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An Exact Renormalization Model for Earthquakes and Material Failure: Statics and Dynamics *

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Abstract: Earthquake events are well-known to possess a variety of empirical scaling laws. Accordingly, renormalization methods offer some hope for understanding why earthquake statistics behave in a similar way over orders of magnitude of energy. We review the progress made in the use of renormalization methods in approaching the earthquake problem. In particular, earthquake events have been modeled by previous investigators as hierarchically organized bundles of fibers with equal load sharing. We consider by computational and analytic means the failure properties of such bundles of fibers, a problem that may be treated exactly by renormalization methods. We show, *independent of the specific properties of an*

individual fiber, that the stress and time thresholds for failure of fiber bundles obey universal, albeit different, scaling laws with respect to the size of the bundles. The application of these results to fracture processes in earthquake events and in engineering materials helps to provide insight into some of the observed patterns and scaling—in particular, the apparent weakening of earthquake faults and composite materials with respect to size, and the apparent emergence of relatively well-defined stresses and times when failure is seemingly assured.

1. Introduction

The failure of natural and man-made materials presents an important class of pattern formation and scaling problems. It is useful to consider failure on a lattice in one, two, or three dimensions where the probability of failure of a given lattice point as a function of the applied load σ (essentially instantaneously in time) or the time t (given some applied load history) is prescribed. A fundamental feature which distinguishes the nature of failure in such systems is the manner in which the load formerly supported by failed elements is redistributed among the survivors. There is substantial evidence in the materials science and, especially, in the geophysical literature that universal scaling laws for failure can possibly emerge. The best known of these relations is the Gutenberg-Richter magnitude-frequency law [1] which relates the “magnitude” of an earthquake (i.e. the logarithm to base 10 of the amplitude of the ground motion or “shaking”) to the relative occurrence

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of such events. [Largely unknown in the West, Ishimoto and Iida [2] established the connection between the amplitude and frequency a number of years before; in Japan, the magnitude-frequency relation is usually referred to as the Ishimoto-Iida law.] Preceding these relatively contemporary discoveries, Omori [3] found that the timing of aftershocks also obeyed a well-defined and seemingly universal power-law distribution. More recently, Kagan and Knopoff [4] established the existence of fractal properties, yet another form of self-similarity, in the geometry of earthquake faults, which had been assumed by geologists by many years to be planar surfaces. These empirical scaling laws are reviewed in Kagan [5]. Taken together, all of these manifestations of the earthquake mechanism provide compelling reasons to believe that seismicity is essentially self-similar, i.e. in the absence of scale information, measurable features appear to be the same at all scales ranging from those characteristic of grains in rocks to those characteristic of plate tectonics or significant thermal variation with depth, a range of at least six orders of magnitude.

Consequently, we wish to understand the pattern and scaling of failure of the macroscopic system given that the microscopic properties are known. This objective has been shared by a number of investigators, primarily from the United States and Russia. We will begin by reviewing some of the important contributions made in the application of renormalization methods to geophysics and, especially, the earthquake mechanism. Then, we will present some recent results that we have obtained relevant to scaling properties for failure as a function of the applied load and of the elapsed time. To pursue this goal, we will investigate the paradigm of "fiber bundles" and employ renormalization methods, which can be shown to be exact, both in the computational and analytic investigations. Renormalization methods that we apply eliminate *all* redundant operations in the computational investigation (in the same spirit as the FFT optimizes the computation of Fourier transforms), thereby making it possible to explore the behavior over many orders of magnitude. Once the scaling laws are identified from the computation, bounding and asymptotic properties can be analytically derived. The relevance of these *universal* scaling

Fig. 1. Illustration of the Allègre *et al.* renormalization model.

laws to materials science and geophysical problems will then be discussed.

2. Renormalization Methods and the Fiber Bundle Paradigm

The first effort to employ renormalization methods to the problem of fracture was that of Allègre, Le Mouel, and Provost [6] who considered the failure of a simple lattice structure (see Figure 1). In their "pillar of strength model," they considered the probability of failure of a large cubic lattice consisting of rigid arms joined at the vertices by bolts which had a certain probability, say p_0 of failure—and that this in turn was proportional to the global stress σ . They proposed that the probability of failure of a cube, say p_1 could be expressed as a function of the probability of one of the vertices, namely $p_1 = \Pi(p_0)$. Proceeding to a larger cube composed of 8 smaller cubes, they then proposed that each of the smaller cubes had the same role as one of the vertices in the earlier situation—thus the probability of the 8-cube, namely p_2 would be approximated by $p_2 \approx \Pi(p_1) = \Pi[\Pi(p_0)] = \Pi^{(2)}(p_0)$.

Similarly, one could construct larger and larger cubes so that, in general, a cube containing 2^n vertices would have a probability of failure $p_n \approx \Pi^{(n)}(p_0)$ which in turn can be regarded as some function of the stress σ . This iteration is the essence of renormalization: each size-scale is seen to mirror the one below it—and above it—and a sequence of similar operations emerges. The function Π was constructed from combinatoric considerations by constructing all physical situations where this structure would fail under a load, and the limit of the sequence was then computed. The outcome of this model was a “critical point for failure,” a probabilistic interpretation which obtains a critical stress level σ_c such that failure will *never* occur for $\sigma < \sigma_c$ and failure will *always* occur for $\sigma > \sigma_c$. Turcotte [7] argued that, while the model was worthwhile, a more realistic physical scenario corresponded to the situation wherein the structure failed under shear. Apart from the geometric details contained within the probabilistic function Π , the outcome of Turcotte’s “plane of weakness model” was the same. Around the same time, Smalley, Turcotte, and Sola [8] argued that the pillar of strength and plane of weakness models simply allowed all of the stress released by failed vertices to disappear rather than redistribute the stress among the surviving vertices. They then developed the hierarchical fiber bundle model described below. Meanwhile, Knopoff and Newman [9] and Newman and Knopoff [10] developed a dynamic version of the Allègre *et al.* model, showing how it preserved some of the character of a critical point while introducing a measure of the time-delay seen in fracture experiments. The role of fractals and renormalization in geology and geophysics is reviewed in Turcotte [11].

The Russian school proceeded in parallel in utilizing renormalization methods: in addition to developing models that describe the emergence of probability distributions, they also explored specific realizations on a discrete hierarchical system or “tree” (Figure 2). Shnirman [12], Narkunskaya [13], Narkunskaya and Shnirman [14], and Tumarkin and Shnirman [15] developed a model similar to that of Knopoff and Newman [9] and Newman and Knopoff [10] in that it contains both dynamics and a (vertex) restoration process

Fig. 2. A Shnirman tree with degree of vertices equal to three.

(known as "healing"); moreover, they compared the model results with the scalings found in earthquake catalogs, observing a significant degree of agreement. A vast array of research on earthquake related phenomena has since emerged utilizing cellular automata and other techniques—which do not exploit renormalization or scaling properties in any particular way—and these will not be described here. However, it was the fiber bundle problem introduced into the geophysical context by Smalley *et al.* [8] that provided many new insights.

The failure of a fiber bundle is an interesting problem for a variety of practical applications. In addition to stranded cables, the analysis is also directly applicable to the failure of composite materials. The failure of fiber bundles is also an analog to the initiation of rupture on a fault. The failure of individual strands is analogous to the cascading failure of asperities as rupture propagates along an earthquake fault.

Early efforts to understand material failure utilized as a paradigm the failure properties of bundles of fibers. The question is how the failure of the bundle (e.g. a cable) is related to the failure of the individual fibers (the cable's strands). We will assume that fiber

bundles have no bonding or friction between fibers but are perfectly secured at their ends—in this way, a lattice is indistinguishable from a fiber bundle. While it is easy to see that a planar tape of fibers and a bundle of fibers are direct analogues of one and two dimensional lattices, respectively, fiber bundles can also be employed to explore the behavior of three dimensional structures. The effective dimensionality of a lattice depends upon the rule for redistributing the load following the failure of individual fibers. There are, essentially, two limiting cases that can be considered: in the first, the stress on a failed group of fibers is transferred equally to *all* surviving fibers, while in the second, the stress on a failed group of fibers is transferred to the nearest surviving group(s) of fibers. The redistribution rule establishes the meaning of “nearest neighbors” and, hence, the effective dimensionality of the system. Although it is useful to visualize fiber bundles as long ropes, it is essential to note that the principle mechanism wherein ropes support a load is through the *friction* that exists between different fibers (allowing local stress transfer at fiber breaks or discontinuities), a process that is absent in lattice models and in the idealization we will now pursue.

The first mathematical investigation of the failure properties of bundles of fibers was carried out by Daniels [16]. Following Daniels, we will assume that m such fibers or filaments are placed in parallel along a line to form a planar tape. This tape is then subject to a load σ which is the *force per fiber*, the total force on the bundle being $m\sigma$. [The term *load* is employed in different ways in different disciplines. Among those who work with fibrous, composite materials, it describes the force per unit area or the force per fiber. In the earth sciences, it describes the *total* force acting on or supported by a system. It is most natural for an audience of physicists to employ the former definition, as it invokes the sense of an intrinsic or specific variable, and accordingly we will speak of the load as the force per fiber.] It is further assumed that these idealized fibers are of equal length, stiffness, and cross-sectional area, and have strengths that are i.i.d. (independent and identically distributed) with respect to load or to time (assuming identical load histories).

2.1. The Static Problem

Daniels [16] considered the static behavior of fibers with a cumulative (failure) distribution function $P(\sigma)$, $\sigma \geq 0$. In other words, $P(\sigma)$ describes the probability that an individual fiber would fail below a load σ . Further, Daniels assumed that the fiber bundle obeyed the so-called "equal-load sharing rule," i.e. all nonfailed fibers share the load transferred equally, and all failed fibers carry no load. Daniels explored the asymptotic distribution for the strength of this ensemble of fibers as m grows large, and also developed recursion relations useful for smaller m . Many other investigators, including for example Smith and Phoenix [17], have explored other aspects of this problem, such as an arrangement of such bundles in series. What becomes clear from these investigations is that the asymptotic behavior depends strongly on the type of load-sharing and overall architecture of how such bundles are arranged. As m grows large, the strength of an equal load-sharing bundle approaches a constant [16] whereas that for a local load-sharing bundle (as defined later) approaches zero albeit extremely slowly (Newman and Gabrielov [18] and Gabrielov and Newman [19]). Moreover, by an insightful application of the central limit theorem, Daniels established that the macroscopic distribution function with respect to the load for the failure of an asymptotically large fiber bundle with equal load-sharing is normally distributed. It is the nature of the macroscopic distribution function for failure under a specific version of *local* load-sharing that we wish to establish here.

Smalley *et al.* [8] were the first to appreciate the connection between fiber bundles and lattice models for failure. They considered fiber bundles with $m = 2$ and $m = 4$ as a metaphor for the strength of a fault in the earth's crust, which was considered to be a hierarchically organized system of asperities. In particular, they assumed that m fibers could be assembled to form a bundle which would behave as if it were itself a fiber. Then, they assumed that m of these fiber bundles could themselves be assembled to form a bundle, and so on. This hierarchical assembly process, which we will refer to as

Fig. 3. Fiber bundles for coordination numbers $m = 2$ and $m = 4$.

hierarchical clustering with coordination number m , can be continued indefinitely. They employed an index n to describe the level within the hierarchy. For example, when $m = 2$, the hierarchical structure corresponds to a successive pairing or twinning of fibers. Then, $n = 0$ implies individual fibers, $n = 1$ implies a pair of such fibers, $n = 2$ implies a pair of pairs of such fibers and so on. Thus, an n -th order fiber bundle with coordination number m would contain m^n individual fibers.

The Smalley *et al.* scheme is a renormalized description of failure and follows in the same spirit as Allègre *et al.* [6]. The description by Allègre *et al.*, however, does not redistribute the released stress among the surviving members, an essential feature of this problem—consequently, the method of Smalley *et al.* is physically more realistic. A main feature of this approach is that the load released by a failed unit of any size in the hierarchy is redistributed onto a regime of comparable spatial extent, so the redistribution of stress is a localized phenomena in contrast to a single Daniels bundle where the load-sharing is global. (This represents a specific class of local load-sharing models; other classes for local load-sharing have also been studied, e.g. Harlow and Phoenix, [20].) This feature is the

hallmark for the Green's function for the stress field [21] around a "crack" where the stress field is localized near the failed sites and influences a region whose size is proportional to the crack size and the square of the applied stress. (This is the usual first approximation for the size of the plastic zone around a crack. See Kanninen and Popelar [22] for example.) A key point that becomes clearer in later discussion is that hierarchical structures become an important concept when catastrophic or weakest link fracture growth stalls as the size scale increases due to some crack blunting mechanism caused, say, by shear limitations.

Smalley *et al.* [8] suggested that an additional renormalization procedure could be applied to this static problem since the bundles were grouped in a hierarchical fashion. They employed a simplifying approximation in their renormalization calculation and obtained a "critical point" like behavior. They observed, for a given hierarchical structure defined by the value of m , that a well-defined load would emerge for large assemblies of fibers such that the fiber bundle would survive if the stress was lower than that critical stress and would fail if the stress was higher than that critical stress. The approximation they employed, however, subtly altered the otherwise exact renormalization calculation, and a number of investigators challenged the validity of the implied critical point. Newman and Gabrielov [18] and Gabrielov and Newman [19] explored the problem formulated by Smalley *et al.* using *exact* computational and analytic means. They employed a renormalization of the computation of the hierarchical system of equations wherein they identified and eliminated all redundant computations. In this way, they were able to simulate the behavior of the hierarchical system *over 90 decades* of system size and to identify the asymptotic nature of the solution. With this information at hand, they were able to derive analytically both bounds and asymptotics for the system, although this required the development of new mathematical techniques. Although a critical point does not formally exist for the static problem, the strength of the bundle declines at an extraordinarily slow rate. Moreover, the width of the distribution function for failure diminishes steadily but remains substantial for any finite-sized realization of a fiber bundle. A simplified version

of these developments is presented in the next section.

2.2. *The Dynamic Problem*

In this manuscript, we present new results we have obtained for the dynamic or time-dependent problem by methods analogous to those used by Newman and Gabrielov [18] in the static problem. We review the static problem in the next section because what we learn from the static problem in the stress variable σ can in part be transferred to the dynamic problem in terms of the time variable t . In particular, we wish to explore (for a given applied load) how the failure properties in time of fiber bundles vary with respect to the size of the bundle. Coleman [23,24] pioneered the investigation of the lifetime of bundles in an equal load-sharing framework, and these models have been used to explain the failure of yarns, ropes, and cables as well as fibrous composites. For a review of previous work on the failure of chains of bundles, see Smith and Phoenix [17]. Lattice models of material failure have been pursued in the physics literature including dielectric breakdown, failure of superconducting and random resistor networks as well as brittle fracture, largely in the setting of percolation theory. For a review, see for example Duxbury [25]. Although large cables often involve hierarchical arrangements of bundles, modeling in this vein appears not to have been pursued by others.

In the next section, we will explore the static properties of hierarchical fiber bundles as well as the methods required to investigate these. In the fourth section, we will extend these methods to develop new computational algorithms and present the results of Monte Carlo simulations which offer a hint as to the nature of dynamic failure in hierarchical systems. The numerical results suggest the possible presence of a “critical point” for failure in time for large systems. We will conclude by discussing the significance of the pattern and scaling properties that are apparent, as well as some of the consequences—in particular the suggestion of a critical point for failure in time could have profound

significance in engineering practice—the failure of isolated individual assemblies could predict an avalanche of failure in statistically similar assemblies. In geophysics, this could help explain the success met by some earthquake prediction schemes predicated on pre-monitory seismicity—the presence of microscopic earthquakes could be an indicator of the catastrophic failure yet to come. Indeed, fiber bundles offer a rich paradigm for the investigation of failure in many varied situations.

3. Statics

We first consider an assembly of fibers where a load is instantaneously applied and we check to see how each of the fibers responds. Although the sequence of failures is in some sense time-dependent in reality, we consider this to be a static problem since we are considering an instantaneous, in contrast with an evolutionary, response to the loading. In the following section, we will discuss the situation where there is a finite probability of failure during an *interval of time* under a uniform load, in contrast with an instantaneous response. Nevertheless, the conceptual link between these two situations often leads to qualitative similarities in the failure properties with respect to the load in the static situation and to the time in the dynamic one.

Consider an assembly of fibers with no bonding or friction between fibers, which have a given cumulative strength distribution $P(\sigma)$ where σ is the applied stress, i.e., a fiber will break under a stress σ with probability $P(\sigma)$. The distribution $P(\sigma)$ is a nondecreasing function with values between 0 and 1 defined for $0 \leq \sigma \leq \infty$ where $P(\sigma) \rightarrow 1$ when $\sigma \rightarrow \infty$. Let us denote individual fibers of the bundle as fibers of order 0. Suppose that these fibers are paired in sequential order (i.e. $m = 2$) and consider each pair of fibers of order 0 as though it were itself a fiber, which we will denote as a fiber or, alternatively, as a fiber bundle of order 1. Suppose that these fibers of order 1 are also paired sequentially and

that these pairs then form fibers of order 2, and so on, with fibers of order n consisting of 2^n individual fibers. Equivalently, the order of the system, namely n , varies as $\log_2(\text{mass})$ where the “mass” refers to the total mass of the system. The two fibers of order n which are paired are called neighbours or twins of order n and form a fiber of order $n + 1$. It is natural to equate $P_0(\sigma)$ with $P(\sigma)$, and we denote by $P_n(\sigma)$ the strength distribution for fiber bundles of order n . We will refer to $P_0(\sigma)$ as the primitive or basic cumulative distribution function. The problem here is to determine the properties of $P_n(\sigma)$ as n becomes large. (As we will see later, scaling becomes evident albeit not conclusive when $n \approx 20$ while, for real samples of materials, we typically find $n \approx 25$ to 30. These values of n are generally inaccessible computationally using conventional approaches.)

Since both fibers in a given bundle share equally the load supported by that bundle, in contrast with Daniels, a given pair of fibers breaks under an applied stress σ only if both constituent fibers fail under that stress σ , or one fiber fails under its load and then redistributes it to the surviving fiber which then fails under a load between σ and 2σ . This concept can be applied to the failure of fibers of any order and is the *sine qua non* of the renormalization method. The probability that both fibers of order $n - 1$ fail simultaneously under a stress σ is $P_{n-1}^2(\sigma)$. The probability that one fiber fails under the stress σ with the surviving fiber failing under a stress between σ and 2σ is $2P_{n-1}(\sigma) [P_{n-1}(2\sigma) - P_{n-1}(\sigma)]$. The factor of two appears here since failure can occur in two ways, according to which one of the two fibers fails first. Thus, the probability of failure of a fiber bundle of order n is equal to the sum of the previous two terms, namely

$$P_n(\sigma) = P_{n-1}(\sigma) [2P_{n-1}(2\sigma) - P_{n-1}(\sigma)] \quad . \quad (1)$$

The problem can be formulated similarly for the m -fold case when a fiber of order n consists of m fibers of order $n - 1$ —however, the elaborate combinatoric nature of the problem for $m > 2$, gives rise to expressions that have the order of 2^m terms!

Smalley *et al.* [8] explored this transformation *assuming* that $P(2\sigma)$ could be expressed as a simple function of $P(\sigma)$, i.e., $P(2\sigma) = F[P(\sigma)]$ where the functional F was determined

Fig. 4. Cumulative distribution function $P_n(\sigma)$, $n = 0, 1, \dots, 10$, for the Weibull exponential distribution [$\rho = 1$].

to give the correct result when the Weibull distribution $P(\sigma) = 1 - \exp(-a\sigma^2)$ was employed. They continued to use this same functional relationship for $n > 0$, which is not strictly correct as the correspondence between the distribution $P_n(\sigma)$ and $P_n(2\sigma)$ depends on n . Although a useful approximation, it forced the solution to have the character of a critical point—indeed, *any* functional relationship of this sort necessarily results in a critical point (Allègre *et al.* [6]; Newman and Knopoff [10]). [Another way to think of this is to recognize that we are converting a functional iteration (1) into a point map.] To provide a sense of how the transformation (1) develops, consider the case of the (two parameter) Weibull distribution

$$P(\sigma) = 1 - \exp[-a\sigma^\rho] \quad , \quad (2)$$

where ρ defines the *modulus* or *shape parameter* and typically ranges from 1 to 20 and the case of $\rho = 1$ is called the “exponential distribution.” In Figure 4, we plot the sequence of cumulative distribution functions $P_n(\sigma)$, for $n = 0, 1, \dots, 10$, generated by (1) using the Weibull exponential distribution ($\rho = 1$ and $a = 1$). The trend that we observe is

one where the distribution becomes more and more like a step-function that migrates steadily toward the origin as n increases. We observe, however, that the rate of approach toward the origin quickly decelerates and that the width of the step—a measure of the uncertainty in the value of the stress which will precipitate a failure—remains large. A simple argument explaining why weakening occurs can be constructed by considering two representative fibers which are then paired together. Since they are representative of the distribution, it follows that one is likely to be weaker than the average and the other fiber stronger than the average. Hence, failure is initiated in the weaker fiber and the surviving fiber will likely fail under the doubled load (this is the so-called “weakest-link” approximation). The end result is that failure in the paired bundle, on average, will occur at a lower threshold.

To better quantify the observed tendency of the cumulative distribution function to approach the origin, it is useful to define a “failure threshold” σ_n as the stress where $P_n(\sigma) = P_T$ where the value of P_T assigns some assumed threshold for failure. In common engineering practice, this can be the 50 percent quantile, which we will generally use, or some other criterion for failure. Note that if the iteration (1) were to result in σ_n converging to a finite non-zero value, we would have a critical point for the iteration. In contrast, suppose that weakest link theory applies, i.e. if one fiber fails, then any surviving fibers in the bundle experiencing an increased load will *always* fail. Thus for $m = 2$ in (1), this is equivalent to assuming that if one fiber fails, its partner automatically fails and $P_n(2\sigma) = 1$. This conjecture can be employed formally to (1) to obtain a bound, namely

$$P_{n+1}(\sigma) \leq \begin{cases} P_n(\sigma) [2 - P_n(\sigma)] \\ 1 - [1 - P_0(\sigma)]^{2^n} \end{cases} \quad (3)$$

(It is worth noting that this expression has the form one would obtain if extreme value statistics were applicable [26] if 2^n elements were involved—this already provides an indication that something peculiar might emerge from this bound.) Using the example of

the Weibull exponential distribution, we find that $P_{n+1} \leq 1 - \exp(-\sigma 2^n)$ leading to the conclusion, if we were to employ the *equality* in (3), that $\sigma_n \propto 2^{-n} \propto 1/\text{mass}$. This implies that (a lower bound to) the strength of the bundle varies *inversely* as its mass. An alternate interpretation of this bound, given that the total applied force goes as $2^n \sigma_n$, is that a finite force would suffice to break an infinitely massive cable! There is no contradiction here, as this is a lower bound, but it does indicate that this bound is not particularly useful. We will return shortly to the question of determining bounds.

In order to obtain the scaling properties of σ_n , we must be able to efficiently evaluate equation (1). The direct evaluation of $P_n(\sigma)$ in equation (1) would require 2^n function evaluations. However, we observe that the only stress values that an individual fiber can experience are the $n + 1$ values $\sigma, 2\sigma, \dots, 2^n \sigma$. Assuming that $P_0(x)$ is computed (and stored) for $x = \sigma, 2\sigma, 2^2\sigma, \dots, 2^n \sigma$, we can employ equation (1) to calculate $P_1(x)$ for $x = \sigma, 2\sigma, 2^2\sigma, \dots, 2^{n-1}\sigma$. Again using (1), we can tabulate $P_2(x)$ for $x = \sigma, 2\sigma, 2^2\sigma, \dots, 2^{n-2}\sigma$. (It follows that this construction can be regarded as a renormalization.) We continue in this way until we are left with $P_n(\sigma)$. The evaluation of $P_n(\sigma)$ required only $n + 1$ evaluations of $P_0(\sigma)$ and only the order of n^2 arithmetic operations. Hence, the calculation of $P_n(\sigma)$ is observed to be optimal in the number of computations taken (since the operations taken are now irreducible) and makes feasible the computation of σ_n for very large n . It is worth observing that this procedure for eliminating redundant computations can be developed for any m , including $m = 4$ which is relevant to earthquake faults [8]. The irreducible character of this computational scheme is reminiscent of the Fast Fourier Transform which, using some simple number theoretic results, is able to reduce the computation of an N point discrete Fourier transform from $O(N^2)$ operations to $O(N \log_2 N)$ operations. (This procedure is described in greater detail in Newman and Gabrielov [18].)

Once we had assembled a large array of σ_n , say for $m = 2$ and for $n = 1, \dots, 100$, we plotted σ_n against a very large number of functions of n (for $P_T = 1/2$) until we found

Fig. 5. Reciprocal of the failure threshold, i.e. σ_n^{-1} , plotted against the logarithm (base 10) of the order index n for Weibull distribution functions with shape parameter $\rho = 1, \dots, 5$. Coordination number $m = 2$.

a functional relationship that appears to be satisfied. To an excellent approximation, we found that

$$\sigma_n^{-1} \propto \ln(n) \propto \ln[\ln(\text{mass})] \quad , \quad (4)$$

for $m = 2$. This is seen in Figure 5 where we plot σ_n^{-1} against $\log_{10} n$ for a variety of Weibull models ($\rho = 1, \dots, 5$)—the robust character of this relation offers an important indication that this result is a global one valid for almost any physically reasonable choice of $P_0(\sigma)$. With the insight gained from the renormalized computation, Newman and Gabrielov [18] showed that (4) provides a bound to the behavior for σ_n and Gabrielov and Newman [19] then showed that (4) was the asymptotic limit for the problem.

The route to this proof is contained within equation (1), namely

$$P_n(\sigma) = P_{n-1}(\sigma) [2P_{n-1}(2\sigma) - P_{n-1}(\sigma)] \quad .$$

If we put the first term inside the bracket to the zero power (i.e. we set it to unity), we recover the weakest link theory which is too strong an inequality or lower bound to

Fig. 6. The quantity $\sigma_n^{-2.40942}$ plotted against the logarithm (base 10) of the order index n for Weibull exponential distribution functions (shape parameter $\rho = 1$). Coordination number $m = 4$.

be useful. If we put the first term inside the bracket to the second power, we can show (Newman and Gabrielov, [18]) that the inequality or upper bound evolves in the wrong direction! The resolution of this problem lies in keeping the first term in the bracket unchanged and ignoring the second term—this shows that the lowest increment of stress to be supported by a surviving fiber dominates the asymptotics including larger values of m . (It is easy to create an inequality, but not all inequalities are created equal.) We found, for general m , that

$$\sigma_n^{-\nu} \propto \ln(n) \propto \ln[\ln(\text{mass})] \quad , \quad (5)$$

where

$$\nu = \frac{\ln 2}{\ln \frac{m}{m-1}} \quad . \quad (6)$$

Figure 6 displays $\sigma_n^{-\nu}$ appropriate to $m = 4$ for over two decades of n —this corresponds to $4^{100} \approx 10^{60}$ fibers!

Thus, our use of renormalization methods and conversion to irreducible form rendered the computational problem both tractable and efficient, and facilitated the determination

of the proper pattern of development and scaling for the failure of hierarchical fiber bundles. Based on this insight, we were able to prove analytically [19] that the asymptotic property (5) is in fact *universal*, valid under very weak restrictions imposed on the original distribution $P_0(\sigma)$. The proof, however, was difficult and was based on new mathematical techniques. The implication to be drawn from the static case is that, although no critical point for failure exists, the asymptotic rate of decrease of bundle strength is extraordinarily slow—a fractional power of a logarithm of a logarithm. Given the intuitive connection that has developed between static and dynamic problems of this sort, we now turn our attention to the time-dependent problem in order to understand how fiber bundles under a given load fail as a function of time.

4. Dynamics

In the time-dependent setting, we assume that fibers are sampled independently from a common source which imparts random physical properties to them. Moreover, we assume that all fibers must break given sufficient time, and that the fundamental determinant of failure is the integral of time multiplied by some measure of the stress-dependent rate of failure. A single fiber subjected to a *known* load history $\sigma(t')$, $t' \geq 0$ has a random failure time t with a distribution function that can be expressed

$$P[t|\sigma(\cdot)] = 1 - \exp \left\{ -\Psi \left[\int_0^t \kappa[\sigma(t')] dt' \right] \right\}, \quad t \geq 0 \quad (7)$$

The quantity $\Psi(x)$, $x \geq 0$ is called the *hazard* or *shape function* and basically defines the distribution function in terms of the integrated time. The time integral introduces a hazard rate $\kappa(\sigma)$, $\sigma \geq 0$, which is generally referred to as the *breakdown rule* in terms of the instantaneous stress level σ . Up to this point, our description of the distribution is quite general so long as Ψ is non-decreasing, and both Ψ and κ are non-negative functions.

Typical examples include the *Weibull shape function*

$$\Psi(x) = x^\beta, \quad x \geq 0 \quad (8)$$

where β is a positive constant—this empirical shape function is in good agreement with observations and mimics the static Weibull distribution. (We will, for demonstration purposes, employ the simple exponential shape function where β is presumed to be unity.) Also, we assume two special forms for the breakdown rule $\kappa(\sigma)$, namely

$$\kappa_e(\sigma) = \phi \exp(\eta\sigma), \quad \sigma \geq 0 \quad (9)$$

known as the *exponential breakdown rule* and

$$\kappa_p(\sigma) = \mu\sigma^\rho, \quad \sigma \geq 0, \quad (10)$$

known as the *power-law breakdown rule* where ϕ , η , μ , and ρ are all positive constants.

These forms were initially proposed by Coleman [24] largely on phenomenological grounds and have since accumulated further theoretical and experimental support. The integral form can be thought of as a representation of damage accumulation within the fibers. The two breakdown rules have also been associated with molecular breakdown events. For the exponential breakdown rule, a common interpretation is that thermodynamic considerations require that the hazard rate or breakdown rule contain a Boltzmann factor in the stress. Accordingly, the power-law rule may be regarded as a local approximation to the latter. For a discussion, see the Appendix in Phoenix and Tierney [27]. It is important to note that the power-law breakdown rule is scale free while the exponential case is not—hence, we might expect that the exponential case will not show any simple scaling while the power-law case may have some special scaling properties.

In the case of hierarchical fiber bundles, the essential feature that we must consider is that the time history can be very complicated. Indeed, in a fiber bundle with n levels and coordination number m , there are $(m^n)!$ routes to failure. The general combinatoric problem, even for $m = 2$, appears at first glance to be impossibly difficult. As in the static case, we want to investigate this problem numerically. Owing to the complicated Bayesian structure of the problem, we choose a Monte Carlo simulation as our first means

of investigation. Suppose we construct according to some distribution function following (7), say for $m = 2$, a set of 2^n fiber lifetimes under fixed load on each fiber. For convenience, we will assume that we have used the power-law breakdown rule $\kappa_p(\sigma) \propto \sigma^\rho$ for, say, $\rho = 2$ —this value of ρ is relevant to many composite materials. We will use this example to develop a methodology and then a simulation for exploring fiber bundle failure.

When we look at the general problem of fiber bundle failure for $m = 2$ we see that the very large number of routes to failure prevents the development of a simple simulation procedure. However, the renormalization approach leads to a simple scheme. In the dynamic case, just as in the static one, a given fiber can experience stresses of $\sigma, 2\sigma, \dots, 2^n\sigma$. Accordingly, the time history of that fiber is defined completely by the times at which its load was doubled. With this fact in mind, consider an arbitrary bundle at some arbitrary level in the hierarchy, and assume that this bundle is composed of two sub-bundles identified as a and b . Further, assume that this bundle has had a time history where its external load doubled at times $t_1 < t_2 < \dots < t_q$. Thus, we wish to determine the time to failure which we denote by $T_{a+b}(t_1, t_2, \dots, t_q)$. Suppose we know both $T_a(t_1, t_2, \dots, t_q)$ and $T_b(t_1, t_2, \dots, t_q)$. Without loss of generality, we assume that $T_a(t_1, t_2, \dots, t_q) < T_b(t_1, t_2, \dots, t_q)$. Then it follows that $T_{a+b}(t_1, t_2, \dots, t_q)$ is just $T_b[t_1, t_2, \dots, t_q, T_a(t_1, t_2, \dots, t_q)]$. This identity can now be converted into an efficient recursive algorithm for computing the time to failure of any bundle given that we know how to compute the lifetime of an individual fiber according to its life history. One can readily show that the number of operations required goes as the total number of elements, say $N = 2^n$ to the power $\ln 3 / \ln 2 \approx 1.5849625$. This is in marked contrast with any existing scheme for pursuing this calculation (based upon table look-up for failed fibers) which requires a minimum of $O(N^2)$ operations.

Consider again the simplest situation where $m = 2$ where an individual fiber experiences load doublings at $t_1, t_2, \dots, t_q$. (Note that if there are 2^n fibers, we can have at most $q = n$ load doublings. Load doublings which take place after a fiber fails are irrelevant.) Further, let us assume that the power-law breakdown rule applies where we define $\alpha \equiv 2^{-\rho}$. If the

nominal lifetime of a fiber is t , then in order to preserve the integral in (7) over the hazard rate, the lifetime with that load history is given by $T = T(t_1, t_2, \dots, t_q)$ and satisfies

$$t = t_1 + 2^\rho(t_2 - t_1) + 2^{2\rho}(t_3 - t_2) + \dots + 2^{(q-1)\rho}(t_q - t_{q-1}) + 2^{q\rho}(T - t_q) \quad (11)$$

(The same principle can be employed to establish the failure rule for the exponential breakdown law. This gives some indication of the complexity of the process and its sensitivity to the precise order of events.) A simple way to compute T is to generate the sequence $t'_1, t'_2, \dots, t'_q$ according to

$$t'_1 = t_1 + \alpha(t - t_1)$$

$$t'_2 = t_2 + \alpha(t'_1 - t_2)$$

$$t'_3 = t_3 + \alpha(t'_2 - t_3)$$

...

$$T(=t'_q) = t_q + \alpha(t'_{q-1} - t_q)$$

(12)

Here, t'_1 is the lifetime after the first load doubling at t_1 , t'_2 is the lifetime after the second load doubling at t_2 , and so on. These equations have the simple explanation that, every time the load on a fiber doubles, its remaining nominal survival time is reduced by the factor α . (Since α is not constant for the exponential breakdown rule, we see that this further complicates the problem. We will confine our attention to the power-law breakdown rule.) We observe that this latter sequence of equations defines a contraction map, offering a hint that the width of the distribution function could be asymptotically decreasing. Moreover, for $\alpha < 1/2$, we note that $T < (t_a + t_b)/2$ —this suggests that the mean of the distribution undergoes a systematic decrease. These features are qualitatively like those obtained in the static problem.

As an illustration of what happens, consider selecting the lifetimes of $2^{15} = 32,768$ fibers according to a Weibull exponential shape function (i.e. $\beta = 1$) and a power-law breakdown rule where $\mu = 1$ and $\rho = 2$. (These parameters are selected, in part, to facilitate comparison with the static case described by Figure 4.) Figure 7 illustrates what

Fig. 7. Cumulative distribution function $P_n[t; \sigma(\cdot)]$, $n = 0, 1, \dots, 10$, for a Monte Carlo realization of fiber bundle failure for $m = 2$ with 32,768 points of the Weibull exponential shape function ($\beta = 1$) and power-law breakdown rule $P_b(t|\sigma) = 1 - \exp \left\{ - \int_0^t \kappa_p(\sigma(t')) dt' \right\}$, $t \geq 0$ where $\kappa_p(\sigma)$ is the breakdown rule which varies as σ^ρ where $\rho = 2$.

happens in this situation—we observe a shortlived drop in the median of the distribution (as n increases) followed by a rapid (possibly exponentially fast) reduction in the variance in the distribution. (With 32,768 fibers, the $n = 10$ case contains only 32 points describing the bundles of containing 1,024 individual fibers. Since we are describing a cumulative distribution function, these points are clustered around the region where transition is most rapid allowing for a smooth description of the emergence of a Heaviside function.) There is some similarity here with the static situation, but an important difference lies in the apparent “critical point” which has now appeared.

One of us (Durand) posed an interesting conjecture relative to the recursive algorithm described above: under what circumstance can we express the lifetime of a bundle according to a formula like (12) for $q = 1$, where t_a and t_b are replaced by $T_a(t_1, t_2, \dots, t_q)$ and $T_b(t_1, t_2, \dots, t_q)$ as above, i.e. when can we write

$$T_{a+b}(t_1, t_2, \dots, t_q) = T_a(t_1, t_2, \dots, t_q) + \alpha \times [T_b(t_1, t_2, \dots, t_q) - T_a(t_1, t_2, \dots, t_q)] \quad ? \quad (13)$$

It is easy to show for the exponential breakdown rule, that the latter simplification can never happen because the value of α changes as more and more fibers break—implying that the *sequence* of events is of critical importance—while the above expression relies implicitly on the ability to interchange operations. The introduction of a scale by the exponential breakdown rule show that this is impossible. However, the power-law breakdown rule has no built-in scale and it appeared that some simplification might apply. This would be a natural extension of the renormalization process, but we had no high expectation that the renormalization would be correct. Durand performed some numerical experiments and found that he obtained identical results for the power-law breakdown rule (but not the exponential breakdown rule) when he used the latter expression instead of the more general one above. The computation required not $O[N^{\ln 3 / \ln 2}]$ but $O(N)$ operations! The proof of this remarkable result proceeds by induction and is given in the appendix.

In order to develop this problem analytically, several vital questions emerge. First, can we find any essential simplification for this problem? In the general case, the effect of failure events is not commutative, as the stress prevailing at different times can alter the rate of failures in a scale dependent way. Second, can we somehow exploit the contraction properties of the map (11) to conclusively *prove* that a critical point will emerge? These and related questions will be the subjects of further investigations.

5. Conclusion

In this paper, we have reviewed some of the uses of renormalization and hierarchical modeling in geophysics which has allowed the investigation of scaling, and have described how hierarchically organized models of fiber bundles could provide a useful description for the failure of lattice structures where the stress released by failed units would be redistributed onto regions of comparable size. We considered geometries where the coor-

dination number m described m fibers being grouped together to form a larger bundle, which would then be placed in a group of m similar bundles, and so on. The case $m = 2$ describes a planar tape in the fiber bundle literature, while the case $m = 4$ can readily be associated with a two-dimensional lattice. We employed n to describe the location in the hierarchy, where $n = 0$ describes the root level, $n = 1$ the first combining of fibers, and so on. The failure rule at a given level in the hierarchy is that, whenever a bundle fails, its load is redistributed uniformly among the surviving fibers in that bundle—when all fibers in a given bundle fail, then their load is transferred to a neighboring bundle at the same hierarchical level.

We briefly derived the combinatoric functional mapping which describes the (cumulative) distribution function P_n on increasingly larger spatial scales, established by n , both as a function of the load (per fiber) σ for the static case and the time t for the dynamic case under load σ . In the static case, we developed a mechanism for eliminating redundant calculations and, thereby, facilitated the calculation of $P_n(\sigma)$ for large n . Accordingly, we computed the variation of σ_n for $m = 2$ defined by $P_n(\sigma_n) = 1/2$ and observed that $\sigma_n^{-1} \propto \ln[n] \propto \ln[\ln(\text{mass})]$. With this insight, we then developed analytic general bounding and universal asymptotic results for the scaling according to m . In the dynamic case, we developed a mechanism for eliminating redundant calculations and, thereby, facilitated the Monte Carlo simulation of $P_n[t; \sigma(\cdot)]$ for a significant range of n . We observed that the loading history typically manifested in one of two standard representations, one where the “failure rate” varied exponentially in stress (mimicking a thermodynamic dependence which we have not provided here) and one where the rate varied as a power law in stress (Phoenix and Tierney, [27]). In both instances, we observed the development of an apparent critical point with respect to time in the distribution function. We believe that this “critical point” behavior will persist in the presence of more complicated load *histories* and plan to explore this question in the future.

In both the static and dynamic case, we observed that the effective strength of large

bundles would systematically decrease but that the rate of decrease would be extraordinarily slow. In the static case, the width of the distribution would remain relatively large while, in the dynamic case, the width of the distribution would shrink to vanishingly small size. The emergence of relatively well defined loads or stresses for the failure of such systems suggests that local load sharing can result in situations where the stress or the time to failure are well established. One important application of this pattern is to the earthquake prediction problem (e.g. $m = 4$) where geophysical folklore has suggested for many years that earthquakes might be predictable based on prevailing stress levels or the time elapsed since a previous earthquake. In engineering applications, the implication here is that, when samples of a product such as a cable having this hierarchical structure begin to fail, we might expect an avalanche of failures in similarly constructed objects. Another important engineering application emerges in situations where a composite material obeys a power law breakdown rule (10) with an index $\rho > 1$. In particular, the mean time to failure for the bundle is always less than the mean time to failure for the individual fiber element, consistent with interpretations of lifetime data for fibers in composites. From the standpoint of reliability, greater caution must be placed on predicting the lifetime of such assemblies. With the insight gained from the Monte Carlo simulations of the time to failure, we will address in a future manuscript the analytic derivation of the emergence of a critical point for failure.

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7. Appendix: Proof of Renormalization Conjecture

We observe that the sequence (12) has the effect of a contraction mapping on the interval (t_1, t) and is fundamentally linear, although it is *not* commutative. The repeated application of this mapping using the various loading times $t_1, t_2, \dots, t_q$ gives rise to a polynomial in α . In a fiber bundle, the computation of the route to failure via this mapping is quite complex, since it is formally necessary to compute the load histories of *all* of the fibers. Since an individual fiber in an n^{th} order fiber bundle (i.e. one having 2^n fibers) can experience at most n load doublings, the polynomial in α that emerges has far fewer terms than is suggested by the size of the system. Is it possible that this polynomial can be computed without keeping track of all of the details of the load history of all fibers in a fiber bundle? That is, could this problem be computationally reducible?

Recall from (12) that an individual fiber with lifetime T and a single load doubling time τ would have its lifetime reduced to

$$T' = \tau + \alpha(T - \tau) \tag{14}$$

and the same mapping could be employed to compute the remaining lifetime following additional load doublings. One of us (Durand) proposed that the mapping (14) could in fact be applied to fiber bundles as though they were individual fibers, and verified the plausibility of this conjecture in a simulation. That is, Durand hypothesized that, if you have two fiber bundles with failure times $T_a \leq T_b$, then the failure time for the bundled

pair would be given by

$$T_{a+b} = T_a + \alpha (T_b - T_a) \quad . \quad (15)$$

We wish to prove that this conjecture is indeed valid. By proving this, we are obtaining an *exact* renormalization valid for *any* realization of this stochastic process. This can now be employed in the probabilistic problem to obtain the relationship between distribution times for failure as a function of system size.

In order to provide the proof, we proceed inductively in two stages. We assume only that the mapping (14) can be applied as many times as we like for a fiber bundle of order n . This assumption is, by definition, correct for order 0 or $2^0 = 1$ individual fibers. First, we must show that a fiber bundle with constant load satisfies equation (15). Then, we must show that a fiber bundle that has had any specified load history and, accordingly, would fail at time T will now fail, if its load is subsequently doubled at time τ , at time T' according to equation (14). By proving these two conjectures, we will have considered *all* possible loadings leading to failure in any fiber bundle.

Theorem I. Suppose that we have a fiber bundle consisting of a pair of fiber bundles with independent failure times T_a and T_b , $T_a \leq T_b$. Then, the fiber bundle will fail at time T_{a+b} defined by

$$T_{a+b} = T_a + \alpha (T_b - T_a) \quad . \quad (16)$$

Proof: By assumption, the load doubles on the surviving (sub) bundle at time T_a and fails at $T_a + \alpha (T_b - T_a)$. Thus, the fiber bundle pair fails at time T_{a+b} as required. Now we must introduce the role of the external load doubling history.

Theorem II: Suppose a fiber bundle pair has been exposed to some load history and would fail at some time T_{a+b} . Further, suppose that the fiber bundle's load is doubled at some $\tau < T_{a+b}$ where the new load doubling time τ is greater than any of the preceding load doubling times contained within the fiber bundle pair's history of externally produced

load doublings. Then, the fiber bundle will now fail at time T'_{a+b} defined by

$$T'_{a+b} = \tau + \alpha (T_{a+b} - \tau) \quad . \quad (17)$$

Remark: The theorem is worded to specifically accommodate the non-commutative character of the mapping. The simplest load history for a fiber bundle pair is one where there are no load doubling events and was treated in Theorem I. By proving this theorem, we will be showing for *any* load doubling history that the lifetime of the fiber bundle obeys the same rule as an individual fiber. Equations (12) then show us how to evaluate the lifetime of an individual fiber given a proper load doubling history, i.e. $t_1 < t_2 < \dots < t_q$. Recalling the recursive algorithm described earlier, we see that the fundamental problem is that of deriving and describing a proper load history. In the proof that follows, we explicitly consider *all* possible subcases, thereby inductively assuring the construction of a proper load history.

Proof: It is necessary that we distinguish between two types of prior load doubling histories that the fiber bundle pair could have had. In the simplest case, all load doubling events occurred prior to the failure of the weakest fiber, while in the more complex case, at least one load doubling event occurred after the failure of the weakest fiber.

Case A: Here we assume that the prior load history took place *before* the first sub-bundle failed. Thus, we define by T_a and T_b , $T_a \leq T_b$, the independent failure times of the two sub-bundles subjected to this prior load history. In this case, the time to failure of the bundled pair is given by

$$T_{a+b} = T_a + \alpha (T_b - T_a) \quad . \quad (18)$$

We must now distinguish between two sub-cases.

Sub-case A1. $\tau \leq T_a$. This load doubling causes both independent fiber bundle failure

times to be reduced according to

$$T'_a = \tau + \alpha (T_a - \tau) \quad (19)$$

$$T'_b = \tau + \alpha (T_b - \tau)$$

Then, the time to failure of the system is given by

$$T'_{a+b} = T'_a + \alpha (T'_b - T'_a) = \tau + \alpha (T_a - \tau) + \alpha^2 (T_b - T_a) \quad (20)$$

$$= \tau + [T_a + \alpha (T_b - T_a) - \tau] = \tau + \alpha (T_{a+b} - \tau)$$

which is the new failure time of the fiber bundle pair, as required.

Sub-case A2. $\tau > T_a$. Since the new load doubling time occurs after the first fiber fails, we first calculate a new failure time for the surviving fiber

$$T'_b = T_a + \alpha (T_b - T_a) \quad (21)$$

which we recognize as being the same as T_{a+b} in equation (16). By introducing the additional loading time τ , we immediately obtain

$$T'_{a+b} = \tau + \alpha (T_{a+b} - \tau) \quad (22)$$

which is the new failure time for the fiber bundle which is equivalent to the desired result (17).

Case B. Here we assume that part of the prior load history took place *after* the first sub-bundle failed. Therefore, the surviving sub-bundle failed at some time $T_{a+b} \geq \tau$. By assumption, the new load doubling time causes the surviving sub-bundle to fail at time

$$T'_{a+b} = \tau + \alpha (T_{a+b} - \tau) \quad (23)$$

as claimed.

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