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Crystalline Beams: The String

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*Abstract **

We study the possibility of storing a *string* of charged particles in a magnetic ring. We define the equilibrium configuration, and examine the confinement conditions. Subsequently, we derive the transfer matrix for motion through a drift, a quadrupole, and a dipole. We finally study the stability of the string as the space-charge force is increased.

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1 Introduction

In this section we study the simplest equilibrium configuration: a *string* of charged particles along the reference orbit. The spacing between the particles is the same along the trajectory and they all have the same reference momentum.

Based on molecular-dynamics simulations [1] and experimental observations [2] we surmise that the string will be the first structure to be observed in a storage ring.

Let us reiterate our approach once more: first (in section 2) we describe the equilibrium configuration. Subsequently, we see if the equilibrium configuration satisfies the equations of motion, or whether any additional *external* fields are needed to maintain it (see section 3). In section 4, we derive the equations of motion and the corresponding matrix notation to study the stability of the motion of a test particle perturbed from its equilibrium position.

The stability itself is discussed in section 5. In section 6 we show that a betatron is not suitable for the storage of a string of ions. In section 7 we show our results.

The notation used in this paper is the same as introduced in [3]. The reader should familiarize him/herself with this reference, since it contains the derivation of the equations used in this paper.

2 Equilibrium Configuration

The equilibrium corresponds to all particles equally spaced on the reference trajectory (i.e., $x = y = 0$) travelling with the same momentum (i.e., $\delta = 0$). The longitudinal spacing is λ , and in the arcs of radius of curvature $\rho = 1/h$ the angular spacing is $\Delta = \lambda h$. We require that the interparticle spacing is much smaller than the

radius of curvature, viz. :

$$\lambda \ll \rho, \quad (1)$$

that is $\Delta \ll 1$. Furthermore we assume that λ is much smaller than l , the length of any element of the storage ring (like a drift, quadrupole, or dipole magnet), viz.:

$$\lambda \ll l. \quad (2)$$

The total number of particles is

$$N = \frac{2\pi R}{\lambda}, \quad (3)$$

where $2\pi R$ is the total path length of one traversal of the reference trajectory.

It is easily seen, that there are no net forces in the longitudinal and transverse direction, except when the trajectory is bent in the arcs. Therefore, the scalar potential due to the space charge contains at most a radial component [3], viz.:

$$\phi_{sc} = \nu_1 x. \quad (4)$$

The radial force has the same magnitude for all particles [4]:

$$\nu_1 = -\frac{Qe}{2\rho^2\gamma^2} \sum_{j=1}^{N/2} \frac{1}{[1 - \beta^2 \sin^2(\theta_j/2)]^{\frac{3}{2}} \sin(\theta_j/2)}, \quad (5)$$

where $\theta_j = j\Delta$. The summation is only over half the beam ahead of the test particle.

For nonrelativistic beams we ignore the term in β and take only the contribution from neighboring particles into account by replacing the sin with its argument, viz.:

$$\nu_1 = -\frac{QeNLh^2}{2\pi\gamma^2}, \quad (6)$$

where

$$L = \ln\left(\frac{\pi N}{2}\right). \quad (7)$$

Clearly, in the straight sections $h = 0$ and therefore $\nu_1 = 0$.

3 Confinement Conditions

The equations of motion for the string in its equilibrium configuration are [3]:

$$y'' + K_v(s)y = 0, \quad (8)$$

$$x'' + K_h(s)\bar{D}x = hf_- - \frac{Qe}{p_0c}B_0 + hf_+\delta, \quad (9)$$

$$\frac{d\sigma}{ds} = hf_+x - \frac{\delta}{\gamma^2}, \quad (10)$$

$$\frac{d\delta}{ds} = 0. \quad (11)$$

$K_{h,v}$ are the external focusing functions, and

$$f_{\pm} = 1 \pm \frac{Qr_0\nu_1}{eA\beta^2\gamma^3h}, \quad (12)$$

$$\bar{D} = 1 + 2\frac{Qr_0h}{eA\beta^2\gamma K_h}\nu_1 \left(1 + \frac{Qr_0}{2eA\beta^2\gamma^3h}\nu_1\right). \quad (13)$$

The equilibrium configuration of the string – with $x = y = \delta = 0$ and $\sigma = \text{const.}$ – satisfies the equations of motion, provided that the generalized Lorentz condition is satisfied:

$$hf_- = \frac{Qe}{p_0c}B_0. \quad (14)$$

The centrifugal force due to the curvature of the path and the repulsion due to the space-charge force are balanced by the Lorentz force of the magnetic field. This is the only confinement condition the equilibrium of the string needs to satisfy.

In the straight sections we have $h = 0$ so that $f_{\pm} = \bar{D} = 1$. In the arcs, for reasonable and realistic beam and storage ring parameters, the term in ν_1 (i.e., the curvature term in the space-charge potential) is very small and can be ignored. Specifically, consider Eq. (12). Inserting the expression for ν_1 , Eq. (6), yields:

$$f_{\pm} = 1 \mp \frac{Q^2e^2NL}{2\pi Am_0c^2\gamma^5\beta^2\lambda}\frac{\lambda}{\rho}. \quad (15)$$

The first part of the second term is the ratio of the potential to the kinetic energy (i.e., $Q^2 e^2 / \gamma \lambda / p_0 c$) and – in the case we consider here of cooled beams dominated by the kinetic energy – small. The second part of the second term (i.e., $\frac{\lambda}{\rho}$) is small as well (see (1)). Therefore, $f_{\pm} \approx 1$, and similarly $\bar{D} \approx 1$. Consequently, we conclude that for kinetic-energy-dominated beams the contribution of the curvature term will be insignificant, and we will ignore the curvature term in the space charge in the following.

4 Motion around the Equilibrium

The stability is determined by slightly displacing a test particle from its equilibrium position, leaving the other particles unchanged, and studying its motion. The displacement is smaller than the interparticle spacing λ . With the perturbation added, and ignoring the curvature term, the space-charge potential is:

$$\phi_{sc} = \nu_2 x^2 + \nu_3 y^2 + \nu_4 \sigma^2. \quad (16)$$

First, we displace the particle in the radial and vertical direction, and calculate the coefficients ν_2 and ν_3 . The scalar potential [3] is:

$$\phi_{sc} = 2Qe\gamma \sum_{i=1}^{\infty} \frac{1}{[x^2 + y^2 + (i\gamma\lambda)^2]^{\frac{1}{2}}}, \quad (17)$$

where the summation is over the contribution of all the particles. Expanding and retaining only up to quadratic terms yields:

$$\nu_2 = \nu_3 = -\nu_0 = -\frac{Qeg_0}{\gamma^2 \lambda^3}, \quad (18)$$

where

$$g_0 = \sum_{i=1}^{\infty} \frac{1}{i^3} \approx 1.2. \quad (19)$$

g_0 has a simple interpretation: $g_0 - 1$ is the contribution of the non-nearest neighbor particles to the mean field with potential (16). It

is easily seen that this no more than 20 %. $1/i^3$ gives the relative contribution of the i th neighbor of the test particle. Furthermore, ν_2 and ν_3 are negative quantities. Consequently, we see that in light of (1), the curvature has very little effect on the space charge.

The last coefficient, ν_4 follows from Laplace's equation, viz.:

$$\frac{\partial^2 \phi_{sc}}{\partial x^2} + \frac{\partial^2 \phi_{sc}}{\partial y^2} + (1 - \beta^2) \frac{\partial^2 \phi_{sc}}{\partial \sigma^2} = 0. \quad (20)$$

In fact, this is a general property of ϕ_{sc} [3], so that:

$$\nu_4 = -\gamma^2(\nu_2 + \nu_3), \quad (21)$$

$$\phi_{sc} = \nu_2 x^2 + \nu_3 y^2 - \gamma^2(\nu_2 + \nu_3)\sigma^2. \quad (22)$$

It is easy to see that in the case of the string:

$$\nu_4 = 2\gamma^2\nu_0 \geq 0. \quad (23)$$

We can now write down the equations of motion of the perturbed test particle. Recall that we perturb the equilibrium configuration, so the generalized Lorentz condition is satisfied. Furthermore, recall that the effect of the curvature can be ignored. Therefore, the equations of motion are:

$$y'' + (K_v - K_{sc})y = 0, \quad (24)$$

$$x'' + (K_h - K_{sc})x = h\delta, \quad (25)$$

$$\sigma' = hx - \frac{\delta}{\gamma^2}, \quad (26)$$

$$\delta' = 2\gamma^2 K_{sc} \sigma, \quad (27)$$

where

$$K_{sc} = 2 \frac{Q^2 r_0 g_0}{A \beta^2 \gamma^5 \lambda^3}. \quad (28)$$

From the equations of motion it is clear that the space charge defocuses the cooled beam with respect to the dilute beam (for which K_{sc} is zero). Note that the space-charge parameter K_{sc} has the

dimension of the inverse of the square of a length. It is convenient to express it in terms of an *equivalent radius of curvature* ρ_{sc} and of an *equivalent curvature* $h_{sc} = 1/\rho_{sc}$, so that:

$$K_{sc} = h_{sc}^2. \quad (29)$$

That way we can immediately compare the effect to the space charge to that of the bending magnet of curvature $h(s)$, and the length of any magnetic element of the storage-ring lattice.

We express the solution of the equations of motion in terms of transfer matrices for the three storage-ring elements considered in this paper: the drift, the quadrupole, and the bending magnet. We represent the state of the test particle as a column vector $\mathbf{V}$ with components $(x, x', y, y', \sigma, \delta)$. The transfer matrix M is a 6×6 matrix, and relates the state vector at the entrance to that at the exit of an element.

A storage ring is defined by specifying the sequence of magnet elements in one period, and the number of periods. The total transfer matrix for one revolution is obtained by multiplying the matrices for the various elements in order of their traversal. Examining the eigenvalues of the total transfer matrix then allows us to determine the stability of the motion.

4.1 The Transfer Matrix for the Drift

In a drift $K_h = K_v = h = 0$ and we have only the space-charge forces with which to contend. Let l_Δ be the length of the drift and call $\zeta = l_\Delta h_{sc}$. The equations of motion are decoupled:

$$y'' - h_{sc}^2 y = 0, \quad (30)$$

$$x'' - h_{sc}^2 x = 0, \quad (31)$$

$$\sigma'' + 2h_{sc}^2 \sigma = 0. \quad (32)$$

The motion is defocused in the transverse directions, but is oscillatory in the longitudinal direction.

Because the motion is decoupled in all directions, the transfer matrix has the general form:

$$M_D = \begin{pmatrix} M_H & 0 & 0 \\ 0 & M_V & 0 \\ 0 & 0 & M_E \end{pmatrix}. \quad (33)$$

Each of the entries in (33) is a 2×2 matrix. Specifically we have:

$$M_H = M_V = \begin{pmatrix} \cosh \zeta & \rho_{sc} \sinh \zeta \\ h_{sc} \sinh \zeta & \cosh \zeta \end{pmatrix}, \quad (34)$$

$$M_E = \begin{pmatrix} \cos \sqrt{2}\zeta & -\frac{\rho_{sc}}{\sqrt{2}\gamma^2} \sin \sqrt{2}\zeta \\ \gamma^2 \sqrt{2} h_{sc} \sin \sqrt{2}\zeta & \cos \sqrt{2}\zeta \end{pmatrix}. \quad (35)$$

4.2 The Transfer Matrix for the Quadrupole

In a quadrupole of length l_q , $K_h = -K_v \equiv K$ and $h = 0$. The equations of motion are still decoupled. The one for the longitudinal motion remains unchanged. Those in the vertical and horizontal direction are modified as follows:

$$y'' - (K + K_{sc})y = 0, \quad (36)$$

$$x'' + (K - K_{sc})x = 0. \quad (37)$$

To obtain a convenient, compact notation, we introduce the variable $\tilde{K} = |K \mp K_{sc}|$, the upper (lower) sign holding for the horizontal (vertical) motion.

The general form of the transfer matrix is once again:

$$M_Q = \begin{pmatrix} M_H & 0 & 0 \\ 0 & M_V & 0 \\ 0 & 0 & M_E \end{pmatrix}. \quad (38)$$

M_E is the same as the one already given for the drift, Eq. (35).

When $\pm K > K_{sc}$, the other two matrices are:

$$M_{V,H} = \begin{pmatrix} \cos \zeta & \frac{1}{\sqrt{K}} \sin \zeta \\ -\sqrt{K} \sin \zeta & \cos \zeta \end{pmatrix}, \quad (39)$$

where $\zeta = l_q \sqrt{K}$. Otherwise, when $\pm K < K_{sc}$ we obtain:

$$M_{V,H} = \begin{pmatrix} \cosh \zeta & \frac{1}{\sqrt{K}} \sinh \zeta \\ \sqrt{K} \sinh \zeta & \cosh \zeta \end{pmatrix}. \quad (40)$$

4.3 The Transfer Matrix for the Dipole

The equations of motion in a dipole with field index n and curvature $h = 1/\rho$ are:

$$y'' + (nh^2 - h_{sc}^2)y = 0, \quad (41)$$

$$x'' + [(1-n)h^2 - h_{sc}^2]x = h\delta, \quad (42)$$

$$\sigma' = hx - \frac{\delta}{\gamma^2}, \quad (43)$$

$$\delta' = 2\gamma^2 h_{sc}^2 \sigma. \quad (44)$$

The motion in the radial and longitudinal directions is coupled due to the curvature. The general transfer matrix is:

$$M_B = \begin{pmatrix} M_H & 0 & M_a \\ 0 & M_V & 0 \\ M_b & 0 & M_E \end{pmatrix}. \quad (45)$$

If $n = 0$, M_V is the same as in a drift, Eq. (34); otherwise it is the same as in the quadrupole, Eqs. (39) and (40), with $K = -nh^2$.

To deal with the coupled radial and longitudinal motion we introduce the 4-dimensional state vector $\mathbf{U} = (x, x', \sigma, \delta)$, and the 4×4 matrix

$$\mathcal{M}_B = \begin{pmatrix} M_H & M_a \\ M_b & M_E \end{pmatrix} \quad (46)$$

as the transfer matrix which applies to $\mathbf{U}$. For convenience, we define the ratio of the bending radius due to the dipole magnet and the space charge:

$$\xi = \frac{h_{sc}}{h}. \quad (47)$$

The equations of motion of the components of $\mathbf{U}$ are:

$$x' = p, \quad (48)$$

$$p' + h^2(1 - n - \xi^2)x = h\delta, \quad (49)$$

$$\sigma' = hx - \frac{\delta}{\gamma^2}, \quad (50)$$

$$\delta' = 2\gamma^2\xi^2h^2\sigma. \quad (51)$$

Solving this system of differential equations is straightforward, albeit tedious. We obtain a general solution of the homogenous equations by inserting $e^{\alpha s}$ and solving the resultant eigenvalue problem. The four eigenvalues for α are the roots of the quartic equation:

$$\alpha^4 + \Gamma^2\alpha^2 - \Upsilon^2 = 0, \quad (52)$$

where

$$\Gamma^2 = h^2(1 - n + \xi^2), \quad (53)$$

$$\Upsilon^2 = 2\xi^2h^4(\gamma^2 + n - 1 + \xi^2). \quad (54)$$

Solving (52) yields:

$$2\alpha^2 = -\Gamma^2 \pm \sqrt{\Gamma^4 + 4\Upsilon^2}. \quad (55)$$

It is easily seen that the term

$$\Gamma^4 + 4\Upsilon^2 > 0. \quad (56)$$

Therefore, the eigenvalues are either real or purely imaginary. In fact, we find:

$$\begin{aligned} \alpha_1 &= \alpha_r \\ &= \frac{1}{\sqrt{2}} \left(-\Gamma^2 + \sqrt{\Gamma^4 + 4\Upsilon^2} \right)^{\frac{1}{2}}, \end{aligned} \quad (57)$$

$$\alpha_2 = -\alpha_1 = -\alpha_r, \quad (58)$$

$$\alpha_3 = i\alpha_i = \frac{i}{\sqrt{2}} \left(\Gamma^2 + \sqrt{\Gamma^4 + 4\Upsilon^2} \right)^{\frac{1}{2}}, \quad (59)$$

$$\alpha_4 = -i\alpha_i. \quad (60)$$

These equations hold as long as $\Upsilon^2 > 0$; otherwise we can interchange α_r and α_i . The dipole has length l , and the matrix elements are linear combinations of the functions:

$$F_n = \sum_{k=1}^4 \frac{\alpha_k^n e^{\alpha_k l}}{2 D(\alpha_k)} \quad n = 1 \dots 3, \quad (61)$$

where

$$D(\alpha) = \alpha(2\alpha^2 + \Gamma^2). \quad (62)$$

Since the roots involve α_r and α_i only, we define the two numbers:

$$D_r = \alpha_r(2\alpha_r^2 + \Gamma^2), \quad (63)$$

$$D_i = \alpha_i(-2\alpha_i^2 + \Gamma^2) \quad (64)$$

It is easily established that:

$$D_r = D(\alpha_1) = -D(\alpha_2), \quad (65)$$

$$iD_i = D(\alpha_3) = -D(\alpha_4). \quad (66)$$

So, we obtain:

$$F_0 = \frac{\sinh(\alpha_r l)}{D_r} + \frac{\sin(\alpha_i l)}{D_i}, \quad (67)$$

$$F_1 = \alpha_r \frac{\cosh(\alpha_r l)}{D_r} + \alpha_i \frac{\cos(\alpha_i l)}{D_i}, \quad (68)$$

$$F_2 = \alpha_r^2 \frac{\sinh(\alpha_r l)}{D_r} - \alpha_i^2 \frac{\sin(\alpha_i l)}{D_i}, \quad (69)$$

$$F_3 = \alpha_r^3 \frac{\cosh(\alpha_r l)}{D_r} - \alpha_i^3 \frac{\cos(\alpha_i l)}{D_i}. \quad (70)$$

The complete transfer matrix now reads:

$$\mathcal{M}_B = \quad (71)$$

$$\begin{pmatrix} F_3 + BC F_1 & F_2 + BC F_0 & hC F_0 & hF_1 \\ -AF_2 - (ABC - h^2 C) F_0 & F_3 + BC F_1 & hC F_1 & hF_2 \\ hF_2 & hF_1 & F_3 + AF_1 & -BF_2 - (AB - h^2) F_0 \\ hC F_1 & hC F_0 & CF_2 + AC F_0 & F_3 + AF_1 \end{pmatrix}$$

where

$$A = h^2(1 - n - \xi^2), \quad (72)$$

$$B = \frac{1}{\gamma^2}, \quad (73)$$

$$C = 2\gamma^2\xi^2h^2. \quad (74)$$

5 Stability of the Test Particle Motion

In order to determine the stability of the motion of the test particle, one calculates the transfer matrix over one superperiod. The resulting matrix is 6×6 , but, as we have seen, the motion in the vertical direction is decoupled from the motion in the radial and the longitudinal direction. Consequently, the transfer matrix consists of a 2×2 submatrix M_V representing the vertical motion and a 4×4 matrix M_H representing the (coupled) longitudinal and radial motion.

5.1 Stability of the Vertical Motion

The vertical motion is completely decoupled from the the motion in the horizontal plane, and can be investigated independently. Our starting point is (24), with K_{sc} given by (28). For a dilute beam the space-charge contribution is small and can be neglected, or treated as a perturbation. The motion is then completely determined by the external focussing parameter K_v . From this we can derive the original tune ν_v of the ring, and the amplitude lattice function β_v , which is periodic with the same periodicity as the ring. As the space-charge forces increase, when the lattice constant λ is reduced, two effects are soon noticeable: β_v increases everywhere, and the betatron tune ν_v decreases. Treating the space-charge force as a

perturbation, the resulting tune shift to first order follows from:

$$\Delta\nu_v = \frac{1}{4\pi} \oint_C \beta_v K_{sc} ds, \quad (75)$$

The integral is taken over the whole circumference, and as an estimate of β_v , we replace it by its average $\bar{\beta}_v = R/\nu_v$. In addition we introduce the critical lattice constant λ_c :

$$K_{sc} = \frac{2}{R^2} \frac{\lambda_c^3}{\lambda^3}, \quad (76)$$

$$\lambda_c = \left(\frac{Q^2 r_0 g_0 R^2}{A \beta^2 \gamma^5} \right)^{\frac{1}{3}}. \quad (77)$$

Calculations done with SYNCH [5, 6], involving equations similar to (24), show that (75) agrees well with the actually observed tune shifts for large values of K_{sc} , and is a good approximation until the limit of stability – the edge of the stopband introduced by the space charge itself – is encountered. This occurs when the shift of the betatron tune is equal to half of the original distance from the stopband just below it, viz.:

$$\delta\nu_v = \frac{1}{2}(\nu_v - \frac{m}{2} N_p), \quad (78)$$

where N_p is the periodicity of the storage ring, and m an integer, which yields the stopband just below ν_v . Therefore, in good approximation, we can expect the motion to remain stable if:

$$\Delta\nu_v < \delta\nu_v \quad (79)$$

or, in terms of the particle spacing λ :

$$\lambda > \lambda_c c_v, \quad (80)$$

where

$$c_v = (\nu_v \delta\nu_v)^{-\frac{1}{3}}. \quad (81)$$

Clearly, to obtain low values of λ , a strong focusing lattice with very high tune and periodicity is required. The actual details of the lattice should not have a significant effect.

5.2 Stability of the Planar Motion

Let us rewrite the equations of motion in the radial and longitudinal direction as follows:

$$\begin{aligned} x'' + (K_h - K_{sc}) &= h \delta, \\ \delta'' + 2K_{sc}\delta &= 2\gamma^2 K_{sc} h x. \end{aligned} \quad (82)$$

Coupling between the radial and longitudinal motion is introduced by the product of the space-charge parameter K_{sc} with the curvature in the bending magnets. ξ , introduced earlier (see (47)), can be used to estimate the magnitude of the coupling: $\xi \ll 1$ corresponds to weak coupling, and $\xi \approx 1$ corresponds to strong coupling.

As in the case of vertical motion, for a dilute beam the space-charge term is small, and is treated as a perturbation. The motion is essentially determined by the focusing parameter K_h , and described by the initial (dilute beam) betatron tune ν_h , and the unperturbed amplitude lattice function β_h . As the space-charge forces increase, the betatron tune ν_h decreases, and β_h increases everywhere. Moreover, the longitudinal motion is also described by an oscillation frequency, with which the longitudinal tune ν_s is associated:

$$\nu_s = R \sqrt{2 K_{sc}} \quad (83)$$

The space-charge force increases ν_s as well. Consequently, the working point (ν_h, ν_s) is going to move around the tune diagram, as one varies the space-charge parameter, until one of the stopbands induced by the space charge itself is encountered, and the string loses stability.

Let

$$M_P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (84)$$

represent the 4×4 total transfer matrix per period for the planar motion, with 2×2 matrices as its elements. M_P has two reciprocal pairs of eigenvalues, and the motion is stable if they lie on the unit circle. The motion is unstable if a) at least one pair lies on the real axis, or b) at least one pair is complex, but modulus one. Case a) corresponds to the uncoupled case. An instability of type b) is induced by the coupling between radial and longitudinal motion.

The stability criterion is a bit more involved. It is obtained [7] by examining:

$$2 \Lambda = (\text{Tr} A + \text{Tr} D) \pm [(\text{Tr} A - \text{Tr} D)^2 + 4 \text{Det}(B + \bar{C})]^{1/2}, \quad (85)$$

where $\bar{C}$ is the symplectic conjugate of C [7]. For stability one requires that Λ is real, and

$$|\Lambda| \leq 2. \quad (86)$$

An instability of type b) due to coupling occurs when

$$\text{Tr} A \approx \text{Tr} D \quad (87)$$

and

$$\text{Det}(B + \bar{C}) < 0 \quad (88)$$

Condition (87) is equivalent to

$$\nu_h - \nu_s = m_- N_P \quad (89)$$

or

$$\nu_h + \nu_s = m_+ N_P \quad (90)$$

where $m_{\pm}$ is an integer. In [7] it is proven that a resonance of type (89) is excluded, because in that case condition (88) is never satisfied.

Denote with $\nu_{h,0}$ the initial, dilute beam tune. Substituting $\nu_h = \nu_{h,0} - \delta\nu_h$ with

$$\begin{aligned}\delta\nu_h &= \frac{1}{4\pi} \oint \beta_h K_{sc} ds, \\ &\approx \frac{R^2 K_{sc}}{2\nu_{v,0}}\end{aligned}\quad (91)$$

and (83) in (89) yields the following stability condition:

$$\lambda > \lambda_c c_c, \quad (92)$$

where

$$c_c = (\nu_h q)^{-\frac{2}{3}} \quad (93)$$

and q is the smallest positive value of

$$q = 1 \pm \sqrt{1 + \frac{(\nu_h - m_+ N_P)}{\nu_h}}. \quad (94)$$

If Λ is real, stability condition (86) may be violated in only two ways, either crossing the radial stopband $2\nu_h = m_h N_P$, or the longitudinal stopband $2\nu_s = m_s N_P$, where again $m_{h,s}$ is an integer. The derivation of the radial is analogous to that of the vertical stopband, and the result is similar to (80) with c_v replaced by:

$$c_h = (\nu_h \delta\nu_h)^{-\frac{1}{3}} \quad (95)$$

where $\delta\nu_h$ is an expression similar to (78). The longitudinal stopband is treated similarly, except that no width of the stopband is assumed. This also gives a stability condition:

$$\lambda > \lambda_c c_s \quad (96)$$

where, with $m_s = 1$,

$$c_s = \left(\frac{4}{N_p} \right)^{\frac{2}{3}}. \quad (97)$$

Thus, in conclusion, there are three ways for the planar motion to become unstable, determined by the constants $c_{s,h,c}$; the largest of them sets the limit, since it corresponds to the first resonance being encountered.

It is interesting to note that we can predict with good approximation the movement of the working point in the tune diagram. Taking advantage of the parametric dependence of ν_h and ν_s on K_{sc} , as shown by (83) and (91), we have

$$\nu_h = \nu_{h,0} \left(1 - \frac{\nu_s^2}{4\nu_{h,0}^2} \right). \quad (98)$$

We can then estimate the way the stability is lost as a function of the original tune value $\nu_{h,0}$. It is sufficient to take $m_s = 1$. Let $\nu_{h,0}$ lie between mN_P and $(m+1)N_P$, where m is any positive integer, or zero. We can distinguish the following cases:

- $mN_P < \nu_h < (m + \sqrt{m^2 + \frac{1}{2}})N_P/2$

The instability is caused by approaching a radial stopband.
The stability limit is given by c_h ((95)).

- $(m + \sqrt{m^2 + \frac{1}{2}})N_P/2 < \nu_h < (m + \frac{1}{2})N_P$

The instability is caused by the longitudinal stopband, and the stability limit is given by c_s (97).

- $(m + \frac{1}{2})N_P < \nu_h < (\frac{(6m+1)+4\sqrt{4(m+1)^2-2m}}{14})N_P$

The instability is again caused by the radial stopband.

- $(\frac{(6m+1)+4\sqrt{4(m+1)^2-2m}}{14})N_P < \nu_h < (m+1)N_P$

The instability is lost due to coupling of the modes, and the limit is set by c_c ((94)).

Of course, in all these cases, the limit of stability of the planar motion needs to be compared to that of the vertical motion, where the limit is set by c_v (see (81)).

6 The Betatron Magnet as a Storage Ring

The betatron, consisting of a single 360° dipole magnet with a field index n is perhaps the simplest storage ring. Can we store a string in it? For the motion to be stable, the eigenvalues of the dipole magnet of field index n , as calculated earlier, all need to be purely imaginary. This means that the following quantity to be real and negative:

$$2\alpha^2 = -\Gamma^2 \pm \sqrt{\Gamma^4 + 4\Upsilon^2}. \quad (99)$$

This in turn requires that:

$$\Gamma^2 + 4\Upsilon^2 > 0. \quad (100)$$

This condition is always satisfied. The other condition is:

$$\Gamma^2 > 0 \quad \text{and} \quad \Upsilon^2 < 0 \quad (101)$$

simultaneously. That is, for the motion in the horizontal and longitudinal direction to be stable, the field index needs to satisfy:

$$n < 1 - \gamma^2 - \xi^2 < 0. \quad (102)$$

On the other hand, for the motion in the vertical plane to be stable, one requires:

$$n > \xi^2 > 0. \quad (103)$$

Clearly, these two conditions cannot be satisfied simultaneously, and therefore the string in a betatron is unstable against small perturbations.

7 Results

We have applied our method to five lattices described elsewhere [5] for the case of partially stripped gold ions, viz. : $Q = 51$, $A = 197$ and $E_0 = 0.93126$ GeV /u. E_0 is the rest energy per nucleon. All cases correspond to the same magnetic rigidity $B\rho = 1.2$ Tm.

In Table 1 we show our results. The stability characteristics have been obtained exactly by inspecting the eigenvalues of the total transfer matrix. The last value for λ for which the motion is stable is listed, as well as K_{sc} , ρ_{sc} and the total number of particles N . In addition we indicate which motion becomes unstable: P for planar and V for the vertical motion. Note, that in some cases we have several “regions” of unstable planar motion, in between stable motion, giving us several values of λ .

In Table 2 we compare the threshold values of λ shown in Table 1 with the estimates presented in section 5. We note that for the FODO16 lattice $\lambda_c = 46 \mu m$; for the CRYSTAL8 lattice it is $63.4 \mu m$; for all other lattices it is $61 \mu m$. Note that $h = 0.25 m^{-1}$ for the lattices, except for the CRYSTAL8 lattice, which has $h = 0.39 m^{-1}$.

Fig. 1 shows the (ν_h, ν_s) -tune diagram normalized to the storage-ring period N_P . The location of the major radial, longitudinal, and coupling resonances are indicated. Using (98) to track the location of the working point with increasing space charge, we show the parabolic curves for the various storage rings and initial tuning mode. CRYSTAL8 is not included in the picture, because it shows a vertical instability. In agreement with the analytical results and predictions, and the computer calculations we recognize three types of instability: radial (R) for the FODO16 lattices; longitudinal (L) for the FODO4 high tune lattices, and coupling (C) for the FODO8 and FODO4R lattice.

Two remarkable facts emerge: First, the results shown in Ta-

ble 1 are in full agreement with the space-charge simulation using SYNCH [5]. This gives a powerful tool to explain intense space-charge behavior. From the previous calculations we derived the model of tune-shift and stopband to explain beam instability.

Secondly, there is close agreement between the calculated stability threshold for λ using the actual total transfer matrices and the estimate based on the analytical stability criterion derived in section 5 (see Table 2). This gives us added confidence in using the stability criterion

$$\lambda > \lambda_c c_{v,h,s,c} \quad (104)$$

and to disregard the details of the storage-ring lattice. In addition, the direction in which the motion becomes unstable is predicted accurately, and without exception.

name	total tune	K_{sc} [m^{-2}]	ρ_{sc} [m]	λ μm	N 10^6	unstab. eigenval.	$2\pi R$ [m]
FODO16	4.8	0.39	1.61	23.2	1.8	P,R	40.60
	4.0	0.28	1.88	25.7	1.6	P,R	
	3.2	0.19	2.27	29.2	1.4	P,R	
	2.4	0.11	3.07	35.7	1.1	P,R	
FODO4	5.8	0.02	7.06	62.6	0.98	P,L	60.91
		0.03	5.52	52.8	1.2	P,C	
		0.10	3.14	36.2	1.7	P,R	
		0.18	2.33	29.7	2.1	P,L	
		0.27	1.94	26.3	2.3	P,C	
	4.8	0.02	6.48	60.9	1.0	P,L	
		0.12	2.93	34.6	1.8	P,C	
		0.17	2.41	30.4	2.0	P,L	
		0.20	2.23	28.8	2.1	P,R	
	3.6	0.007	39.3	195.3	0.3	P,C	
		0.02	6.64	59.7	1.0	P,L	
		0.07	3.38	41.4	1.5	P,R	
	7.2	0.003	19.5	122.4	0.50	P,C	
		0.09	3.39	38.1	1.6	P,L	
		0.25	2.00	26.8	2.3	P,R	
		0.33	1.74	24.4	2.5	P,L	
FODO8	6.0	0.02	6.76	60.4	1.0	P,C	60.91
		0.09	3.33	37.7	1.6	P,L	
		0.28	1.90	25.9	2.4	P,C	
FODO4R	7.30	0.003	18.8	119.6	0.51	P,C	60.91
		0.02	6.77	60.5	1.0	P,L	
		0.05	4.68	47.3	1.3	P,R	
CRYSTAL8		0.01	9.56	76.1	0.85	V	64.82

Table 1: Overview of the calculations for five different lattices, for partially stripped gold ions. For CRYSTAL8 $\nu_{h,0} = 1.91$ and $\nu_{v,0} = 1.09$; for other lattices $\nu_{h,0} \approx \nu_{v,0}$. The dilute beam tune, the highest K_{sc} obtainable, the inter-ion spacing λ , and the number of particles stored are shown.

name	total tune	λ_{obs} μm	λ_{est} μm	unstab. eigenval.
FODO16	4.8	23	21	P,R
	4.0	26	23	P,R
	3.2	29	27	P,R
	2.4	36	33	P,R
FODO4	5.8	62	61	P,L
		36	35	P,R
	4.8	61	61	P,L
	3.6	195	174	P,C
		60	60	P,R
		41	43	P,L
FODO8	7.2	122	111	P,C
		38	38	P,L
		27	27	P,R
	6.0	60	61	P,C
		38	38	P,L
FODO4R	7.30	120	122	P,C
		61	61	P,L
		47	36	P,R
CRYSTAL8		76	75	V

Table 2: We compare the critical lattice constant determined with the computer with their estimates.

Figure 1. Tracking of the working point for different lattices and different initial tuning modes versus space charge strength. FODO16 lattices reach the limit of stability by approaching the radial (R) stopband. FODO8 and FODO4R lattices become unstable by crossing the coupling resonance (C). The FODO4 lattices display the longitudinal stopband (L). There is full agreement with both the analytical estimates as well as the computer inspection of the eigenvalues.

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