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**RADIATION PATTERNS OF APODISED ACOUSTIC
LENSES WITH OFF-AXIS IMAGE-PLANE FEEDS**

S. Frankel
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Menlo Park, California
For
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SUMMARY

The radiation pattern of an acoustic lens is a function of the aperture pressure distribution, which in turn is a product of the primary-feed pressure pattern and the lens apodisation function. Starting with an arbitrary* primary pattern from the axial, image-plane feed point, and an appropriate apodisation function to yield a desired distribution over the aperture, a general formulation is derived for the radiation pattern resulting when the feed is moved, within the image plane, a distance off the axis large compared to a wavelength but small compared to the aperture radius. The result requires machine computation of two integrals. Patterns** are plotted for various values of significant parameters but only in the azimuth plane defined by the feed point and the acoustic axis. Results indicate that theoretical side lobe levels are not seriously deteriorated by image-plane shifts appropriate to certain previously designated design objectives. However, practical field experience dictates that these patterns probably cannot be realized within the present state of the art.

* within limitations

** Since we plot the curves on a log scale, the dips between side lobes should extend to $-\infty$ because positive and negative phasing between alternate side lobes cause the curve to pass through zero. The plots were generated by an SC-4020 High Speed Microfilm Recorder with values calculated by a CDC-6600 computer, and it was decided to leave them as generated and let the reader keep in mind the above information.

GLOSSARY

a: Aperture Radius

$B(\rho)$, $C(\rho)$: Eqs. (30), p. 11

$b(r)$, $c(r)$: Eqs. (34), p. 13

D , D_0 : Eqs. (26), p. 10; Fig. 1, p. 8

E : Eqs. (10), p. 4

F : Lens/Image-Plane Distance: Fig. 1, p. 8

$F_d(r)$: Lens/Apodiser Pressure Distribution: Eq. (32), p. 12

$f(\rho)$: Eqs. (5) and (6), p. 3

$f_a(\rho)$: Eq. (5), p. 3

f_e : Eq. (20), p. 7; Eqs. (21) and (22), p. 9

$f_{eo}(\rho)$: Eq. (1), p. 2

I : Eq. (14), p. 5

I_0 : Eq. (15), p. 5

I_1 , I_2 : Eqs. (40), p. 14

$J_k(u)$: Bessel Function of First Kind, k^{th} order, $k = 0, 1, 2, \dots$

$\bar{K}$: Eq. (58), p. 22

K_1 , K_2 : Eqs. (57), p. 22

L/A: Lens/Apodiser System

$\ell(r)$: Eqs. (34), p. 13

M : Eq. (2), p. 2

$N = F/a = 2 \times F/\#$ of Aperture

$n = \rho_1/a$

$P(u)$: Eq. (19), p. 6

$P(u, \phi)$: Eq. (32), p. 12

$p(\rho)$: Eq. (6), p. 3

$p_e(\rho, \phi_\ell; \rho_i, \phi_i)$: Eq. (20), p. 7; Eq. (22), p. 9

p_{eo} : Eq. (1), p. 2

$r = \rho/a$

$u = \frac{2\pi a}{\lambda} \sin \theta$

v : Eqs. (55), p. 21

$\alpha_g(\rho)$: Eq. (11), p. 5

$\beta = \phi - \phi_i$: Eqs. (34), p. 13

γ_i, γ_r : Eqs. (38), p. 14

$\delta(r^2) = \delta(s)$: Eqs. (13), p. 5

$\mathcal{E}(\rho)$: Eqs. (10), p. 4

θ = Field Point Meridian Angle (Spherical Coordinates)

$\Lambda_p(u)$: Eq. (18a), p. 6

λ : Wavelength of Acoustic Radiation

μ, μ_0 : Eqs. (20), p. 7; Eqs. (21), p. 9

μ_ℓ : Eq. (2), p. 2

$u = \phi - \phi_\ell$: Eqs. (34), p. 13

ξ : Eq. (24), p. 9

ρ = Radial Distance From Acoustic Axis to a Point in L/A Aperture Plane (Fig. 1)

ρ' : Eq. (21), p. 9

ρ_i = Radial Distance in Image Plane (Fig. 1) from Acoustic Axis to Source

ϕ = Field Point Azimuth Angle (Spherical Coordinates)

ϕ_i = Azimuth Angular Position of a Point in Image Plane (Fig. 1)

ϕ_ℓ = Azimuth Angular Position of a Point in the L/A Aperture Plane (Fig. 1)

$\phi_\ell' = \phi_\ell - \phi_i$: Eq. (21), p. 9

CONTENTS

	<u>Page</u>
1. Introduction	1
2. Aperture Distribution: General	2
2.1 Axial Source	4
2.2 Off-Axis Source	7
3. Radiation Pattern	12
4. Practical Limitations	25
5. Conclusions	25
References	26
Appendix	55
Table I. Side-Lobe Maxima vs. m	23
 <u>Figures</u>	
1. Geometry for Off-Axis Feed	8
2. $\bar{K}$ vs. u ; $m = 2$, $v = 0$	27
3. $\bar{K}$ vs. u ; $m = 3$, $v = 0$	28
4. $\bar{K}$ vs. u ; $m = 4$, $v = 0$	29
5. $\bar{K}$ vs. u ; $m = 4$, $v = 1$, $N = 2$, $n = 0.01$	30
6. $\bar{K}$ vs. u ; $m = 4$, $v = 1$, $N = 2$, $n = 0.03$	31
7. $\bar{K}$ vs. u , $m = 4$, $v = 1$, $N = 2$, $n = 0.1$	32
8. $\bar{K}$ vs. u ; $m = 4$, $v = 3$, $N = 2$, $n = 0.01$	33

<u>Figures (Continued)</u>	<u>Page</u>
9. $\bar{K}$ vs. u ; $m = 4$, $v = 3$, $N = 2$, $n = 0.03$	34
10. $\bar{K}$ vs. u ; $m = 4$, $v = 3$, $N = 2$, $n = 0.1$	35
11. $\bar{K}$ vs. u ; $m = 4$, $v = 10$, $N = 2$, $n = 0.01$	36
12. $\bar{K}$ vs. u ; $m = 4$, $v = 10$, $N = 2$, $n = 0.03$	37
13. $\bar{K}$ vs. u ; $m = 4$, $v = 10$, $N = 2$, $n = 0.1$	38
14. $\bar{K}$ vs. u ; $m = 4$, $v = 30$, $N = 2$, $n = 0.01$	39
15. $\bar{K}$ vs. u ; $m = 4$, $v = 30$, $N = 2$, $n = 0.03$	40
16. $\bar{K}$ vs. u ; $m = 4$, $v = 30$, $N = 2$, $n = 0.1$	41
17. $\bar{K}$ vs. u ; $m = 5$, $v = 0$	42
18. $\bar{K}$ vs. u ; $m = 5$, $v = 1$, $N = 2$, $n = 0.01$	43
19. $\bar{K}$ vs. u ; $m = 5$, $v = 1$, $N = 2$, $n = 0.03$	44
20. $\bar{K}$ vs. u ; $m = 5$, $v = 1$, $N = 2$, $n = 0.1$	45
21. $\bar{K}$ vs. u ; $m = 5$, $v = 3$, $N = 2$, $n = 0.01$	46
22. $\bar{K}$ vs. u ; $m = 5$, $v = 3$, $N = 2$, $n = 0.03$	47
23. $\bar{K}$ vs. u ; $m = 5$, $v = 3$, $N = 2$, $n = 0.1$	48
24. $\bar{K}$ vs. u ; $m = 5$, $v = 10$, $N = 2$, $n = 0.01$	49
25. $\bar{K}$ vs. u ; $m = 5$, $v = 10$, $N = 2$, $n = 0.03$	50
26. $\bar{K}$ vs. u ; $m = 5$, $v = 10$, $N = 2$, $n = 0.1$	51
27. $\bar{K}$ vs. u ; $m = 5$, $v = 30$, $N = 2$, $n = 0.01$	52
28. $\bar{K}$ vs. u ; $m = 5$, $v = 30$, $N = 2$, $n = 0.03$	53
29. $\bar{K}$ vs. u ; $m = 5$, $v = 30$, $N = 2$, $n = 0.1$	54
<u>Appendix Figures</u>	
30. $ K_1 $ vs. u ; $m = 4$, $v = 1$, $N = 2$	57
31. $ K_2 $ vs. u ; $m = 4$, $v = 1$, $N = 2$	58

Appendix Figures (Concluded)

		<u>Page</u>
32.	K_1 vs. u; m = 4, v = 3, N = 2	59
33.	K_2 vs. u; m = 4, v = 3, N = 2	60
34.	K_1 vs. u; m = 4, v = 10, N = 2	61
35.	K_2 vs. u; m = 4, v = 10, N = 2	62
36.	K_1 vs. u; m = 4, v = 30, N = 2	63
37.	K_2 vs. u; m = 4, v = 30, N = 2	64
38.	K_1 vs. u; m = 5, v = 1, N = 2	65
39.	K_2 vs. u; m = 5, v = 1, N = 2	66
40.	K_1 vs. u; m = 5, v = 3, N = 2	67
41.	K_2 vs. u; m = 5, v = 3, N = 2	68
42.	K_1 vs. u; m = 5, v = 10, N = 2	69
43.	K_2 vs. u; m = 5, v = 10, N = 2	70
44.	K_1 vs. u; m = 5, v = 30, N = 2	71
45.	K_2 vs. u; m = 5, v = 30, N = 2	72

1. Introduction

In a previous report we discussed the radiation patterns of circular-aperture projectors wherein the aperture-excitation was assumed to be precisely controlled in specified ways [1] . In particular, it was assumed that an aperture excited from a point source in the image plane slightly off the acoustic axis would radiate the same pattern as the axially-excited aperture, but tilted slightly with respect to the axial pattern.

In practice a variety of causes result, generally, in small deviations of the aperture distribution, both in amplitude and in phase, from the presumed distribution. In addition to that mentioned above, namely pattern deviation for an off-axis source, we may mention (1) effects of random variations in system geometry and homogeneities of acoustic propagating media (e.g., lens materials, apodisation media) [2] * ; (2) failure of amplitude to go to zero in appropriate fashion at aperture boundary because of uncontrollable non-random physical factors**; (3) local reflections due to supports and enclosures may be difficult to control; (4) acoustic system supports may introduce strains in the system which result in slight internal refractions and consequent phase errors [3, 11] .

Of immediate concern in this report is investigation of the first of these, namely, the effect of associated undesired shifts in aperture distribution resulting from off-axis excitation.

* Results of scalar electromagnetic theory are applicable, with appropriately drawn analogies [1] .

** For example, sufficiently large attenuation by apodisation may be unrealizable.

2. Aperture Distribution: General

Until we understand more about the practical limitations involved in the design of apodised imaging systems, it seems inadvisable to attempt full delineation of a rational design procedure. Rather, we can, perhaps, contribute building blocks for such a procedure by studying specific kinds of situations.

To analyze the effect of off-axial feed consider, first, the case of an on-axis feed exciting a circular lens/apodiser (L/A) arrangement. The feed is assumed designed to excite the arrangement symmetrically, (i.e., independently of azimuth angle, ϕ) so that the excitation of the arrangement has the form

$$p_{eo} = f_{eo} e^{-j\mu_o(\rho)} \quad (1)$$

f_{eo} , in general, lacks the desired behavior because of acoustic limitations of the source, and because of other structural factors in the vicinity of the acoustic aperture; $\mu_o(\rho)$, which should be independent of ρ , generally is not, because of the appreciable variation of path length from source to points of the L/A entrance pupil.

The purpose of the lens is, as usual, to introduce a correction in the phase behavior, so that the overall phase shift is independent of ρ ; i.e., the lens introduces a shift

$$\mu_l = M - \mu_o(\rho) \quad (2)$$

where M is a constant, sufficiently large that $\mu_l > 0$ for all $\rho \leq a$, where a is the pupil radius. Since the largest path length occurs for $\rho = a$, it is frequently convenient to take

$$\mu_l(a) = 0 = M - \mu_o(a)$$

that is

$$M = \mu_o(a) \quad (3)$$

so that

$$\mu_\ell(\rho) = \mu_o(a) - \mu_o(\rho) \quad (4)$$

The purpose of the apodiser is to adjust the aperture amplitude to a desired function, $f(\rho)$, essentially with phase independent of ρ and ϕ . Obviously the apodising function is*

$$f_a(\rho) = \frac{f(\rho)}{f_{eo}(\rho)} \quad (5)$$

and the overall effective aperture distribution is

$$\begin{aligned} p(\rho) &= \begin{bmatrix} f_{eo}(\rho) e^{-j\mu_o(\rho)} \end{bmatrix} \begin{bmatrix} e^{-j\mu_\ell(\rho)} \end{bmatrix} \begin{bmatrix} f_a(\rho) \end{bmatrix} \\ &= f(\rho) e^{-j\mu_o(a)} \end{aligned} \quad (6)$$

in which the constant phase shift, $\mu_o(a)$, can be ignored.

The possibility that apodisation without relative phase shift is unattainable is not to be considered a fundamental limitation, since in principle, this may be corrected by further adjustment of the lens function; and, hopefully, vice versa.

When the source is moved off axis, the excitation, in general, no longer has the desired distribution, and, in fact, is, in general, no longer independent of ϕ . With

ρ_i = distance from source to acoustic axis

ϕ_i = azimuth angle of source with respect to reference azimuth

ϕ_ℓ = azimuth angle of a point in the lens plane with respect to reference azimuth

* See discussion following Eq. (31), Section 2.2, below.

We can write, for the pressure distribution at the entrance of the L/A system

$$p_e = p_e(\rho, \phi_\ell; \rho_1, \phi_1) = f_e(\rho, \phi_\ell; \rho_1, \phi_1) e^{-j\mu(\rho, \phi_\ell; \rho_1, \phi_1)} \quad (7)$$

which for an axial source reduces to (1) as

$$p_e(\rho, \phi_\ell; 0, 0) = p_{eo}(\rho) = f_{eo}(\rho) e^{-j\mu_o(\rho)} \quad (8)$$

The effect of the L/A system on (7) is to transform it to

$$p(\rho, \phi_\ell; \rho_1, \phi_1) = f(\rho) E(\rho, \phi_\ell; \rho_1, \phi_1) e^{-j\mathcal{E}(\rho, \phi_\ell; \rho_1, \phi_1)} \quad (9)$$

where $f(\rho)$ is the desired distribution [Eq. (6)] with the factor $e^{-j\mu(a)}$ omitted, and

$$\left. \begin{aligned} E(\rho, \phi_\ell; \rho_1, \phi_1) &= \frac{f_e(\rho, \phi_\ell; \rho_1, \phi_1)}{f_{eo}(\rho)}, \text{ provided } f_{eo}(\rho) \neq 0 \\ \mathcal{E}(\rho, \phi_\ell; \rho_1, \phi_1) &= \mu(\rho, \phi_\ell; \rho_1, \phi_1) - \mu_o(\rho) \end{aligned} \right\} \quad (10)$$

E and $\mathcal{E}$ measure deviation of amplitude and phase functions from the axial "norm".

2.1 Axial Source

Eq. (5) specifies the required apodiser transfer function when the source is axial. Conversely, the reciprocal of f_a measures the required attenuation function. In db,

$$\alpha_a(\rho) = 20 \log \frac{f_{eo}(\rho)}{f(\rho)} \quad (11)$$

As an example, if we wish to realize the type of distribution discussed in [1], namely

$$f(\rho) = \left[1 - \left(\frac{\rho}{a} \right)^2 \right]^m \quad (12)$$

where a is the radius of the lens, then we require that $f(a) = 0$, and in fact, that the derivatives, $f^{(k)}(a) = 0$, $k = 1, \dots, (m-1)$.

Thus, if $f_{e0}(a) \neq 0$, an infinite amount of attenuation is required to realize the desired function. Since this is impossible, the question immediately reduces to: "What is the minimum acceptable attenuation?"

Let the actual distribution be

$$F(r) = f(r) + \delta(s), \quad s = r^2, \quad r = \frac{\rho}{a} \quad (13)$$

where $\delta(r^2)$ represents a small deviation from $f(r^2)$. Then the radiation pattern can be described essentially by [1]

$$I = I_0 + \int_0^1 r \delta(s) J_0(ur) dr \quad (14)$$

where

$$I_0 = \int_0^1 r f(r) J_0(ur) dr, \quad (15)$$

$u = \frac{2\pi a}{\lambda} \sin \theta$, J_0 is the Bessel function of the first kind, zero order, and

a = radius of aperture

λ = wavelength of radiation

The integral in (14) may be expanded asymptotically [1]; we have

$$I = I_0 + \left\{ \frac{J_1(u)}{u} \delta(1) - \frac{2 J_2(u)}{u^2} \delta'(1) + \frac{2^2 J_3(u)}{u^3} \delta''(1) - \dots \right\} \quad (16)$$

$$\text{where } \delta^{(k)}(1) \equiv \left. \frac{d\delta}{ds} \right|_{s=1} \quad (17)$$

and J_k ($k = 1, 2, 3, \dots$) are Bessel functions of the first kind.

In case $f(r) = (1 - r^2)^m$, we have [1]

$$I_0 = \frac{\Lambda_{m+1}(u)}{2(m+1)} \quad (18)$$

$$\text{where } \Lambda_p(u) = (2/u)^p (p!) J_p(u) \quad (18a)$$

Thus, the deviated pattern, normalized to unit main lobe of the desired pattern is

$$P(u) = \Lambda_{m+1}(u) + 2(m+1) \left\{ \frac{J_1(u)}{u} \delta(1) - \frac{2J_2(u)}{u^2} \delta'(1) + \dots \right\} \quad (19)$$

For example, suppose that all derivatives of δ are zero at $s = r^2 = 1$, but $\delta(1) \neq 0$. Take $m = 5$ and demand that the error introduced by δ not exceed -85 db for $u > 10$. Thus the requirement is

$$\left| 2(m+1) \frac{J_1(u)}{u} \delta(1) \right| \Bigg|_{\substack{u=10 \\ m=5}} < 10^{-4.25}$$

Asymptotically we can write [5]

$$\frac{J_1(u)}{u} \sim \sqrt{\frac{2}{\pi}} \frac{\cos\left(u - 3\frac{\pi}{4}\right)}{u^{3/2}}$$

Since the peak value of the cosine function is unity the borderline case is

$$\frac{2 \times 6 \sqrt{\frac{2}{\pi}} \delta(1)}{(10)^{3/2}} = 10^{-4.25}$$

$$\text{or } \delta(1) = \sqrt{\frac{\pi}{2}} \frac{10^{-2.75}}{12} = \frac{1}{5,360}, \text{ or } -75 \text{ db.}$$

For instance, if the primary excitation is 30 db down at the aperture rim, apodisation must contribute 45 db additional attenuation on this locus.

2.2 Off-Axis Source

Fig. 1 shows the geometry for a source at a point (ρ_1, ϕ_1) in the image plane. Let the distribution in the lens aperture plane for an axial source $(\rho_1 = 0)$ be

$$\left. \begin{aligned} p_e(\rho, \phi_l; 0, 0) &= f_e(\rho; 0, 0) e^{-j\mu(\rho; 0, 0)} \\ &= f_{eo}(\rho) e^{-j\mu_o(\rho)} \end{aligned} \right\} \quad (20)$$

that is, the excitation is assumed independent of ϕ_l .

Figure 1. Geometry for Off-Axis Feed

Moving the source to (ρ_1, ϕ_1) without changing the orientation is equivalent to a coordinate transformation in the lens plane:

$$\left. \begin{aligned} \rho'^2 &= \rho^2 + \rho_1^2 - 2 \rho \rho_1 \cos(\phi_\ell - \phi_1) \\ \phi_\ell' &= \phi_\ell - \phi_1 \end{aligned} \right\} \quad (21)$$

so that the distribution in the lens plane is

$$p_e(\rho, \phi_\ell; \rho_1, \phi_1) = f_e(\rho', \phi_\ell') e^{-j\mu(\rho')} \quad (22)$$

with ρ' and ϕ_ℓ' given by (21)

To express (22) in terms of first-order effect of non-axial feed, write

$$\rho' = (\rho^2 + \xi)^{\frac{1}{2}} \quad (23)$$

where

$$\xi = \rho_1^2 - 2 \rho \rho_1 \cos \phi_\ell' \quad (24)$$

Then

$$\frac{\partial f_e}{\partial \xi} = \frac{\partial f_e}{\partial \rho'} \frac{d\rho'}{d\xi} = \frac{1}{2\rho'} \frac{\partial f_e}{\partial \rho'}$$

and

$$\left(\frac{\partial f_e}{\partial \xi} \right)_{\xi=0} = \frac{1}{2\rho} \left(\frac{\partial f_e}{\partial \rho'} \right)_{\xi=0} = \frac{1}{2\rho} \left[\frac{\partial f_{ea}(\rho)}{\partial \rho} \right]$$

where $f_{eo}(\rho)$ is the amplitude of the aperture distribution resulting from axial feed, assumed independent of ϕ_l .

A Taylor's expansion to two terms yields

$$\begin{aligned} f_e(\rho', \phi_l') &= f_{eo}(\rho) + \frac{1}{2\rho} \frac{\partial f_{eo}(\rho)}{\partial \rho} \cdot \xi \\ &\approx f_{eo}(\rho) - \left[\frac{\partial f_{eo}(\rho)}{\partial \rho} \cos \phi_l' \right] \rho_1 \end{aligned} \quad (25)$$

where ρ_1^2 has been dropped from (24) as contributing only a second-order effect.

To calculate the effect of non-axial feed on the phase, μ , note that in the axial case, the path length from source to a point in the aperture plane is (Fig. 1)

$$D(\rho; 0, 0) = (F^2 + \rho^2)^{\frac{1}{2}} = D_0 \quad (26)$$

and

$$\mu_0(\rho) = kD_0, \quad k = \frac{2\pi}{\lambda} \quad (27)$$

For a non-axial feed the path length is

$$\begin{aligned} D(\rho, \phi_l; \rho_1, \phi_1) &= (F^2 + \rho'^2)^{\frac{1}{2}} \\ &= \left[(F^2 + \rho^2) + \xi \right]^{\frac{1}{2}} \\ &\approx D_0 + \frac{\xi}{2D_0} \approx D_0 - \frac{\rho\rho_1}{D_0} \cos \phi_l' \end{aligned}$$

and

$$\mu(\rho') = kD \approx kD_0 - \frac{k\rho\rho_1}{D_0} \cos \phi_l' \quad (28)$$

After transmission through the L/A system we have, using Eqs. (4) and (5),

$$\begin{aligned}
 p(\rho, \phi_l; \rho_1, \phi_1) &= p_e(\rho', \phi') f_a(\rho) e^{-j\mu_l(\rho)} \\
 &= \left\{ f_{eo}(\rho) - \left[\frac{\partial f_{eo}(\rho)}{\partial \rho} \cos \phi'_l \right] \rho_1 \right\} \left\{ \exp \left[-j \left(k D_o(\rho) - \frac{k \rho \rho_1}{D_o(\rho)} \cos \phi'_l \right) \right] \right\} \\
 &\quad \left\{ \frac{f(\rho)}{f_{eo}(\rho)} \right\} \left\{ \exp \left[-jk \left(D_o(a) - D_o(\rho) \right) \right] \right\} \\
 &= \left\{ f(\rho) - \left[\frac{f(\rho)}{f_{eo}(\rho)} \frac{\partial f_{eo}(\rho)}{\partial \rho} \cos \phi'_l \right] \rho_1 \right\} \left\{ \exp \left[-jk \left(D_o(a) - \frac{\rho \rho_1}{D_o(\rho)} \cos \phi'_l \right) \right] \right\}
 \end{aligned}$$

or, dropping the unimportant phase term $D_o(a)$,

$$p = \left\{ f(\rho) - \left[f_a(\rho) \frac{\partial f_{eo}(\rho)}{\partial \rho} \cos \phi'_l \right] \rho_1 \right\} \left\{ \exp \left[+j \frac{k \rho \rho_1}{D_o(\rho)} \cos \phi'_l \right] \right\} \quad (29)$$

Clearly this reduces to the norm, Eq. (6), except for a constant phase factor, when $\rho_1 = 0$.

$$\begin{aligned}
 \text{Write } B(\rho) &= f_a(\rho) \frac{\partial f_{eo}(\rho)}{\partial \rho} \\
 C(\rho) &= \frac{k \rho}{D_o(\rho)}
 \end{aligned} \quad (30)$$

$$p = \left[f(\rho) - B(\rho) \cdot \rho_1 \cdot \cos \phi'_l \right] e^{jC(\rho) \rho_1 \cos \phi'_l} \quad (31)$$

An important limitation must be noted at this point. The apodising function, $f_a(\rho)$, Eq. (5) is not realizable if $f_{eo}(\rho) = 0$ for any $\rho_0 \leq a$, unless we also require $f(\rho_0) = 0$. In that case,

$$f_a(\rho_0) = \lim_{\rho \rightarrow \rho_0} \left[\frac{f(\rho)}{f_{eo}(\rho)} \right] = \frac{f'(\rho_0)}{f'_{eo}(\rho_0)}$$

This point will be discussed further in connection with an example later in the report [see Eqs. (50) ff]

3. Radiation Pattern

The radiation pattern is given, essentially, by [1]

$$P(u, \phi) = \int_0^{2\pi} \int_0^1 F_d(r) e^{j u r \cos(\phi - \phi_\ell)} r dr d\phi_\ell \quad (32)$$

where

$$r = \rho/a$$

a = radius of L/A aperture

$$u = \frac{2\pi a}{\lambda} \sin \theta$$

ϕ = radiation space azimuth angle

θ = radiation space co-latitude

= angle between a line to a point in radiation space and acoustic axis

$F_d(r)$ = L/A pressure distribution

Putting $\rho = ar$ in Eqs. (30) and (31) we have

$$p(\rho) = F_d(r) = \left[f(ar) - B(ar) \rho_1 \cos \phi'_l \right] e^{jC(ar) \rho_1 \cos \phi'_l}$$

Then (32) is

$$\begin{aligned} P(u, \phi) &= \int_0^{2\pi} \int_0^1 f(ar) e^{j \left[ur \cos (\phi - \phi_l) + C \rho_1 \cos \phi'_l \right]} r dr d\phi_l \\ &- \rho_1 \int_0^{2\pi} \int_0^1 B(ar) \cos \phi'_l e^{j \left[ur \cos (\phi - \phi_l) + C \rho_1 \cos \phi'_l \right]} r dr d\phi_l \end{aligned} \quad (33)$$

Write

$$\left. \begin{aligned} f(ar) &= l(r) \\ C(ar) &= rc(r) \\ B(ar) &= b(r) \\ v &= \phi - \phi_l \\ \beta &= \phi - \phi_1 \end{aligned} \right\} \quad (34)$$

where

$$c(r) = \frac{ka}{D_0(ar)} \quad (35)$$

Then

$$\begin{aligned} \cos \phi'_l &= \cos (-\phi'_l) = \cos (\phi_1 - \phi_l) = \cos (v - \beta) \\ &= \cos \beta \cos v + \sin \beta \sin v \end{aligned} \quad (36)$$

Then the phase factor common to the integrals in (33) becomes

$$\begin{aligned}
 ur \cos (\phi - \phi'_\ell) + Cp_i \cos \phi'_\ell &= ur \cos v + rc(r)\rho_i \cos \beta \cos v \\
 &\quad + rc(r)\rho_i \sin \beta \sin v \\
 &= r \left[\gamma_r \cos v + \gamma_i \sin v \right]
 \end{aligned} \tag{37}$$

where

$$\left. \begin{aligned} \gamma_r &= u + c(r)\rho_i \cos \beta \\ \gamma_i &= c(r)\rho_i \sin \beta \end{aligned} \right\} \tag{38}$$

and (33) becomes

$$\begin{aligned}
 P(u, \phi) &= \int_0^{2\pi} \int_0^1 r l(r) \exp \left[jr (\gamma_r \cos v + \gamma_i \sin v) \right] dr d\phi_\ell \\
 &\quad - \rho_i \int_0^{2\pi} \int_0^1 r b(r) \exp \left[jr (\gamma_r \cos v + \gamma_i \sin v) \right] \cos \phi'_\ell dr d\phi_\ell \\
 &= \int_0^1 r l(r) dr \int_0^{2\pi} \exp \left[jr (\gamma_r \cos v + \gamma_i \sin v) \right] d\phi_\ell \\
 &\quad - \rho_i \int_0^1 r b(r) dr \int_0^{2\pi} \exp \left[jr (\gamma_r \cos v + \gamma_i \sin v) \right] \cos \phi'_\ell d\phi_\ell
 \end{aligned} \tag{39}$$

Write

$$\left. \begin{aligned} I_1 &= \int_0^{2\pi} \exp \left[jr (\gamma_r \cos v + \gamma_i \sin v) \right] d\phi_\ell \\ I_2 &= \int_0^{2\pi} \exp \left[jr (\gamma_r \cos v + \gamma_i \sin v) \right] \cos \phi'_\ell d\phi_\ell \end{aligned} \right\} \tag{40}$$

so that Eq. (39) is

$$P(u, \phi) = \int_0^1 r \ell(r) I_1(r; u, \phi) dr - \rho_1 \int_0^1 r b(r) I_2(r; u, \phi) dr \quad (41)$$

We consider next evaluation of the integrals I_1, I_2 , Eqs. (40). First transform ϕ_ℓ according to (34):

$$\phi_\ell = \phi - v; d\phi_\ell = -dv$$

and use Eq. (36) for $\cos \phi'_\ell$. Then we have

$$\begin{aligned} I_1 &= - \int_{\phi}^{\phi-2\pi} \exp \left[jr(\gamma_r \cos v + \gamma_1 \sin v) \right] dv \\ &= \int_{-\phi}^{2\pi-\phi} \exp \left[jr(\gamma_r \cos v - \gamma_1 \sin v) \right] dv \end{aligned} \quad (42)$$

and

$$\begin{aligned} I_2 &= - \int_{\phi}^{\phi-2\pi} \exp \left[jr(\gamma_r \cos v + \gamma_1 \sin v) \right] \left\{ \cos \beta \cos v + \sin \beta \sin v \right\} dv \\ &= \cos \beta \int_{-\phi}^{2\pi-\phi} \exp \left[jr(\gamma_r \cos v - \gamma_1 \sin v) \right] \cos v \cdot dv \\ &\quad - \sin \beta \int_{-\phi}^{2\pi-\phi} \exp \left[jr(\gamma_r \cos v - \gamma_1 \sin v) \right] \sin v dv \end{aligned} \quad (43)$$

Next, it is clear that all integrands in Eqs. (42) and (43) are periodic with periodicity 2π , and are capable of Fourier expansion in the form

$$\sum_{s=0}^{\infty} b_s \cos (sv - \phi_s)$$

where b_s and ϕ_s are independent of v . It is also clear that integration term-by-term of these series in the interval $-\phi < v < 2\pi - \phi$ yields zero for all $s \neq 0$. Thus, only the terms in the expansions independent of v are of significance in the integrals.

The required expansions are in principle, easily obtained, but the results are generally unwieldy. We have [6]

$$e^{j\gamma_r r \cos v} = J_0(\gamma_r r) + 2jJ_1(\gamma_r r) \cos v + 2j^2J_2(\gamma_r r) \cos 2v + \dots$$

$$\begin{aligned} e^{-j\gamma_i r \sin v} &= e^{-j\gamma_i r \cos \left(\frac{\pi}{2} - v\right)} \\ &= J_0(-\gamma_i r) + 2jJ_1(-\gamma_i r) \cos \left(\frac{\pi}{2} - v\right) + 2j^2J_2(-\gamma_i r) \cos \left[2\left(\frac{\pi}{2} - v\right)\right] + \dots \\ &= J_0(\gamma_i r) - 2jJ_1(\gamma_i r) \sin v - 2j^2J_2(\gamma_i r) \cos 2v + \dots \end{aligned}$$

Thus

$$\begin{aligned} \exp \left[jr(\gamma_r \cos v - \gamma_i \sin v) \right] &= \left(e^{j\gamma_r r \cos v} \right) \left(e^{-j\gamma_i r \sin v} \right) \\ &= \left[J_0(\gamma_r r) + 2jJ_1(\gamma_r r) \cos v + 2j^2J_2(\gamma_r r) \cos 2v + \dots \right] \\ &\quad \left[J_0(\gamma_i r) - 2jJ_1(\gamma_i r) \sin v - 2j^2J_2(\gamma_i r) \cos 2v + \dots \right] \\ &= J_0(\gamma_r r) J_0(\gamma_i r) + \left[-2jJ_0(\gamma_r r) J_1(\gamma_i r) \sin v + 2jJ_1(\gamma_r r) J_0(\gamma_i r) \cos v \right] \\ &\quad + \left\{ \left[-2j^2J_0(\gamma_r r) J_2(\gamma_i r) + 2j^2J_2(\gamma_r r) J_0(\gamma_i r) \right] \cos 2v + 4J_1(\gamma_r r) J_1(\gamma_i r) \right. \\ &\quad \left. \sin v \cos v \right\} \\ &\quad + \left[-4j^3J_1(\gamma_r r) J_2(\gamma_i r) \cos v \cos 2v - 4j^3J_2(\gamma_r r) J_1(\gamma_i r) \cos 2v \sin v \right] \\ &\quad - 4J_2(\gamma_r r) J_2(\gamma_i r) \cos^2 2v + \dots \end{aligned}$$

$$\begin{aligned}
&= J_0(\gamma_r r) J_0(\gamma_i r) + 2j \left[J_1(\gamma_r r) J_0(\gamma_i r) \cos v - J_0(\gamma_r r) J_1(\gamma_i r) \sin v \right] \\
&- 2 \left[J_0(\gamma_r r) J_2(\gamma_i r) - J_2(\gamma_r r) J_0(\gamma_i r) \right] \cos 2v + 2J_1(\gamma_r r) J_1(\gamma_i r) \sin 2v \\
&+ 2jJ_1(\gamma_r r) J_2(\gamma_i r) \cos v + 2jJ_1(\gamma_r r) J_2(\gamma_i r) \cos 3v \\
&+ 2jJ_2(\gamma_r r) J_1(\gamma_i r) \sin v - 2jJ_2(\gamma_r r) J_1(\gamma_i r) \sin 3v \\
&- 2 J_2(\gamma_r r) J_2(\gamma_i r) - 2J_2(\gamma_r r) J_2(\gamma_i r) \cos 4v + \dots
\end{aligned}$$

For I_1 we need the constant term:

$$J_0(\gamma_r r) J_0(\gamma_i r) - 2 J_2(\gamma_r r) J_2(\gamma_i r) + \dots$$

For I_2 we need the coefficients of $\cos v$ and of $\sin v$

$$\cos v: 2jJ_1(\gamma_r r) J_0(\gamma_i r) + 2jJ_1(\gamma_r r) J_2(\gamma_i r) + \dots$$

$$\sin v: 2jJ_0(\gamma_r r) J_1(\gamma_i r) + 2jJ_2(\gamma_r r) J_1(\gamma_i r) + \dots$$

Since $\gamma_i r = rc(r)\rho_1 \sin \beta = \frac{kar}{D_0(\rho)} \rho_1 \sin \beta$, one is tempted to seek justification for neglecting terms in ρ_1^2 , and therefore taking $J_0(\gamma_i r) = 1$,

$J_1(\gamma_i r) = \gamma_i r$, $J_n(\gamma_i r) = 0$, $n > 1$. However, we are interested in values of γ_i which are large even where ρ_1 is small, except for the special cases $\beta = 0, \pi$, corresponding to cuts in the radiation pattern in the plane containing the acoustic axis and the displaced image point ($\phi = \phi_1$). This case is of major interest. It yields $\gamma_i = 0$, and, consequently

$$\begin{aligned}
I_1 &= \int_{-\phi}^{2\pi-\phi} J_0(\gamma_r r) \cdot dv = 2\pi J_0(\gamma_r r), \quad \beta = 0, \pi \\
I_2 &= \int_{-\phi}^{2\pi-\phi} \left[2jJ_1(\gamma_r r) \cos v \right] \cos v dv \\
&= j 2\pi J_1(\gamma_r r), \quad \beta = 0, \pi
\end{aligned}$$

and (41) becomes

$$\begin{aligned} \frac{P(u, \phi)}{2\pi} &= \int_0^1 r l(r) J_0(\gamma_r r) dr \\ &- j\rho_1 \int_0^1 r b(r) J_1(\gamma_r r) dr; \beta = 0, \pi \end{aligned} \quad (44)$$

and

$$\gamma_r = u \pm \rho_1 c(r), \beta = \begin{cases} 0 \\ \pi \end{cases} \quad (44a)$$

More generally, we get

$$I_1 = 2\pi \left\{ J_0(\gamma_r r) J_0(\gamma_1 r) - 2 J_2(\gamma_r r) J_2(\gamma_1 r) + \dots \right\} \quad (45)$$

$$\begin{aligned} I_2 = 2\pi \left\{ \left[jJ_1(\gamma_r r) J_0(\gamma_1 r) + jJ_1(\gamma_r r) J_2(\gamma_1 r) + \dots \right] \cos \beta \right. \\ \left. + \left[jJ_0(\gamma_r r) J_1(\gamma_1 r) - jJ_2(\gamma_r r) J_1(\gamma_1 r) + \dots \right] \sin \beta \right\} \quad (46) \end{aligned}$$

Returning to Eq. (44) note from Eqs. (38) and (35) that with $\beta = \begin{cases} 0 \\ \pi \end{cases}$ we have

$$\begin{aligned} \gamma_r r &= ur \pm \frac{k\rho_1}{D_0(\rho)} r \\ &= kar \left[\sin \theta \pm \frac{\rho_1}{D_0(\rho)} \right] \end{aligned} \quad (47)$$

Since $0 \leq \theta \leq \pi$, $\gamma_r r$ has a zero only for the lower sign in (47), for $\rho_1 > 0$, or only for the upper sign for $\rho_1 < 0$. In either case the zero, which corresponds to the direction of the main lobe*, is given by

$$\sin \theta = \frac{|\rho_1|}{D_0(\rho)} \quad (48)$$

* i.e., to $u = 0$ in the axial-source case

D_0 is given by (26). If $\frac{a}{F} \ll 1$, $D_0 \approx F$ and θ is independent of ρ and has the value usually given by geometrical optics. In that case, the first integral in (44) approximates the pattern of an axial source with the main lobe displaced by an angle given approximately by

$$\theta = \frac{\rho_1}{F} \quad (49)$$

Returning to Eqs. (44), (44a), note that for $\beta = \pi$, we have

$$\gamma_r = u - \rho_1 c(r), \quad \beta = \pi$$

or

$$-\gamma_r = -u + \rho_1 c(r)$$

Thus, $J_0(\gamma_r r) = J_0(-\gamma_r r) = J_0 \left[-u + \rho_1 c(r) \right]$, which is equivalent to taking $\beta = 0$ [Eq. (44a)] and reversing the sign of u . That is, the first integral in (44) can be studied with $\beta = \pi$ alone provided negative as well as positive values of u are admitted. This is equivalent to admitting negative values of θ with $\beta = \pi$ in place of positive values only, with $\beta = 0$.

In the second integral, using $J_1(-\gamma_r r) = -J_1(\gamma_r r)$ reverses the sign of the integral, thus reversing the phase of the out-of-phase component of the pressure pattern. However, when the power pattern is calculated, this phase reversal disappears, and the calculation remains correct. Therefore, in what follows, we use $\beta = \pi$ only, and admit negative as well as positive values of u , with the understanding that if phase details in the pressure pattern are of interest, the sign of the second integral must be reversed for $u < 0$.

In order to calculate a typical set of patterns we now choose some appropriate functions for the integrands in (44). First, take

$$\left. \begin{aligned} \ell(r) = f(\rho) &= (1 - r^2)^m \\ f_{eo}(\rho) &= \frac{\sin \pi r}{\pi r} = \frac{\sin \chi}{\chi}, \quad \chi = \frac{\pi \rho}{a} \end{aligned} \right\} \quad (50)$$

Note that f_{eo} has been chosen to simulate, approximately, the pattern from a small, uniformly insonified primary aperture in the image plane, that it goes to zero at $r = 1$, but has no zero in $0 \leq r < 1$.

From the second of (50),

$$f'_{eo}(\rho) = \frac{\pi}{a} \frac{d}{d\chi} \left(\frac{\sin \chi}{\chi} \right) = \frac{\pi}{a} \frac{\pi r \cos \pi r - \sin \pi r}{\pi^2 r^2} \quad (51)$$

Then, by Eqs. (5), (30) and (34),

$$\begin{aligned} b(r) &= \frac{f(\rho)}{f_{eo}(\rho)} f'_{eo}(\rho) \\ &= \frac{\pi}{a} (1 - r^2)^m \frac{\pi r \cot \pi r - 1}{\pi r} \end{aligned} \quad (52)$$

Eq. (52) is indeterminate at $r = 1$. However, we can define $b(1)$ as $\lim_{r \rightarrow 1} b(r)$.

This limit is

$$\begin{aligned} b(1) &= \frac{\pi}{a} \lim_{r \rightarrow 1} \left[\frac{(1 - r^2)^m (\pi r \cos \pi r - \sin \pi r)}{\pi r \sin \pi r} \right] \\ &= \frac{\pi}{a} \left[\frac{\pi \cos \pi - \sin \pi}{\pi} \right] \lim_{r \rightarrow 1} \left[\frac{(1 - r^2)^m}{\sin \pi r} \right] \\ &= - \frac{\pi}{a} \lim_{r \rightarrow 1} \left[\frac{(1 - r^2)^m}{\sin \pi r} \right] \\ &= \left(- \frac{\pi}{a} \right) \left(- \frac{2m}{\pi} \right) \left[(1 - r^2)^{m-1} \right]_{r=0} \end{aligned}$$

$$\left. \begin{aligned} &= \frac{2}{a} \text{ for } m = 1 \\ &= 0 \text{ for } m > 1 \end{aligned} \right\}$$

By Eqs. (35) and (38) we have, for $\beta = \pi$

$$\begin{aligned} \gamma_r &= u - \rho_1 c(r) \\ &= u - \frac{k a \rho_1}{D_0(ar)} \\ &= u - \frac{k a \rho_1}{\left(F^2 + a^2 r^2\right)^{\frac{1}{2}}} \end{aligned} \quad (53)$$

by (26).

Write

$N = \frac{F}{a} = 2 \frac{F}{D} = 2 \times F/\#$ of lens when object is at infinity. Also write

$$\gamma_r = u - v \left(1 + \frac{r^2}{N^2} \right)^{-\frac{1}{2}} \quad (54)$$

where

$$v = \frac{k a \rho_1}{F} \quad (55)$$

Then, using Eqs. (50 - 55) in (44), we get

$$\frac{P(u)}{2\pi} = K_1 - j n K_2 \quad (56)$$

where

$$\left. \begin{aligned} K_1 &= \int_0^1 r(1-r^2)^m J_0 \left\{ \left[u - v \left(1 + \frac{r^2}{N^2} \right)^{-\frac{1}{2}} \right] r \right\} dr \\ K_2 &= \int_0^1 (1-r^2)^m (\pi r \cot \pi r - 1) J_1 \left\{ \left[u - v \left(1 + \frac{r^2}{N^2} \right)^{-\frac{1}{2}} \right] r \right\} dr \end{aligned} \right\} \quad (57)$$

$$n = \frac{\rho_1}{a} ; \quad -\frac{2\pi a}{\lambda} \leq u \leq \frac{2\pi a}{\lambda} ; \quad \beta = \pi$$

and v is given by (55).

K_1 and K_2 were programmed for the CDC 6600 computer at Sandia Laboratories by W. Kenneth Ream of Division 7261. These were plotted, along with the function

$$\bar{K} = \left| \frac{P(u)}{2\pi} \right| = \left(K_1^2 + n^2 K_2^2 \right)^{\frac{1}{2}} \quad (58)$$

as functions of u , for various values of the parameters m , v , N , n .

Figures 2-29 show the plots of a representative set of computations on Eq. (58).

All of these curves are plotted for $N = 2$ ($F/\# = 1$) which may be considered a representative figure. In accordance with conclusions derived in Reference 1, our primary interest is in aperture distributions represented by $m = 4$ and 5. v is taken in the range $0 \leq v \leq 30$, while n is limited to the range $0 \leq n \leq 0.1$, since the foregoing theory assumes, essentially, that $n \ll 1$. Note that for $v = 0$ (which, for practical purposes, implies $\rho_1 = 0$), Eqs. (57) and (58) are independent of N .

Figures 2 and 3 for $m = 2, 3$; $v = 0$; are included for interest. The side-lobe peaks for these were determined analytically in Reference 1 and listed in Table III of that report. That table is reproduced here for comparison and re-labeled Table I. (Plots for $m = 4, 5$; $v = 0$; are given in Figures 4 and 17 respectively). Figures 5 through 16 are plots for $m = 4$, $N = 2$, and various values of v and n . Figures 5 - 7 are for $v = 1$, $n = 0.01, 0.03, 0.1$. Figures 8-10 change to $v = 3$ and repeat the range of n . In the same way, Figures 11-13 are for $v = 10$, while Figures 14-16 are plotted for $v = 30$. As mentioned above, Figure 17 is the

Table I. Side-Lobe Maxima vs. m

m	Side Lobe Number									
	1		2		3		4		5	
	Ψ	db	Ψ	db	Ψ	db	Ψ	db	Ψ	db
1	3.4	24.6	5.2	33.6	7.0	39.8	8.7	44.4	10.4	48.3
2	3.5	30.6	5.1	42.7	6.7	49.6	8.2	55.6	9.7	60.6
3	3.6	36.0	5.1	48.9	6.5	58.0	7.9	65.4	9.2	71.5
4	3.8	40.9	5.1	55.4	6.4	65.8	7.7	74.3	8.9	81.6
5	3.9	47.7	5.2	61.1	6.4	72.9	7.6			
6	4.0	50.1	5.3	66.8	6.4	79.6				
7	4.1	54.3	5.3	74.0						

axial-source plot for $m = 5$, while Figures 18-29 repeat, for $m = 5$, the sequence of plots described above for $m = 4$.

Appended to this report is an additional set of plots (Figures 30-45) showing K_1 and K_2 vs. u for various sets of parameters. See Table of Contents for identification of the several plots.

Examination of Figures 5-29 shows that, other parameters being constant, the patterns do not vary greatly with n in the plotted range. Reference to the plots in the appendix shows why this is so. Values of K_2 tend to be of the same order of magnitude as values of K_1 for a given n , so that adding a quantity no larger than $0.1 K_2$ to K_1 at a phase angle of $\frac{\pi}{2}$ must have little effect. In essence the curves of $K_1(u)$ are adequate for values of $n \leq 0.1$.

Next, if we confine attention to the curves of $K_1(u)$, we note a progressive deterioration in the side lobe level with increasing v . At far-out side lobes (say no closer than the fifth) this deterioration becomes serious for $v > 10$. Now

$$v = \frac{k a \rho_1}{F} = 2\pi \frac{a \rho_1}{\lambda F} = \frac{2\pi}{N} \frac{\rho_1}{\lambda}$$

As indicated in earlier reports, $\rho_i \approx i \frac{\lambda}{2}$, $i = 0, 1, 2, \dots$, so that an upper limit of $v = 10$, with $N = 2$, implies

$$10 = \frac{2\pi}{2} \frac{i\lambda}{2\lambda} = i \frac{\pi}{2}$$

That is,

$$i = \frac{20}{\pi} \approx 7$$

For a matrix of 10 x 10 image elements, $i \leq 5$, so that v would be within adequate bounds for the stipulated conditions.

Thus, it would appear that, theoretically at least, the realization of adequate discrimination against diffraction glare should be at least a borderline possibility. Problems of realization are discussed briefly in the next section.

4. Practical Limitations

Without knowing specifically what limits the performance capabilities of radiation apertures, we are aware, generally, that in the field of radar apertures, the practical limit of first side-lobe realizability appears to be of the order of 35 db. Our objective of 75 db for fifth side-lobes indicates a desired idealized first side-lobe level of about 40 db. What happens to the relationship between first and fifth side-lobes when the pattern is degraded by uncontrollable factors is not clear, but it seems safe to guess that if it is hard to avoid a given deterioration at -40 db, it will not be easier to avoid the same deterioration at -75 db.

One of the major factors limiting the realization of design side-lobe levels is that of random manufacturing error. Ruze has discussed this problem in an early paper [2]. The predicted results appear to check well with actual experience.

Meager reports from the field indicate very considerable difficulty in obtaining even moderate apodisation success. The prospects of achieving the rigid objectives set down in Section 2 of this report appear quite dismal at the present time.

5. Conclusions

The foregoing analysis indicates that adequate suppression of diffraction lobes from target highlights is a theoretical possibility. To date, the available evidence, meager except by analogy with electromagnetic diffraction problems, is that the theoretical possibilities are not practically realizable within the present state-of-the-art. Consequently, underwater acoustic imaging of typical machined objects by means of acoustic lens imaging systems, while theoretically possible, appears for the present, to be practically unrealizable.

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Figure 2. $\bar{K}$ vs u ; $m = 2$, $v = 0$

Figure 3. $\bar{K}$ vs u ; $m = 3$, $v = 0$

Figure 4. $\bar{K}$ vs u ; $m = 4$, $v = 0$

Figure 5. $\bar{K}$ vs u ; $m = 4$, $v = 1$, $N = 2$, $n = 0.01$

Figure 6. $\bar{K}$ vs u ; $m = 4$, $v = 1$, $N = 2$, $n = 0.03$

Figure 7. $\bar{K}$ vs u ; $m = 4$, $v = 1$, $N = 2$, $n = 0.1$

Figure 8. K vs u ; $m = 4$, $v = 3$, $N = 2$, $n = 0.01$

Figure 9. $\bar{K}$ vs u ; $m = 4$, $v = 3$, $N = 2$, $n = 0.03$

Figure 10. $\bar{K}$ vs u ; $m = 4$, $v = 3$, $N = 2$, $n = 0.1$

Figure 11. $\bar{K}$ vs u ; $m = 4$, $v = 10$, $N = 2$, $n = 0.01$

Figure 12. $\bar{K}$ vs u ; $m = 4$, $v = 10$, $N = 2$, $n = 0.03$

Figure 13. $\bar{K}$ vs u ; $m = 4$, $v = 10$, $N = 2$, $n = 0.1$

Figure 14. $\bar{K}$ vs u ; $m = 4$, $v = 30$, $N = 2$, $n = 0.01$

Figure 15. $\bar{K}$ vs u ; $m = 4$, $v = 30$, $N = 2$, $n = 0.03$

Figure 16. $\bar{K}$ vs u ; $m = 4$, $v = 30$, $N = 2$, $n = 0.1$

Figure 17. $\bar{K}$ vs u ; $m = 5$, $v = 0$

Figure 18. $\bar{K}$ vs u ; $m = 5$, $v = 1$, $N = 2$, $n = 0.01$

Figure 19. $\bar{K}$ vs u ; $m = 5$, $v = 1$, $N = 2$, $n = 0.03$

Figure 20. $\bar{K}$ vs u ; $m = 5$, $v = 1$, $N = 2$, $n = 0.1$

Figure 21. $\bar{K}$ vs u ; $m = 5$, $v = 3$, $N = 2$, $n = 0.01$

Figure 22. $\bar{K}$ vs u ; $m = 5$, $v = 3$, $N = 2$, $n = 0.03$

Figure 23. $\bar{K}$ vs u ; $m = 5$, $v = 3$, $N = 2$, $n = 0.1$

Figure 24. $\bar{K}$ vs u ; $m = 5$, $v = 10$, $N = 2$, $n = 0.01$

Figure 25. $\bar{K}$ vs u ; $m = 5$, $v = 10$, $N = 2$, $n = 0.03$

Figure 26. $\bar{K}$ vs u ; $m = 5$, $v = 10$, $N = 2$, $n = 0.1$

Figure 27. $\bar{K}$ vs u ; $m = 5$, $v = 30$, $N = 2$, $n = 0.01$

Figure 28. $\bar{K}$ vs u ; $m = 5$, $v = 30$, $N = 2$, $n = 0.03$

Figure 29. $\bar{K}$ vs u ; $m = 5$, $v = 30$, $N = 2$, $n = 0.1$

APPENDIX

Figures 30 - 45 Inclusive

Figure 30. $|K_1|$ vs u ; $m = 4$, $v = 1$, $N = 2$

Figure 31. $|K_2|$ vs u ; $m = 4$, $v = 1$, $N = 2$

Figure 32. $|K_1|$ vs u ; $m = 4$, $v = 3$, $N = 2$

Figure 33. $|K_2|$ vs u ; $m = 4$, $v = 3$, $N = 2$

Figure 34. $|K_1|$ vs u ; $m = 4$, $v = 10$, $N = 2$

Figure 35. $|K_2|$ vs u ; $m = 4$, $v = 10$, $N = 2$

Figure 36. $|K_1|$ vs u ; $m = 4$, $v = 30$, $N = 2$

Figure 37. $|K_2|$ vs u ; $m = 4$, $v = 30$, $N = 2$

Figure 38. $|K_1|$ vs u ; $m = 5$, $v = 1$, $N = 2$

Figure 39. $|K_2|$ vs u ; $m = 5$, $v = 1$, $N = 2$

Figure 40. $|K_1|$ vs u ; $m = 5$, $v = 3$, $N = 2$

Figure 41. $|K_2|$ vs u ; $m = 5$, $v = 3$, $N = 2$

Figure 42. $|K_1|$ vs u ; $m = 5$, $v = 10$, $N = 2$

Figure 43. $|K_2|$ vs u ; $m = 5$, $v = 10$, $N = 2$

Figure 44. $|K_1|$ vs u ; $m = 5$, $v = 30$, $N = 2$

Figure 45. $|K_2|$ vs u ; $m = 5$, $v = 30$, $N = 2$

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