \\ a_{20} &= (4\pi/5)^{1/2} \left(\sigma \cos \gamma - \frac{1}{14} \sigma^2 \cos 2\gamma\right), \\ a_{2-1} &= a_{21} = 0, \\ a_{2-2} &= a_{22} = (2\pi/5)^{1/2} \left(\sigma \sin \gamma + \frac{1}{14} \sigma^2 \sin 2\gamma\right). \end{aligned} \quad (4.16)$$

To this order, the two notations are related by

$$\begin{aligned}\beta^2 &= (a_{20})^2 + 2(a_{22})^2 = \frac{4\pi}{5} \sigma^2 \left(1 - \frac{1}{7} \sigma \cos 3\gamma\right), \\ \tan \gamma_{B,hr} &= \frac{\sqrt{2} a_{22}}{a_{20}} = \tan \gamma \left(1 + \frac{\sigma}{7} \frac{\sin 3\gamma}{\sin 2\gamma}\right), \\ \cos 3\gamma_{B,hr} &= \cos 3\gamma - \frac{3}{14} \sigma (\sin 3\gamma)^2.\end{aligned}\tag{4.17}$$

The deformations found in nuclei are sufficiently small that the second term of

$$\sigma = 0.631 \beta \left(1 + 0.045 \beta \cos 3\gamma + \dots\right)\tag{4.18}$$

may ordinarily be neglected.

It is clear from Eqs.(4.13) that any function of a , b , and c is a periodic function of γ with period 2π and, if square integrable, can be represented by a Fourier series in γ ; furthermore, if the function is symmetric in a , b , and c , Eqs.(4.14) show that its Fourier series can contain only terms of the type $\cos 3n\gamma$. Since increasing γ by π is formally equivalent to changing the sign of σ , the coefficient of $\cos 3n\gamma$ must be an even function of σ if n is even and an odd function if n is odd. Finally we note that a , b , and c are expressible in terms of $\sigma \cos \gamma$ and $\sigma \sin \gamma$; any function of these variables that is proportional to an integral power m of σ cannot contain Fourier harmonics of order higher than m . Therefore, if the coefficient of $\cos 3n\gamma$ can be represented by a power series in σ , the lowest power that occurs will be σ^{3n} . These arguments show that a symmetric function such as $\langle r_1^2 \rangle$, $\mathcal{F}$, or η can be developed in a series of the form

$$\begin{aligned}\alpha_0 &+ \beta_0 \sigma^2 + \gamma_0 \sigma^4 + \delta_0 \sigma^6 + \dots \\ &+ (\alpha_3 \sigma^3 + \beta_3 \sigma^5 + \dots) \cos 3\gamma \\ &+ (\alpha_6 \sigma^6 + \dots) \cos 6\gamma \\ &+ \dots\end{aligned}\tag{4.19}$$

Herein lies a principal advantage of the present choice of parameters; $\langle r_1^2 \rangle$ is independent of γ to order σ^2 , is a linear function of $\cos 3\gamma$ to order σ^5 , and is a quadratic function of $\cos 3\gamma$ to order σ^8 .

In order to find the coefficients in the series (4.19) for $\langle r_1^2 \rangle$, we first consider the series

$$\xi + 3 = a^2 + b^2 + c^2 = \sum_{n=0}^{\infty} g_n(\sigma) \cos 3n\gamma, \quad (4.20)$$

$$g_n(\sigma) = (2 - \delta_{n0}) \frac{1}{2\pi} \int_{-\pi}^{\pi} (a^2 + b^2 + c^2) \cos 3n\gamma \, d\gamma.$$

The three terms in parentheses make equal contributions to the integral, for they differ from one another only by a phase angle in γ of $2\pi/3$, which is a period of $\cos 3n\gamma$. Replacing the terms in parentheses by $3c^2$, we obtain¹⁴

$$\begin{aligned} g_n(\sigma) &= (2 - \delta_{n0}) \frac{3}{\pi} \int_0^{\pi} e^{2\sigma \cos \gamma} \cos 3n\gamma \, d\gamma \\ &= (2 - \delta_{n0}) 3 I_{3n}(2\sigma), \end{aligned}$$

where

$$I_{3n}(2\sigma) = \sum_{m=0}^{\infty} \frac{\sigma^{3n+2m}}{m! (3n+m)!} \quad (4.21)$$

is a Bessel function of imaginary argument. The Fourier expansion

¹⁴ G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, 1948), p. 181.

of ξ is therefore

$$\xi = 3 I_0(2\sigma) - 3 + 6 \sum_{n=1}^{\infty} I_{3n}(2\sigma) \cos 3n\gamma \quad (4.22)$$

$$\begin{aligned} &= 3\sigma^2 + \sigma^3 \cos 3\gamma + \frac{3}{4} \sigma^4 \\ &\quad + \frac{\sigma^5}{4} \cos 3\gamma + \frac{\sigma^6}{120} (10 + \cos 6\gamma) + \dots \end{aligned} \quad (4.23)$$

Since η differs from ξ only by a phase angle of π in γ , we have

$$\eta = 3 I_0(2\sigma) - 3 + 6 \sum_{n=1}^{\infty} (-1)^n I_{3n}(2\sigma) \cos 3n\gamma. \quad (4.24)$$

Finally we substitute these expansions in Eq.(4.12) and obtain, to terms of order σ^6 ,

$$\begin{aligned} \langle r_1^2 \rangle = & 1 - \frac{1}{5} \sigma^2 - \frac{1}{105} \sigma^3 \cos 3\gamma + \frac{1}{28} \sigma^4 \\ & + \frac{13}{4620} \sigma^5 \cos 3\gamma - \frac{203}{28,600} \sigma^6 + \frac{379}{900,900} \sigma^6 (\cos 3\gamma)^2 + \dots \end{aligned} \quad (4.25)$$

Inspection of the coefficients shows that $\langle r_1^2 \rangle$ should be very nearly a linear function of $\cos 3\gamma$ for $\sigma \lesssim 1$. This expectation is borne out by Table I, which was computed by use of Eqs. (3.7), (3.8), (3.4), and (4.13). Even for σ as large as 2, the value of $\langle r_1^2 \rangle$ at $\cos 3\gamma = 0$ differs by less than 1 % from the average of its values at $\cos 3\gamma = \pm 1$. This nearness to linearity, which is shown graphically in Fig. 1, is impressive when one realizes that $\sigma = 2$ corresponds to elliptic integrals with argument $\phi > 87^\circ$ and to prolate or oblate ellipsoids of revolution with one axis twenty times longer than another.

The dependence of $\langle r_1^2 \rangle$ on σ is illustrated in Fig. 2; curves for other values of γ would lie between the prolate and oblate curves. The behavior at large σ can be obtained from Eq.(3.9) in the prolate case:

$$\begin{aligned} \cos 3\gamma = 1, \quad e^{-\sigma/2} = a = b < c = e^\sigma, \\ \langle r_1^2 \rangle = e^{-\sigma} (1 - e^{-3\sigma})^{-1/2} \cosh^{-1} e^{3\sigma/2} \\ = e^{-\sigma} \left(\frac{3}{2} \sigma + \ln 2 \right) + e^{-4\sigma} \frac{1}{2} \left(\frac{3}{2} \sigma + \ln 2 - \frac{1}{2} \right) + \dots \end{aligned} \quad (4.27)$$

From Eq.(3.10) the corresponding relations for the oblate case are

$$\begin{aligned} \cos 3\gamma &= -1, \quad e^{-\sigma} = a < b = c = e^{\sigma/2}, \\ \langle r_1^2 \rangle &= e^{-\sigma/2} (1 - e^{-3\sigma})^{-1/2} \cos^{-1} e^{-3\sigma/2} \\ &= \frac{\pi}{2} e^{-\sigma/2} - e^{-2\sigma} + \frac{\pi}{4} e^{-7\sigma/2} - \dots \end{aligned} \quad (4.28)$$

One would like to have a simple formula giving a good approximation to $\langle r_1^2 \rangle$ for all values of $\cos 3\gamma$ when σ is large; Eq. (4.30) below falls just a little short of this mark. From an expansion of $F(\phi, k)$ in the neighborhood of its singularity,¹⁵ we find

$$\begin{aligned} F(\phi, k) &\approx \ln 4 - \ln [\cos \phi + (1 - k^2 \sin^2 \phi)^{1/2}] \\ &\quad (\cos \phi \ll 1, 1 - k \ll 1). \end{aligned} \quad (4.29)$$

¹⁵ E. L. Kaplan, J. Math. and Phys. 27, 11 (1948).

As σ tends to infinity, $\cos \phi$ tends to zero uniformly in γ , but k tends to unity (non-uniformly in γ) for $\cos 3\gamma \neq -1$ while remaining always zero in the oblate case. Excluding the oblate case, we substitute Eq.(4.29) in Eq.(3.7) and obtain

$$\begin{aligned} \langle r_1^2 \rangle &\approx \frac{1}{c} \ln \frac{4c}{a+b}, \\ \langle r_1^2 \rangle &\approx e^{-\sigma \cos \gamma} \left[\ln 4 + \sqrt{3} \sigma \sin \left(\frac{\pi}{3} - \gamma \right) - \ln (1 + e^{-\sqrt{3} \sigma \sin \gamma}) \right] \\ &\quad (0 \leq \gamma < \frac{\pi}{3}). \end{aligned} \quad (4.30)$$

For $\sigma = 2$ the error of this approximation increases from 0.2% to 4% as $\cos 3\gamma$ decreases from 0.8 to -0.8; for $\sigma = 5$ the corresponding errors are 0.02% and 0.5%.

V. TWO EXAMPLES

The deformations found in atomic nuclei ($\sigma \lesssim 0.3$) are small enough so that values of Coulomb deformation energy obtained from Eq.(2.5) do not differ significantly from those obtained from the assumption of a uniform charge distribution with quadrupole deformation. The point here is not the value but the assumption on which it is based; we have shown that the results are valid for a nucleus with a diffuse surface if the charge distribution is ellipsoidal in the sense of Sec.I.

Perhaps as reasonable an assumption as one can make at present about the charge distribution in a heavy nucleus is that it has a deformed Fermi shape:

$$\rho(R) = \rho_0 \left[1 + \exp \frac{R - 1.07 A^{\frac{1}{3}} f}{2.40 f / 4.39} \right]^{-1}, \quad (5.1)$$

where R is defined by Eq.(2.1) and ρ_0 is determined by the total charge. We shall now use this model to calculate the Coulomb deformation energy of Gd^{160} . The corresponding spherical Fermi distribution, with R replaced by r , has a Coulomb energy and a mean square radius given by¹⁶

$$\begin{aligned} U &= 560 \text{ Mev} , \\ \overline{r^2} &= 24.4 f^2 . \end{aligned} \quad (5.2)$$

¹⁶ Reference 7 : Eqs.(4.11), (4.12), and (4.24).

The amount of deformation is determined by the intrinsic quadrupole moment, found experimentally from the Coulomb excitation cross section

to have magnitude¹⁷

$$|Q_0| = 10 \times 10^{-24} \text{ cm}^2 ; \quad (5.3)$$

the sign of Q_0 is positive if the nucleus is a prolate ellipsoid of revolution. The relation between Q_0 and σ is easily found

¹⁷ Alder, Bohr, Huus, Mottelson, and Winther, Revs. Modern Phys. 28, 432 (1956), Table V.2 .

from the equation

$$\int (2z^2 - x^2 - y^2) \rho(R) d\underline{r} = (2c^2 - a^2 - b^2) \frac{1}{3} \int \rho(R) R^2 d\underline{R} . \quad (5.4)$$

The integral on the right-hand side is the total charge times the mean square radius of the corresponding spherical distribution.

For a prolate ellipsoid of revolution we thus have

$$\begin{aligned} Q_0 &= \frac{2}{3} (e^{2\sigma} - e^{-\sigma}) Z \overline{r^2}_{\text{spher.}} \\ &= Z \overline{r^2}_{\text{spher.}} 2\sigma (1 + \frac{1}{2}\sigma + \dots) . \end{aligned} \quad (5.5)$$

In the oblate case the sign of σ is reversed in Eq.(5.5).

Substitution of Eqs.(5.2) and (5.3) leads to

$$\begin{aligned} \sigma &= 0.27 , \\ \langle r_1^2 \rangle &= 0.985 , \end{aligned} \quad (5.6)$$

where the latter value is obtained from Eq.(4.25). Thus the energy of the deformed distribution (5.1) is lower than 560 Mev by 1.5% or 8.3 Mev. Since the quantum-mechanical exchange corrections amount to approximately 5 % of the classical Coulomb energy, according to the statistical estimate, a better figure for the Coulomb deformation

energy would be 7.9 Mev .

Two remarks may be made in support of the assumption (5.1).

First, an ellipsoidal distribution satisfies the reasonable condition¹⁸

¹⁸ D. L. Hill, "Matter and Charge Distribution within Atomic Nuclei,"
Handbuch der Physik (Springer, Berlin, 1957), Vol.XXXIX, p. 181 .

that the dependence of density on radius should be the same in all directions aside from an angle-dependent radial scale factor, for this condition is equivalent to requiring that the surfaces of constant density be simple enlargements of one another. Second, a widely used nuclear model is that of nucleons moving in an anisotropic harmonic oscillator potential, which may be written in the form $V = \frac{1}{2} s R^2$; the ellipsoidal surfaces of constant nuclear potential are, according to the Thomas-Fermi statistical approximation, also surfaces of constant density.

Assumptions about the mass distribution in stellar galaxies must be even more tentative. For elliptical galaxies the observed luminosity contours are indeed approximately elliptical, although their eccentricity tends to increase and then go through a broad maximum as one proceeds outwards from the center.¹⁹ Furthermore,

¹⁹ G. de Vaucouleurs, "General Physical Properties of External Galaxies," Handbuch der Physik (Springer, Berlin, 1959), Vol.LIII, p.322 .

there are marked departures from elliptical shape in galaxies with a high degree of flattening. However, the ratio of mass to luminosity may not be constant throughout a galaxy.

A more direct source of information about mass distribution is the measurement of rotational velocity as a function of distance from the galactic center. If the motion is assumed to be circular, the gravitational force can then be deduced as a function of distance. A detailed analysis of this kind has been carried out for radio observations of the Large Magellanic Cloud by Kerr and de Vaucouleurs.²⁰

²⁰ F. J. Kerr and G. de Vaucouleurs, Australian J. Phys. 9, 90 (1956).

Although this is not an elliptical galaxy, they have fitted the data with an oblate ellipsoidal distribution of the form

$$\rho = \begin{cases} \rho_0 \left(1 - \frac{R^2}{A^2}\right) & R \leq A \\ 0 & A \leq R \end{cases} \quad (5.7)$$

$$a = b = 5c ,$$

$$A = 5^{-1/3} \times 2.8 \text{ kiloparsec} = 1.64 \text{ kpc}$$

$$\rho_0 = 0.30 \text{ suns/pc}^3 .$$

The total mass, $M = 2.2 \times 10^9$ suns, includes their estimated correction for random motions, but not an additional correction for mass in the outer fringe of the galaxy, beyond $R = A$. From Eqs.(2.11) and (4.13) and Table I, we find

$$\begin{aligned} \sigma &= 1.07 , \quad \gamma = \pi , \\ \langle r_1^2 \rangle &= 0.82 , \\ \tau(R) &= \begin{cases} -\pi G \rho_0 A^2 \left(1 - \frac{R^2}{A^2}\right)^2 & R \leq A \\ 0 & A \leq R \end{cases} \end{aligned} \quad (5.8)$$

At the center of the galaxy, the potential is $\langle r_1^2 \rangle \tau(0)$, according to Sec. II, and the escape velocity is found from this to be 120 km/sec. The spherical distribution corresponding to (5.7) has, by Eq.(2.14), a gravitational energy

$$U(1,1,1) = -\frac{5}{7} \frac{G M^2}{A} ; \quad (5.9)$$

on dividing by the squared velocity of light, the mass equivalent of this energy turns out to be 81 suns. It should be noted that the gravitational energy of the galaxy, as defined here, does not include the gravitational self-energies of the individual stars. Since the energy of the deformed distribution is $\langle r_1^2 \rangle U(1,1,1)$, the gravitational energy of deformation has a mass equivalent of 15 suns.

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APPENDIX

The incomplete elliptic integrals $F(\phi, k)$ required for Table I have values of k that are often very close to unity. Except for the elementary case $k = 0$, they were computed by recursion relations derived from King's "hyperbolic scale of increasing amplitudes." ²¹ The recursion process terminates

²¹ L. V. King, On the Direct Numerical Calculation of Elliptic Functions and Integrals (Cambridge University Press, Cambridge, 1924).

with the first step when k is unity and converges rapidly unless $1 - \sin \phi \ll 1 - k$.

One calculates first a succession of arithmetic and geometric means:

$$\begin{aligned} \alpha_0 &= 1, & \gamma_0 &= k \neq 0, \\ \alpha_{n+1} &= \frac{1}{2} (\alpha_n + \gamma_n), & \gamma_{n+1} &= (\alpha_n \gamma_n)^{\frac{1}{2}}, \\ \omega_n &= \frac{1}{4} (\alpha_{n+1} / \alpha_{n+2})^2. \end{aligned} \tag{A.1}$$

Four iterations always sufficed to determine α_∞ . The main recursion relations are then

$$\begin{aligned} H &= (1 + \sin \phi) / \cos \phi, \\ P_0 &= \frac{1-k}{1+k} H^{-2}, & R_0 &= \frac{1-k}{1+k} H^2, \\ Q_n &= \frac{1 + R_n}{1 + P_n}, \\ P_{n+1} &= \omega_n P_n^2 Q_n, & R_{n+1} &= \omega_n R_n^2 / Q_n. \end{aligned} \tag{A.2}$$

The elliptic integral is given by

$$F(\phi, k) = (\ln H - \sum_{n=0}^{\infty} 2^{-n-2} \ln Q_n) / \alpha_{\infty} . \quad (A.3)$$

The value of n at which the terms of the series became less than 10^{-8} was approximately 6 as an average but increased to 12 for $\sigma = 5$, $\cos 3\gamma = -0.8$.

The connection between King's recursion relations and ours is given by

$$\begin{aligned} \ln H &= \phi_0 , \\ \ln Q_n &= 4\phi_n - 2\phi_{n+1} . \end{aligned} \quad (A.4)$$

Caption for Table I, which is given on the following page:

Table I. The mean square radius $\langle r_1^2 \rangle$ of the ellipsoid $(x_1/a)^2 + (y_1/b)^2 + (z_1/c)^2 = 1$, where $abc = 1$. The mean is an average over angles, defined by Eq.(2.8). The parameters σ and $\cos 3\gamma$, defined by Eqs.(4.14), are symmetric functions of a , b , and c ; the magnitude of the deformation from sphericity is measured by σ and the shape of the deformation by $\cos 3\gamma$. The value of $\cos 3\gamma$ is 1 for a prolate ellipsoid of revolution, -1 for an oblate one, and 0 for a triaxial ellipsoid with one semi-axis of unit length. The ratio of axis lengths for an ellipsoid of revolution is $\exp(3\sigma/2)$. The relation of $\langle r_1^2 \rangle$ to the incomplete elliptic integral of the first kind is given by Eq.(3.7). The ratio of the Coulomb energy of an ellipsoidal charge distribution to the energy of the corresponding spherical distribution is equal to $\langle r_1^2 \rangle$.

Table I. (Table caption on preceding page)

$\frac{\cos 3\gamma}{\sigma}$	-1.0	- 0.8	- 0.6	- 0.4	- 0.2	0.0	0.2	0.4	0.6	0.8	1.0
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.05	0.99950	0.99950	0.99950	0.99950	0.99950	0.99950	0.99950	0.99950	0.99950	0.99950	0.99950
0.10	0.99801	0.99801	0.99801	0.99801	0.99801	0.99800	0.99800	0.99800	0.99800	0.99800	0.99799
0.15	0.99555	0.99554	0.99554	0.99553	0.99552	0.99552	0.99551	0.99551	0.99550	0.99549	0.99549
0.20	0.99213	0.99212	0.99210	0.99209	0.99207	0.99206	0.99204	0.99203	0.99201	0.99200	0.99198
0.25	0.98778	0.98775	0.98773	0.98770	0.98767	0.98764	0.98761	0.98758	0.98755	0.98752	0.98749
0.30	0.98254	0.98248	0.98243	0.98238	0.98233	0.98228	0.98223	0.98218	0.98213	0.98208	0.98203
0.35	0.97642	0.97634	0.97626	0.97618	0.97610	0.97602	0.97594	0.97587	0.97579	0.97571	0.97563
0.40	0.96947	0.96935	0.96924	0.96912	0.96900	0.96889	0.96877	0.96865	0.96854	0.96842	0.96831
0.45	0.96173	0.96156	0.96140	0.96124	0.96107	0.96091	0.96074	0.96058	0.96042	0.96026	0.96009
0.50	0.95324	0.95302	0.95279	0.95257	0.95235	0.95213	0.95191	0.95169	0.95146	0.95124	0.95103
0.6	0.93419	0.93382	0.93344	0.93307	0.93269	0.93232	0.93195	0.93158	0.93121	0.93085	0.93048
0.7	0.91270	0.91212	0.91154	0.91096	0.91039	0.90982	0.90925	0.90869	0.90813	0.90757	0.90702
0.8	0.88914	0.88830	0.88747	0.88664	0.88581	0.88500	0.88419	0.88338	0.88258	0.88179	0.88100
0.9	0.86391	0.86275	0.86160	0.86047	0.85934	0.85823	0.85713	0.85603	0.85495	0.85389	0.85283
1.0	0.83736	0.83583	0.83431	0.83282	0.83135	0.82989	0.82845	0.82703	0.82564	0.82425	0.82289
1.2	0.78169	0.77924	0.77684	0.77448	0.77218	0.76991	0.76770	0.76553	0.76341	0.76132	0.75929
1.4	0.72452	0.72096	0.71749	0.71412	0.71084	0.70766	0.70457	0.70156	0.69864	0.69581	0.69307
1.6	0.66774	0.66290	0.65824	0.65374	0.64941	0.64524	0.64123	0.63737	0.63366	0.63009	0.62666
1.8	0.61268	0.60646	0.60052	0.59485	0.58945	0.58429	0.57937	0.57469	0.57023	0.56599	0.56197
2.0	0.56024	0.55258	0.54535	0.53852	0.53206	0.52597	0.52021	0.51478	0.50967	0.50486	0.50035
2.5	0.44343	0.43228	0.42205	0.41265	0.40399	0.39601	0.38866	0.38189	0.37567	0.36998	0.36481
3.0	0.34804	0.33400	0.32155	0.31043	0.30045	0.29147	0.28338	0.27609	0.26955	0.26371	0.25857
3.5	0.27206	0.25605	0.24236	0.23050	0.22014	0.21102	0.20297	0.19585	0.18960	0.18414	0.17947
4.0	0.21225	0.19521	0.18122	0.16949	0.15949	0.15089	0.14343	0.13696	0.13136	0.12657	0.12259
4.5	0.16544	0.14820	0.13467	0.12368	0.11455	0.10686	0.10031	0.09471	0.08994	0.08593	0.08269
5.0	0.12889	0.11212	0.09956	0.08968	0.08169	0.07509	0.06957	0.06492	0.06100	0.05776	0.05520

FIGURE CAPTIONS

Fig. 1. The anisotropy factor $\langle r_1^2 \rangle$ as a function of the shape parameter for fixed values of the deformation magnitude. The symbols are explained in the caption of Table I.

Fig. 2. The anisotropy factor $\langle r_1^2 \rangle$ as a function of the deformation magnitude for ellipsoids of revolution. The symbols are explained in the caption of Table I.