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Asymptotics of a Free Boundary Problem*

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Abstract

This article is concerned with free boundary problems for the differential equation

$$u'' + \frac{2\nu + 1}{r}u' + u - u^q = 0, \quad r > 0,$$

where $0 \leq q < 1$ and $\nu \geq 0$. As was shown by Kaper and Kwong [*Differential and Integral Equations* 3, 353–362], there exists a unique $R > 0$, such that the equation admits a classical solution u that is positive and monotone on $(0, R)$ and that satisfies the boundary conditions

$$u'(0) = 0, \quad u(R) = u'(R) = 0.$$

This article is concerned with the behavior of R and $u(0)$ as $q \rightarrow 1$.

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1 The Problem

We consider free boundary problems for the reaction-diffusion equation $\Delta u + u - u^q = 0$ ($0 \leq q < 1$) in $\mathbf{R}^N$ ($N = 2, 3, \dots$). In particular, we are interested in *radial* solutions—that

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is, solutions that depend only on the radial variable $r = |x|$. Such solutions satisfy the differential equation

$$u'' + \frac{2\nu + 1}{r}u' + u - u^q = 0, \quad r > 0, \quad (1.1)$$

where $\nu = \frac{1}{2}(N - 2)$. Without loss of generality we will consider ν as a real number, not necessarily half-integer ($\nu \geq 0$).

As was shown by Kaper and Kwong [1], there exists, for each $q \in [0, 1)$, a unique $R > 0$, such that (1.1) admits a classical solution u that is positive and monotone on $(0, R)$ and that satisfies the boundary conditions

$$u'(0) = 0, \quad u(R) = u'(R) = 0. \quad (1.2)$$

In this article, we investigate the behavior of R and $u(0)$ as $q \rightarrow 1$.

2 The Result

We prove the following result.

Theorem 1 *For each $q \in [0, 1)$, there is a unique $R > 0$ such that (1.1), (1.2) admits a (classical) solution u that is positive everywhere on $(0, R)$. The function u is monotonically decreasing on $(0, R)$; $\|u\|$ is bounded, but R grows beyond bounds as $q \rightarrow 1$.*

In the special case $\nu = \frac{1}{2}$ ($N = 3$) we have a lower bound on R ,

$$R > \sqrt{\frac{2}{1-q}}, \quad 0 \leq q < 1, \quad (2.1)$$

but, as we do not have a comparable upper bound, we cannot conclude that $R = O((1 - q)^{-1/2})$ as $q \rightarrow 1$.

3 The Proof

Using a shooting argument, we replace the boundary value problem (1.1), (1.2) by the initial value problem

$$u'' + \frac{2\nu + 1}{r}u' + u - u^q = 0, \quad r > 0, \quad (3.1)$$

$$u(0) = \gamma, \quad u'(0) = 0. \quad (3.2)$$

The results of [1] imply that, for any $q \in [0, 1)$, there is a unique $\gamma > 1$, such that the solution of (3.1), (3.2) decreases from γ to meet the r -axis with zero slope at some value $R > 0$. Denoting this solution by $u(\cdot, \gamma)$, we have

$$u(R, \gamma) = 0, \quad u'(R, \gamma) = 0. \quad (3.3)$$

The lower bound on γ can be sharpened to $(2/(1+q))^{1/(1-q)}$, but 1 suffices for our purpose. The proof consists of a detailed investigation of the behavior of $u(\cdot, \gamma)$.

3.1 Down to 1.

We begin by showing that $u(r, \gamma)$ decreases monotonically from the value γ at $r = 0$ to the value 1 at some finite point r_0 .

Lemma 1 *There exists a point $r_0 < j_{\nu,1}/(1-q)^{1/2}$, such that $u(\cdot, \gamma)$ is monotonically decreasing on $(0, r_0)$, with $u(r_0, \gamma) = 1$ and $u'(r_0, \gamma) < 0$. Here, $j_{\nu,1}$ is the first positive zero of J_ν —the Bessel function of the first kind of order ν .*

Proof. As long as $u > 1$, we have $u - u^q > (1 - q)u$, so $u(\cdot, \gamma)$ oscillates faster than the solution v of the equation

$$v'' + \frac{2\nu + 1}{r}v' + (1 - q)v = 0. \quad (3.4)$$

In particular, $u(\cdot, \gamma)$ reaches the value 1 before v does. Now, $v(r)$ is a constant multiple of $r^{-\nu}J_\nu(r(1-q)^{1/2})$, where J_ν is the Bessel function of the first kind of order ν —see, for example, [2]. Hence, $v(r) = 1$ for some value $r < j_{\nu,1}/(1-q)^{1/2}$, where $j_{\nu,1}$ is the first positive zero of J_ν . We conclude that there must be a point $r_0 < j_{\nu,1}/(1-q)^{1/2}$, such that $\gamma > u(r, \gamma) > 1$ for $0 < r < r_0$ and $u(r_0, \gamma) = 1$.

Since $u'(0, \gamma) = 0$ and $u''(r, \gamma) < 0$ near 0, it must be the case that $u'(r, \gamma) < 0$ near 0.

Suppose $u(\cdot, \gamma)$ were not monotone on $(0, r_0)$. Then there exists a value $r_1 \in (0, r_0)$ where $u(r, \gamma)$ has a local minimum, with $u(r_1, \gamma) > 1$. Because $u(r, \gamma)$ reaches the value 1 at r_0 , there must then exist a value $r_2 \in (r_1, r_0)$ such that $u(r_2, \gamma) = u(r_1, \gamma)$ and $u'(r_2, \gamma) \leq 0$. Multiplying the differential equation (3.1) by u' and integrating over (r_1, r_2) , we find that

$$\frac{1}{2}(u'(r_2, \gamma))^2 = -(2\nu + 1) \int_{r_1}^{r_2} \frac{(u'(r, \gamma))^2}{r} dr. \quad (3.5)$$

But here we have a contradiction, as the two sides of this identity have opposite signs. It must therefore be the case that $u(\cdot, \gamma)$ is monotone on $(0, r_0)$.

The monotonicity of $u(\cdot, \gamma)$ on $(0, r_0)$ implies that $u'(r_0, \gamma) \leq 0$. If $u'(r_0, \gamma) = 0$, then it follows from the Lipschitz continuity of the function $u - u^q$ for $u > 0$ and the consequential uniqueness of the solution of the initial value problem for (3.1) in the direction of decreasing r starting at $r = r_0$ that $u(r, \gamma) = 1$ for all $r \in (0, r_0)$. But then we have a contradiction, as $u(0, \gamma) = \gamma > 1$. We conclude that $u'(r_0, \gamma) < 0$. ■

3.2 Beyond r_0 .

From Lemma 1 we know that $u(r, \gamma)$ decreases monotonically until it reaches the value 1 with a negative slope at $r = r_0$. Beyond r_0 , $u(r, \gamma)$ decreases further until it either reaches the value 0 with a negative or zero slope, or it bottoms out at some finite value of r with a minimum value between 0 and 1.

Let r_1 be the point where $u(r, \gamma)$ ceases to be positive,

$$r_1 = \sup \{r > r_0 : u(s, \gamma) > 0, 0 < s < r\}. \quad (3.6)$$

If r_1 is finite and $u(r_1, \gamma) = 0$, we do not consider $u(\cdot, \gamma)$ beyond r_1 . In this case, we can use the same argument as in the proof of Lemma 1 to show that $u(\cdot, \gamma)$ is monotonically decreasing on the entire interval $(0, r_1)$. In particular, if γ is such that not only $u(r_1, \gamma) = 0$, but also $u'(r_1, \gamma) = 0$, then $u(\cdot, \gamma)$ defines the (unique) solution u of the free boundary problem (3.1), (3.3), where $R = r_1$.

If $r_1 = \infty$, then $u(r, \gamma)$ has a positive minimum at some finite value of r , after which it oscillates with decreasing amplitude around the constant value 1.

Lemma 2 For $0 < r < r_1$, we have $0 < u(r, \gamma) < \gamma$.

Proof. The lemma is true for $0 < r \leq r_0$ (cf. Lemma 1). Beyond r_0 , we use a simple energy argument. The energy E of any solution u of (3.1), defined by the expression

$$E(r) = \frac{1}{2}(u'(r))^2 + \frac{1}{2}(u(r))^2 - \frac{1}{q+1}(u(r))^{q+1}, \quad (3.7)$$

is a monotonically decreasing function of its argument, as $E'(r) = -((2\nu+1)/r)(u'(r))^2 \leq 0$ for all $r \geq 0$.

Suppose the lemma were false for $r_0 < r < r_1$. Then $u(r_2, \gamma) = \gamma$ for some $r_2 \in (r_0, r_1)$, where $E(r_2) \geq \gamma^2/2 - \gamma^{q+1}/(q+1) = E(0)$, and we have a contradiction. ■

Let w be defined in terms of $u(\cdot, \gamma)$ by the expression

$$w(r) = \frac{ru(r, \gamma)}{\gamma}. \quad (3.8)$$

This function is a solution of the initial value problem

$$w'' + \frac{2\nu-1}{r}w' + \left(1 - \frac{1}{(u(r, \gamma))^{1-q}} - \frac{2\nu-1}{r^2}\right)w = 0, \quad r > 0, \quad (3.9)$$

$$w(0) = 0, \quad w'(0) = 1. \quad (3.10)$$

It vanishes when u vanishes, while its derivative vanishes when both u and u' vanish. Furthermore,

$$0 < w(r) < r, \quad 0 < r < r_1. \quad (3.11)$$

The following lemma gives a lower bound for r_1 .

Lemma 3 We have $r_1 > j_{\nu,1}/\delta$, where

$$\delta = \sqrt{1 - \gamma^{-(1-q)}}. \quad (3.12)$$

Proof. Because $u(r, \gamma) < \gamma$, w oscillates less than the solution v of the initial value problem

$$v'' + \frac{2\nu - 1}{r}v' + \left(\delta^2 - \frac{2\nu - 1}{r^2}\right)v = 0, \quad r > 0; \quad v(0) = 0, \quad v'(0) = 1, \quad (3.13)$$

at least as long as v is positive. Since $v(r) = 2^\nu \Gamma(\nu + 1) \delta^{-\nu} r^{1-\nu} J_\nu(\delta r)$, the first zero of v occurs at $j_{\nu,1}/\delta$. It must therefore be the case that $r_1 > j_{\nu,1}/\delta$. ■

3.3 Bounds on $(0, j_{\nu,1}/\delta)$.

We rewrite the equation (3.9) in the form

$$w'' + \frac{2\nu - 1}{r}w' + \left(\delta^2 - \frac{2\nu - 1}{r^2}\right)w = f(w), \quad (3.14)$$

where

$$f(w) = \frac{1}{\gamma^{1-q}} \left\{ \left(\frac{r}{w} \right)^{1-q} - 1 \right\} w. \quad (3.15)$$

Using the method of variation of parameters, we obtain the integral equation for w ,

$$w(r) = rg(\delta r) + \frac{\pi}{2} \int_0^r r^{1-\nu} s^\nu \{J_\nu(\delta s)Y_\nu(\delta r) - Y_\nu(\delta s)J_\nu(\delta r)\} f(w(s)) ds, \quad (3.16)$$

where

$$g(\rho) = 2^\nu \Gamma(\nu + 1) \rho^{-\nu} J_\nu(\rho). \quad (3.17)$$

J_ν and Y_ν are the Bessel functions of the first and second kind, respectively, of order ν . The expression (3.16) holds for all $r \in [0, r_1]$ or, if r_1 is finite, for all $r \in [0, r_1]$. We now restrict r to the interval $[0, j_{\nu,1}/\delta]$.

Lemma 4 For $0 < r < j_{\nu,1}/\delta$, we have

$$0 < rg(\delta r) < w(r) < r \left(g(\delta r) + \frac{\phi(\delta r)}{\log \gamma} \right), \quad (3.18)$$

where g is defined in (3.17) and

$$\phi(\rho) = \frac{\rho^2 (g(\rho))^{-1} \log(g(\rho))^{-1}}{4(\nu + 1)}. \quad (3.19)$$

Proof. Take any $r \in (0, j_{\nu,1}/\delta)$. It follows from the Kneser-Sommerfeld expansion—see [2, Section 15.42]—that

$$J_\nu(\delta s)Y_\nu(\delta r) - Y_\nu(\delta s)J_\nu(\delta r) = \frac{4\delta r J_\nu(\delta r)}{\pi J_\nu(\delta s)} \sum_{n=1}^{\infty} \frac{(J_\nu(j_{\nu,n}s/r))^2}{(j_{\nu,n}^2 - (\delta r)^2) j_{\nu,n} J_\nu'^2(j_{\nu,n})}, \quad (3.20)$$

for $0 \leq s \leq r$. All the terms in the right member are positive, so the expression in the left member is positive. Furthermore, $f(w(s))$ is positive for $0 \leq s \leq r$. Therefore, the integral in (3.16) is positive. Obviously, $g(\delta r)$ is positive, so

$$w(r) > rg(\delta r) > 0, \quad 0 < r < j_{\nu,1}/\delta. \quad (3.21)$$

From (3.21) and the fact that g is decreasing on $(0, j_{\nu,1})$ we deduce that

$$\frac{s}{w(s)} < \frac{1}{g(\delta s)} \leq \frac{1}{g(\delta r)}, \quad 0 \leq s \leq r. \quad (3.22)$$

Therefore,

$$f(w(s)) < \frac{(g(\delta r))^{-(1-q)} - 1}{\gamma^{1-q}} w(s), \quad 0 \leq s \leq r. \quad (3.23)$$

Furthermore, $w(s) \leq s$, cf. (3.11). These inequalities lead to the estimate

$$\begin{aligned} & \frac{\pi}{2} \int_0^r r^{1-\nu} s^\nu \{J_\nu(\delta s)Y_\nu(\delta r) - Y_\nu(\delta s)J_\nu(\delta r)\} f(w(s)) ds \\ & \leq r \frac{(g(\delta r))^{-(1-q)} - 1}{\gamma^{1-q} - 1} \left[\rho^{-\nu} \frac{\pi}{2} \int_0^\rho \{J_\nu(z)Y_\nu(\rho) - Y_\nu(z)J_\nu(\rho)\} z^{\nu+1} dz \right]_{\rho=\delta r}. \end{aligned} \quad (3.24)$$

The expression in square brackets can be evaluated by means of the recurrence formulae for Bessel functions—see [2, Section 3.2]—and the resulting expression can be simplified further by means of the Wronskian—see [2, Section 3.63]—,

$$\rho^{-\nu} \frac{\pi}{2} \int_0^\rho \{J_\nu(z)Y_\nu(\rho) - Y_\nu(z)J_\nu(\rho)\} z^{\nu+1} dz = 1 - g(\rho). \quad (3.25)$$

We estimate this expression by substituting the series expansion for the Bessel function J_ν and truncating after the first term,

$$1 - g(\rho) = 1 - 2^\nu \Gamma(\nu + 1) \rho^{-\nu} J_\nu(\rho) \leq \frac{\rho^2}{4(\nu + 1)}. \quad (3.26)$$

To estimate the factor in front of the square brackets in (3.24), we observe that $0 < g(\delta r) < 1$ on $(0, j_{\nu,1}/\delta)$ and $\gamma > 1$. Furthermore, one readily verifies that

$$\frac{1 - x^{1-q}}{\log x^{-1}} \leq \frac{y^{1-q} - 1}{\log y}.$$

for any pair (x, y) with $0 < x \leq 1 \leq y$. Therefore,

$$\begin{aligned} & \frac{(g(\delta r))^{-(1-q)} - 1}{\gamma^{1-q} - 1} = (g(\delta r))^{-(1-q)} \frac{1 - (g(\delta r))^{1-q}}{\gamma^{1-q} - 1} \\ & \leq \frac{(g(\delta r))^{-(1-q)} \log(g(\delta r))^{-1}}{\log \gamma} \leq \frac{(g(\delta r))^{-1} \log(g(\delta r))^{-1}}{\log \gamma}. \end{aligned} \quad (3.27)$$

The lemma now follows from (3.16), (3.21), (3.24), (3.25), (3.26), and (3.27). ■

In terms of u , we have the following bounds:

$$0 < \gamma g(\delta r) < u(r, \gamma) < \gamma \left[g(\delta r) + \frac{\phi(\delta r)}{\log \gamma} \right], \quad 0 < r < \frac{j_{\nu,1}}{\delta}. \quad (3.28)$$

Because $\phi(\rho)$ increases beyond bounds as $g(\rho)$ decreases to 0, the upper bound in (3.18) or (3.28) increases indefinitely as r approaches the right endpoint of the interval $(0, j_{\nu,1}/\delta)$.

In the following analysis we also need an estimate of the quantity $r^{1-2\nu}(r^{2\nu-1}w)'(r)$. It is given by the expression

$$r^{1-2\nu}(r^{2\nu-1}w)'(r) = h(\delta r) + \delta \frac{\pi}{2} \int_0^r r^{1-\nu} s^\nu \{J_\nu(\delta s)Y_{\nu-1}(\delta r) - Y_\nu(\delta s)J_{\nu-1}(\delta r)\} f(w(s)) ds, \quad (3.29)$$

where

$$h(\rho) = 2^\nu \Gamma(\nu + 1) \rho^{1-\nu} J_{\nu-1}(\rho). \quad (3.30)$$

Like (3.16), (3.29) holds for all $r \in [0, r_1)$ or, if r_1 is finite, for all $r \in [0, r_1]$. The following lemma gives an estimate on $(0, j_{\nu,1}/\delta)$.

Lemma 5 For $0 < r < j_{\nu,1}/\delta$, we have

$$\left| r^{1-2\nu}(r^{2\nu-1}w)'(r) - h(\delta r) \right| < 2(\nu + 1) \frac{\phi(\delta r)}{\log \gamma}, \quad (3.31)$$

where h is defined in (3.30) and ϕ is defined in (3.19).

Proof. The proof is similar to, although slightly more involved than the proof of Lemma 4. Instead of (3.16), we use (3.29). The analog of (3.24) is

$$\begin{aligned} & \delta \frac{\pi}{2} \int_0^r r^{1-\nu} s^\nu \{J_\nu(\delta s)Y_{\nu-1}(\delta r) - Y_\nu(\delta s)J_{\nu-1}(\delta r)\} f(w(s)) ds \\ & \leq \frac{(g(\delta r))^{-(1-q)} - 1}{\gamma^{1-q} - 1} \left[\rho^{1-\nu} \frac{\pi}{2} \int_0^\rho \{J_\nu(z)Y_{\nu-1}(\rho) - Y_\nu(z)J_{\nu-1}(\rho)\} z^{\nu+1} dz \right]_{\rho=\delta r} \end{aligned} \quad (3.32)$$

The expression in square brackets can again be evaluated; instead of (3.25) we have

$$\rho^{1-\nu} \frac{\pi}{2} \int_0^\rho \{J_\nu(z)Y_{\nu-1}(\rho) - Y_\nu(z)J_{\nu-1}(\rho)\} z^{\nu+1} dz = 2(\nu + 1) - h(\rho), \quad (3.33)$$

where

$$2(\nu + 1) - h(\rho) = 2(\nu + 1) - 2^\nu \Gamma(\nu + 1) \rho^{1-\nu} J_{\nu-1}(\rho) \leq \frac{1}{2} \rho^2. \quad (3.34)$$

The lemma follows from (3.29), (3.32), (3.33), (3.34), and (3.27). ■

3.4 Estimates at r_0 .

We use the results of Lemmas 4 and 5 to estimate r_0 and $r^{1-2\nu}(r^{2\nu-1}w)'$ at r_0 .

Lemma 6 *Let $a \in (j_{\nu-1,1}, j_{\nu,1})$ be fixed. Then there exists a constant $\gamma_1 > 1$ that does not depend on q , such that*

$$r_0^{1-2\nu}(r^{2\nu-1}w)'(r_0) < -\frac{1}{2}|h(a)| \quad (3.35)$$

and

$$\frac{a}{\delta} < r_0 < \left(1 + \frac{4\nu}{|h(a)|}\right)^{1/(2\nu)} \frac{a}{\delta}, \quad (3.36)$$

for all $\gamma \geq \gamma_1$.

Proof. With the choice of a indicated in the statement of the lemma we have $g(a) > 0$ and $h(a) < 0$. These inequalities follow from the interlacing property of the zeros of Bessel functions,

$$0 < j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \dots,$$

cf. [2, Section 15.22].

We begin by observing that w oscillates less than v , where $v(r) = rg(\delta r)$, g defined by (3.17). Therefore r_0 , which is defined by the identity $w(r) = r/\gamma$, is certainly beyond the point r_2 , where $g(\delta r_2) = 1/\gamma$. Therefore, if

$$\gamma_0 = 1/g(a), \quad (3.37)$$

then $g(\delta r_2) \leq g(a)$ for all $\gamma \geq \gamma_0$. Now, g is monotonically decreasing between a and $j_{\nu-1,1}$, so then we also have $\delta r_2 \geq a$ for all $\gamma \geq \gamma_0$. Since $r_0 > r_2$, we have thus achieved that

$$a/\delta < r_0. \quad (3.38)$$

for all $\gamma \geq \gamma_0$.

With $r_3 = a/\delta$, it follows from (3.31) that

$$r_3^{1-2\nu}(r^{2\nu-1}w)'(r_3) < -|h(a)| + 2(\nu+1)\frac{\phi(a)}{\log \gamma}. \quad (3.39)$$

Here, $h(a)$ and $\phi(a)$ do not depend on q or γ . Therefore, if we now define γ_1 ,

$$\gamma_1 = \min\left\{\gamma_0, \exp\left(2(\nu+1)\frac{\phi(a)}{|h(a)|}\right)\right\}, \quad (3.40)$$

then γ_1 is independent of q and

$$r_3^{1-2\nu}(r^{2\nu-1}w)'(r_3) < -\frac{1}{2}|h(a)| \quad (3.41)$$

for all $\gamma \geq \gamma_1$. Writing the differential equation (3.9) in the form

$$\left(r^{1-2\nu}(r^{2\nu-1}w)'\right)' = -\left(1 - u^{-(1-q)}\right), \quad (3.42)$$

we observe that the function $r^{1-2\nu}(r^{2\nu-1}w)'$ is decreasing as long as $u(r, \gamma) > 1$ —that is, up to r_0 . Therefore, the bound (3.41) extends to the entire interval $[r_3, r_0]$ and we have

$$r^{1-2\nu}(r^{2\nu-1}w)'(r) < -\frac{1}{2}|h(a)|, \quad r_3 \leq r \leq r_0. \quad (3.43)$$

for all $\gamma \geq \gamma_1$. In particular, the inequality holds at r_0 , as asserted in (3.35).

Multiplying both sides of the inequality (3.43) by $r^{2\nu-1}$ and integrating over the interval (r_3, r_0) , we find

$$\left(\frac{w(r_3)}{r_3} + \frac{|h(a)|}{4\nu}\right)r_3^{2\nu} - \frac{|h(a)|}{4\nu}r_0^{2\nu} > \frac{w(r_0)}{r_0}r_0^{2\nu}. \quad (3.44)$$

Here, we estimate the expression in the right member from below by 0. In the left member, we estimate the ratio $w(r_3)/r_3$ from above by 1; cf. (3.11). Thus,

$$r_0^{2\nu} < \left(1 + \frac{4\nu}{|h(a)|}\right)r_3^{2\nu}. \quad (3.45)$$

The inequalities (3.36) now follow from (3.38) and (3.45). ■

3.5 Down to 0.

We are now in a position to prove that the continuation of u beyond r_0 decreases to 0 for all sufficiently large γ , independently of q .

Lemma 7 *There exists a constant γ_2 that does not depend on q ($\gamma_2 \geq \gamma_1$, where γ_1 is the constant introduced in Lemma 6), such that $r_1 < \infty$ for all $\gamma \geq \gamma_2$.*

Proof. The proof is by contradiction, where we assume that, for some $\gamma \geq \gamma_1$, the solution $u(\cdot, \gamma)$ of (3.1), (3.2) is positive for all $r \geq 0$.

Consider the function w defined by (3.8). By assumption, w is positive for all $r > 0$. Because $(r^{1-2\nu}(r^{2\nu-1}w)')' = (r/\gamma)(u^q - u)$ and $u^q - u < 1 - q$ for $u > 0$, we have

$$(r^{1-2\nu}(r^{2\nu-1}w)')'(r) < \frac{(1-q)r}{\gamma}, \quad r > 0. \quad (3.46)$$

Integrating (3.46) from r_0 to any point $r > r_0$, and using the estimate (3.35) at r_0 , we find

$$r^{1-2\nu}(r^{2\nu-1}w)'(r) < -\frac{1}{2}|h(a)| + \frac{(1-q)r^2}{2\gamma}, \quad r > r_0, \quad (3.47)$$

for all $\gamma \geq \gamma_1$. Because $\gamma\delta^2 = \gamma - \gamma^q > \gamma^{1-q} - 1 > (1-q)\log \gamma$, it follows that

$$r^{1-2\nu}(r^{2\nu-1}w)'(r) < -\frac{1}{2}|h(a)| + \frac{r^2\delta^2}{2\log \gamma}, \quad r > r_0, \quad (3.48)$$

for all $\gamma \geq \gamma_1$.

Now we restrict r to a compact interval $[r_0, r_2]$, where

$$r_2 = b/\delta, \quad (3.49)$$

and $b > a$ is a suitably chosen constant. Defining the constant γ_2 by

$$\gamma_2 = \min\{\gamma_1, e^{2b^2/|h(a)|}\}, \quad (3.50)$$

we then have

$$\frac{r^2 \delta^2}{2 \log \gamma} \leq \frac{1}{4} |h(a)|, \quad r_0 \leq r \leq r_2, \quad (3.51)$$

for all $\gamma \geq \gamma_2$, so (3.48) reduces to

$$r^{1-2\nu} (r^{2\nu-1} w)'(r) < -\frac{1}{4} |h(a)|, \quad r_0 \leq r \leq r_2, \quad (3.52)$$

for all $\gamma \geq \gamma_2$. Hence,

$$w(r_2) < \left[(a/b)^{2\nu} \frac{w(r_0)}{r_0} - (1 - (a/b)^{2\nu}) \frac{|h(a)|}{8\nu} \right] r_2. \quad (3.53)$$

Using (3.36) to estimate $w(r_0)/r_0$ and writing the inequality in terms of u , we thus find that

$$u(r_2, \gamma) < (a/b)^{2\nu} \left(1 + \frac{4\nu}{|h(a)|} \right)^{1/(2\nu)} - \gamma \left(1 - (a/b)^{2\nu} \right) \frac{|h(a)|}{8\nu}, \quad (3.54)$$

for all $\gamma \geq \gamma_2$.

But now we have a contradiction, as the expression in the right member of this inequality certainly becomes negative for sufficiently large values of γ . We conclude therefore that $u(\cdot, \gamma)$ reaches the value 0 at some finite point r_1 , as claimed. ■

3.6 Completion of the Proof.

According to Lemma 7, $u(\cdot, \gamma)$ ceases to be positive at a finite point r_1 for all $\gamma \geq \gamma_2$, where γ_2 is a constant that does not depend on q . Obviously, r_1 depends on the value of γ ; in fact, it decreases as γ increases. Let

$$\Gamma = \inf\{\gamma > 1 : r_1 < \infty\}. \quad (3.55)$$

If $\gamma = \Gamma$, then $u(\cdot, \gamma)$ reaches the r -axis with a horizontal slope, so $u(\cdot, \Gamma)$ defines the unique solution u of the free boundary problem (1.1).¹ (2), where

$$R = r_1(\Gamma). \quad (3.56)$$

As we observed at the beginning of Section 3.2, the function u is monotone everywhere where it is positive. Hence, $\|u\| = \Gamma$. Obviously, Γ depends on q . However, it follows from Lemma 7 that $1 < \Gamma \leq \gamma_2$, so $\|u\|$ is bounded as $q \rightarrow 1$ ($q < 1$).

It remains to investigate the behavior of R as $q \rightarrow 1$ ($q < 1$). Because Γ is bounded, $\lim_{q \rightarrow 1} \Gamma^{1-q} = 1$. Then it follows from (3.12) that $\lim_{q \rightarrow 1} \delta = 0$ and therefore, by Lemma 3, $\lim_{q \rightarrow 1} R = \infty$. Thus, the proof of the theorem is complete. ■

3.7 Special Case: $N = 3$.

In the special case $N = 3$ ($\nu = \frac{1}{2}$), it is actually possible to find a lower bound for R that shows that R grows beyond bounds as $q \rightarrow 1$.

A simple energy argument gives the inequality

$$0 = E(R) < E(0) = \frac{\Gamma^2}{2} - \frac{\Gamma^{1+q}}{1+q}, \quad (3.57)$$

cf. (3.7). Hence,

$$\Gamma^{1-q} > \frac{2}{1+q}. \quad (3.58)$$

Next, we use an energy argument for (3.9). If $\nu = \frac{1}{2}$, this equation reduces to

$$w'' + w - \Gamma^{-(1-q)} r^{1-q} w^q = 0. \quad (3.59)$$

Hence,

$$\left(w'^2 + w^2 - \frac{2}{1+q} \Gamma^{-(1-q)} r^{1-q} w^{1+q} \right)' = -2 \frac{1-q}{1+q} \Gamma^{-(1-q)} r^{-q} w^{1+q}. \quad (3.60)$$

Upon integration over $(0, R)$, the left member yields -1 ; in the right member we use the inequality $w(r) < r$ to obtain the estimate

$$\int_0^R r^{-q} w^{1+q} dr < \frac{1}{2} R^2. \quad (3.61)$$

Thus, using (3.58), we find that

$$R > \sqrt{\frac{2}{1-q}}. \quad (3.62)$$

References

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- [2] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge University Press (1966).

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