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The Generalized Erlangen Program and Setting a Geometry for Four-Dimensional Conformal Fields

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1 The Physics of the 20th Century: A Personal Evaluation

The new results presented in this paper represent joint work by three authors, as listed above; but as the first three sections were mostly a personal reaction of the first-named author, who attended the Beersheva conference and is grateful for the hospitality displayed by Ben Gurion University and was drawn into making these remarks by the challenging questions posed by R. Sen, the organizer, in his convocation document – that author (YN) should be solely blamed for any errors, bad judgement or pretentiousness in their presentation. This is also the reason for the compound title.

In this Fin de Siècle spirit, the mood may well border on the apocalyptic. As you will see, we tend to steer away from such millenary fever. Whatever the approach, however, it is instructive to go back to April 1900 and read the famous (and much maligned) lecture delivered by William Thomson, first Baron Kelvin (and indeed better known by the peerage title he chose, i.e. Lord Kelvin) at the meeting of the Royal Institution of Great Britain (an organisation similar to the AAAS in the USA) [1]. The title was **“Nineteenth Century Clouds over the Dynamical Theory of Heat and Light”**; notice the emphasis on the **clouds** rather than on the harmony of the whole, which these clouds were marring (Kelvin is generally presented as having taken the opposite view and of poohpoohing the “clouds”):

... The beauty and clearness of the dynamical theory, which asserts heat and light to be modes of motion, is at present obscured by two clouds. The first came into existence with the undulatory theory of light ... it involved the question ‘How could the Earth move through an elastic solid, such as essentially is the luminiferous ether?’ [It was supposed to carry transverse oscillations - YN]. The second is the Maxwell-Boltzmann current doctrine regarding the partition of energy ... a cloud that has obscured the brilliance of the molecular theory of heat and light during the last quarter of the Nineteenth Century.

Kelvin is referring to the Michelson-Morley experiment (behind the null results of which the Special Theory of Relativity was lurking) and to the specific heat of the solids, experimentally contradicting the equipartition theorem (an issue to be resolved by Quantum Theory). Kelvin, a great believer in the ether, had used much imagination in inventing for it structures that could be reconciled with the Michelson-Morley experiment. He had worked very hard and yet had failed to find a way of making the classical theory compatible with these two paradoxical results. He continues,

And here, I am afraid, I must end by saying that the difficulties are so great in forming anything like a comprehensive theory, that we cannot

Abstract

This is the text of a talk at the International Symposium on "Mathematical Physics towards the XXI Century" held in March 1993 at Beersheva, Israel. In the first part we attempt to summarize XXth Century Physics, in the light of Kelvin's 1900 speech "Dark Clouds over XIXth Century Physics". Contrary to what is usually said, Kelvin predicted that the "clouds" (relativity and quantum mechanics) would revolutionize physics and that one hundred years might be needed to harmonize them with classical physics. Quantum Gravity can be considered as a leftover from Kelvin's program - so are the problems with the interpretation of quantum mechanics. At the end of the XXth Century, the Standard Model is the new panoramic synthesis, drawn in gauge-geometric lines - realizing the Erlangen program beyond F. Klein's expectations. The hierarchy problem and the smallness of the cosmological constant are our "clouds", generations and the Higgs sector are to us what radioactivity was in 1900. In the second part we describe Metric-Affine spacetimes. We construct the Noether machinery and provide expressions for the conserved energy and hypermomentum. Superimposing conformal invariance over the affine structure induces the Virasoro-like infinite constraining algebra of diffeomorphisms, applied with constant parameters and opening the possibility of a 4DCFT, similar to 2DCFT.

even imagine a finger-post pointing a way that can lead us towards an explanation. I only say we cannot **now** imagine it. But this time next year - this time ten years - **this time one hundred years** - probably it will be just as easy as we think it is to understand that glass of water, which now seems so plain and simple. I cannot doubt but that ... **we shall learn to look on things in a different way** - when that which is now a difficulty will be the only commonsense and intelligible way of looking at the subject.

The usual rendering of this talk describes Kelvin as claiming that the entirety of physics was now described by the grand synthesis of Maxwell's electrodynamics and of thermodynamics with the kinetic theory of gases, with only two minor difficulties still marring the view, but soon to be resolved and integrated in that grand scheme. Instead, we find Kelvin convinced of the impossibility of integrating these two persistent difficulties within the 1900 picture, estimating that harmony will require "learning to look on things in a different way" and that it could take anything between one ... to one hundred years to achieve that harmony!

Indeed, within a year, there was Planck's quantization as an ad-hoc hypothesis; within five years there was Einstein creating Special Relativity and inventing the photon. Note that four years earlier, Becquerel had first hit upon the unknown world of the nucleus - the facts apparently still appearing vague enough, to the point of escaping Kelvin's attention; but within eleven years after the speech, Rutherford was discovering the nucleus and presenting the explicit challenge of a new layer of matter with its interactions - to be included in the next grand synthesis!

The story has indeed unfurled at the rate Kelvin estimated. Within fifteen years Einstein ushered in General Relativity, needed for the harmonization of Special Relativity with Newtonian physics. Within twenty-five years, Heisenberg, Born and Schrödinger had replaced the Planck-Einstein ad-hoc theory by Quantum Mechanics; so much for the clarification of Kelvin's two "clouds" and for the understanding of their relationship to the classical picture of the 19th Century. Note, though, that it is only in the last few years that one has been able to go beyond the residual ad-hoc elements in the "Copenhagen interpretation" of Quantum Mechanics, with a first glimpse of how the classical picture could emerge as a decoherence limit in a fundamentally "quantum" world (see the work of Zeh, Joos, Zureck, Hartle, etc.). Note, however, that this program is just as incomplete as statistical irreversibility: only by having the number of decohering scatterings "reach" infinity, do we arrive at a classical apparatus - just as the Second Law does not generate true irreversibility without having the number of particles "reach" infinity. This is a mystery that we inherited from Boltzmann and appear to hand over "doubled" to the next generations.

But what about marrying the two new theories, in order to attain the new grand synthesis as expected by Kelvin? Forty-eight years after the speech, Tomonaga, Schwinger, Feynman and Dyson indeed managed to reconcile Quantum Mechanics

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with Special Relativity, but only for the case of QED. There were now new forces that had also to be taken care of. Together with Gell-Mann and Zweig, YN had had the privilege in 1960-63 of identifying a deeper layer in the underlying structure of nuclear matter (quarks), the level at which the phenomena ushered in by Becquerel and Rutherford seem to originate. Yang and Mills provided the foundations for the new mechanism in 1953, Veltman and 't Hooft welded it to Special Relativity in 1970-71 and the way was open for a new synthesis. Seventy-five years from that speech, the work of 't Hooft, Gross, Wilczek and Politzer, of Weinberg, Fritzsche and Gell-Mann (with previous installments in the sixties by Nambu and YN himself) now led to the formulation of QCD for the description of 90% of what had been defined as the Strong nuclear interaction; Weinberg, Salam and Glashow had constructed the Electroweak Hamiltonian, experimentally corroborated in 1978-82. The **"Standard Model"** was born, the first presentation of a new grand synthesis.

2 Quantum Gravity as the Missing Piece in Kelvin's Program for the 20th Century

Yet twenty years later, very close to the 21st Century, even the task defined by Kelvin has not yet been accomplished: **Quantum Gravity** still eludes us! The marriage of Quantum Mechanics with Relativity, the final merger and integration of the two "dark clouds", has not been accomplished. Note that it is precisely a conceptual gap which is lurking, as long as we have no theory of Quantum Gravity: Leon Rosenfeld had suggested in the fifties that General Relativity might be left as a classical theory, thus just providing the "arena" for quantum physics (a view which would have given the Copenhagen interpretation a "fundamental" status). However, it was immediately shown that one could then have easily violated and neutralized Heisenberg's Uncertainty Relations, by using measurements based on Gravity. **This is thus the unresolved leftover from Kelvin's 1900 "project", even after almost one hundred years!** If we do not manage to formulate the correct theory of Quantum Gravity by the year 2000, we shall have failed in clearing up the **"Nineteenth Century Clouds"** of Kelvin's title, humbly passing on the task to the 21st....

What is the situation with Quantum Gravity? First, we know for sure that Einstein's Lagrangian is not (perturbatively) renormalizable. 't Hooft and Veltman had evaluated in 1974 [2] the gravitational one-loop amplitude in the vacuum and had found that it is finite. There was therefore a suspicion (or should we say a hope?) that even though dimensional arguments made it most probable that the theory would diverge, some miraculous hidden symmetry might yet cancel the infinities, as had happened for the one-loop case. However, the calculation of the two-loop diagram was very complicated. It was thus only ten years later, when Goroff and Sagnotti [3] had access to a supercomputer and evaluate the two-loop diagram, that we could have an

answer: There is no miracle and the end result is plain infinity..

There are two possible paths one might follow:

- a) Keep Einstein's Lagrangian, but use a non-perturbative approach (and prove that the theory has finite quantum solutions); this is the approach followed by Ashtekar [4], for example, who will speak about it at this meeting; as a matter of fact, his method is the great hope for this approach at this moment and the candidate solutions [5] appear to be very interesting — but there is as yet no indication with respect to finiteness. As to these “loop space” solutions for the quantum Hamiltonian constraints equations for both Maxwell and Einstein theories, they appear to vindicate Kelvin's vortex atoms, made of “knots tied in the vortex lines of the aether”. An important advantage of Ashtekar's Hamiltonian formulation is that with his variables, General Relativity acquires the structure of a complex Yang-Mills theory [6] of gauged (or local) translations.
- b) Modify the Lagrangian, adding new short range contributions so that Einstein's Lagrangian would reemerge as the long range (i.e. larger than Planck length) effective contribution. Just adding terms quadratic in the curvature already makes the theory renormalizable [7], but it is then not unitary. The results we are presenting in the next sections of this paper are partly motivated by an alternative modification, involving Metric-Affine Gravity [8]. Spacetime is assumed to be non-Riemannian underneath Planck dimensions (or above Planck energies); a Goldstone-Higgs mechanism [9] breaks the $SL(4, R)$ local gauge symmetry, yielding Einstein's Riemannian theory as the low-energy effective theory. We have shown that this model is also renormalizable [9], but we have not yet clarified whether or not it is unitary. Supergravity [10], especially the $N = 8$ version, was another such possible route; the result however has only provided an improvement in the renormalizability features (less divergences) but not a cure. At the end, we may have to start with both a non-Riemannian (perhaps Metric-Affine) geometry and supersymmetry (which would then be far more complicated) or alternatively conformal symmetry (this relates to the ideas presented in the next section).

Another massive attempt to resolve the riddle of Quantum Gravity has been the Quantum Superstring (QSS) [11]. It does provide finite gravitational amplitudes - although the perturbation series yields a divergent sum, most probably just indicating the existence of non-perturbative solutions. Physics (i.e. the true, nonlinearized theory) requires the embedding manifold (“target space” is the professional appellation) to be curved; it can then be regarded as the equivalent of the Einstein Lagrangian with all higher order counterterms, plus a dilaton and an axial-vector interaction, etc. Above Planck mass, it goes over to a symmetry limit corresponding to having

the diffeomorphisms in the 2-dimensional string world sheet turning into an active (global) symmetry [12]. However, it requires 10 (or 26) dimensions. Although the vanishing of anomalies severely constrains the internal (gauged) symmetry to being either $E(8) \times E(8)$ or $Spin(32)$ – at the level of the 10 or 26 dimensions – the further reduction to 4-dimensional spacetime spoils it all – in the sense of introducing almost total freedom. Since Gravity does not stand by itself here – this was the additional appeal of the QSS, the “Theory of Everything” (TOE) – we have to know what precisely happens to the “folded up” dimensions. Since “anything goes”, the TOE loses its predictive power – including a certain arbitrariness in the related Quantum Gravity sector.

Physicists who persist to work on the theory (as a TOE) cherish the hope of developing a String Field Theory. This goes presumably beyond the concept of locality – or generalizes it in a peculiar way – so dear to most participants in this meeting, which emphasizes the achievements of Axiomatic Quantum Field Theory. In String Quantum Field Theory, one would write an equation for the Superstring Functional $\Psi[g_{\alpha\beta}(\zeta^\gamma), X^\mu(\zeta^\gamma)]$ in which the argument X^μ is the embedding coordinate, which is itself a local Quantum Field over the ζ^γ coordinates of the string’s world sheet (or of the world hypersheet for a higher dimensional “extendon” – see next paragraph). The lowest energy solution for Ψ should select the true vacuum [13]. This, however, still appears as a distant dream. Meanwhile, another spin-off relates to **non-critical strings**, i.e. not in the dimensions removing the conformal anomalies; as a result, the 2-dimensional sheet metric $g_{\alpha\beta}(\zeta^\gamma)$ preserves one field component, namely 2-dimensional gravity. The embedding manifold is taken care of by collapsing it to 0 or 1 dimension, treating it as a set of internal degrees of freedom. Non-critical strings have thus been used as a theoretical laboratory for quantum gravity, providing us for the first time with the opportunity to look at “almost physical” (only 2 dimensions!) black hole-dilaton [14] quantum solutions of a gravitational theory [15].

There is also another departure stemming from the work on the string. Strings were the first extended structures to be studied as elementary quantum objects, i.e. one-dimensional “**extendons**”. They contain this spin 2 massless state and therefore caused all this excitement. However, may be this is a red herring; should we go on to higher-dimensional extendons [16]? This route has begun to be explored; one model happens to represent a quantum membrane whose low energy truncation is 11-dimensional Supergravity [10] (or $N = 8$ in $D = 4$) which had generated such expectations in 1976-82.

Note that I have dealt here with the operational issues and have not touched upon “technical” problems of Quantum Field Theory, such as the adaptation of the local formalism constructed for Minkowski space – to either just curved spacetime or even to non-Riemannian manifolds. Again, should the answer come from the string, we

shall have to deal with third quantization, etc.. and also ensure that the truncation underneath Planck energy yields a local (massless) field theory. However, we are not yet there.

3 New Physics for the 21st Century

Quantum Gravity represents unfinished business from the legacy of the two Nineteenth Century clouds. What about new, Twentieth Century, fresh ones? Here are two such "clouds", plus what is perhaps the tip of an iceberg, somewhat like radioactivity was in 1900.

The two clouds may, as a matter of fact, also relate to Quantum Gravity; if they end up being resolved by it, it will mean that they too are still part of cleaning up the Nineteenth Century sky. The (perhaps new) clouds are known as the **Hierarchy Problem** and the **Mystery of the Cosmological Constant**. The "iceberg" is what is known as the **Higgs Sector** and which YN called, back in 1964, the **Fifth Interaction** [17]; it may also relate to the problem of the "**Generations**" and of the hypothetical **constituents** of quarks and leptons, as we shall see.

Note "en passant" that we still have the unfinished business of GUTs (Grand Unified Theories), after the experimental failure of (at least the direct application of) $SU(5)$ - no proton decay at the 10^{31} years' lifetime level. There is no doubt that we need an algebraically constraining structure, to explain the cancellation of the chiral anomalies between electrically charged leptons and quarks. The candidates are other forms of $SU(5)$, or instead, $SO(10)$ or $E(6)$ - the latter fits with a superstring TOE. YN and collaborators - Thierry-Mieg and Sternberg, and independently Fairlie, have suggested a gauged internal superalgebra [18] as a possible alternative to a GUT. This would be $SU(2)_L \times U(1)_W \subset SU(2/1) \subset SU(7/1)$. The geometric features are interesting, coincide with Quillen's theory of the "superconnection" and are related to Connes' Non-commutative Geometry [19].

The hierarchy issue derives from the renormalization features of scalar fields. These enter into present theories as the spontaneous symmetry breakdown (Higgs) fields. Their interaction with Gravity - or alternatively their interaction with the gauge fields in GUTs - all seem to indicate unstable masses, or masses tending to inflate via radiative corrections [20]. This is catastrophic in a theory that has more than one spontaneous symmetry breakdown, e.g. one at 10^{16} GeV to separate QCD from $SU(2)_L \times U(1)_W$ and one at 100 GeV to break the Electroweak gauge. The lower energy Higgs fields tend to acquire masses at the 10^{16} GeV level too, within a GUT, and as a matter of fact, all Higgs fields appear to climb to Planck energies because of gravity. The answers that have been suggested are either overall Minkowski

Supersymmetry (with the partner particles in the 500-1000 GeV region), or **Technicolor**, i.e. a new type of matter, with “**techniquarks**” making up the Higgs mesons as “**technipions**”, glued by “**technigluons**”. Supersymmetry would simply extend to bosons the zero-mass condition imposed by Chiral Symmetry on fermions, and 100-1000 GeV represent a good approximation to zero, on a 10^{16} GeV scale.. Both solutions would exhibit some samples of new particles around 500-1000 GeV. The answer thus awaits the results of future experiments at the SSC or LHC. Let me express the hope that these machines will indeed be built in this last decade of the century.

Let us now turn to the **cosmological constant**. The present experimental bounds put its density ρ_Λ [21] at $-2\rho_{obs} < \rho_\Lambda < 4\rho_{obs}$, where $\rho_{obs} \simeq 10^{-29} \text{ g/cm}^3$ is the observed mean energy density of the universe. On the other hand, as the Planck values represent the only numerical input in the interactions with gravity, it is not surprising that the calculation of radiative corrections predict for the density of the cosmological constant ($\hat{=}$ the vacuum energy) a value of the order of Planck density,

$$\rho_\Lambda \simeq \rho_{\text{Planck}}, \quad \rho_{\text{Planck}} = \frac{10^{-5} g}{(10^{-33} \text{ cm})^3} \simeq 10^{94} \text{ g/cm}^{-3}.$$

There is thus a discrepancy of 123 orders of magnitude between theoretical and experimental values ... Coleman and others have conceived of various mechanisms [22] that could “make it easier” to generate a ‘baby universe’ than carry a large Λ . (This is in the spirit of color confinement, where emitting a meson serves as an emergency drainage for extra energy in a hadron, thus evading the emission of a quark or other color-carrying particles). Note that as far as the phenomenology goes, it is possible that the experimental value of Λ will end up being around the present observational upper bound and that this vacuum energy will thus resolve the missing-mass problem in cosmology (to which we have as yet not given the status of a “Kelvin cloud” — perhaps we should have...). This would still allow the energy density in the first stages of the Big Bang to be very much larger than the cosmological vacuum energy density (which is constant in time, whereas the energy density decreases very fast with time). Returning to the theoretical versus experimental Λ dilemma, there are problems with the “baby universe” solution and the situation is at best confusing, at worst highly paradoxical — a perfect Kelvin cloud.

We now deal with the “iceberg”. Several of the participants at this meeting are friends who were already “in the field” thirty years ago. They will remember that when Gell-Mann and YN identified $SU(3)$ (the “Eightfold Way”) as the group classifying the hadrons, with highly successful perturbatively derived mass formulae for masses (including both Gell-Mann/Okubo and Coleman/Glashow, etc.), magnetic moments, weak amplitudes, etc. — and roughly symmetric (only slightly broken)

strong-interactions amplitudes, Oakes and Yang [23] raised a pertinent question: How can a strong-interaction symmetry be broken and yet yield successful perturbative results? With unitarity allowing us to cut anywhere and insert a complete set of intermediate states, we can have the strong breaking multiplying itself and propagating all over the group manifold. A year later, YN [17] offered an answer, which is the one that was adopted twenty years later, with the advent of QCD. The answer was that the traditional “Strong Interaction” should be split into two interactions, a non-perturbative one (QCD in the 1973 Standard Model), generating the main characteristics (Regge trajectories, etc...) and a somewhat weaker force, admitting a perturbative description, which YN called “The Fifth Interaction”¹, responsible for $SU(3)$ -breaking (the mass of the s quark), for the muon mass, etc. It thus couples to “flavors” such as strangeness, whereas QCD couples to color. The Fifth has not yet been identified at all, and it could well be that it should couple at the level of the hypothetical constituents of quarks and leptons - for instance through strangeness and muon number representing the presence of a specific charge at that level, with zero value for the first generation fields (u, d, e, ν_e), one for the second (c, s, μ, ν_μ) and two for the third (t, b, τ, ν_τ).

A segregation of $SU(3)$ breaking has thus indeed been adopted, which is why YN described QCD as taking care of 90% of the Strong Interactions only. In the present picture, all quark and lepton masses are inputs - at most they are “blamed” on Yukawa interactions with scalar fields, generally one interaction for each mass.. Clearly, this is just a way of hiding our ignorance. In addition, there are the masses of the Higgs-Goldstone fields involved in breaking $SU(2)_L \times U(1)_W$ and perhaps of yet another Higgs set, at 10^{16} GeV. Together with the Cabibbo angle and other parameters of the Kobayashi-Maskawa mixing matrix, the Fifth has to explain some 15 - 20 parameters.

Another unfinished business is represented by the mystery of the three (or four, the last one with a very massive neutrino) **generations** - perhaps in itself an indication of the existence of a further structure layer with **rishons, quinks, preons**, ... as constituents. There would then also have to be new forces operating at the new level. This is on top of the “GUT” group or supergroup containing the Standard Model symmetries in an irreducible constraining structure, which we mentioned earlier.

As you see, the 21st Century physicists have quite a job in store. Whether or not in the future (do not expect this to happen before another fifty years) there will

¹The appellation “Fifth Interaction” was plagiarized in the Seventies, when for a while it looked as if there is an additional medium range component to Gravity. This appears to have been yet another red herring.

be a “Final Theory” [24] is, in our opinion an open question. Chew thought we had reached it in the Sixties (the Bootstrap) and there was nothing that could be said against the idea from first principles, it had to be tried out... . Hawking’s inaugural lecture at Cambridge in 1979 was entitled “The End of Physics” – he thought Supergravity and the GUT would provide all the answers. Weinberg’s recent book “Dreams of a Final Theory” is based on the hope that the TeV accelerators will do it, but he stresses that this is only a possibility, with the alternatives involving a Pandora’s box at 0.5 – 1 TeV. This would be just as interesting – may be more so, in our view..

So much for the 21st Century. Let us now return to our immediate worries...

4 Metric-affine Spacetime - Potentials and Field Strengths

We now move on to the gravitational theory I mentioned, in explaining that Quantum Gravity program in which one starts with a Post-Riemannian Theory which then reduces to Einstein’s for distances much larger than the Planck length (10^{-33} cm) and for energies below Planck energy (10^{19} GeV). We have hypothesized [8] a Metric-Affine spacetime, i.e. one in which the homogeneous local symmetry is $\overline{GL}(4, R)$, instead of the Lorentz $\overline{SO}(1, 3)$, the inhomogenous local symmetry thus being given by the Affine Group $\overline{A}(4, R)$, instead of the Poincaré group (the overline bar denotes [25] double-coverings, $\overline{SO}(1, 3) = SL(2, C)$, etc.). Should we find that scale invariance is not essential at the primordial stage, we would stay with the unimodular versions $\overline{SL}(4, R)$ and $\overline{SA}(4, R)$.

We start with differential forms [8] in which the α, β indices represent anholonomic ($\hat{=}$ tangent space) variables, carrying the action of anholonomic $\overline{GL}(4, R)$:

potential	deg. p	p- form	A(4,R) variation	field strength	(p+1)-form	Bianchi identity
metric	0	$g_{\alpha\beta}$	$2\epsilon_{(\alpha\beta)}$	nonmetricity	$Q_{\alpha\beta} = -Dg_{\alpha\beta}$	$DQ_{\alpha\beta} = 2R_{(\alpha\beta)}$
coframe	1	ϑ^α	$-\epsilon_\beta^\alpha \vartheta^\beta$	torsion	$T^\alpha = D\vartheta^\alpha$	$DT^\alpha = R_\gamma^\alpha \wedge \vartheta^\gamma$
connection	1	Γ_α^β	$D\epsilon_\alpha^\beta$	generalized curvature $\rightarrow$ both $(\alpha\beta) \& [\alpha\beta]$	$R_\alpha^\beta =$ $d\Gamma_\alpha^\beta + \Gamma_\gamma^\beta \wedge \Gamma_\alpha^\gamma$	$DR_\alpha^\beta = 0$

(4.1)

The matter fields are [25] spinorial manifields Ψ , with given density character w . In the anholonomic presentation, they can be multiplicity-free, e.g. based on

$[(\frac{1}{2}, 0) + (0, \frac{1}{2})]$ and with a highly restrictive field equation [26]. Upon symmetry breakdown to a Riemannian manifold, in the flat tangent space, they reduce to an infinite sum of Lorentz spinors $\frac{1}{2} + \frac{5}{2} + \frac{9}{2} + \dots$. Going over to a holonomic picture [27] they cannot be multiplicity free any longer [26,28,29].

The action of the $\overline{GL}(4, R)$ gauge-covariant derivative can be split in two irreducible pieces: that of the traceless $\overline{SL}(4, R)$ (4-volume preserving) with connection ${}^\dagger\Gamma$ and a rescaling piece involving the trace, represented by the Weyl one-form $Q := (1/4)Q^\alpha{}_\alpha$:

$$D\Psi = d\Psi + {}^\dagger\Gamma_\alpha{}^\beta \wedge \rho(L^\alpha{}_\beta)\Psi + \frac{1}{2}wQ \wedge \Psi. \quad (4.2)$$

For a given Ψ , the generators $L_\alpha{}^\beta$ of $\overline{SL}(4, R)$ act on the fields via their representation $\rho(L_\alpha{}^\beta)$ and $w = \rho(L_\gamma{}^\gamma)$. Note that although $Q_{\alpha\beta}$ is listed in (4.1) as a field strength, it plays the role of a Yang-Mills like potential here, because of an identity relating $\Gamma_{\alpha\beta}$ to metric (the Riemannian piece, given by the Christoffel formula), torsion and nonmetricity [30].

5 The Framework of Metric-Affine Dynamics: Noether Currents and Identities

In a first order formalism we assume that the material Lagrangian 4-form depends generally on Ψ , its gradient $d\Psi$, on the potentials of (4.1) and on their gradients or the relevant field strengths. Minimal coupling would not require the last two, but Pauli-type terms might arise through some additional constraint or at the symmetry breaking stage; also, we may encounter such terms as a result of the redundancy in “potential” p -forms in our list (4.1): $10 + 4 + 16 = 30$, as against e.g. $16 + 4 = 20$ parameters of $\overline{A}(4, R)$. As a matter of fact we know a priori that the status of the metric in a Metric-Affine theory is somewhat contrived. We thus write

$$L = L(g_{\alpha\beta} dg_{\alpha\beta}, \vartheta^\alpha, d\vartheta^\alpha, \Gamma_\alpha{}^\beta, d\Gamma_\alpha{}^\beta, \Psi, d\Psi). \quad (5.1)$$

We define the variational derivative with respect to a p -form — which is identical to a covariant variational derivative,

$$\frac{\delta L}{\delta \Psi} := \frac{\partial L}{\partial \Psi} - (-1)^p d \frac{\partial L}{\partial (\partial \Psi)} = \frac{\partial L}{\partial \Psi} - (-1)^p D \frac{\partial L}{\partial (D\Psi)}. \quad (5.2)$$

The matter currents are the metrical stress-energy $\sigma^{\alpha\beta}$, the canonical energy-momentum Σ_α , and the hypermomentum Δ_β^α , which is asymmetric in α and β . They are defined by

$$\sigma^{\alpha\beta} := 2 \frac{\delta L}{\delta g_{\alpha\beta}} = 2 \frac{\partial L}{\partial g_{\alpha\beta}} + 2 D \frac{\partial L}{\partial Q_{\alpha\beta}}, \quad (5.3a)$$

$$\Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha} = \frac{\partial L}{\partial \vartheta^\alpha} + D \frac{\partial L}{\partial T^\alpha} \quad (5.3b)$$

$$\Delta^\alpha{}_\beta := \frac{\delta L}{\delta \Gamma_\alpha{}^\beta} = \rho(L^\alpha{}_\beta)\Psi \wedge \frac{\partial L}{\partial(D\Psi)} + 2g_{\beta\gamma} \frac{\partial L}{\partial Q_{\alpha\gamma}} + \vartheta^\alpha \wedge \frac{\partial L}{\partial T^\beta} + D \frac{\partial L}{\partial R_\alpha{}^\beta} \quad (5.4)$$

What is meant by gauging the translations is somewhat ambivalent [31]. We have introduced the gauged $\overline{A}(4, R)$ and shall return to it; but we can also explicitly evaluate the variation of the Lagrangian under L_ξ , a Lie-derivative, i.e. a local anholonomized analytical diffeomorphism, also yielding a “translation current” — since local diffeomorphisms are identical to a local gauging of translations, except for the non Lie algebra aspects of Lie derivatives. We obtain the canonical energy-momentum current [32].

$$\begin{aligned} \Sigma_\alpha = e_\alpha \rfloor L - (e_\alpha \rfloor D\Psi) \wedge \frac{\partial L}{\partial D\Psi} - (e_\alpha \rfloor \Psi) \wedge \frac{\partial L}{\partial \Psi} \\ - (e_\alpha \rfloor Q_{\beta\gamma}) \frac{\partial L}{\partial Q_{\beta\gamma}} - (e_\alpha \rfloor T^\beta) \wedge \frac{\partial L}{\partial T^\beta} + D \frac{\partial L}{\partial T^\alpha} - (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \frac{\partial L}{\partial R_\beta{}^\gamma} \end{aligned} \quad (5.3c)$$

where $\rfloor$ implies “interior” product ($\hat{=}$ contracting a 1-form with a “vector-field”, i.e. an anti-derivative operator). The expression in the first line of (5.3c) survives the transition to Special Relativity and, quite generally, loses its third contribution for a 0-form Ψ . This transition to Special Relativity can be accomplished by successively enforcing $Q_{\beta\gamma} = 0$, $T^\beta = 0$, $R_\beta{}^\gamma = 0$ via Lagrange multipliers.

Obtaining Noether identities for curved manifolds corresponds to evaluating the effect of local gauging. Here we gauge $\overline{A}(4, R) = R^4 \ltimes \overline{GL}(4, R)$, where gauging R^4 is equivalent to applying $\text{diff}(4, R)$ as explained. We find

$$\begin{aligned} D\Sigma_\alpha &\equiv (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma - \frac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma})\sigma^{\beta\gamma} + \\ &+ (e_\alpha \rfloor D\Psi) \frac{\delta L}{\delta \Psi} + (-1)^p (e_\alpha \rfloor \Psi) \wedge D \frac{\delta L}{\delta \Psi} + \\ &\simeq (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma - \frac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma})\sigma^{\beta\gamma} \end{aligned} \quad (5.5)$$

$$D\Delta^\alpha{}_\beta + \vartheta^\alpha \wedge \Sigma_\beta - g_{\beta\gamma} \sigma^{\alpha\gamma} \equiv -\rho(L^\alpha{}_\beta)\Psi \wedge \frac{\delta L}{\delta \Psi} \simeq 0. \quad (5.6a)$$

where $\equiv$ and $\simeq$ respectively denote “strong” and “weak” [$\hat{=}$ on mass shell, here requiring satisfaction of the matter field equations $\frac{\delta L}{\delta \Psi} = 0$]. Taking the trace in (5.6a) yields the scale (dilation) identity,

$$D\Delta + \vartheta^\alpha \wedge \Sigma_\alpha - \sigma^\alpha{}_\alpha \equiv -\rho(L^\gamma{}_\gamma)\Psi \wedge \frac{\delta L}{\delta \Psi} \simeq 0. \quad (5.6b)$$

we observe the role of the traces in $\vartheta^\alpha \wedge \Sigma_\alpha$ and $\sigma^\alpha{}_\alpha$, which becomes important when, e.g. for renormalizability purposes, one has to construct “improved” expressions [33]. Moreover, if we define a metrical energy-momentum 3-form

$$\sigma_\alpha := e_\beta \rfloor \sigma_\alpha{}^\beta, \quad (5.7a)$$

the 2nd Noether identity (5.6a) allows us to rewrite it quite generally as

$$\sigma_\beta \equiv \Sigma_\beta + e_\alpha \rfloor D \Delta^\alpha{}_\beta + \rho(L^\alpha{}_\beta) e_\alpha \rfloor (\Psi \wedge \frac{\delta L}{\delta \Psi}) \quad (5.7b)$$

Imposing invariance of metric and connection under Lie derivatives we use the generalized Killing equation for vector fields,

$$\mathcal{L}_\xi g = 0, \quad \mathcal{L}_\xi \Gamma_\alpha{}^\beta = 0 \quad (5.7c)$$

We define a new current 3-form

$$\epsilon_{MA} := \xi^\alpha \Sigma_\alpha + (e_\beta \rfloor \widehat{D} \xi^\gamma) \Delta^\beta{}_\gamma, \quad (5.8a)$$

where

$$\widehat{D} = d + \widehat{\Gamma}, \quad \widehat{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta \quad (5.8b)$$

which is “weakly” conserved,

$$d\epsilon_{MA} \simeq 0 \quad (5.8c)$$

6 Superimposing Conformal Invariance

We now go beyond the general Metric-Affine spacetimes and impose in addition conformal symmetry. We introduce a conformal Killing vector ζ , so that (5.7) is replaced by

$$\mathcal{L}_\zeta g = \omega g, \quad \mathcal{L}_\zeta \Gamma_\alpha{}^\beta = \frac{1}{2} \delta_\alpha^\beta d\omega \quad (6.1a)$$

with the scaling transformations now involving the metrical weight λ ,

$$g \rightarrow g' = \exp(\lambda\omega) g = \Omega^\lambda g$$

where the conformal function $\Omega = \exp\{\omega(x)\}$ is determined by

$$\omega = \frac{1}{2} e_\gamma \rfloor \widehat{D} \zeta^\gamma - \zeta \rfloor Q \quad (6.1b)$$

which is obtained [31] from (6.1a). This modifies (5.8a) which is replaced by

$$d[\epsilon_{MA} - \frac{1}{2} \omega \Delta] \simeq \frac{1}{2} \omega (\vartheta^\alpha \wedge \Sigma_\alpha) \quad (6.2)$$

which becomes an exact conservation law in the case of traceless $\vartheta^\alpha \wedge \dagger \Sigma_\alpha = 0$.

We can also rewrite the expression in the square bracket in (6.2) while separating the traceless pieces $\dagger \Sigma_\alpha$ and $\dagger \Delta^\alpha_\beta$ from the traces in Σ_α and Δ^α_β . Denoting it by ϵ_C we obtain [32],

$$\epsilon_C = \epsilon_{MA} - \frac{1}{2} \omega \Delta = \zeta^\alpha \dagger \Sigma_\alpha + (e_\beta] \widehat{D} \zeta^\gamma) \dagger \Delta^\beta_\gamma + \frac{1}{4} \zeta] (\vartheta^\alpha \wedge \Sigma_\alpha) + \frac{1}{2} (\zeta] Q) \Delta. \quad (6.3)$$

Alternately, using the “strong” form (5.6b) of the local-gauge Noether identities [34], we have been able [8] to derive, for scalar fields, a “strongly improved” version, this time for the metrical energy-momentum tensor current, for scalar fields,

$$\vartheta^\alpha \wedge \dagger \sigma_\alpha = \vartheta^\alpha \wedge \sigma_\alpha + 3 D \Delta = \left[4 U(|\sigma|) - \sigma \frac{\partial U(|\sigma|)}{\partial \sigma} \right] \eta, \quad (6.4)$$

where we use the “improved”

$$\dagger \sigma_\alpha := \sigma_\alpha + \frac{3}{4} e_\alpha] D \Delta \quad (6.4a)$$

in terms of (5.3a) and (5.7a). The potential $U(|\sigma|)$ is a polynomial type, and η is the volume 4-form.

Imposing conformal invariance on a Metric-Affine spacetime is, however, an extremely strong condition. To realize why this is so, we turn to the flat spacetime “orbital” representation of the relevant Noether currents and will use a component notation. For the shears in $sl(4, R)$, we have the currents (underlining denotes the flat case),

$$\underline{\dagger \Delta}^{\alpha(\beta\gamma)} = x^{(\beta} \underline{\tilde{t}}^{\gamma)\alpha} - \frac{1}{4} g^{\beta\gamma} x^\lambda \underline{\tilde{t}}_\lambda^\alpha \quad (6.5a)$$

for the dilations

$$\underline{\Delta}^\alpha = x^\gamma \underline{\tilde{t}}_\gamma^\alpha, \quad (6.5b)$$

and for the proper conformal currents

$$\underline{K}^{\alpha\beta} = (2 x^\beta x^\gamma - g^{\beta\gamma} x^2) \underline{\tilde{t}}_\gamma^\alpha. \quad (6.5c)$$

Here the “improved” $\underline{\tilde{t}}_\gamma^\alpha$ turns out to be the Hodge dual of the improved version of the metric energy-momentum 4-form (5.3a),

$$\underline{\tilde{t}}_\alpha^\gamma := {}^* \dagger \sigma_\alpha^\gamma, \quad \text{with} \quad \dagger \sigma_\alpha^\gamma := \vartheta^\gamma \wedge \dagger \sigma_\alpha. \quad (6.5d)$$

Using this geometrical defining representation, Ogievetsky [35] proved for the Lie algebraic generators (i.e. the $\int d^3x$ integrals) of (6.5), under recurring commutation,

$$[\int \eta_\alpha \dagger \Delta^{\alpha(\beta\gamma)}, \int \eta_\mu K^{\mu\nu}] \dots = \text{diff}(4, R) \quad (6.6)$$

(η_α is the hypersurface 3-form) i.e. the commutation relations never close and generate instead the infinite algebra of analytical diffeomorphisms, with generators,

$$L_\beta^{<p^0 \dots p^3>\mu\nu\rho\kappa} = \int \prod_{i=0}^3 (x^i)^{p_i} \tilde{t}_\beta^\alpha \eta_\alpha \quad (6.7)$$

The algebraic relations are preserved anholonomically. Thus, a metric-affine space-time which admits a conformal symmetry will have its frames locally invariant (in the **active** operational sense) under the group of analytical diffeomorphisms. This result overlaps with the fact that we have included in our affine gauge approach local translations, i.e. active diffeomorphisms, except that whereas the latter are only infinitesimal (their generators do not form a Lie algebra anyhow), the Ogievetsky transformations can be integrated to finite diffeomorphisms, without involving an infinite-component connection, since the algebraic generators are multiplied by constant parameters solely: the local dependence has already been taken care of through the generating Taylor expansion itself.

The emergence of an explicit infinite-dimensional Lie algebra may make it possible to treat conformal fields in 4-dimensions similarly to what is done in the special case of two dimensions. In $n = 2$, there is an infinite-dimensional conformal algebra which is isomorphic to the algebra of analytic two-dimensional diffeomorphisms [36]. In two dimensions, this feature constrains the fields and leads to the highly restrictive “fusion” rules [36], which have recently put 2-dimensional conformal field theory into the focus of interest of statistical mechanics and string theory. Note that the Ogievetsky algebra in 4 dimensions should possess a quantum extension with central charges as in the case of its 2-dimensional analog, the Virasoro algebra. Neither this extension nor the representation theory have been investigated to date.

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