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DANCOFF FACTORS EVALUATION FOR SPHERES

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ABSTRACT

The possible use of fuel in the shape of spheres, in a fusion-fission hybrid reactor blanket due to the prospect of an improved refueling mechanism and irradiation uniformity as compared to a blanket composed of fuel rods, requires the development of new methods and computational capabilities for the related neutronic calculations. In this paper, the evaluation of Dancoff factors for a collection of spheres is studied, first for two identical spheres and then for an infinite lattice of identical spheres. The chord distribution function for two identical spheres is presented, from which the Dancoff factors are calculated. To further simplify the resonance calculations, simple and accurate approximations for the Dancoff factors of two spheres are evaluated. The many spheres system is analyzed using approximation methods, due to uncertainties concerning its geometrical arrangement. A close packed lattice of spheres is assumed and the corresponding Dancoff factors are estimated utilizing the two spheres results and integral properties of the spheres lattice. The estimated maximum error is no more than a few percent.

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1.0 INTRODUCTION

When dealing with the escape probability from a fuel lump, which is part of a periodic lattice in the evaluation process of effective energy group cross sections, it is customary to introduce the Dancoff correction concept. This correction takes into account the probability of a neutron, born in a fuel lump, to be absorbed in another fuel lump rather than in the moderator. The possible use of fuel in the shape of spheres, in a fusion-fission hybrid reactor blanket due to the prospect of an improved refueling mechanism and irradiation uniformity as compared to a blanket composed of fuel rods, requires the development of new methods and computational capabilities for the related neutronic calculations. In this paper, the evaluation of Dancoff factors for two spheres and for a collection of many spheres is presented.

The Dancoff factors for two spheres had been calculated in the past,¹ using the original expression for two lumps of arbitrary geometry, given by Dancoff and Ginsburg². The method presented in this paper uses the chord distribution function concept. The analytic cord distribution function for two identical spheres has been evaluated and used for the calculation of the Dancoff factors. Further simplification has been achieved by introducing approximation methods, which eliminate the need for numerical integration. The numerical results of the various methods and comparison between them are presented.

The many spheres system was analyzed using approximation methods, due to the uncertainties concerning its geometrical arrangement. A possible approach can be based on the Bell approximation,³ which is simple but generally inaccurate. A new method had been developed, to overcome this deficiency, utilizing the two spheres calculational results and integral properties of the many sphere collection (e.g., average chord length). The numerical results' error had been checked in several ways (in the absence of exact numerical calculations) and was found to be no more than a few percent. The same method was applied to infinite hexagonal and square lattices of cylindrical rods, as an additional verification, yielding a maximum error of a few percent.

This new model, as well as the Bell approximation, was incorporated into the BONAMI code,⁴ a computer code which performs a resonance self-shielding calculation with an inclusion of a heterogeneous effects option. The new input instructions are presented in the appendix.

2.0 THEORETICAL REVIEW

A short review of the theoretical material needed for the following sections is presented. Further discussion can be found elsewhere.⁵

When dealing with the escape probability from a fuel lump, which is part of a periodic lattice of fuel lumps, it is customary to introduce the Dancoff correction concept. This correction takes into account the probability of a neutron, born in a fuel lump, to be absorbed in another fuel lump rather than in the moderator, more precisely:

$$C = 1 - P_m^o \quad (2-1)$$

where P_m^o is the probability that a neutron incident on the moderator, with no previous traversal of fuel, will collide in the moderator.* The Dancoff correction is given by

$$C = \frac{\iint \exp(-\Sigma \ell) \hat{n} \cdot \hat{\Omega} \, ds \, d\Omega}{\iint \hat{n} \cdot \hat{\Omega} \, ds \, d\Omega} \quad (2-2)$$

where Σ is the moderator macroscopic total cross-section, ds is the fuel lump surface element, $\hat{n}$ is the normal to the surface, ℓ is the cord length (any line segment traversing the moderator and connected, at the two ends only, to fuel lumps), oriented in the direction of the unit vector $\hat{\Omega}$. Ω covers only half its full range (2π) and it will be assumed from now on that $\hat{n} \cdot \hat{\Omega} \geq 0$

Next we define the chord distribution function

$$f(\ell) = \frac{1}{\pi S} \iint \hat{n} \cdot \hat{\Omega} \delta(\ell - \ell'(\hat{n}, \hat{\Omega})) \, ds \, d\Omega \quad (2-3)$$

*The Dancoff factor and Dancoff correction have not been used consistently for C and $D=1-C$. Only C will be used in this paper as defined in equation 2-1.

If we use

$$\iint \hat{n} \cdot \hat{\Omega} \, ds \, d\Omega = \pi S \quad (2-4)$$

where S is the fuel lump surface area, C can be written as

$$C = \int \exp(-\Sigma \ell) f(\ell) \, d\ell \quad (2-5)$$

We define the average cord length as

$$\bar{\ell} = \int \ell f(\ell) \, d\ell \quad (2-6)$$

For infinite periodic lattice of convex fuel lumps the average cord length in the moderator is given by the simple expression (as for convex bodies)

$$\bar{\ell} = \frac{4V}{S} \quad (2-7)$$

where V and S are the moderator volume and the fuel surface, respectively, within a unit cell. For a finite number of fuel lumps (e.g. two fuel lumps system) equation 2-7 is generally not applicable, and can be used only in the large average cord length limit.

3.0 CORD DISTRIBUTION FUNCTION OF TWO SPHERES

This section will be devoted to the analytic evaluation of the chord distribution function of two identical spheres. Recall from Section 2 that the chord distribution function is given by

$$f(\ell) = \frac{1}{\pi S} \iint \hat{n} \cdot \hat{\Omega} \delta(\ell - \ell'(\hat{n}, \hat{\Omega})) ds d\Omega \quad (3-1)$$

The two spheres geometry used for the evaluation of $f(\ell)$ is shown in Figure 3-1. $\vec{r}_1$ specifies the reference point on sphere 1 surface, from which the various cords, ℓ , are connected to the surface of sphere 2. $\hat{\Omega}$, the chord direction, is measured from x (the angle θ) and the plane formed by x and P (the angle ϕ). $\vec{r}_2$ and $\vec{r}'_2$ give the location of $\vec{\ell}$ on sphere 2 with reference to the center of spheres 1 and 2 respectively. The following symmetries and observations can be used to simplify 3-1.

1. Only cords from sphere 1 to sphere 2 will be considered because of the reflection symmetry.
2. The surface integration depends only on θ_1 ($\theta_1: 0 \rightarrow \pi/2$)
3. For given θ_1 and θ , ℓ is independent of ϕ , also, ϕ can be integrated up to π (instead of 2π), by using the axial symmetry. However, ϕ is bounded by the condition $\hat{n} \cdot \hat{\Omega} \geq 0$, therefore, the integration limits for ϕ are

$$\phi: \begin{cases} 0 \rightarrow \pi & 0 \leq \ell \leq \sqrt{p^2 - 4a^2} \\ \phi_m \rightarrow \pi & \sqrt{p^2 - 4a^2} \leq \ell \leq p \end{cases}$$

where ϕ_m can be obtained from the condition $\hat{n} \cdot \hat{\Omega} = 0$.

4. One of the three lengths, ℓ , p or a can be eliminated. The natural choice is a and from now on the various lengths will be given in units of a (except those cases where a is shown explicitly).

From 1

$$S = 4\pi \quad (3-2)$$

FIGURE 3-1. ELEMENTS OF TWO SPHERES GEOMETRY

From 2

$$dS = -2\pi d\mu_1 \quad (3-3)$$

where $\mu_1 = \cos\theta_1$. (This convention will be understood from now on.) From Figure 3-1

$$d\Omega = -d\phi d\mu = -d\phi \frac{\ell'^2 - x^2 + 1}{2\ell'^2 x} d\ell' \quad (3-4)$$

Using the above results and equation 3-1

$$f(\ell) = \frac{1}{2\pi} \iint \hat{n} \cdot \hat{\Omega} \frac{\ell^2 - x^2 + 1}{2\ell^2 x} d\phi d\mu_1 \quad (3-5)$$

The evaluation of $\hat{n} \cdot \hat{\Omega}$ utilizes Figure 3-1. Only the final result will be presented.

$$\hat{n} \cdot \hat{\Omega} = I(x^2, \ell) + \Pi(x^2, \ell) \cos\phi \quad (3-6)$$

$$I(x^2, \ell) = - \frac{[x^2 - (P^2 - 1)] \cdot [x^2 + (\ell^2 - 1)]}{4\ell x^2} \quad (3-6a)$$

$$\Pi(x^2, \ell) = - \sqrt{\frac{[x^2 - (P+1)^2][x^2 - (P-1)^2][x^2 - (\ell+1)^2][x^2 - (\ell-1)^2]}{4\ell x^2}} \quad (3-6b)$$

From the discussion on ϕ , equations 3-5 and 3-6, the substitution $d(x^2) = -2Pd\mu_1$ and the definition $Y \equiv x^2$, it follows

$$f(\ell) = \frac{1}{2\pi P} \int dY \frac{\ell^2 - Y + 1}{2\ell^2 \sqrt{Y}} \left\{ (\phi_m - \pi) I(Y, \ell) + \sin\phi_m \Pi(Y, \ell) \right\} \quad (3-7)$$

ϕ_m can be found by solving the condition $\hat{n} \cdot \hat{\Omega} = 0$ (for the case $\phi_m \neq 0$)

$$\cos\phi_m = \frac{(P^2 - 1) - Y}{\sqrt{[Y - (P+1)^2][Y - (P-1)^2]}} \cdot \frac{(\ell^2 - 1) + Y}{\sqrt{[Y - (\ell+1)^2][Y - (\ell-1)^2]}} \quad (3-8)$$

The limits of integration, for given ℓ , deserve some discussion. If there are no geometrical constraints

$$\ell^2 + 1 \leq Y \leq (\ell + 1)^2 \quad (3-9)$$

However, the actual limits of Y also depend on P . For small ℓ values $(P-2)^2 \leq \ell^2 \leq P(P-2)$, $Y \geq (P-1)^2$, and for large ℓ values $P^2 - 2\sqrt{P^2 - 1} \leq \ell^2 \leq P$, $Y \leq P^2 + 1$. (The cord length limits are shown in Figure 3-2.) For some intermediate range of ℓ there are upper and lower limits imposed on Y whenever $\hat{n} \cdot \hat{\ell} = 0$. These will be denoted by $Y_{\pm}$.

$$Y_{\pm} = \frac{1 + P^2 \ell^2 \pm \ell \sqrt{4P^2 - (P^2 - \ell^2)^2}}{1 + \ell^2} \quad (3-10)$$

The Y integration limits, corresponding to the various cord length ranges, for $\phi_m = 0$ and $\phi_m \neq 0$, are given in Table 3-1. Those limits were used for the integration in equation 3-7.

TABLE 3-1
CHORD DISTRIBUTION FUNCTION INTEGRATION LIMITS
FOR THE DIFFERENT CHORD LENGTH RANGES

ϕ RANGE	ℓ^2 RANGE LIMITS		Y INTEGRATION LIMITS	
	Lower	Upper	Lower	Upper
$[0, \pi]$	$(P-2)^2$	$P^2 - 2P$	$(P-1)^2$	$(\ell+1)^2$
	$P^2 - 2P$	$P^2 - 4$	$\ell^2 + 1$	Y_-
	$P^2 - 2P$	$P^2 - 2\sqrt{P^2 - 1}$	Y_+	$(P+1)^2$
$[\phi_m, \pi]$	$P^2 - 2P$	$P^2 - 4$	Y_-	Y_+
	$P^2 - 4$	P^2	$\ell^2 + 1$	Y_+

The five different cord length ranges were contracted to three by adding the results corresponding to the two different ϕ ranges (see Figure 3-2). The final expression for the chord distribution function of two identical spheres is

$$l_1 = P - 2$$

$$l_2 = \sqrt{P^2 - 2P}$$

$$l_3 = \sqrt{P^2 - 4}$$

$$l'_3 = \sqrt{P^2 - 1} - 1$$

$$l_4 = P$$

P = pitch/sphere radius
 l = cord length/sphere radius
 Figure is drawn with unit sphere radius

FIGURE 3-2. LIMITS OF THE THREE CHORD LENGTH RANGES
 FOR THE CHORD DISTRIBUTION FUNCTION OF
 TWO SPHERES.

a

(3-11)

$$P-2 \leq \ell \leq \sqrt{P^2-2P}$$

$$f(\ell) = \frac{1}{60P\ell^3} \left\{ 6\ell^5 - P^5 - 15P\ell^4 + 10P^2\ell^3 + 20P^3 - 20\ell^3 - 40P^2 + 32 \right\}$$

b

$$\sqrt{P^2-2P} \leq \ell \leq \sqrt{P^2-4}$$

$$f(\ell) = \frac{1}{60P\ell^3} \left\{ (6\ell^5 + P^5 + 15P\ell^4 + 10P^2\ell^3 - 20P^3 - 20\ell^3) - 4\sqrt{\ell^2+1} [3\ell^4 + (5P^2-4) \cdot (\ell^2-2)] \right\}$$

c

$$\sqrt{P^2-4} \leq \ell \leq P$$

$$f(\ell) = \frac{1}{60P\ell^3} \left\{ (6\ell^5 + P^5 + 15P\ell^4 + 10P^2\ell^3 - 20P^3 - 20\ell^3) - 4\sqrt{\ell^2+1} [3\ell^4 + (5P^2-4)(\ell^2-2)] \cdot \right. \\ \left. \frac{1}{\pi} \cos^{-1} \left(\frac{\ell(\ell^2-P^2+2)}{\sqrt{4P^2-(P^2-\ell^2)^2}} \right) + \frac{2}{\pi} \ell(P^2+3\ell^2+4) \sqrt{(P^2-\ell^2)(\ell^2-P^2+4)} - \right. \\ \left. - \frac{1}{2\pi} \sum_{i=1}^4 \phi_i U_i \cos^{-1} \psi_i \right\}$$

where ϕ , U and ψ stand for

$$\psi = \begin{cases} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} & \sqrt{P^2-4} \leq \ell \leq \sqrt{P^2-1}-1 \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \text{otherwise} \end{cases}$$

$$U = \begin{bmatrix} P^5 + 15P\ell^4 - 20P^3 + 20P^2 - 16 \\ P^5 + 15P\ell^4 - 20P^3 - 20P^2 + 16 \\ 6\ell^5 + 10P^2\ell^3 - 20\ell^3 + 20P^2 - 16 \\ 6\ell^5 + 10P^2\ell^3 - 20\ell^3 - 20P^2 + 16 \end{bmatrix}$$

$$\psi = \frac{1}{\sqrt{4P^2-\alpha^2}} \left\{ \ell(\alpha-2) - \alpha(\alpha-4) \cdot \begin{bmatrix} \ell/(\alpha-2P) \\ \ell/(\alpha+2P) \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$$

where

$$\alpha = P^2 - \ell^2$$

The chord distribution function is a smooth function. It vanishes on the minimum and maximum values of ℓ ($P-2$ and P respectively) except for the case $P=2$ (two touching spheres). In this case, the chord distribution function is given only by part C in Equation 3-11; setting $P=2$, and $f(0)=1/6$. The chord distribution function of two spheres, for a few representative pitch values, shifted by the minimum cord length and normalized to one, are shown in Figure 3-3.

FIGURE 3-3. CHORD DISTRIBUTION FUNCTION FOR TWO SPHERES.

4.0 DANCOFF FACTORS FOR TWO SPHERES

Two methods were developed to calculate the Dancoff factors for two spheres:

1. Numerical integration using equation 2-5 and the explicit cord distribution function from Section 3
2. Approximation methods based on the data from the first method

The approximation methods will be discussed first followed by the various numerical results.

4.1 APPROXIMATION METHODS

The use of the analytic cord distribution function for the Dancoff factors calculations, although accurate, demands some calculational effort due to the need of numerical integration. Next, we will use approximation methods to calculate the Dancoff factors for two spheres. The Dancoff factor, for a given pitch, is

$$C(\Sigma) = \int_{P-2}^P \exp(-\Sigma \ell) f(\ell) d\ell \quad (4-1)$$

where Σ is the moderator total macroscopic cross-section. Define the normalized cord distribution function $\hat{f}(\ell)$

$$\int_{P-2}^P \hat{f}(\ell) d\ell = 1 \quad (4-2)$$

in terms of $\hat{f}(\ell)$, $C(\Sigma)$ can be expressed as

$$C(\Sigma) = C(0) \int_{P-2}^P \exp(-\Sigma \ell) \hat{f}(\ell) d\ell \quad (4-3)$$

Two approximation methods were used:

1. Approximated chord distribution function⁶
2. Moment expansion⁷

In both methods, $C(0)$ and $\bar{\ell}$, are required, therefore, we will start by approximating these two. In the limit $P \gg 1$

$$\bar{\ell} \rightarrow P - \frac{4}{3} \quad (4-4a)$$

$$C(0) \rightarrow \frac{1}{4P^2} \quad (4-4b)$$

Using these limits and the data available for $\bar{\ell}$ and $C(0)$ the following approximations were found adequate

$$\bar{\ell} \approx P - \frac{4}{3} - \frac{1}{18} \cdot \frac{9P^2 - 11P + 8}{(P-1)P^2} \quad (4-5a)$$

$$C(0) \approx \frac{1}{4P^2} + \frac{1}{7.4P^4 - 42} \quad (4-5b)$$

4.1.1 APPROXIMATED CHORD DISTRIBUTION FUNCTION

Inspection of the chord distribution function shapes, in Figure 3-3, indicates that a possible candidate to approximate those functions⁶ is*

$$f_g(t) = \frac{1}{t(g-1)!} \left(\frac{gt}{t} \right)^g \exp \left(-\frac{gt}{t} \right) \quad (4-6)$$

where g is a free parameter and t is the cord length shifted by $P-2$

$$t = \ell - P + 2 \quad (4-7)$$

combining equations 4-3 and 4-6 yields

$$C(\Sigma) \approx C(0) \frac{\exp\{-\Sigma(P-2)\}}{[1+\Sigma\bar{\ell}/g]^g} \quad (4-8)$$

by using the numerical data a linear fit was found adequate.

$$g = 12.0\bar{\ell} - 3.25 \quad (4-9)$$

Collecting all the results $C(\Sigma)$ is given by

$$C(\Sigma) \approx C(0) \cdot \frac{\exp\{-\Sigma(P-2)\}}{[1+\Sigma\bar{\ell}/g]^g} \quad (4-10)$$

*This is the function of the form $t^n \exp(t/\bar{\ell})$ properly normalized such that $\int \hat{f}_g(t) dt = 1$ and $\int \hat{f}_g(t) t dt = \bar{\ell}$

$$g = 12.0 \bar{t} - 3.25 \quad (4-10a)$$

$$\bar{t} = \frac{2}{3} - \frac{1}{18} \cdot \frac{9P^2 - 11P + 8}{P^2(P-1)} \quad (4-10b)$$

$$C(0) = \frac{1}{4P^2} + \frac{1}{7.4P^4 - 42} \quad (4-10c)$$

4.1.2 MOMENT EXPANSION

In this method, one expands the exponent in equation 4-3 in a power series of Σt . If only terms up to the second order are retained, $C(\Sigma)$ is approximated by

$$C(\Sigma) \approx C(0)\exp(-\Sigma \bar{t}) \left\{ 1 + A(\Sigma \bar{t})^2 \right\} \quad (4-11)$$

where A is a constant. If equation 4-11 is viewed as an equality, A is no longer constant, and it depends on Σt and P . A possible approximation for $C(\Sigma)$ is

$$C(\Sigma) \approx C(0)\exp(-\Sigma \bar{t}) \left\{ 1 + 0.339 \cdot (\Sigma \bar{t})^2 \cdot \frac{1 + 0.112(\Sigma \bar{t})^2}{1 + 0.298(\Sigma \bar{t})} \cdot \frac{1.186P^2 - 1}{3.031P^2 - 8.38} \right\} \quad (4-12)$$

where $C(0)$, $\bar{t}$ and $\bar{t}$ are given in equations 4-5 and 4-8.

4.2 NUMERICAL RESULTS

The exact cord distribution function was used in equation 4-1 to generate Table 4-1, which represents the Dancoff factors for two spheres, as a function of the pitch and moderator total cross-section (in unit of the sphere radius). The various values agree with those published previously¹ to the accuracy shown in the table. In Table 4-2 a comparison between selected values of the exact and approximate Dancoff factors is shown. From this table, the nature of the approximation for the specific case of two spheres (especially in view of Figure 3-3) and the expression compactness, the use of the cord distribution function approximation is preferred compared to the moment expansion approach. However, in both cases the absolute error is no larger than two parts in 10^4 as compared to the exact values or less than 0.5% relative error.

In the numerical calculations, FORTRAN IV single precision was used on a CDC-7600 computer. The numerical integration utilized the Simpson's method with second degree parabola segments.

TABLE 4-1
DANCOFF FACTORS FOR TWO SPHERES

P	Σ 0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	$\bar{L}_1$
2.00*	.07559	.07295	.07047	.06813	.06592	.06383	.06185	.05997	.05819	.05651	.35928
2.10	.06650	.06338	.06044	.05769	.05511	.05268	.05039	.04824	.04621	.04429	.42800
2.20	.05932	.05586	.05263	.04964	.04684	.04424	.04181	.03955	.03743	.03544	.50622
2.30	.05340	.04969	.04628	.04314	.04024	.03757	.03509	.03280	.03068	.02872	.59272
2.40	.04840	.04453	.04100	.03779	.03485	.03216	.02970	.02745	.02539	.02349	.68670
2.50	.04412	.04014	.03655	.03331	.03038	.02772	.02532	.02314	.02116	.01936	.79481
2.60	.04042	.03637	.03275	.02952	.02662	.02403	.02170	.01962	.01774	.01606	.90947
2.70	.03719	.03310	.02948	.02628	.02345	.02093	.01870	.01672	.01496	.01339	1.04496
2.80	.03435	.03024	.02665	.02350	.02074	.01831	.01618	.01431	.01267	.01122	1.27749
2.90	.03183	.02772	.02417	.02108	.01841	.01608	.01406	.01230	.01077	.00944	1.58323
3.00	.02959	.02550	.02199	.01894	.01639	.01417	.01226	.01061	.00919	.00797	1.94228
3.10	.02759	.02352	.02007	.01714	.01465	.01253	.01072	.00918	.00787	.00675	1.59675
3.20	.02578	.02175	.01836	.01552	.01312	.01111	.00941	.00797	.00676	.00574	1.70470
3.30	.02416	.02016	.01685	.01409	.01179	.00987	.00827	.00694	.00582	.00489	1.81022
3.40	.02268	.01874	.01549	.01282	.01061	.00879	.00729	.00605	.00503	.00418	1.91534
3.50	.02134	.01744	.01427	.01169	.00948	.00765	.00624	.00529	.00435	.00358	2.02011
3.60	.02012	.01627	.01318	.01068	.00846	.00703	.00571	.00464	.00377	.00307	2.12436
3.70	.01900	.01521	.01219	.00977	.00764	.00630	.00506	.00407	.00328	.00264	2.22874
3.80	.01797	.01424	.01129	.00896	.00712	.00586	.00450	.00358	.00285	.00228	2.33267
3.90	.01703	.01335	.01048	.00823	.00647	.00509	.00401	.00316	.00249	.00196	2.43636
4.00	.01616	.01254	.00974	.00757	.00589	.00459	.00357	.00279	.00217	.00170	2.53985
4.10	.01535	.01179	.00906	.00697	.00537	.00414	.00319	.00246	.00190	.00147	2.64314
4.20	.01461	.01110	.00845	.00643	.00490	.00374	.00285	.00218	.00167	.00128	2.74626
4.30	.01391	.01047	.00788	.00594	.00448	.00338	.00256	.00193	.00146	.00111	2.84920
4.40	.01327	.00988	.00737	.00550	.00410	.00307	.00229	.00172	.00129	.00096	2.95201
4.50	.01267	.00934	.00689	.00509	.00375	.00278	.00206	.00152	.00113	.00084	3.05469
4.60	.01211	.00884	.00645	.00472	.00345	.00253	.00185	.00136	.00100	.00073	3.15724
4.70	.01159	.00837	.00603	.00438	.00317	.00230	.00167	.00121	.00088	.00064	3.25966
4.80	.01110	.00793	.00568	.00406	.00291	.00209	.00150	.00108	.00077	.00056	3.36196
4.90	.01064	.00753	.00533	.00378	.00268	.00190	.00135	.00096	.00068	.00049	3.46420
5.00	.01021	.00715	.00501	.00352	.00247	.00174	.00122	.00086	.00061	.00043	3.56633
5.10	.00981	.00680	.00472	.00328	.00228	.00158	.00110	.00077	.00054	.00037	3.66835
5.20	.00943	.00647	.00444	.00305	.00210	.00145	.00100	.00069	.00047	.00033	3.77032
5.30	.00907	.00616	.00419	.00285	.00194	.00132	.00090	.00062	.00042	.00029	3.87219
5.40	.00873	.00587	.00395	.00266	.00179	.00121	.00082	.00058	.00037	.00025	3.97399
5.50	.00841	.00560	.00373	.00249	.00166	.00111	.00074	.00050	.00033	.00022	4.07572
5.60	.00810	.00534	.00352	.00232	.00154	.00101	.00067	.00044	.00029	.00020	4.17738
5.70	.00782	.00510	.00333	.00217	.00142	.00093	.00061	.00040	.00026	.00017	4.27899
5.80	.00755	.00487	.00315	.00204	.00132	.00085	.00055	.00036	.00023	.00015	4.38053
5.90	.00729	.00466	.00298	.00191	.00122	.00078	.00050	.00032	.00021	.00013	4.48202
6.00	.00704	.00446	.00282	.00179	.00113	.00072	.00046	.00029	.00019	.00012	4.58346
6.10	.00681	.00427	.00267	.00168	.00105	.00066	.00042	.00026	.00017	.00010	4.68485
6.20	.00659	.00409	.00254	.00157	.00098	.00061	.00038	.00024	.00015	.00009	4.78619
6.30	.00638	.00392	.00241	.00148	.00091	.00056	.00035	.00021	.00013	.00008	4.88749
6.40	.00618	.00375	.00228	.00139	.00085	.00052	.00031	.00019	.00012	.00007	4.98875
6.50	.00599	.00360	.00217	.00131	.00079	.00048	.00029	.00017	.00011	.00006	5.08997
6.60	.00581	.00346	.00206	.00123	.00073	.00044	.00026	.00016	.00009	.00006	5.19115
6.70	.00563	.00332	.00196	.00116	.00068	.00040	.00024	.00014	.00008	.00005	5.29229
6.80	.00547	.00319	.00186	.00109	.00064	.00037	.00022	.00013	.00008	.00004	5.39340
6.90	.00531	.00307	.00177	.00103	.00059	.00034	.00020	.00012	.00007	.00004	5.49447

P = Pitch/sphere radius

Σ = Moderator total cross-section/sphere radius

$\bar{L}_1$ = Average cord length/sphere radius

*The values of $C(\Sigma)$ and $\bar{L}_1$ for the case $P = 2$ are accurate to the fourth digit only.

TABLE 4.1 (cont.)

P	Σ	0.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	$\mathcal{L}_1$
7.00	.00516	.00295	.00169	.00097	.00055	.00032	.00018	.00010	.00006	.00003	.00001	5.59558
7.10	.00501	.00284	.00161	.00091	.00052	.00029	.00017	.00010	.00005	.00003	.00001	5.59659
7.20	.00487	.00273	.00153	.00086	.00048	.00027	.00015	.00009	.00005	.00003	.00001	5.59758
7.30	.00474	.00263	.00146	.00081	.00045	.00025	.00014	.00008	.00004	.00002	.00001	5.59853
7.40	.00461	.00253	.00139	.00077	.00042	.00023	.00013	.00007	.00004	.00002	.00001	5.59946
7.50	.00448	.00244	.00133	.00072	.00039	.00021	.00012	.00006	.00004	.00002	.00001	5.60037
7.60	.00437	.00235	.00127	.00066	.00037	.00020	.00011	.00006	.00003	.00002	.00001	5.60125
7.70	.00425	.00227	.00121	.00064	.00034	.00018	.00010	.00005	.00003	.00002	.00001	5.60210
7.80	.00414	.00219	.00115	.00061	.00032	.00017	.00009	.00005	.00003	.00001	.00001	5.60294
7.90	.00404	.00211	.00110	.00058	.00030	.00016	.00008	.00004	.00002	.00001	.00001	5.60375
8.00	.00394	.00204	.00105	.00055	.00028	.00015	.00008	.00004	.00002	.00001	.00001	5.60454
8.10	.00384	.00196	.00101	.00052	.00026	.00014	.00007	.00004	.00002	.00001	.00001	5.60531
8.20	.00375	.00190	.00096	.00049	.00025	.00013	.00006	.00003	.00002	.00001	.00001	5.60607
8.30	.00366	.00183	.00092	.00046	.00023	.00012	.00006	.00003	.00001	.00001	.00001	5.60680
8.40	.00357	.00177	.00088	.00044	.00022	.00011	.00005	.00003	.00001	.00001	.00001	7.00752
8.50	.00348	.00171	.00084	.00041	.00020	.00010	.00005	.00002	.00001	.00001	.00001	7.00822
8.60	.00340	.00166	.00081	.00039	.00019	.00009	.00005	.00002	.00001	.00001	.00001	7.00890
8.70	.00333	.00160	.00077	.00037	.00018	.00009	.00004	.00002	.00001	.00001	.00001	7.00957
8.80	.00325	.00155	.00074	.00035	.00017	.00008	.00004	.00002	.00001	.00001	.00001	7.01020
8.90	.00318	.00150	.00071	.00034	.00016	.00008	.00004	.00002	.00001	.00001	.00001	7.01084
9.00	.00311	.00145	.00068	.00032	.00015	.00007	.00003	.00002	.00001	.00001	.00001	7.01146
9.10	.00304	.00141	.00065	.00030	.00014	.00006	.00003	.00001	.00001	.00001	.00001	7.01207
9.20	.00297	.00136	.00062	.00029	.00013	.00006	.00003	.00001	.00001	.00001	.00001	7.01266
9.30	.00291	.00132	.00060	.00027	.00012	.00006	.00003	.00001	.00001	.00001	.00001	7.01324
9.40	.00285	.00128	.00057	.00026	.00012	.00005	.00002	.00001	.00001	.00001	.00001	8.01381
9.50	.00279	.00124	.00055	.00025	.00011	.00005	.00002	.00001	.00001	.00001	.00001	8.01437
9.60	.00273	.00120	.00053	.00023	.00010	.00005	.00002	.00001	.00001	.00001	.00001	8.01492
9.70	.00267	.00116	.00051	.00022	.00010	.00004	.00002	.00001	.00001	.00001	.00001	8.01545
9.80	.00262	.00113	.00049	.00021	.00009	.00004	.00002	.00001	.00001	.00001	.00001	8.01598
9.90	.00256	.00109	.00047	.00020	.00009	.00004	.00002	.00001	.00001	.00001	.00001	8.01649
10.00	.00251	.00106	.00045	.00019	.00008	.00003	.00001	.00001	.00001	.00001	.00001	8.01699
10.10	.00246	.00103	.00043	.00018	.00008	.00003	.00001	.00001	.00001	.00001	.00001	8.01748
10.20	.00241	.00100	.00041	.00017	.00007	.00003	.00001	.00001	.00001	.00001	.00001	8.01797
10.30	.00237	.00097	.00040	.00016	.00007	.00003	.00001	.00001	.00001	.00001	.00001	8.01844
10.40	.00232	.00094	.00038	.00016	.00006	.00003	.00001	.00001	.00001	.00001	.00001	9.01890
10.50	.00228	.00092	.00037	.00015	.00006	.00002	.00001	.00001	.00001	.00001	.00001	9.01936
10.60	.00223	.00089	.00035	.00014	.00006	.00002	.00001	.00001	.00001	.00001	.00001	9.01981
10.70	.00219	.00086	.00034	.00013	.00005	.00002	.00001	.00001	.00001	.00001	.00001	9.02024
10.80	.00215	.00084	.00033	.00013	.00005	.00002	.00001	.00001	.00001	.00001	.00001	9.02067
10.90	.00211	.00082	.00032	.00012	.00005	.00002	.00001	.00001	.00001	.00001	.00001	9.02110
11.00	.00207	.00079	.00030	.00012	.00004	.00002	.00001	.00001	.00001	.00001	.00001	9.02122
11.10	.00204	.00077	.00029	.00011	.00004	.00002	.00001	.00001	.00001	.00001	.00001	9.02163
11.20	.00200	.00075	.00028	.00011	.00004	.00001	.00001	.00001	.00001	.00001	.00001	9.02202
11.30	.00197	.00073	.00027	.00010	.00004	.00001	.00001	.00001	.00001	.00001	.00001	9.02241
11.40	.00193	.00071	.00026	.00010	.00004	.00001	.00001	.00001	.00001	.00001	.00001	10.02280
11.50	.00190	.00069	.00025	.00009	.00003	.00001	.00001	.00001	.00001	.00001	.00001	10.02318
11.60	.00186	.00067	.00024	.00009	.00003	.00001	.00001	.00001	.00001	.00001	.00001	10.02355
11.70	.00183	.00065	.00023	.00008	.00003	.00001	.00001	.00001	.00001	.00001	.00001	10.02391
11.80	.00180	.00064	.00022	.00008	.00003	.00001	.00001	.00001	.00001	.00001	.00001	10.02427
11.90	.00177	.00062	.00022	.00008	.00003	.00001	.00001	.00001	.00001	.00001	.00001	10.02462

TABLE 4.1 (cont.)

P	Σ	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
2.00		.05490	.05338	.05193	.05055	.04923	.04797	.04677	.04563	.04453	.04348
2.10		.04248	.04077	.03915	.03762	.03616	.03478	.03347	.03223	.03105	.02992
2.20		.03358	.03184	.03021	.02868	.02724	.02588	.02461	.02341	.02228	.02121
2.30		.02690	.02521	.02364	.02218	.02082	.01956	.01838	.01728	.01626	.01530
2.40		.02175	.02016	.01869	.01734	.01609	.01494	.01388	.01291	.01201	.01117
2.50		.01773	.01625	.01489	.01366	.01254	.01152	.01058	.00973	.00895	.00824
2.60		.01454	.01318	.01195	.01084	.00984	.00894	.00813	.00739	.00672	.00612
2.70		.01200	.01075	.00965	.00866	.00777	.00699	.00628	.00565	.00508	.00458
2.80		.00994	.00882	.00782	.00695	.00617	.00549	.00488	.00434	.00387	.00344
2.90		.00828	.00726	.00637	.00560	.00492	.00433	.00381	.00335	.00295	.00260
3.00		.00691	.00600	.00521	.00453	.00394	.00343	.00299	.00260	.00227	.00198
3.10		.00580	.00498	.00428	.00368	.00317	.00273	.00235	.00203	.00175	.00151
3.20		.00487	.00414	.00352	.00300	.00255	.00218	.00185	.00158	.00135	.00115
3.30		.00411	.00346	.00291	.00245	.00206	.00174	.00147	.00124	.00105	.00089
3.40		.00348	.00289	.00241	.00201	.00167	.00140	.00117	.00097	.00081	.00066
3.50		.00295	.00243	.00200	.00165	.00136	.00112	.00093	.00077	.00064	.00053
3.60		.00250	.00204	.00166	.00136	.00111	.00091	.00074	.00061	.00050	.00041
3.70		.00213	.00172	.00139	.00112	.00091	.00073	.00059	.00048	.00039	.00032
3.80		.00182	.00145	.00116	.00093	.00074	.00059	.00048	.00038	.00031	.00025
3.90		.00155	.00123	.00097	.00077	.00061	.00048	.00038	.00030	.00024	.00019
4.00		.00133	.00104	.00081	.00064	.00050	.00039	.00031	.00024	.00019	.00015
4.10		.00114	.00088	.00068	.00053	.00041	.00032	.00025	.00019	.00015	.00012
4.20		.00098	.00075	.00057	.00044	.00034	.00026	.00020	.00015	.00012	.00009
4.30		.00084	.00064	.00048	.00037	.00028	.00021	.00016	.00012	.00009	.00007
4.40		.00072	.00054	.00041	.00031	.00023	.00017	.00013	.00010	.00007	.00006
4.50		.00062	.00046	.00034	.00026	.00019	.00014	.00011	.00008	.00006	.00004
4.60		.00054	.00040	.00029	.00021	.00016	.00012	.00009	.00006	.00005	.00003
4.70		.00046	.00034	.00025	.00018	.00013	.00010	.00007	.00005	.00004	.00003
4.80		.00040	.00029	.00021	.00015	.00011	.00008	.00006	.00004	.00003	.00002
4.90		.00035	.00025	.00018	.00013	.00009	.00006	.00005	.00003	.00002	.00002
5.00		.00030	.00021	.00015	.00011	.00008	.00005	.00004	.00003	.00002	.00001
5.10		.00026	.00018	.00013	.00009	.00006	.00004	.00003	.00002	.00002	.00001
5.20		.00023	.00016	.00011	.00008	.00005	.00004	.00003	.00002	.00001	.00001
5.30		.00020	.00013	.00009	.00006	.00004	.00003	.00002	.00001	.00001	.00001
5.40		.00017	.00012	.00008	.00005	.00004	.00003	.00002	.00001	.00001	.00001
5.50		.00015	.00010	.00007	.00005	.00003	.00002	.00001	.00001	.00001	.00000
5.60		.00013	.00009	.00006	.00004	.00003	.00002	.00001	.00001	.00001	.00000
5.70		.00011	.00007	.00005	.00003	.00002	.00001	.00001	.00001	.00000	.00000
5.80		.00010	.00006	.00004	.00003	.00002	.00001	.00001	.00000	.00000	.00000
5.90		.00009	.00006	.00004	.00002	.00001	.00001	.00001	.00000	.00000	.00000
6.00		.00008	.00005	.00003	.00002	.00001	.00001	.00001	.00000	.00000	.00000
6.10		.00007	.00004	.00003	.00002	.00001	.00001	.00000	.00000	.00000	.00000
6.20		.00006	.00004	.00002	.00001	.00001	.00001	.00000	.00000	.00000	.00000
6.30		.00005	.00003	.00002	.00001	.00001	.00000	.00000	.00000	.00000	.00000
6.40		.00004	.00003	.00002	.00001	.00001	.00000	.00000	.00000	.00000	.00000
6.50		.00004	.00002	.00001	.00001	.00001	.00000	.00000	.00000	.00000	.00000
6.60		.00003	.00002	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000
6.70		.00003	.00002	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000
6.80		.00003	.00002	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000
6.90		.00002	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000
7.00		.00002	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000
7.10		.00002	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000
7.20		.00002	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
7.30		.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
7.40		.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
7.50		.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

TABLE 4.1 (cont.)

p	Σ	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
2.00		.04248	.04151	.04059	.03971	.03885	.03804	.03725	.03649	.03577	.03506
2.10		.02485	.02483	.02486	.02593	.02594	.02420	.02339	.02252	.02188	.02117
2.20		.02021	.01926	.01836	.01752	.01671	.01596	.01524	.01456	.01391	.01330
2.30		.01441	.01357	.01279	.01206	.01138	.01074	.01014	.00957	.00904	.00855
2.40		.01040	.00969	.00903	.00842	.00785	.00733	.00684	.00639	.00597	.00558
2.50		.00758	.00699	.00644	.00594	.00548	.00505	.00467	.00431	.00398	.00368
2.60		.00557	.00508	.00463	.00422	.00385	.00352	.00321	.00294	.00268	.00245
2.70		.00413	.00372	.00335	.00303	.00273	.00247	.00223	.00201	.00182	.00165
2.80		.00307	.00274	.00244	.00218	.00195	.00174	.00156	.00139	.00124	.00111
2.90		.00230	.00203	.00179	.00158	.00140	.00123	.00109	.00097	.00086	.00076
3.00		.00173	.00151	.00132	.00115	.00101	.00088	.00077	.00067	.00059	.00052
3.10		.00130	.00112	.00097	.00084	.00073	.00063	.00055	.00047	.00041	.00036
3.20		.00099	.00084	.00072	.00062	.00053	.00045	.00039	.00033	.00029	.00024
3.30		.00075	.00063	.00054	.00045	.00038	.00033	.00028	.00023	.00020	.00017
3.40		.00057	.00048	.00040	.00034	.00028	.00024	.00020	.00017	.00014	.00012
3.50		.00044	.00036	.00030	.00025	.00021	.00017	.00014	.00012	.00010	.00008
3.60		.00033	.00027	.00022	.00018	.00015	.00012	.00010	.00008	.00007	.00006
3.70		.00025	.00021	.00017	.00014	.00011	.00009	.00007	.00006	.00005	.00004
3.80		.00020	.00016	.00013	.00010	.00008	.00007	.00005	.00004	.00003	.00003
3.90		.00015	.00012	.00010	.00008	.00006	.00005	.00004	.00003	.00002	.00002
4.00		.00012	.00009	.00007	.00006	.00005	.00004	.00003	.00002	.00002	.00001
4.10		.00009	.00007	.00006	.00004	.00003	.00003	.00002	.00002	.00001	.00001
4.20		.00007	.00005	.00004	.00003	.00002	.00002	.00001	.00001	.00001	.00001
4.30		.00005	.00004	.00003	.00002	.00002	.00001	.00001	.00001	.00001	.00000
4.40		.00004	.00003	.00002	.00002	.00001	.00001	.00001	.00001	.00000	.00000
4.50		.00003	.00002	.00002	.00001	.00001	.00001	.00001	.00000	.00000	.00000
4.60		.00003	.00002	.00001	.00001	.00001	.00001	.00000	.00000	.00000	.00000
4.70		.00002	.00001	.00001	.00001	.00001	.00000	.00000	.00000	.00000	.00000
4.80		.00002	.00001	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000
4.90		.00001	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000	.00000
p	Σ	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
2.00		.03439	.03374	.03311	.03250	.03191	.03135	.03080	.03027	.02976	.02926
2.10		.02049	.01984	.01922	.01862	.01804	.01749	.01696	.01645	.01597	.01550
2.20		.01272	.01217	.01165	.01115	.01068	.01023	.00980	.00940	.00901	.00864
2.30		.00808	.00764	.00723	.00684	.00648	.00614	.00581	.00551	.00522	.00495
2.40		.00521	.00488	.00456	.00427	.00400	.00374	.00351	.00329	.00308	.00289
2.50		.00340	.00315	.00291	.00270	.00250	.00231	.00214	.00199	.00184	.00171
2.60		.00224	.00205	.00188	.00172	.00158	.00145	.00133	.00122	.00111	.00102
2.70		.00149	.00135	.00122	.00111	.00100	.00091	.00083	.00075	.00068	.00062
2.80		.00100	.00089	.00080	.00072	.00064	.00058	.00052	.00047	.00042	.00038
2.90		.00067	.00059	.00053	.00047	.00042	.00037	.00033	.00029	.00026	.00023
3.00		.00045	.00040	.00035	.00031	.00027	.00024	.00021	.00018	.00016	.00014
3.10		.00031	.00027	.00023	.00020	.00018	.00015	.00013	.00012	.00010	.00009
3.20		.00021	.00018	.00016	.00013	.00011	.00010	.00009	.00007	.00006	.00005
3.30		.00014	.00012	.00010	.00009	.00008	.00006	.00005	.00005	.00004	.00003
3.40		.00010	.00008	.00007	.00006	.00005	.00004	.00004	.00003	.00003	.00002
3.50		.00007	.00006	.00005	.00004	.00003	.00003	.00002	.00002	.00002	.00001
3.60		.00005	.00004	.00003	.00003	.00002	.00002	.00001	.00001	.00001	.00001
3.70		.00003	.00003	.00002	.00002	.00001	.00001	.00001	.00001	.00001	.00001
3.80		.00002	.00002	.00001	.00001	.00001	.00001	.00001	.00001	.00000	.00000
3.90		.00002	.00001	.00001	.00001	.00001	.00001	.00000	.00000	.00000	.00000
4.00		.00001	.00001	.00001	.00001	.00000	.00000	.00000	.00000	.00000	.00000

TABLE 4.1 (cont.)

p Σ	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
2.00	.02878	.02831	.02786	.02742	.02700	.02658	.02618	.02579	.02542	.02505
2.10	.01504	.01461	.01419	.01378	.01339	.01302	.01265	.01230	.01197	.01164
2.20	.00829	.00795	.00763	.00733	.00704	.00676	.00649	.00624	.00600	.00577
2.30	.00469	.00445	.00423	.00401	.00381	.00362	.00344	.00326	.00310	.00295
2.40	.00271	.00254	.00239	.00224	.00210	.00196	.00186	.00174	.00164	.00154
2.50	.00159	.00147	.00137	.00127	.00118	.00110	.00102	.00095	.00088	.00082
2.60	.00094	.00086	.00079	.00073	.00067	.00061	.00057	.00052	.00048	.00044
2.70	.00056	.00051	.00046	.00042	.00038	.00035	.00032	.00029	.00026	.00024
2.80	.00034	.00030	.00027	.00025	.00022	.00020	.00018	.00016	.00015	.00013
2.90	.00020	.00018	.00016	.00014	.00013	.00011	.00010	.00009	.00008	.00007
3.00	.00012	.00011	.00010	.00009	.00007	.00007	.00006	.00005	.00005	.00004
3.10	.00008	.00007	.00006	.00005	.00004	.00004	.00003	.00003	.00003	.00002
3.20	.00005	.00004	.00003	.00003	.00003	.00002	.00002	.00002	.00001	.00001
3.30	.00003	.00002	.00002	.00002	.00002	.00001	.00001	.00001	.00001	.00001
3.40	.00002	.00002	.00001	.00001	.00001	.00001	.00001	.00001	.00000	.00000
3.50	.00001	.00001	.00001	.00001	.00001	.00000	.00000	.00000	.00000	.00000

TABLE 4-2
COMPARISON BETWEEN EXACT AND APPROXIMATE DANCOFF FACTORS
FOR TWO SPHERES

$\frac{\Sigma}{P}$	0.0	0.2	0.5	1.0	2.0	5.0
2.0	.07559* .07558** .07558***	.07048 .07045 .07043	.06383 .06384 .06375	.05490 .05501 .05480	.04248 .04269 .04235	.02468 .02462 .02459
2.5	.04412 .04405 .04405	.03655 .03649 .03649	.02772 .02768 .02768	.01773 .01771 .01769	.00758 .00757 .00754	.00076 .00075 .00076
3.0	.02959 .02957 .02957	.02199 .02197 .02197	.01417 .01415 .01415	.00691 .00689 .00689	.00173 .00171 .00171	.00004 .00003 .00003
4.0	.01616 .01616 .01616	.00974 .00974 .00974	.00459 .00458 .00458	.00133 .00132 .00132	.00012 .00012 .00012	.0 .0 .0
6.0	.00704 .00705 .00705	.00282 .00282 .00282	.00072 .00072 .00072	.00008 .00007 .00008	.0 .0 .0	.0 .0 .0
10.0	.00251 .00251 .00251	.00045 .00045 .00045	.00003 .00003 .00003	.0 .0 .0	.0 .0 .0	.0 .0 .0

*Exact (using equation 3-11)
**Approximated $f(\ell)$
***Moment Expansion

P = Pitch/sphere radius
 Σ = Moderator Total Cross-section X Sphere Radius

5.0 DANCOFF FACTORS FOR MANY SPHERES

Up to this point only the two-spheres system has been analyzed. In this section the many spheres problem will be considered, and a model for calculating the Dancoff factors for many spheres, will be developed. Due to uncertainties concerning the spheres relative location, and the mathematical difficulties to evaluate the analytic cord distribution function for many spheres, an approximation method approach was taken.

5.1 APPROXIMATION METHODS

The simplest approximation that can be used is the Bell approximation. It is free from the geometry details except of the moderator average cord length (identical spheres are assumed).

$$C = \frac{1}{1 + \Sigma \frac{4}{3} \frac{V_m}{V_f}} \quad (5-1)$$

Σ is the moderator total cross-section in units of sphere radius where V_m/V_f is moderator to fuel volume ratio. Although simple, this approximation is very crude, therefore, a better approximation will be pursued.

To describe the spheres collection, the following model is assumed.

1. All the spheres are identical
2. The spheres can be viewed as an infinite closed pack lattice, where each sphere center is located on a lattice point

The second assumption can be justified by the observation that one of the goals of shaping the fuel in the form of spheres is to maximize the fuel to total volume ratio. Table 5-1 represents some basic data on a close packed lattice from which it is clear that only pitch/sphere radius ≥ 2 will be used. (The pitch here is the distance between the centers of nearest neighbors spheres). Choosing a reference sphere in the lattice, there are 12 close neighbors spheres, whose contribution to the Dancoff factor can be found to a desired accuracy.*

*To be exact, for touching spheres situation, there is some interference between the 12 spheres, however, this effect is negligible in view of the accuracy pursued.

TABLE 5-1
PRELIMINARY DATA ON THE MANY SPHERES COLLECTION

P	$V_{\text{fuel}}/V_{\text{total}}$	$\bar{L}_1$	$\bar{L}$
2.0	0.740	0.359	0.467
2.1	0.640	0.486	0.751
2.2	0.556	0.606	1.063
2.3	0.487	0.723	1.405
2.5	0.379	0.949	2.184
2.8	0.270	1.277	3.608
4.0	0.093	2.540	13.072
6.0	0.027	4.583	47.284

P = distance between two nearest sphere centers/sphere radius

V = volume

$\bar{L}_1$ = average cord length of two nearest neighbors spheres/sphere radius

$\bar{L}$ = average cord length for the whole sphere collection/sphere radius

The data is for a close packed lattice.

As the lattice opens (as the pitch increases) more and more remote spheres are completely seen by the reference sphere. However, it is important to note that many of the spheres are partially shadowed by closer spheres and their exact contribution to the overall Dancoff factor is not known. Next, we develop a general approximation for the Dancoff factor of many lumps of fuel, in the case where only a few two lumps Dancoff factor are able to be calculated.

The Dancoff factor is given by

$$C(\Sigma) = \int e^{-\Sigma \ell} f(\ell) d\ell \quad (5-2)$$

where Σ is the moderator total macroscopic cross-section, and a given pitch is assumed. $f(\ell)$ is normalized such that

$$\int f(\ell) d\ell = 1 \quad (5-3)$$

We define

$$f(\ell) \equiv f_1(\ell) + f_2(\ell) \quad (5-4)$$

where $f_1(\ell)$, or the Dancoff correction and the averaged cord length corresponding to it, is known, and $f_2(\ell)$ represents the rest of $f(\ell)$. Define also the normalized partial cord distribution function

$$\hat{f}_i(\ell) = \frac{f_i(\ell)}{\int f_i(\ell) d\ell} = \frac{f_i(\ell)}{C_i(0)} \quad (5-5)$$

In the last step equation 5-2 was used for the partial cord distribution function. Utilizing equations 5-2 to 5-5, the Dancoff factor can be represented in the following manner

$$C(\Sigma) = C_1(\Sigma) + [1 - C_1(0)] e^{-\Sigma \ell_2} \int_0^{\infty} e^{-\Sigma \ell'} \hat{f}_2(\ell' + \ell_2) d\ell' \quad (5-6)$$

where ℓ_2 is the shortest cord length such that $f_2(\ell) \neq 0$. Equation 5-6 is still exact, however, to find $C(\Sigma)$, one has to approximate the integral. The rational approximation will be used, (although more sophisticated approaches can be used) thus setting

$$\hat{f}_2(\ell' + \ell_2) = \frac{1}{\bar{\ell}_2 - \ell_2} \exp \{-\ell' / (\bar{\ell}_2 - \ell_2)\} \quad (5-7)$$

where

$$\bar{\ell}_2 = \int \hat{f}_2(\ell) \ell d\ell \quad (5-8)$$

yields for $C(\Sigma)$

$$C(\Sigma) = C_1(\Sigma) + [1 - C_1(0)] \frac{e^{-\Sigma \bar{\ell}_2}}{1 + \Sigma(\bar{\ell}_2 - \ell_2)} \quad (5-9)$$

To find $\bar{\ell}_2$, note that $\bar{\ell}$ is known (see equation 2-7, see also Table 5-1) using equations 5-4 and 5-5

$$\bar{\ell} = C_1(0)\bar{\ell}_1 + (1 - C_1(0))\bar{\ell}_2 \quad (5-10)$$

or rearranging

$$\bar{\ell}_2 = \frac{\bar{\ell} - C_1(0)\bar{\ell}_1}{1 - C_1(0)} \quad (5-11)$$

Equation 5-6 (or equation 5-9) is general and can be applied under the conditions mentioned previously. To implement those general results in the many spheres case, note that the known part of the Dancoff factor consists of the contributions from the twelve nearest neighbors spheres. Denoting the two spheres Dancoff factor by $C_{T.S.}$ we have

$$C_1(\Sigma) = 12 \cdot C_{T.S.}(\Sigma) \quad (5-12a)$$

$$\ell_2 = \sqrt{2} P-2 \quad (5-12b)$$

$$\bar{\ell} = \frac{P^3}{\sqrt{2} \pi} - \frac{4}{3} \quad (5-12c)$$

$\bar{\ell}_1$ can be found numerically using equation 2-6 and the known cord distribution function (see Table 4-1), or using equation 4-5a.

5.2 NUMERICAL RESULTS

The Dancoff factors for many spheres were calculated using equations 5-1 and 5-9. The results are shown in Table 5-2. To the expected error (see below), the use of the exact distribution function for two spheres or the approximate

TABLE 5-2
DANCOFF FACTORS FOR MANY SPHERES

Σ P	0.0	0.2	0.4	0.7	1.0	2.0	3.0	4.0	5.0
2.0	1.000*	0.915	0.842	0.754	0.682	0.517	0.415	0.346	0.296
	1.000**	0.915	0.842	0.753	0.682	0.517	0.416	0.349	0.300
2.1	1.000	0.868	0.764	0.644	0.552	0.357	0.249	0.181	0.136
	1.000	0.869	0.769	0.655	0.571	0.400	0.307	0.250	0.210
2.2	1.000	0.821	0.691	0.550	0.449	0.252	0.155	0.100	0.067
	1.000	0.825	0.702	0.573	0.485	0.320	0.239	0.190	0.158
2.3	1.000	0.775	0.624	0.470	0.366	0.181	0.099	0.057	0.038
	1.000	0.781	0.640	0.504	0.416	0.262	0.192	0.151	0.125
2.5	1.000	0.685	0.505	0.344	0.246	0.095	0.041	0.019	0.009
	1.000	0.696	0.534	0.395	0.314	0.186	0.132	0.103	0.084
3.0	1.000	0.486	0.294	0.161	0.096	0.021	0.006	0.002	0.000
	1.000	0.513	0.345	0.231	0.174	0.095	0.066	0.050	0.040
4.0	1.000	0.231	0.103	0.040	0.018	0.001	0.000	0.000	0.000
	1.000	0.277	0.161	0.099	0.071	0.037	0.025	0.019	0.015

*From equation 5-9
**Bell approximation (equation 5-1)

P = Pitch between nearest neighbors/
sphere radius

Σ = Moderator total cross-sectionX
sphere radius

one in equation 5-9 is identical (the maximum difference compared to 1 is two parts in a thousand). To get an estimate of the error, two methods were used. The first uses upper and lower limits of $C(\Sigma)$, e.g.

$$C_1(\Sigma) \leq C(\Sigma) \leq C_1(\Sigma) + [1 - C_1(0)] e^{-\Sigma \ell_2} \quad (5-13)$$

These limits can easily be refined. The second method deals with those pitch values which allow for more than the nearest neighbors spheres to be seen fully by the reference sphere. Equation 5-9 was utilized where $C_1(\Sigma)$ stood for different combinations of spheres (FCC lattice was assumed). Four combinations were used, each of which included the next nearest neighbors. Table 5-3 shows the distances and number of spheres used (e.g. the last combination includes all 74 spheres). This method, in the range of applicability, gave a good convergence pattern.

TABLE 5-3
FCC LATTICE POINTS SEPARATION

GROUP NO.	DISTANCE TO SPHERE CENTER	NUMBER OF SPHERES
1	P	12
2	$\sqrt{2}$ P	6
3	$\sqrt{3}$ P	24
4	$\sqrt{5}$ P	32

The errors are negligible for $\Sigma a \ll 1$, $\Sigma a \gg 1$, $P/a \gg 1$ and from the numerical results $\Sigma P \geq 5$, and the maximum error can be estimated to be no more than a few percent. It has been felt that better accuracy is not needed because of higher uncertainties concerning the lattice.

This method was applied to hexagonal and square lattices of infinite cylinders to check its validity. Two cylinders Dancoff factor⁸ and infinite lattice Dancoff factor values⁹ were tested in equation 5-9, while using the parameters given in Table 5-4.

TABLE 5-4
PARAMETERS FOR HEXAGONAL AND SQUARE LATTICES OF CYLINDERS

EXACT TWO NEAREST NEIGHBORS DANCOFF FACTORS	HEXAGONAL LATTICE	SQUARE LATTICE
	6	4
ℓ_2	$\sqrt{3}P-2$	$\sqrt{2}P-2$
$\bar{\ell}$	$\frac{2}{\pi} \left(\frac{\sqrt{3}}{2} P^2 - \pi \right)$	$\frac{2}{\pi} (P^2 - \pi)$

The values of $\bar{\ell}_1$ (the average cord length between two cylinders) had been estimated from reference 8 by using the relation

$$\bar{\ell}_1 = - \frac{d\{\ln[C(\Sigma)]\}}{d\Sigma} \Big|_{\Sigma=0} \quad (5-14)$$

and the asymptotic limit for large average cord length values

$$\bar{\ell}_1 \rightarrow \frac{4P}{\pi} - 2 \quad (5-15)$$

which serves also as an upper limit for a given pitch.

The numerical results are presented in Tables 5-5 and 5-6.

The maximum error for the various Dancoff factors is no higher than a few percent and the asymptotic behavior is correct. The Bell approximation results are also shown for comparison. These considerations support the validity of the method developed in this section.

TABLE 5-5

DANCOFF FACTORS FOR SQUARE LATTICE OF CYLINDERS

Σ P	0.1	0.2	0.5	1.0	2.0	3.0	$\bar{\ell}_1$
2.0	0.948* 0.949** 0.948***	0.902 0.903 0.901	0.789 0.791 0.785	0.661 0.664 0.647	0.519 0.521 0.478	0.443 0.444 0.379	0.28
2.1	0.925 0.925 0.925	0.860 0.860 0.861	0.707 0.710 0.712	0.541 0.546 0.553	0.362 0.366 0.382	0.267 0.269 0.292	0.45
2.2	0.902 0.902 0.902	0.819 0.820 0.822	0.633 0.638 0.649	0.446 0.454 0.480	0.260 0.264 0.316	0.169 0.169 0.236	0.60
2.5	0.832 0.833 0.835	0.704 0.709 0.716	0.459 0.467 0.503	0.259 0.264 0.336	0.103 0.105 0.202		1.06
4.0	0.518 0.531 0.550	0.319 0.328 0.379	0.106 0.109 0.196	0.025 0.025 0.109			3.05
10.0	0.085 0.083 0.140	0.024 0.024 0.075	0.001 0.001 0.031				10.73

*From Equation 5-9

**From References 8, 9

***Bell Approximation

P = Pitch between nearest neighbors/
cylinder radius Σ = Moderator total cross section X
cylinder radius $\bar{\ell}_1$ = Estimated average cord length of
two nearest neighbors cylinders/
cylinder radius

TABLE 5-6
DANCOFF FACTORS FOR HEXAGONAL LATTICE OF CYLINDERS

Σ P	0.1	0.2	0.5	1.0	2.0	3.0	$\bar{L}_1$
2.2****	0.938* 0.938** 0.937***	0.882 0.881 0.882	0.743 0.741 0.750	0.575 0.576 0.599	0.368 0.370 0.428	0.247 0.247 0.333	0.60
2.5	0.874 0.873 0.874	0.770 0.769 0.776	0.550 0.550 0.580	0.340 0.340 0.409	0.148 0.149 0.257		1.06
4.0	0.567 0.579 0.594	0.366 0.378 0.423	0.135 0.138 0.227	0.035 0.035 0.128			3.05
10.0	0.099 0.100 0.158	0.031 0.031 0.086	0.002 0.002 0.036				10.73

*From Equation 5-9

**From Reference 8, 9

***Bell Approximation

****For this case, the interference effect is not negligible because of the infinite cylinders length

P = Pitch between nearest neighbors/
cylinder radius

Σ = Moderator total cross section $\times$
cylinder radius

$\bar{L}_1$ = Estimated average cord length
of two nearest neighbors
cylinders/cylinder radius

6.0 SUMMARY

The evaluation of Dancoff factors for spheres has been considered throughout this paper. The two spheres system was analyzed first, via the chord distribution function concept. The existence of an analytic solution for the chord distribution function of two spheres results in a simple and fast calculation method, computer-wise, for the evaluation of the exact Dancoff factors. Two approximation methods were used to eliminate the need for numerical integration, both yielding absolute error of only a few parts in 10^4 . The many spheres system was analyzed next, using two approaches: a) the simple but generally inaccurate Bell approximation; b) a new method which approximates the Dancoff factor by adding the known calculable contribution of the close neighbors spheres and an estimated contribution from the rest of the spheres. This method predicts the Dancoff factors for close packed spheres lattice to within a few percent. It was felt that better accuracy is not needed at the present time due to the uncertainties concerning the actual relative spheres' locations.

The work presented in this paper can be extended in a few possible directions.

1. The Dancoff factor evaluation for two spheres can be simplified further by, at least, partial explicit integration of equation 2-5 using the result of Section 3.
2. The evaluation of the cord distribution function and Dancoff factors for two unidentical spheres can be pursued.
3. Better accuracy for the many spheres system, if needed, can be achieved by better estimates of the tail contribution (e.g. effective shortest cord length, approximated cord distribution function with free parameter.⁶)
4. The general problem of a many spheres system with the inclusion of different sphere sizes.

7.0 REFERENCES

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Appendix: Additions to BONAMI Code

Two new options were added to the BONAMI code's heterogeneous effects calculation capabilities, both dealing with fuel in the shape of spheres:

1. Spherical cell - Bell Approximation (ISSOPT=8)

This option uses equation 5-1.

2. Spherical cell - New approach (ISSOPT=9)

This option uses equation 5-9 to calculate the Dancoff factor for the identical spheres collection. Equation 4-10 is used for the two spheres Dancoff factor calculations. Under the assumption of a close packed lattice, the relation between the pitch P (in units of the sphere radius) and the moderation to fuel volume ratio V_1/V_0 is given by

$$\frac{V_1}{V_0} = \frac{3P^3}{4\sqrt{2}\pi} - 1 \quad (A-1)$$

For both cases spherical geometry is used (IGR=3), and the effective outer radii of the different zones (Block 2 array 7*), should be evaluated using a spherical cell.