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Turbulent Transport Across Invariant Canonical Flux Surfaces

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Net transport due to a combination of Coulomb collisions and turbulence effects in a plasma is investigated using a fluid moment description that allows for kinetic and nonlinear effects via closure relations. The model considered allows for “ideal” turbulent fluctuations that distort but preserve the topology of species-dependent canonical flux surfaces $\psi_{\#,s} \equiv \int dF \cdot \mathbf{B}_{\#,s}$, where $\mathbf{B}_{\#,s} \equiv \nabla \times [\mathbf{A} + (m_s/q_s)\mathbf{u}_s]$ in which $\mathbf{u}_s$ is the flow velocity of the fluid species. Equations for the net transport relative to these surfaces due to “nonideal” dissipative processes are found for the total number of particles and total entropy enclosed by a moving canonical flux surface. The corresponding particle transport flux is calculated using a toroidal axisymmetry approximation for the ideal surfaces. The resulting Lagrangian transport flux includes classical, neoclassical-like, and anomalous contributions and shows for the first time how these various contributions should be summed to obtain the total particle transport flux.

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I. Introduction

In calculating “radial” transport of particles and energy in a toroidal magnetically confined plasma, one must make an assumption about the magnetic field structure. The magnetic field topologies used include closed, nested, toroidally axisymmetric magnetic flux surfaces,¹ in which only flow across the magnetic field yields net cross surface transport, destroyed flux surfaces with a stochastic magnetic field^{2,3} that wanders randomly in the radial direction, for which flow along the field provides an additional transport path, or a combination thereof.⁴ As another intermediate possibility, we consider closed, nested canonical flux surfaces (see below). These surfaces are allowed to distort and fluctuate with the plasma turbulence but are required to remain topologically toroidal (invariant). Such an assumption is appealing because of the simplification provided, particularly when combined with approximate toroidal axisymmetry of these surfaces.

Previous authors have explored some aspects of a closed flux surface model in the presence of turbulent plasma fluctuations. Kadomtsev and Pogutse⁵ considered a periodic cylinder model with a strong longitudinal magnetic field. They showed that in the absence of dissipation with constant electron temperature on each flux surface the surfaces and electron fluid move together in the direction normal to the surfaces. Boozer^{6,7} explored magnetic flux surface motion by deriving a condition on the velocity field of the magnetic field structure. He also discussed some implications of this magnetic flux invariance condition for particle transport. However, the overall transport implications of such a model were not considered.

In this paper a general closed canonical flux surface model for turbulent plasma transport is explored. Specifically, we will assume the existence of a set of “ideal” closed, nested canonical magnetic flux surfaces at all times in some region of a toroidal plasma. The canonical flux $\psi_{\#,s} \equiv \int d\mathbf{F} \cdot \mathbf{B}_{\#,s}$ is that due to a species dependent (subscript s)

canonical magnetic field defined by $\mathbf{B}_{\#,s} \equiv \nabla \times [\mathbf{A} + (m_s/q_s)\mathbf{u}_s]$, which is comprised of the magnetic field plus a vorticity term. A closed canonical flux surface assumption represents a constraint on the admissible plasma turbulence. With this assumption, ideal turbulent fluctuations in the canonical magnetic field result in small distortions of the surfaces away from axisymmetry which preserve the topology of the surfaces; hence, only “nonideal” relative flow between the (possibly moving) surfaces and plasma fluids yields net transport. We thus adopt a Lagrangian-like approach for calculating the transport relative to the moving surfaces. We show that in the ideal or reversible case where there are no dissipative processes $\psi_{\#,s}$ is frozen into the plasma species fluid, and thus $\mathbf{B}_{\#,s}$ moves with the fluid. Dissipative or irreversible effects in the turbulent fluctuations are nonideal. They lead to net transport relative to the ideal surfaces and can lead to changes in topology (which we have neglected).

The outline of this paper is as follows. Section II describes the fluid moment description used in this paper. Section III elucidates the general properties of the canonical magnetic field $\mathbf{B}_{\#,s}$ and its canonical flux surfaces, many of which are used in subsequent transport calculations. Section IV presents the condition for canonical fluctuating flux surface motion, known as flux invariance, which is analogous to the familiar “frozen flux” condition of ideal magnetohydrodynamics. Section V gives the net transport equations and their corresponding transport fluxes. To find expressions for the net transport fluxes, a toroidal axisymmetry approximation is used for the ideal surfaces which is relevant for some cases, particularly in a tokamak plasma. Based on an ordering of small fluctuations relative to equilibrium quantities, Section VI discusses electron transport in a tokamak plasma. Section VII provides concluding comments.

II. Fluid moment description and equations

The fluid moment description of plasmas consists of a hierarchy of equations that are different velocity space moments of the plasma kinetic equation. The kinetic equation that governs the $\mathbf{x}$, $\mathbf{v}$ phase space distribution function f_s for a plasma species s in the presence of electric and magnetic fields is⁸

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_s = C_s(f_s). \quad (1)$$

Taking velocity space moments of the kinetic equation ($\int d^3v$, $\int \mathbf{v} d^3v$, and so on) gives an infinite hierarchy of equations which is entirely equivalent to the kinetic equation.

To be useful, the hierarchy of equations must be truncated. This paper adopts a 13-moment approach. Equations for the density, momentum, and energy are used together with closure relations for the viscous stress and conductive heat flux moments which are in principle calculated including kinetic effects.^{9,10} These equations are as follows for each plasma species.

Particle continuity:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = 0. \quad (2)$$

Momentum balance:

$$(m_s/q_s)n_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) = n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) - (1/q_s) (\nabla p_s + \nabla \cdot \underline{\underline{\Pi}}_s - \mathbf{R}_s). \quad (3)$$

Temperature evolution:

$$\frac{3}{2} n_s \left(\frac{\partial T_s}{\partial t} + \mathbf{u}_s \cdot \nabla T_s \right) = -p_s \nabla \cdot \mathbf{u}_s - \nabla \cdot \mathbf{q}_s - \underline{\underline{\Pi}}_s : \nabla \mathbf{u}_s + Q_s. \quad (4)$$

Entropy evolution (derived from density and temperature evolution):

$$\frac{\partial (n_s s_s)}{\partial t} + \nabla \cdot \left(n_s s_s \mathbf{u}_s + \frac{\mathbf{q}_s}{T_s} \right) = -\frac{1}{T_s} (\mathbf{q}_s \cdot \nabla \ln T_s + \underline{\underline{\Pi}}_s : \nabla \mathbf{u}_s) + \frac{Q_s}{T_s}. \quad (5)$$

Definitions of the quantities appearing in these equations are as follows (with $\mathbf{v}' \equiv \mathbf{v} - \mathbf{u}_s$):

Particle density:

$$n_s = \int d^3v f_s. \quad (6)$$

Pressure:

$$p_s = \int d^3v' (m_s/3) v'^2 f_s \equiv n_s T_s. \quad (7)$$

Collisional heat generation rate:

$$Q_s = \int d^3v' (m_s/2) v'^2 C_s(f_s). \quad (8)$$

Entropy:

$$s_s = \ln(T_s/n_s^{2/3}). \quad (9)$$

Flow velocity:

$$\mathbf{u}_s = \frac{1}{n_s} \int d^3v \mathbf{v} f_s. \quad (10)$$

Conductive heat flux:

$$\mathbf{q}_s = \int d^3v' (m_s/2) \mathbf{v}' v'^2 f_s. \quad (11)$$

Collisional friction force:

$$\mathbf{R}_s = \int d^3v m_s \mathbf{v} C_s(f_s). \quad (12)$$

Viscous stress:

$$\underline{\underline{\Pi}}_s = \int d^3v' m_s (\mathbf{v}' \mathbf{v}' - v'^2 \underline{\underline{\mathbf{I}}}/3) f_s, \quad (13)$$

in which $\underline{\underline{\mathbf{I}}}$ is the identity tensor.

III. Properties of the canonical magnetic field $\mathbf{B}_{\#,s}$

A charged particle in a magnetic field has a canonical momentum $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$.

In a toroidally axisymmetric system, the toroidal component of this particle canonical

momentum will be conserved. For a fluid moment description of a plasma, we consider the velocity space moment of the related quantity

$$\mathbf{A}_{\#,s} \equiv (1/n_s) \int d^3v (\mathbf{A} + [m_s/q_s] \mathbf{v}) f_s. \quad (14)$$

The curl of this quantity is equal to the magnetic field plus the fluid vorticity weighted by the mass to charge ratio of the fluid species:

$$\mathbf{B}_{\#,s} \equiv \nabla \times \mathbf{A}_{\#,s} = \mathbf{B} + (m_s/q_s) \nabla \times \mathbf{u}_s. \quad (15)$$

This field will be referred to as the canonical magnetic field because of its origin from the curl of the velocity space moment of the canonical momentum. The canonical flux is defined by

$$\begin{aligned} \psi_{\#,s} &\equiv \int d\mathbf{F} \cdot \mathbf{B}_{\#,s} = \int d\mathbf{F} \cdot \nabla \times [\mathbf{A} + (m_s/q_s) \mathbf{u}_s] \\ &= \int d\mathbf{F} \cdot \mathbf{B} + (m_s/q_s) \int d\mathbf{F} \cdot \nabla \times \mathbf{u}_s \end{aligned} \quad (16)$$

in which $d\mathbf{F}$ is a differential surface element and the integral is taken over an arbitrary surface. The first term is the magnetic flux whereas the second term is the vorticity flux weighted by the mass to charge ratio of the fluid species being considered.

Various properties of the canonical magnetic field are straightforwardly determined. The field is divergence free

$$\nabla \cdot \mathbf{B}_{\#,s} = 0 \quad (17)$$

and its time evolution is given by a generalized Faraday's law

$$\frac{\partial \mathbf{B}_{\#,s}}{\partial t} = -\nabla \times [\mathbf{E} - (m_s/q_s) \frac{\partial \mathbf{u}_s}{\partial t}]. \quad (18)$$

For electrons in a typical strongly magnetized laboratory plasma, one has $\mathbf{B}_{\#,e} \simeq \mathbf{B}$ as long as $(c/\omega_{p,e}) \nabla_{\perp} \ll 1$. For ions, however, the vorticity contribution becomes substantial for large flow velocity fluctuations at short wavelengths ($k_{\perp} \rho_i \sim 1$), although the magnetic field is still dominant for typical parameters.

The canonical magnetic field also has the interesting property that it moves with the fluid velocity in the ideal case where there is no dissipation and an isentropic assumption is used for the pressure.¹¹⁻¹³ [Actually, the isentropic assumption is overly restrictive; all that is required is that the pressure be a function of the density, $p=p(n)$.^{12,13}] This result is analogous to the frozen flux theorem in ideal MHD⁸ in which the magnetic field and fluid move together, and the proof here proceeds similarly. Alternatively, it is analogous to Kelvin's theorem in neutral fluids for constancy of the vortex circulation in the absence of dissipation.¹⁴

For an arbitrary surface F which moves with the fluid velocity at all points, the time rate of change of the canonical flux through the surface is given by

$$\frac{d\psi_{\#,s}}{dt} = \int_F \frac{\partial \mathbf{B}_{\#,s}}{\partial t} \cdot d\mathbf{F} + \oint_C \mathbf{B}_{\#,s} \cdot (\mathbf{u}_s \times d\mathbf{l}), \quad (19)$$

where C is the surface boundary. The first term represents the contribution from the changing field over the fixed surface F , and the second term represents the additional flux swept out by the motion of the boundary. Using the generalized Faraday's law, Eq. (18), in the first term along with Stokes' theorem and rearranging the second term with a vector identity $[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}]$ gives

$$\frac{d\psi_{\#,s}}{dt} = \oint_C [\mathbf{E} - \left(\frac{m_s}{q_s}\right) \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \times \mathbf{B}_{\#,s}] \cdot d\mathbf{l}. \quad (20)$$

The ideal momentum balance equation in the absence of dissipation is obtained by setting $\mathbf{\Pi}_s$ and $\mathbf{R}_s$ to zero in Eq. (3):

$$m_s n_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) = n_s q_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) - \nabla p_s. \quad (21)$$

The convective part of the inertial term may be rewritten using the vector identity $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}$ as

$$\mathbf{u}_s \cdot \nabla \mathbf{u}_s = -\mathbf{u}_s \times \nabla \times \mathbf{u}_s - \nabla(u_s^2/2), \quad (22)$$

giving

$$m_s n_s \frac{\partial \mathbf{u}_s}{\partial t} = n_s q_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}_{\#,s}) - \nabla p_s - m_s n_s \nabla (u_s^2/2). \quad (23)$$

With an isentropic pressure condition $p_s = c n_s^\gamma$, where c is a constant, the integral in Eq. (20) is zero since it is taken around a closed path. Under these ideal conditions, the canonical flux enclosed by an arbitrary surface is preserved and the field lines move with the fluid; i.e.,

$$\frac{d\psi_{\#,s}}{dt} = 0 \quad (24)$$

for the ideal case where $\mathbf{\Pi}_s = 0$, $\mathbf{R}_s = 0$, and the pressure is isentropic. Note in particular that flow inertia ($m_s n_s \partial \mathbf{u}_s / \partial t$) does not directly cause the fluid to move relative to the $\mathbf{B}_{\#,s}$ field lines.

The fact that the $\mathbf{B}_{\#,s}$ field lines move with the ideal fluid under this frozen flux condition follows from an argument like that given in Schmidt.⁸ Consider a material line constructed from two intersecting surfaces which are everywhere tangent to the $\mathbf{B}_{\#,s}$ field at time $t=0$. Since they are tangent to $\mathbf{B}_{\#,s}$ there is no canonical flux through the surfaces. Due to the canonical flux preservation theorem, they will remain flux free in the course of their motion and thus their intersection remains a field line. Since there is no contradiction in calling it the same field line, the $\mathbf{B}_{\#,s}$ field lines may be said to move with the ideal fluid.

Up to now we have been considering general surfaces for the $\psi_{\#}$ flux. We now turn to consider closed, nested canonical magnetic flux surfaces that are assumed to exist at all times in some region of the plasma. We specifically consider the case of a field-line invariant $\mathbf{B}_{\#,s} \cdot \nabla \psi_{\#,s} = 0$, where $\mathbf{B}_{\#,s}$ lies on surfaces of constant $\psi_{\#,s}$. The properties associated with the canonical flux surfaces in such a case are similar to those of the usual magnetic flux surfaces. For a thorough exploration of magnetic flux surfaces and flux coordinates, refer to the book by D'haeseleer et al.¹⁵ Relevant results are reviewed in the

remainder of this section.

For any closed canonical flux surface, the condition

$$\mathbf{B}_{\#,s} \cdot \nabla \Omega = f(\mathbf{x}), \quad (25)$$

where f is a function defined on the surface, will be referred to as a canonical differential equation. The function f determines the function Ω within the canonical flux surface to within an arbitrary constant which can be singular (delta-function-like) on rational surfaces where $\mathbf{B}_{\#,s} \cdot \nabla$ goes to 0 in fully three-dimensional geometries. For simplicity, here and henceforth, we will assume that we are considering either axisymmetric or irrational surface cases for which the constant of integration is a simple surface constant. In the special case that $f(\mathbf{x}) = 0$, one has $\Omega = \Omega(\psi_{\#,s})$, where $\psi_{\#,s}$ is a coordinate that labels the flux surface. In this case Ω is referred to as a flux function. An analogous condition establishes the total pressure as a magnetic flux function in ideal MHD: $\mathbf{J} \times \mathbf{B} = \nabla p$ gives $\mathbf{B} \cdot \nabla p = 0$ and thus $p = p(\psi)$.

The canonical flux surface average of a quantity f is defined by

$$\langle f \rangle \equiv \lim_{\delta\psi_{\#,s} \rightarrow 0} \frac{\int_{\psi_{\#,s}}^{\psi_{\#,s} + \delta\psi_{\#,s}} d^3x f}{\int_{\psi_{\#,s}}^{\psi_{\#,s} + \delta\psi_{\#,s}} d^3x}. \quad (26)$$

The preceding statements in this chapter have been general in nature. The following statements apply only to a toroidally axisymmetric system but are also used when the toroidally axisymmetric approximation is made. In a toroidally axisymmetric system, the surfaces are closed and nested, which facilitates the calculation of plasma transport for such systems. The Jacobian of a coordinate system based on these surfaces is independent of toroidal angle ζ , which gives the result

$$\langle R^2 \nabla \zeta \cdot \nabla \Omega \rangle = 0 \quad (27)$$

for an arbitrary function Ω . Here, R is the major radius for a given location, which is the distance from the central axis of the torus to a given point. Writing the canonical

magnetic field as $\mathbf{B}_{\#,s} = I_{\#,s} \nabla \zeta + \nabla \psi_{\#,s} \times \nabla \zeta$, one finds the geometric identity

$$\frac{\nabla \psi_{\#,s} \times \mathbf{B}_{\#,s}}{B_{\#,s}^2} = -\frac{I_{\#,s} \mathbf{B}_{\#,s}}{B_{\#,s}^2} + R^2 \nabla \zeta. \quad (28)$$

This equation is in the same form as the corresponding one for the magnetic field for an axisymmetric system.¹⁶

IV. Canonical flux invariance

In the previous section it was shown that the canonical magnetic field lines move with the fluid in the absence of dissipative processes, i.e., in an ideal fluid. Including the nonideal effects of dissipation, however, means that the field line and fluid motions decouple. In this section (and Appendix A) we derive the condition for the surface motion known as canonical flux invariance. The model we will consider below supposes that the canonical field lines trace out closed, nested surfaces in the region of interest.

The general condition which the surface velocity must satisfy to be consistent with closed, nested canonical flux surfaces may be found by considering a generalization of magnetic flux invariance, as described by Boozer,^{6,7} to the canonical magnetic field. Flux invariance follows by using $\psi_{\#,s}, \theta_s, \zeta_s$ coordinates for closed, nested canonical flux surfaces and imposing the restriction that the flow velocity field for the surface motion $\mathbf{u}_{s,f}$ is such that the toroidal flux through an arbitrary constant ζ_s surface is constant in time. Here $\psi_{\#,s}$ is the toroidal flux, θ_s is a poloidal angle around the flux surface, and ζ_s is a toroidal angle. For transport calculations, flux invariance is equivalent to the closed surface assumption. This equivalence evidently arises because only surface motion in the direction locally normal to the canonical flux surface is physically meaningful, and thus the restriction imposed on the velocity field $\mathbf{u}_{s,f}$ to yield canonical flux invariance does not introduce additional assumptions. Appendix A contains details of the canonical flux invariance derivation and shows the equivalence of canonical flux invariance and a closed

canonical flux surface assumption for transport calculations.

From Eq. (A7) in Appendix A, the surface velocity $\mathbf{u}_{sf}$ satisfies the canonical flux invariance condition

$$\mathbf{E} - (m_s/q_s)\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_{sf} \times \mathbf{B}_{\#,s} = V_{\#,s}(\psi_{\#,s})\nabla\zeta_s - \nabla\Phi_s, \quad (29)$$

in which $V_{\#,s}$ is the generalized loop voltage and Φ_s is a scalar function. The dot product with $\mathbf{B}_{\#,s}$ and a canonical flux surface average gives

$$V_{\#,s} = \frac{\langle \mathbf{B}_{\#,s} \cdot [\mathbf{E} - (m_s/q_s)\partial \mathbf{u}_s/\partial t] \rangle}{\langle \mathbf{B}_{\#,s} \cdot \nabla\zeta \rangle}. \quad (30)$$

The function Φ_s is determined to within a constant within a flux surface by a differential equation in the form of Eq. (25). To help understand its role, consider an electron fluid with only electrostatic fluctuations. Neglecting the typically small inertial term and generalized loop voltage, the parallel component of Eq. (29) gives

$$\mathbf{B}_{\#,e} \cdot \nabla \tilde{\phi} = \mathbf{B}_{\#,e} \cdot \nabla \tilde{\Phi}_e, \quad (31)$$

so that

$$\tilde{\phi} = \tilde{\Phi}_e + \tilde{g}(\psi_{\#,e}). \quad (32)$$

In this case variations in Φ_e due to fluctuations within a canonical flux surface are equivalent to variations in the electric potential for determining transport. This equivalence arises because $\nabla g(\psi_{\#,e})$ is normal to the canonical flux surface and hence only affects the flow within the surface. We will see explicitly in the next section that fluctuations in $\tilde{\phi}$ and $\tilde{\Phi}_e$ can lead to net cross-surface transport flows, but ∇g does not.

V. Net transport equations and fluxes

Transport equations for the total number of particles of a given species and total entropy of that species are determined in this section. Then the corresponding particle transport flux is found.

We determine the net transport equation for a quantity β by finding the time rate of change of it integrated over the flux surface, and then substituting for $\partial\beta/\partial t$ from the fluid moment equations. This procedure is elucidated in Appendix B. The resulting net transport equations for the total number of particles N and total entropy $S \equiv \int_V d^3x n_s s_s$ inside a given canonical flux surface volume V bounded by a surface F are

$$\frac{\partial N_s}{\partial t} = - \int_F n_s (\mathbf{u}_s - \mathbf{u}_{sf}) \cdot d\mathbf{F}, \quad (33)$$

$$\begin{aligned} \frac{\partial S_s}{\partial t} = & - \int_F \frac{\mathbf{q}_s}{T_s} \cdot d\mathbf{F} - \int_F n_s s_s (\mathbf{u}_s - \mathbf{u}_{sf}) \cdot d\mathbf{F} - \\ & \int_V d^3x \frac{1}{T_s} \mathbf{q}_s \cdot \nabla \ln T_s - \int_V d^3x \frac{1}{T_s} \underline{\Pi}_s : \nabla \mathbf{u}_s + \int_V d^3x \frac{Q_s}{T_s}. \end{aligned} \quad (34)$$

In the particle transport equation, the term on the right side shows that it is the relative velocity between fluid and surface which gives a net particle flux through the surface. The entropy transport equation is more complicated. In addition to the conductive and convective entropy fluxes $\mathbf{q}_s/T_s$ and $n_s s_s (\mathbf{u}_s - \mathbf{u}_{sf})$, note that there is a heat conduction dissipation $\mathbf{q}_s \cdot \nabla \ln T_s$, a viscous dissipation of energy $\underline{\Pi}_s : \nabla \mathbf{u}_s$, and collisional heating Q_s .

Note that fluid flows along the $\mathbf{B}_{\#,s}$ field produce no net transport because such flows lie within the canonical flux surfaces. Only the components of the flows normal to the canonical flux surfaces produce net transport.

To find an expression for determining the particle flux relative to a canonical flux surface, consider first the full momentum balance equation, Eq. (3). For this expression to be useful for the canonical magnetic field, the $\mathbf{u}_s \times \mathbf{B}$ is converted to a $\mathbf{u}_s \times \mathbf{B}_{\#,s}$ as in Section III by applying a vector identity to the convective part of the inertial term using Eq. (22). Combining the first of the terms from the vector identity with $\mathbf{u}_s \times \mathbf{B}$ gives the desired result, rearranged as

$$\mathbf{E} - (m_s/q_s) \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \times \mathbf{B}_{\#,s} = (1/n_s q_s) [\nabla p_s + \nabla \cdot \underline{\Pi}_s - \mathbf{R}_s]. \quad (35)$$

The expression for the relative fluid and surface flows results from the difference between this equation and the flux invariance condition, Eq. (29):

$$(\mathbf{u}_s - \mathbf{u}_{sf}) \times \mathbf{B}_{\#,s} = (1/n_s q_s)(\nabla p_s + \nabla \cdot \underline{\Pi}_s - \mathbf{R}_s) - V_{\#,s}(\psi_{\#,s}) \nabla \zeta_s + \nabla \alpha_s. \quad (36)$$

Here, we have defined

$$\alpha_s \equiv \Phi_s + \frac{m_s u_s^2}{2q_s}. \quad (37)$$

To within a constant, the scalar function α_s is determined within a canonical flux surface by the parallel component of the momentum balance, in the form of Eq. (25):

$$\mathbf{B}_{\#,s} \cdot \nabla \alpha_s = (1/n_s q_s) \mathbf{B}_{\#,s} \cdot [\nabla p_s + \nabla \cdot \underline{\Pi}_s - \mathbf{R}_s] - \mathbf{B}_{\#,s} \cdot [V_{\#,s}(\psi_{\#,s}) \nabla \zeta_s]. \quad (38)$$

The lowest order terms in the expression in Eq. (36) for the relative flow consist of parts of ∇p_s and $\nabla \alpha_s$ for fluctuations that are isotropic in velocity space; i.e., for $\nabla \cdot \tilde{\underline{\Pi}}_s = 0$. These effects combine to produce no relative cross surface flow if the temperature is constant on a canonical flux surface to lowest order, which is a generalization of the result of Kadomtsev and Pogutse.⁵ To see this result, we decompose the temperature and density into parts which are constant on the canonical flux surfaces and parts which vary within the canonical flux surfaces:

$$T_s = \bar{T}_s(\psi_{\#,s}) + \tilde{T}_s(\psi_{\#,s}, \theta_s, \zeta_s), \quad (39)$$

$$n_s = \bar{n}_s(\psi_{\#,s}) + \tilde{n}_s(\psi_{\#,s}, \theta_s, \zeta_s). \quad (40)$$

Note that the fluctuating quantities are not necessarily small compared to the average ones. We then decompose the relative flow of Eq. (36) into explicit ideal and nonideal parts (subscripts i and n, respectively) with the following definitions

$$(\mathbf{u}_s - \mathbf{u}_{sf})_i \times \mathbf{B}_{\#,s} = (1/q_s)(\nabla T_s + \bar{T}_s \nabla \ln n_s + (\tilde{T}_s/n_s) \nabla \bar{n}_s) + \nabla \alpha_{s,i}. \quad (41)$$

$$(\mathbf{u}_s - \mathbf{u}_{sf})_n \times \mathbf{B}_{\#,s} = (1/n_s q_s)(\tilde{T}_s \nabla \tilde{n}_s + \nabla \cdot \underline{\Pi}_s - \mathbf{R}_s) - V_{\#,s}(\psi_{\#,s}) \nabla \zeta_s + \nabla \alpha_{s,n}. \quad (42)$$

Here α_s has been separated into an ideal part $\alpha_{s,i}$ and a nonideal part $\alpha_{s,n}$. The designation ideal refers to the fact that the corresponding ideal flow yields no relative motion between the fluid and the canonical surfaces in the direction normal to the surfaces, and hence no net transport. To see this result, note that since $\bar{T}_s$ is a canonical flux function, the parallel (to $\mathbf{B}_{\#,s}$) component of Eq. (41) gives

$$(1/q_s)(T_s + \bar{T}_s \ln n_s) + \alpha_{s,i} = h(\psi_{\#,s}), \quad (43)$$

where h is some canonical flux function. Using this result to determine $\nabla \alpha_{s,i}$ yields the equation for the relative ideal flow

$$(\mathbf{u}_s - \mathbf{u}_{sf})_i \times \mathbf{B}_{\#,s} = \nabla h - (1/q_s)(\ln n_s) \nabla \bar{T}_s + (\tilde{T}_s/n_s q_s) \nabla \bar{n}_s. \quad (44)$$

Because h , $\bar{T}_s$, and $\bar{n}_s$ are all canonical flux functions, the right side of this expression is everywhere normal to the canonical flux surfaces; thus, the difference between ideal fluid and surface flows must lie within the canonical flux surfaces.

In a toroidally axisymmetric magnetic confinement device such as a tokamak, approximating closed canonical flux surfaces as toroidally axisymmetric will suffice in cases where the surfaces do not distort substantially. This approximation is valid if the surface velocity due to fluctuations is sufficiently small or if the frequency of the fluctuations is high enough that even a fast surface velocity yields only small net movement of the canonical flux surfaces.

The quantity of interest for particle transport is the canonical flux surface average of the relative particle flux normal to the canonical flux surface, defined by

$$\Gamma_{\psi_{\#,s}} \equiv \langle n_s (\mathbf{u}_s - \mathbf{u}_{sf}) \cdot \nabla \psi_{\#,s} \rangle. \quad (45)$$

This expression is proportional to the integral of the relative particle flux over the flux surface as given earlier in Eq. (33). Note that since the ideal relative flows lie within the

$\psi_{\#,s}$ surfaces, they do not contribute to the particle flux. The net particle flux results solely from the nonideal relative flows normal to the canonical flux surfaces.

The equation for the nonideal relative flow between the fluid and surfaces, Eq. (42), multiplied by the density is thus used to determine the net particle flux. This equation yields

$$n_s(\mathbf{u}_s - \mathbf{u}_{sf})_{\perp,n} = (1/B_{\#,s}^2)\mathbf{B}_{\#,s} \times \{(1/q_s)[\tilde{T}_s \nabla \tilde{n}_s + \nabla \cdot \underline{\Pi}_s - \mathbf{R}_s] - n_s V_{\#,s}(\psi_{\#,s}) \nabla \zeta + n_s \nabla \alpha_{s,n}\}. \quad (46)$$

Using the vector identity $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B}$ to rearrange the cross and dot products in the resulting expression for the particle flux and applying the toroidally axisymmetric approximation for $\nabla \psi_{\#,s} \times \mathbf{B}_{\#,s}$ in the representation given in Eq. (28), yields

$$\Gamma_{\psi_{\#,s}} = \langle R^2 \nabla \zeta \cdot [(1/q_s)(\tilde{T}_s \nabla \tilde{n}_s - \mathbf{R}_s) - \bar{n}_s V_{\#,s} \nabla \zeta + \tilde{n}_s \nabla \alpha_{s,n}] \rangle, \quad (47)$$

in which we have used the result of appendix C that the flux surface average of the lowest order toroidal viscous force vanishes.

We will now show that this result for the particle flux can be rewritten in a form that more clearly displays the classical, neoclassical-like, and anomalous transport flux terms. Using the direct form that results from Eqs. (45,46), we have

$$\Gamma_{\psi_{\#,s}} = \langle (\nabla \psi_{\#,s} \times \mathbf{B}_{\#,s} / B_{\#,s}^2) \cdot (1/q_s)[\tilde{T}_s \nabla \tilde{n}_s + \nabla \cdot \underline{\Pi}_s - \mathbf{R}_s] - n_s V_{\#,s} \nabla \zeta + n_s \nabla \alpha_{s,n} \rangle. \quad (48)$$

Using the geometric identity in Eq. (28), the result of appendix C, and the approximate equivalence of magnetic and canonical magnetic flux surfaces gives

$$\begin{aligned} \Gamma_{\psi_{\#,s}} \simeq & \langle (\nabla \psi_{\#,s} \times \mathbf{B}_{\#,s} / B_{\#,s}^2) \cdot (-[\mathbf{R}_s/q_s]) \rangle + \\ & \langle (I_{\#,s}/B_{\#,s}^2)([-\mathbf{B}_{\#,s} \cdot \nabla \cdot \underline{\Pi}_s/q_s] - \mathbf{B}_{\#,s} \cdot \bar{n}_s \nabla \alpha_{s,n}) \rangle - \end{aligned}$$

$$\begin{aligned}
& \langle (\nabla \psi_{\#,s} \times \mathbf{B}_{\#,s} / B_{\#,s}^2) \cdot n_s V_{\#,s} \nabla \zeta \rangle + \\
& \langle (\nabla \psi_{\#,s} \times \mathbf{B}_{\#,s} / B_{\#,s}^2) \cdot ([\tilde{T}_s \nabla \tilde{n}_s / q_s] + \tilde{n}_s \nabla \tilde{\alpha}_{n,s}) \rangle.
\end{aligned} \tag{49}$$

The first line here is the classical transport.¹⁸ The second line is similar to the neoclassical (Pfirsch-Schlüter plus banana-plateau¹⁶) particle flux contribution. The third line is analogous to the grid velocity effect.¹⁶ Anomalous transport appears in the fourth line from the out of phase density and temperature fluctuations and density fluctuations combined with $\nabla \tilde{\alpha}_{s,n}$. If the topology of the canonical magnetic flux surfaces is broken by the fluctuations (e.g., due to reconnection of canonical field lines), then this effect would lead to additional contributions to $\Gamma_{\psi_{\#,s}}$. However, in the absence of such a topology breaking effect, Eq. (49) indicates all of the possible particle transport effects. When the canonical flux surfaces bifurcate because of a slow reconnection process, the average net transport should be similar to that which occurs when magnetic islands are present in plasmas.¹⁹ When the plasma has a mixed topology of islands and anomalous transport regions the “slower” transport in the anomalous regions usually dominates.⁴

VI. Electron transport in a tokamak plasma

We now consider electron transport in a tokamak plasma. Some of the following considerations will also be relevant for a reversed field pinch (RFP).¹⁷

Adopting the drift ordering, $\omega \ll \Omega$, and introducing the small parameter $\delta \sim 10^{-2}$, for typical experimentally observed fluctuations in a tokamak plasma we order various relevant quantities as follows:

$$\frac{\tilde{f}}{f_0} \sim \delta \tag{50}$$

$$k_{\perp} \rho_e \sim \delta \tag{51}$$

$$\frac{k_{\parallel}}{k_{\perp}} \sim \delta. \tag{52}$$

The dominant driving terms in $\nabla\alpha_{n,e}$ along the $\mathbf{B}_{\#,s}$ field will be the parallel thermal force or the parallel viscous stress, depending on the collisionality regime. The approximation $\mathbf{B}_{\#,e} \simeq \mathbf{B}$ is used for simplicity in these estimates. From Braginskii,¹⁸ the parallel thermal force takes the form

$$R_{t,\parallel,e} = -0.71n_e\nabla_{\parallel}T_e \quad (53)$$

in the collisional fluid limit. Chang and Callen¹⁰ calculate the fluid moment closure relations kinetically in a slab model using a drift-kinetic Chapman-Enskog-like equation. In the adiabatic limit ($\omega \ll k_{\parallel}v_{t,e}$), they find

$$\hat{\mathbf{b}} \cdot \nabla \cdot \tilde{\mathbf{\Pi}}_{\parallel,e} = -0.40n_e\nabla_{\parallel}T_e, \quad (54)$$

where we have neglected the term $m_e\mu_{\parallel,e}\nabla_{\parallel}^2\tilde{u}_{\parallel,e}$ because it is smaller by a factor of order δ provided $\tilde{T}_e/\bar{T}_e \sim \delta$ and $\tilde{u}_{\parallel,e}/\bar{u}_{\parallel,e} \sim \delta$. (The latter condition follows from the empirical observation that $\tilde{B}/B \sim \delta^2$.²⁰) These terms are the lowest order terms in the collisional fluid and adiabatic limits if $\tilde{T}_e/\bar{T}_e \sim \delta$ for the turbulent fluctuations. Note that both of these terms have the form of density multiplied by parallel temperature gradient. We are thus motivated to consider

$$\mathbf{B}_{\#,e} \cdot \nabla \tilde{\alpha}_{n,e} = \frac{\lambda}{e} \mathbf{B}_{\#,e} \cdot \nabla \tilde{T}_e \quad (55)$$

where $\lambda = -0.71$ in the collisional fluid limit and $\lambda = 0.40$ in the adiabatic limit. In this case

$$\tilde{\alpha}_{n,e} = \frac{\lambda}{e} \tilde{T}_e + f(\psi_{\#,e}) \quad (56)$$

and the resulting ensemble-averaged particle flux from $\nabla\tilde{\alpha}_{n,e}$ is

$$\Gamma_{\psi_{\#,e}}^{\alpha_n} = \langle R^2 \nabla \zeta \cdot (\lambda/e) \tilde{n}_e \nabla \tilde{T}_e \rangle. \quad (57)$$

Using approximate axisymmetry,

$$\langle R^2 \nabla \zeta \cdot \nabla (\tilde{n}_e \tilde{T}_e) \rangle = 0, \quad (58)$$

yields

$$\langle R^2 \nabla \zeta \cdot (\tilde{T}_e \nabla \tilde{n}_e) \rangle + \langle R^2 \nabla \zeta \cdot (\tilde{n}_e \nabla \tilde{T}_e) \rangle = 0. \quad (59)$$

Using these expressions together with the expression for the anomalous particle flux, the lowest order anomalous contribution to the total particle flux is

$$\Gamma_{\psi_{\#,e}}^{anom} = -\frac{\gamma}{e} \langle \tilde{T}_e \frac{\partial \tilde{n}_e}{\partial \zeta} \rangle \quad (60)$$

with the coefficient $\gamma = 1 + \lambda$ having the value 0.29 in the collisional fluid limit and 1.4 in the adiabatic limit. The anomalous electron particle flux can thus be written in terms of a correlation of $\tilde{T}_e$ and $\nabla \tilde{n}_e$ in the collisional fluid and adiabatic limits.

This last result can be compared to the usual approximate expressions for estimating transport fluxes in an Eulerian, slab model framework where $\Gamma_x = \langle \tilde{n} \tilde{u}_x \rangle = \langle \tilde{n} \partial \tilde{\phi} / \partial y \rangle / B \equiv -D dn/dx$. Using the fact that $\nabla \psi_{\#,s} \simeq RB_p \hat{x}$, we can write Eq. (60) in the form

$$\Gamma_r = -\frac{\gamma}{eRB_p} \langle \tilde{T}_e \partial \tilde{n}_e / \partial \zeta \rangle \sim -D dn/dr, \quad (61)$$

where the brackets indicate an ensemble average as well as a flux surface average. From this equation we find

$$D \sim -\frac{\Gamma_r}{dn/dr} \sim -\frac{\Gamma_r a}{\bar{n}}. \quad (62)$$

Thus, we obtain

$$D \sim \frac{\gamma a}{eB_p \bar{n}} \langle \tilde{T}_e k_\zeta \tilde{n}_e \rangle. \quad (63)$$

To find an upper bound on D , we use the case where the fluctuations $\tilde{n}_e$ and $\tilde{T}_e$ are 180 degrees out of phase and the fluctuation spectrum is odd in k_ζ . Using $k_\zeta \simeq B_p k_\theta / B$ then yields

$$D \sim \frac{\gamma a v_{t,e}}{2} \langle (k_\theta \rho_e) \frac{\tilde{T}_e}{\bar{T}_e} \frac{\tilde{n}_e}{\bar{n}_e} \rangle. \quad (64)$$

For TFTR parameters²¹ $a \simeq 1$ m, $\tilde{n}_e/\bar{n}_e \simeq 10^{-2}$, $k_\perp \rho_e \simeq 2 \times 10^{-3}$, and $v_{t,e} \simeq 5 \times 10^7$ m/s, and assuming $\tilde{T}_e/\bar{T}_e \sim \tilde{n}_e/\bar{n}_e$, we find $D \sim 7$ m²/s for an approximate upper bound.

However, for realistic cases where $\tilde{n}_e$ and $\tilde{T}_e$ may not be totally out of phase and the k_ζ spectrum is nearly even in k_ζ this estimate will be much smaller.

VII. Conclusion

We have explored a model of transport in which the ideal canonical flux surfaces associated with $\mathbf{B}_{\#,s} = \mathbf{B} + (m_s/q_s)\nabla \times \mathbf{u}_s$ fluctuate but remain closed and nested in the presence of ideal plasma fluctuations. Net particle transport relative to the ideal canonical flux surfaces has been shown to result only from nonideal, dissipative processes. Net transport equations for particles and entropy were derived, with the result that only the component of fluxes normal to the surfaces of the relative motion between the fluids and surfaces produces net particle and entropy transport. Net entropy can also be produced by heat conduction dissipation, viscous dissipation, and collisional heating.

We have also shown explicitly, for the particle flux, how to add the classical, neo-classical, and fluctuation-induced (anomalous) transport fluxes. The anomalous particle flux has been calculated relative to these ideal moving canonical flux surfaces, with the result that, for electrons, density and temperature fluctuations which are out of phase drive a net, cross-surface particle flux. The net anomalous electron particle flux can be written in terms of electron density and temperature fluctuations in both the collisional fluid and adiabatic limits, as shown in Eq. (60). This new formula provides a way to estimate the anomalous transport relative to a set of fluctuating ideal canonical surfaces; i.e., it represents anomalous transport in a Lagrangian rather than Eulerian formalism.

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Appendix A. Derivation of canonical flux invariance

The derivation of canonical flux invariance given here is similar to the derivation given by Boozer for the magnetic field.^{6,7} Following that derivation, this appendix considers the question of whether the existence of a closed, nested set of flux surfaces at all times is sufficient to yield the canonical flux invariance condition. The conclusion is that the closed canonical flux surface condition is equivalent to canonical flux invariance for determining transport.

With closed canonical flux surfaces, one can define a $\psi_{\#,s}, \theta_s, \zeta_s$ coordinate system, in which $\psi_{\#,s}$ is the flux surface label, θ_s a poloidal angle, and ζ_s a toroidal angle. For the canonical flux through an arbitrary constant ζ_s surface S ,

$$\psi_{\#,s} \equiv \int_S \mathbf{B}_{\#,s} \cdot d\mathbf{a}_{\zeta_s}, \quad (\text{A1})$$

the time rate of change for movement at velocity $\mathbf{u}_{sf}$ is

$$\frac{d\psi_{\#,s}}{dt} = \int_S \frac{\partial \mathbf{B}_{\#,s}}{\partial t} \cdot d\mathbf{a}_{\zeta_s} + \int_{\partial S} \mathbf{B}_{\#,s} \cdot (\mathbf{u}_{sf} \times d\mathbf{l}). \quad (\text{A2})$$

Using the generalized Faraday's law, Eq. (18), for the first term and a vector identity and Stokes' theorem for the second term yields

$$\frac{d\psi_{\#,s}}{dt} = \int_S [\nabla \times (\mathbf{E} - (m_s/q_s) \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_{sf} \times \mathbf{B}_{\#,s})] \cdot d\mathbf{a}_{\zeta_s}. \quad (\text{A3})$$

In order that $d\psi_{\#,s}/dt = 0$ for an arbitrary constant ζ_s surface, the integrand must vanish. Using the fact that a constant ζ_s surface element is along $\nabla \zeta_s$, recalling the vector identity $\nabla \times (f \nabla g) = \nabla f \times \nabla g$, and writing $\mathbf{E} - (m_s/q_s) \partial \mathbf{u}_s / \partial t + \mathbf{u}_{sf} \times \mathbf{B}_{\#,s}$ in the general covariant form

$$\mathbf{E} - (m_s/q_s) \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_{sf} \times \mathbf{B}_{\#,s} = c_1 \nabla \psi_{\#,s} + c_2 \nabla \theta_s + c_3 \nabla \zeta_s, \quad (\text{A4})$$

the condition required for canonical flux invariance is written

$$[\nabla c_1 \times \nabla \psi_{\#,s} + \nabla c_2 \times \nabla \theta_s + \nabla c_3 \times \nabla \zeta_s] \cdot \nabla \zeta_s = 0. \quad (\text{A5})$$

Since the third term in brackets vanishes, the other terms combine to give the condition $\partial c_1 / \partial \theta_s = \partial c_2 / \partial \psi_{\#,s}$, which enables one to write

$$c_1 \nabla \psi_{\#,s} + c_2 \nabla \theta_s + c_3 \nabla \zeta_s = V_{\#,s} \nabla \zeta_s - \nabla \Phi_s. \quad (\text{A6})$$

Boozer⁶ has shown for the analogous condition for the magnetic field that the loop voltage term can always be written as a flux function for some $\nabla \Phi_s$. The same is true for the $\mathbf{B}_{\#,s}$ field because its form and the form of the flux invariance condition are analogous to those for the magnetic field, and hence the proof proceeds analogously to that given by Boozer. The canonical flux invariance condition thus becomes

$$\mathbf{E} - (m_s/q_s) \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_{sf} \times \mathbf{B}_{\#,s} = V_{\#,s}(\psi_{\#,s}) \nabla \zeta_s - \nabla \Phi_s. \quad (\text{A7})$$

The canonical flux invariance condition was obtained by imposing the restriction that $d\psi_{\#,s}/dt$ vanish for any constant ζ_s surface. Since this constraint does not follow directly from the existence of a set of closed canonical flux surfaces, the question arises as to whether closed canonical flux surfaces could exist without canonical flux invariance being satisfied. Although the condition found from closed canonical flux surfaces only is more general than the canonical flux invariance condition, the two are equivalent for determining transport, and this result is shown in the following.

For a moving closed canonical flux surface, any piece of the surface itself must remain flux free during the course of the surface motion. (To visualize a piece of the surface, think of a patch on an inner tube.) This observation means that an arbitrary piece of a closed canonical flux surface must satisfy $d\psi_{\#,s}/dt = 0$. Writing the canonical flux invariance condition plus a vector field $\mathbf{T}_s$ which accounts for possible departures from canonical flux invariance, we have

$$\mathbf{E} - (m_s/q_s) \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_{sf} \times \mathbf{B}_{\#,s} = V_{\#,s}(\psi_{\#,s}) \nabla \zeta_s - \nabla \Phi_s + \mathbf{T}_s. \quad (\text{A8})$$

Substituting this equation into Eq. (20) gives

$$\frac{d\psi_{\#,s}}{dt} = \oint_C \mathbf{T}_s \cdot d\mathbf{l}, \quad (\text{A9})$$

where the closed curve C lies in a flux surface. The general form for $\mathbf{T}_s$ required in order for the expression to vanish is

$$\mathbf{T}_s = \nabla f + g(\mathbf{x}) \nabla \psi_{\#,s}. \quad (\text{A10})$$

The gradient term has already been included as $\nabla \Phi_s$, and $g(\mathbf{x}) \nabla \psi_{\#,s}$ is normal to the canonical flux surface and thus yields only extra flow within the surface, which is not important for transport calculations. Hence, the closed canonical flux surface assumption is the same as the canonical flux invariance condition for calculating transport.

Appendix B. Transport relative to moving surfaces

Consider a quantity B given by the volume integral of a quantity β :

$$B \equiv \int_V d^3x \beta. \quad (\text{B1})$$

Its time rate of change at a time t_o is given by

$$\frac{\partial B}{\partial t} = \lim_{\Delta t \rightarrow 0} \frac{B(t_o + \Delta t) - B(t_o)}{\Delta t}. \quad (\text{B2})$$

Substituting for B gives

$$\frac{\partial B}{\partial t} = \lim_{\Delta t \rightarrow 0} \left[\int_{V(t_o + \Delta t)} d^3x \beta(t_o + \Delta t) - \int_{V(t_o)} d^3x \beta(t_o) \right] / \Delta t. \quad (\text{B3})$$

Let $V(t_o + \Delta t) = V(t_o) + \Delta V$, so that

$$\frac{\partial B}{\partial t} = \lim_{\Delta t \rightarrow 0} \left[\int_{V(t_o)} d^3x \frac{\beta(t_o + \Delta t) - \beta(t_o)}{\Delta t} + \frac{1}{\Delta t} \int_{\Delta V} d^3x \beta(t_o + \Delta t) \right]. \quad (\text{B4})$$

The change in volume ΔV results from the surface motion at velocity $\mathbf{u}_{sf}$. Dividing the surface into pieces ΔF , in a time Δt one of these pieces moves a distance $\Delta x_n = \mathbf{u}_{sf} \cdot \hat{\mathbf{n}} \Delta t$, where $\hat{\mathbf{n}}$ is the unit normal to the surface. Thus, we obtain

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{\Delta V} d^3x \beta(t_o + \Delta t) &= \int_F \lim_{\Delta t \rightarrow 0} [\beta(t_o + \Delta t) \frac{\Delta x_n}{\Delta t} \Delta F] \\ &= \int_F \beta \mathbf{u}_{sf} \cdot d\mathbf{F}. \end{aligned} \quad (\text{B5})$$

Using this result in Eq. (B4), we find

$$\frac{\partial B}{\partial t} = \int_{V(t_o)} d^3x \frac{\partial \beta}{\partial t} + \int_{F(t_o)} \beta \mathbf{u}_{sf} \cdot d\mathbf{F}. \quad (\text{B6})$$

When β is taken to be the density n_s or the entropy density $n_s s_s$, we find equations for the time rate of change of the total number of particles N_s and total entropy S_s inside a moving flux surface by using Eqs. (2) and (5):

$$\frac{\partial N_s}{\partial t} = - \int_{F(t_o)} n_s (\mathbf{u}_s - \mathbf{u}_{sf}) \cdot d\mathbf{F} \quad (\text{B7})$$

$$\begin{aligned} \frac{\partial S_s}{\partial t} &= - \int_{F(t_o)} \frac{\mathbf{q}_s}{T_s} \cdot d\mathbf{F} - \int_{F(t_o)} n_s s_s (\mathbf{u}_s - \mathbf{u}_{sf}) \cdot d\mathbf{F} - \\ &\int_{V(t_o)} d^3x \frac{1}{T_s} \mathbf{q}_s \cdot \nabla \ln T_s - \int_{V(t_o)} d^3x \frac{1}{T_s} \underline{\Pi}_s : \nabla \mathbf{u}_s + \int_{V(t_o)} d^3x \frac{Q_s}{T_s}. \end{aligned} \quad (\text{B8})$$

Appendix C. Toroidal viscous force

Chang showed in his PhD thesis²² that the lowest order viscous stress always takes on the Chew, Goldberger, Low²³ form. Here we show that the flux surface average of the toroidal component of a viscous force in this form vanishes. To lowest order the average distortion of the distribution function from a Maxwellian is gyrophase independent, giving

$$\underline{\Pi} = (p_{\parallel} - p_{\perp})(\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{I}/3), \quad (\text{C1})$$

in which $\hat{\mathbf{b}} \equiv \mathbf{B}/B$ is a unit vector along the magnetic field. From this equation, we find

$$\nabla \cdot \underline{\mathbf{II}} = \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla(p_{\parallel} - p_{\perp}) - (1/3) \nabla(p_{\parallel} - p_{\perp}) + (p_{\parallel} - p_{\perp})(\nabla \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}} + (p_{\parallel} - p_{\perp}) \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}. \quad (\text{C2})$$

Taking the toroidal component, using Eq. (28) for $R^2 \nabla \zeta$, vector identities, and Ampere's Law gives

$$R^2 \nabla \zeta \cdot \hat{\mathbf{b}} = I/B \quad (\text{C3})$$

and

$$R^2 \nabla \zeta \cdot (\hat{\mathbf{b}} \cdot \nabla) \hat{\mathbf{b}} = \frac{\nabla \psi \cdot \mu_o \mathbf{J}}{B^2} + \frac{I \mathbf{B}}{B} \cdot \nabla(1/B). \quad (\text{C4})$$

Using $\nabla \psi \cdot \mathbf{J} = 0$ for an ideal MHD equilibrium, we have

$$\begin{aligned} R^2 \nabla \zeta \cdot \nabla \cdot \underline{\mathbf{II}} &= (I/B) \hat{\mathbf{b}} \cdot \nabla(p_{\parallel} - p_{\perp}) - (1/3) R^2 \nabla \zeta \cdot \nabla(p_{\parallel} - p_{\perp}) \\ &\quad + 2(p_{\parallel} - p_{\perp})(I/B) \mathbf{B} \cdot \nabla(1/B) \\ &= I \mathbf{B} \cdot \nabla[(p_{\parallel} - p_{\perp})/B^2] - (1/3) R^2 \nabla \zeta \cdot \nabla(p_{\parallel} - p_{\perp}). \end{aligned} \quad (\text{C5})$$

Taking the flux surface average of this equation gives, with the help of Eq. (27) and $I = I(\psi)$,

$$\langle R^2 \nabla \zeta \cdot \nabla \cdot \underline{\mathbf{II}} \rangle = 0. \quad (\text{C6})$$

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