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**USING COMPUTERIZED TOMOGRAPHY TO DETERMINE
IONOSPHERIC STRUCTURES**

**Part II: A Method Using Curved Paths to Increase Vertical
Resolution**

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Abstract

A method is presented to unfold the two-dimensional vertical structure in electron density by using data on the total electron content for a series of paths through the ionosphere. The method uses a set of orthonormal basis functions to represent the vertical structure and takes advantage of curved paths and the eikonal equation to reduce the number of iterations required for a solution. Curved paths allow a more thorough probing of the ionosphere with a given set of transmitter and receiver positions. The approach can be directly extended to more complex geometries.

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Acronyms

CT	computer tomography
GPS	Global Positioning System
HF	high frequency
LANL	Los Alamos National Laboratory
LDRD	Laboratory Directed Research and Development
RF	radiofrequency
SI units	International System of Units
SNL	Sandia National Laboratories
TEC	total electron content
UHF	ultrahigh frequency
UFO	a code consisting of interactive utility routines
VHF	very high frequency
1D, 2D, ...	one dimensional, two dimensional, etc.

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USING COMPUTERIZED TOMOGRAPHY TO DETERMINE IONOSPHERIC STRUCTURES

Part II: A Method Using Curved Paths to Increase Vertical Resolution

Introduction

Our simple example of a horizontally layered ionosphere that is treated in Part I illustrates the problem that straight-line paths in rectangular pixels give only one independent equation regardless of how many transmitters and receiving stations are used.¹ The change to pixels with curved upper and lower boundaries added a second independent equation that is of little help for the multiple unknowns. Here we present a method to treat curved paths that adds additional independent equations to help unfold the structure for simple, horizontally layered ionospheres. If the curved paths give a less-singular tomography matrix, the unfolding is more straightforward and accuracy improves.

At wave frequencies <100 MHz, the curvature of propagation paths in the ionosphere can be appreciable. This curvature makes the path lengths larger, increases the TEC value for each path, and gives a more sensitive diagnostic for the electron density along the path. We expect that the curved paths will increase our capability to unfold the vertical structure in the ionosphere. The added capability allows lower-frequency data to be used for ionospheric reconstruction. With the ability to treat curved paths, possibilities are also created for the transmitters to launch signals at a few frequencies. At the receivers, the signals at each frequency will measure a TEC different from all the others because of the differing paths. Separate detection of TEC for waves with right- and left-circular polarizations at one frequency can also give different TECs when geomagnetic field effects are significant. The added measurements give a more thorough probing of the ionosphere and improve resolution of the distribution of electron density in the ionosphere.

Horizontally layered ionosphere, altitude basis functions

We treat the simple case of an electron density that varies only with the altitude, independent of the latitude and longitude. Even though the variations with altitude dominate those with latitude or longitude, this is not a realistic case for the ionosphere; it does allow us to test a capability to unfold the dominant vertical structure that is the most troublesome to reconstruct.

The electron density is expanded in an orthogonal set of basis functions, $H_k(z)$, where $1 \leq k < \kappa$.

$$n_e(z) = \sum_{k=1}^{\kappa} \alpha_k H_k(z) . \quad (1)$$

The z represents the altitude while the α_k are the unknown coefficients to be determined. The $H_k(z)$ are numerical basis functions illustrated in Figure 1. They are orthogonal on the interval $0 \leq z \leq 1000$ km. With z measured in km,

$$\int_0^{1000 \text{ km}} H_k(z) H_{k'}(z) dz = \delta_{kk'} . \quad (2)$$

With a given $n_e(z)$, the coefficients are obtained by integration.

$$\alpha_k = \int_0^{1000 \text{ km}} n_e(z) H_k(z) dz .$$

The basis functions were developed by James A. Secan at Northwest Research Associates, Inc., Bellevue, Washington, to represent the vertical structure in the ionosphere with enough generality to span the variations that have been detected in years of study, while at the same time using as few coefficients as possible to give reasonable accuracy.² At Northwest Research Associates, Secan and co-workers have treated the straight-path case with the additional variation with latitude represented in terms of a Fourier sine and cosine series. Our present motives are less ambitious for horizontal structure, yet more ambitious for the vertical structure--testing for improved detection of vertical structure by using curved paths. The numerical functions provided by Secan have been smoothed by a cubic-spline fit and normalized as in Equation 2 by using a simple

trapezoidal-rule integration scheme.³ Because of the numerics, the orthogonality is sensitive to the integration procedure. Using the simple trapezoidal rule, the integrals in Equation 2 for $k \neq k'$ are presently ≤ 0.0039 rather than being identically zero. Compared to the normalized value 1, the orthogonality error is less than 0.4% and is sufficiently small for present purposes.

Figure 1. Basis functions for vertical structure in the ionosphere, $\kappa=7$.

The ionosphere typically has a peak electron density near 300 km, consistent with each H_k illustrated in Figure 1. The altitude range 150-500 km is called the F region and the altitude of maximum density there is called the F peak. The F region contains the largest values of electron number density that occur in the ionosphere. All of the basis functions in Figure 1 have their maximum variation in this altitude region. Altitudes from 90 to 150 km are called the E region. Ionization below 90 km is called the D region of the ionosphere. Our

present basis functions start at 100 km; they do not model the D region.

In our example of a horizontally layered ionosphere, once the α_k are determined, the electron density is completely specified. In addition, initial guesses are available for the α_k because of prior knowledge of the ionosphere. Each α_k has a reasonable range of variation that adds constraints to the unfolding problem. As an example of guesses appropriate for a Chapman layer having a peak n_e at 350 km, the values $\alpha_k = (1.3333e13, 4.846e12, 2.819e11, -1.936e11, 1.108e12, -5.409e11, -7.610e11)$ yield the electron-density profile illustrated in Figure 2.

Figure 2. A sample electron-density profile approximating a Chapman layer.

As a guide to the procedure being developed, we introduce the matrix equation $R\mathbf{f} = \mathbf{T}$, that was discussed in Part I.¹ $\mathbf{f}$ is the column vector of unknown coefficients to be determined:

$$\mathbf{f} = (\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \dots \quad \alpha_\kappa)^T ,$$

where the superscript T indicates the transpose of the vector. $\mathbf{T}$ is a column vector, with elements equal to the TEC values measured for each of the paths through the ionosphere. R is the tomography matrix, with one row for each path and one column for each of the κ unknowns. In Part I¹ the paths are straight lines, the R matrix is constant throughout the iterations to

improve f , and the unknowns are treated as the electron density inside each pixel. Here the paths are curved, the elements of the tomography matrix vary as the iterations change the elements of f , and the unknowns sought by the iterations are the coefficients, a_k .

Ionospheric Index of Refraction

Treatment of curved propagation paths in the ionosphere requires that we momentarily divert our attention away from the determination of the electron density (n_e) profile (that is our primary goal) and toward the electron-density-driven index of refraction (n) that controls the path. The n and n_e are directly related; once one profile is known, the other can be determined.

The formula for n is often called the Appleton formula. Davies gives a readily accessible derivation.⁴

$$n^2 = 1 - \frac{X}{(1 - iZ) - \frac{Y_T^2}{2(1 - X - iZ)} \pm \left[\frac{Y_T^4}{4(1 - X - iZ)^2} + Y_L^2 \right]^{1/2}} \quad (3)$$

Quantities are defined in the following list.

X is the square of the ratio of electron plasma frequency to frequency

$X = n_e e^2 / (\epsilon_0 m \omega^2) = \omega_p^2 / \omega^2 = 3182.5 n_e / \omega^2$ in SI units.

e = electronic charge = 1.6022×10^{-19} C

m = electron mass = 9.1094×10^{-31} kg

ϵ_0 = permittivity of vacuum = 8.8542×10^{-12} F/m

ω = 2π times the propagating wave frequency

Z = ratio of collision frequency to frequency = ν / ω .

ν = the frequency of electron collisions with heavy particles

$i = (-1)^{1/2}$ (There is an assumed $\exp(i\omega t)$ time dependence in the propagating wave.)

$Y_L = (eB/m) \cos \theta_B / \omega$ = longitudinal gyrofrequency/ ω

θ_B = the angle between the propagation direction and the magnetic field

B = the geomagnetic field, in Tesla

$Y_T = (eB/m) \sin \theta_B / \omega$ = transverse gyrofrequency/ ω

The electron speeds are not relativistic, so the electron rest mass is appropriate in the plasma frequency and gyrofrequency. The wave with the + sign is usually called the ordinary wave

because of properties of the following special case. For propagation orthogonal to B , the plus sign makes n as it would be in an ordinary case without a geomagnetic field. The minus sign refers to the extraordinary wave. As long as $X \neq 1$, the ordinary and extraordinary waves propagate independently; the ordinary wave with its right-circular polarization, and the extraordinary wave with its left-circular polarization.⁴

The n varies with altitude because of the n_e , B , θ_B , and ν variations. At altitudes ≥ 100 km, collision frequencies are $\leq 10^5/s$. For frequencies ≥ 10 MHz, the largest Z can possibly be is 0.0016, with much smaller values as ν decreases with altitude. Here, the Z term is neglected because it is $\ll 1$. In our application, we later find that the X is also small compared to 1.

The magnetic field terms depend on the electron gyrofrequency, eB/m. By taking 0.5×10^{-4} Tesla as an upper limit to the geomagnetic field and 10 MHz as a lower limit to the frequency, $Y_L \leq 0.17$. The Y_L and Y_T terms can be significant.

The electron plasma frequency reaches its peak at the altitude of maximum n_e (typically near 2×10^{12} electrons/m³ at sunspot maximum, daytime conditions). For frequencies > 10 MHz, we find the bound $X < 1.5$. Values of $X > 1$ give an imaginary n , and indicate that the ionosphere is absorbing and reflecting the signal rather than allowing propagation to a satellite receiver above the ionosphere. Thus for propagation problems, $X < 1$. With this bound, we can neglect only the collision frequency:

$$n^2 = 1 - \frac{X}{1 - \frac{Y_T^2}{2(1-X)} \pm \left[\frac{Y_T^4}{4(1-X)^2} + Y_L^2 \right]^{1/2}} \quad (4)$$

For our example, we let the frequency increase beyond 10 MHz and assume that the frequency is large enough so that the magnetic field can be neglected (or equivalently, that our receiver detects only the component with right-circular polarization and that θ_B is near 90°), yet is sufficiently small so that the X term in the numerator is still significant. This retains some dependence $n(z)$ and allows us to test the hypothesis that curved paths increase vertical resolution in ionospheric tomography.

$$n^2 = 1 - \omega_p^2/\omega^2. \quad (5)$$

At $z > 1000$ km, $n_e \leq 10^{10}$ electrons/m³, and $1 \geq n^2 \geq 0.99968$. Thus, at transmitter altitudes near ground level (where the electron density is even smaller) and at a satellite altitude ≥ 1000 km, the index of refraction is taken as $n = 1$. These values furnish convenient boundary values for our tomography problem.

Path geometry

The path geometry is diagrammed in Figure 3. The origin for the r, θ coordinate system is at the center of the earth. The angle θ is measured from the equator and (r, θ) are the coordinates of a point on the ray path. The ray starts at the transmitter point (R_t, θ_t) with path-zenith angle ϕ_0 measured from the $\mathbf{e}_r$ direction. The $\mathbf{e}_r$ and $\mathbf{e}_\theta$ represent the usual unit vectors in the directions of increasing polar coordinates r and θ at the position (r, θ) . The $\mathbf{e}_\theta$ is not shown in the figure; it is orthogonal to $\mathbf{e}_r$ and in the direction of increasing θ . The zenith angle, ζ , is measured at the transmitter as the angle between the vertical and the positions along the path. The $\mathbf{e}_\rho$ and $\mathbf{e}_\zeta$ are unit vectors along the direction from the transmitter to a point (ρ, ζ) on the path and in the direction of increasing zenith angle, ζ . As implied in Figure 3, the path is presently confined to a plane. In addition to r, θ , the path is also specified by the polar coordinates ρ, ζ , with the origin of the polar coordinates at the transmitter.

The altitude is $z = r - R_e$. The earlier assumption that the electron density varies only with z gives several simplifications. A ray starting in the propagation plane specified by the point (R_t, θ_t) and the launch angle ϕ_0 remains in that plane. Figure 3 is drawn as if the ray were in a plane of constant longitude. An arbitrary ray will have exactly the same diagram, except for a new "definition" of the equator. In the examples treated here, all the propagation is treated in 2D, latitude-altitude space. If the vertical resolution of the ionosphere is enhanced by treating curved paths in this planar geometry, the improvement should also apply to a 3-D geometry.

Path length and optical path length

The path length for the propagation path from (R_t, θ_t) to (r, θ) is given by

$$S = \int_0^r ds = \int_{R_t}^r (1 + r^2 \theta'^2)^{1/2} dr , \quad (9)$$

where S represents the path length. For a straight-line path, $S = \rho$ as indicated in Figure 3. The ρ is easily evaluated in terms of r, θ .

$$\rho = [r^2 + R_t^2 - 2rR_t \cos(\theta_t - \theta)]^{1/2} . \quad (10)$$

Evaluation of S by the integration over r in Eq. 9 is less straightforward. In polar coordinates, a straight line is given by

$$r \cos(\theta - \theta_0) = p . \quad (11)$$

The p and θ_0 correspond to the two parameters (y-intercept and slope) that define a straight line in a cartesian system. At the initial point where the transmitter is located,

$$R_t \cos(\theta_t - \theta_0) = p . \quad (12)$$

At the receiver position (r_1, θ_1) ,

$$r_1 \cos(\theta_1 - \theta_0) = p . \quad (13)$$

These two equations can be solved for θ_0 , with back substitution giving the constant p .

$$\theta_0 = \tan^{-1} \frac{R_t \cos \theta_t - r_1 \cos \theta_1}{r_1 \sin \theta_1 - R_t \sin \theta_t} . \quad (14)$$

The equation for the straight-line dependence $\theta(r)$ is $\theta = \theta_0 + \cos^{-1}(p/r)$, or

$$\theta' = \frac{\pm p}{r (r^2 - p^2)^{1/2}} . \quad (15)$$

For $p > 0$, the upper (lower) sign is used when the transmitter at (R_t, θ_t) shown in Figure 3 lies southward (northward) of the satellite receiver. The straight-line path length may now be written as

$$S = \int_{R_t}^r \left[\frac{r^2 - 2p^2}{r^2 - p^2} \right]^{1/2} dr . \quad (16)$$

Equation 16 is an awkward form for direct analytical integration. The result must be the earlier ρ in Eq. 10, where θ is written in terms of r .

$$S = \rho = [r^2 + R_t^2 - 2rR_t\lambda]^{1/2} , \quad (17)$$

where

$$\begin{aligned} \lambda &= \cos[\theta_t - \theta_0 - \cos^{-1}(p/r)] \\ &= (p/r) \cos(\theta_t - \theta_0) + (1 - p^2/r^2)^{1/2} \sin(\theta_t - \theta_0) . \end{aligned}$$

The awkwardness of the integrand suggests that numerical integration is a convenient, versatile method to evaluate curved path lengths when the result is not available from simple rules such as the Law of Cosines used earlier. In the curved-path case, substitution of the correct $\theta'(r)$ in Eq. 9 allows direct numerical integration for S .

The optical path length for the propagation path from (R_t, θ_t) to (r, θ) is given by

$$P = \int_0^S n ds = \int_{R_t}^r n(r, \theta) (1 + r^2 \theta'^2)^{1/2} dr . \quad (18)$$

The optical path length, P , is sometimes called the phase function. In propagating an optical distance P , the phase delay time is $\tau_p = P/c_1$, where c_1 is the speed of light in a vacuum. (The subscript on c distinguishes it from a later variable, also

represented by c because of conventions chosen by other authors.) The propagating wave has the phase dependence $\exp[-ik \int n ds]$, where $k=2\pi/\lambda_v$ (wavelength in vacuum).

We let the path be represented by $r = f(s)$ and $\theta = g(s)$, with

$$(ds)^2 = (dr)^2 + r^2 (d\theta)^2, \quad (19)$$

and

$$\left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 = 1. \quad (20)$$

The optical path length is constrained by Fermat's principle: The optical path length is stationary with respect to infinitesimal variations in path. We examine path variation in $g(s)$ and in $f(s)$ to find resulting constraints on our integrand. We alter the path to $r = f(s)$ and $\theta = g(s) + \epsilon g_1(s)$, where the beginning and end points are not changed (such as for fixed transmitter and receiver positions). $g_1(0) = 0$, $g_1(S) = 0$. The infinitesimal element of path length is given by

$$ds_1^2 = \left(\frac{df}{ds}\right)^2 ds^2 + r^2 \left(\frac{dg}{ds} + \epsilon \frac{dg_1}{ds}\right)^2 ds^2. \quad (21)$$

Keeping only first-order terms in g_1 (Eq. 21) gives:

$$ds_1 = \left(1 + r^2 \epsilon \frac{dg}{ds} \frac{dg_1}{ds}\right) ds. \quad (22)$$

The resulting (first order) variation in optical path length is

$$\delta P = \int_0^S [n(f, g + \epsilon g_1) \left(1 + r^2 \epsilon \frac{dg}{ds} \frac{dg_1}{ds}\right) - n(f, g)] ds. \quad (23)$$

An integration by parts with application of the boundary conditions on g_1 gives

$$\delta P = \int_0^S \epsilon g_1 \left[\frac{\partial n}{\partial \theta} - \frac{d}{ds} \left[\frac{dg}{ds} nr^2 \right] \right] ds. \quad (24)$$

Fermat's principle states that this variation must be zero for arbitrary (but small) values of ϵg_1 . Thus

$$\frac{\partial n}{\partial \theta} = \frac{d}{ds} \left[nr^2 \frac{d\theta}{ds} \right]. \quad (25)$$

Let us apply the same procedure to a path variation where r and θ are both allowed to change: $r = f(s) + \epsilon r f_1(s)$ and $\theta = g(s) + \epsilon g_1(s)$. As before, the beginning and end points are not changed. The resulting first-order variation in optical path length is

$$\delta P = \int_0^S \epsilon_{\theta} g_1 \left(\frac{\partial n}{\partial \theta} - \frac{d}{ds} \left[n r^2 \frac{d\theta}{ds} \right] \right) ds$$

$$+ \int_0^S \epsilon_r f_1 \left(\frac{\partial n}{\partial r} - \frac{d}{ds} \left[n \frac{dr}{ds} \right] + n r \left[\frac{d\theta}{ds} \right]^2 \right) ds .$$

Because this variation must be zero for arbitrary values of $\epsilon_r f_1$ and $\epsilon_{\theta} g_1$, we obtain two equations that constrain the path, Eq. 25 and

$$\frac{\partial n}{\partial r} = \frac{d}{ds} \left[n \frac{dr}{ds} \right] - n r \left[\frac{d\theta}{ds} \right]^2 . \quad (26)$$

Rather than being independent of Eq. 25, Eq. 26 can also be derived by a combination of Eqs. 20 and 25. Just from Eq. 20 we find:

$$\frac{dn}{ds} = \frac{d}{ds} \left[n \left\{ r^2 \left[\frac{d\theta}{ds} \right]^2 + \left[\frac{dr}{ds} \right]^2 \right\} \right] .$$

Expanding the right-hand side, rearranging terms, simplifying, use of Eq. 25 and the rule $dn/ds = \partial n/\partial \theta \, d\theta/ds + \partial n/\partial r \, dr/ds$ leads directly to Eq. 26.

As pointed out by a colleague, D.J. Riley, there is another source for Eqs. 25 and 26.⁵ S. Cornbleet⁶ uses a variational problem in cartesian coordinates and a transformation to polar coordinates to derive the equations. (For sign consistency in Eq. 25, Cornbleet's sign convention must be altered.)

If we apply the variational procedure presented in Eqs. 19-25 to a path alteration of r alone (rather than to θ alone) then the last term is missing in Eq. 26 (the term that involves $d\theta$) and the result is in error. It is indeed easy to be misled.

The two differential equations for $\partial n/\partial \theta$ and $\partial n/\partial r$ represent constraints between the index of refraction and the actual path taken by the propagating waves. We take the case where the index of refraction varies only with altitude, $n = n(r)$. Because $\partial n/\partial \theta = 0$, we have $n r^2 d\theta/ds$ is constant along the path. Earlier we found that the path geometry gives a path-zenith angle given by $\tan \phi = r \, d\theta/dr$. The sine of the path-zenith angle is $\sin \phi = r \, d\theta/ds$. Our constraint becomes

$$c = n r \sin \phi = \text{constant} \quad (27)$$

along the propagation path. An initial value of unit index of refraction at the launch position (R_t, θ_t) and an initial path-zenith angle ϕ_0 gives the constant $c = R_t \sin \phi_0$ that constrains all the other n, r, ϕ along the path. As the ray propagates to altitudes above 1000 km where the n approaches 1, the $r \sin \phi$ approaches the impact parameter, labeled c .

Equation 27 is traditionally found by applying Snell's Law and the Law of Sines to a spherically-symmetric, layered index of refraction. We have found some insight on the constraint by using Fermat's principle.

$$nr \sin \phi = nr^2 d\theta/ds = c . \quad (28)$$

The rate of change of θ with path length is

$$d\theta/ds = \frac{c}{nr^2} . \quad (29)$$

Because $ds = (1 + r^2 \theta'^2)^{1/2} dr$, we have

$$\frac{dr}{ds} = (1 - \frac{c^2}{n^2 r^2})^{1/2} . \quad (30)$$

The element of path length can now be written as

$$ds = [1 + r^2 \theta'^2]^{1/2} dr = [1 - \frac{c^2}{n^2 r^2}]^{-1/2} dr , \quad (31)$$

consistent with the $r\theta' = \tan \phi$ from the path tangent, and $r\theta' = c(n^2 r^2 - c^2)^{-1/2}$ from $d\theta/ds$ and ds/dr .

The replacement of the θ' term (in ds) allows the optical path length to be written as

$$P = \int_0^S n ds = \int_{R_t}^r \frac{n^2 r}{(n^2 r^2 - c^2)^{1/2}} dr . \quad (32)$$

θ' also allows evaluation of the path $\theta = \theta(r)$.

$$\theta' = \frac{d\theta}{dr} = \frac{c}{r(n^2 r^2 - c^2)^{1/2}} , \quad (33)$$

$$\theta - \theta_t = c \int_{R_t}^r \frac{dr}{r(n^2 r^2 - c^2)^{1/2}} . \quad (34)$$

All the components are now in place for a ray-tracing solution for the optical path length. A given initial guess for $n(r)$ with given values of transmitter position (R_t, θ_t) and receiver position (r_1, θ_1) allows $\theta(r)$ to be evaluated once the

constant c is determined. If the solution is unique, only one value of c (corresponding to a unique initial path-zenith angle, $\sin \phi_0 = c/n_0 R_t$) will result in a ray that reaches (r_1, θ_1) . The c may be determined from iteration on the launch angle until the correct ray is discovered. Numerical integration is convenient for the chosen c and $n(r)$ and allows considerable freedom in the choice of r dependence for the index of refraction. Fortunately, the characteristics of the ionosphere and its electron-density distribution limit that freedom.

In this application, after each iteration to find the launch angle for a specific receiver, a second iteration is required to improve on the guess for $n(r)$. This second iteration is an attempt to make the optical path lengths (that are calculated) consistent with the measurements made at the receiver. Once these are consistent, the index of refraction is known as accurately as the receiver data will allow. These nested iterations require considerable computer resources. Even with interactive computers having rapid, impressive graphical displays to aid the iterations, the fun of such searching would soon grow tedious. An additional warning is in order. There are some distributions of $n(r)$ such that no rays will reach a given receiver position. Thus, some regions of space may not be traversed at all by rays from a specific transmitter. Extra care is required in the nested iterations to avoid a fruitless, lengthy search.

In the next section, we follow a method suggested by Enright et al., and try to make one iteration suffice to improve the index of refraction as well as to launch the ray at the correct path-zenith angle.⁷

Determination of n and c from the characteristic function

The basic tomography problem has become: Given values of n at the boundaries and given measurements of $\int n ds$ for a sufficient number of emitter-receiver points on the boundaries, reconstruct the n for all points between the boundaries.

Suppose the optical path lengths were given as an expansion of orthogonal functions, with initial estimates for each of the coefficients. Because of the known optical path length, fundamental optics allows direct evaluation of the constant c , allows direct evaluation of the index of refraction, and

eliminates the iteration to select the correct path. Iteration of the coefficients to make the optical paths most consistent with measurements at the receivers solves the problem. We first examine the fundamental optics and later form the expansion and the iteration procedure.

Born and Wolfe⁸ refer to the optical length of the ray between (R_t, θ_t) and (r_1, θ_1) as the characteristic function.

$$P(\mathbf{r}_0, \mathbf{r}_1) = \int_{R_t}^{r_1} n \, ds . \quad (35)$$

The $\mathbf{r}_0$ specifies the transmitter location (R_t, θ_t) ; $\mathbf{r}_1$ specifies the position on the path S , (r_1, θ_1) . They also introduce the geometrical wavefronts, described by a function $L(r, \theta) = \text{constant}$.⁸ Because the wavefronts are orthogonal to the trajectories,

$$\text{grad } L = n \, \mathbf{e}_S . \quad (36)$$

In polar coordinates, our components of grad are $\mathbf{e}_r \, \partial/\partial r$ and $r^{-1}\mathbf{e}_\theta \, \partial/\partial \theta$. The projection of grad L on S is

$$\frac{dL}{ds} = \mathbf{e}_S \cdot \text{grad } L = n , \quad (37)$$

or

$$dL = n \, ds . \quad (38)$$

The characteristic function can be expressed in terms of the wavefront function:

$$P(\mathbf{r}_0, \mathbf{r}_1) = L(\mathbf{r}_1) - L(\mathbf{r}_0) = \int_{R_t}^{r_1} n \, ds . \quad (39)$$

The wavefront function also leads to the eikonical equation. From Eq. 36,

$$\text{grad } L \cdot \text{grad } L = n^2 . \quad (40)$$

When the grad operator is applied at the launch position, we find

$$\text{grad } P(\mathbf{r}_0, \mathbf{r}) \Big|_{\mathbf{r}=\mathbf{r}_0} = - \text{grad } L(\mathbf{r}_0) = - n \mathbf{e}_s \Big|_{\mathbf{r}=\mathbf{r}_0}. \quad (41)$$

At the other positions on the ray, we find

$$\text{grad } P(\mathbf{r}_0, \mathbf{r}) = + \text{grad } L(\mathbf{r}) = + n \mathbf{e}_s \quad (42)$$

and

$$\text{grad } P \cdot \text{grad } P = + n^2. \quad (43)$$

In common with Eq. 40, Eq. 43 is also referred to as the eikonical equation.

With a known $P(\mathbf{r}_0, \mathbf{r})$ and a known $n(\mathbf{r}_1)$, the equation

$$\text{grad } P(\mathbf{r}_0, \mathbf{r}_1) = n \mathbf{e}_{s1} \quad (44)$$

gives an immediate estimate of $\mathbf{e}_s$ at the receiver position (r_1, θ_1) . Because $\cos \phi_1 = \mathbf{e}_r \cdot \mathbf{e}_{s1}$ and because $n(r_1) = 1$, $n(r_1)r_1 \sin \phi_1$ is evaluated to obtain the constant c . If the transmitter point is used to evaluate c , we find that the unit vector $\mathbf{e}_s$ is also available from

$$\text{grad } P(\mathbf{r}_0, \mathbf{r}) \Big|_{\mathbf{r}=\mathbf{r}_0} = - n \mathbf{e}_s \Big|_{\mathbf{r}=\mathbf{r}_0}. \quad (45)$$

At $r=R_t$, $\theta = \theta_t$, $\sin \phi_0$ is also estimated, c follows as $n(R_t)R_t \sin \phi_0 = c$ where $n(R_t) = 1$. Because (R_t, θ_t) and (r_1, θ_1) are sufficiently far from the distribution being probed so that $n(R_t) = 1$ and $n(r_1) = 1$, we have $R_t \sin \phi_0 = r_1 \sin \phi_1$. Figure 3 labels this constant as the impact parameter, c .

Once the characteristic function is known, we also know the index of refraction from the eikonical equation. Thus the fundamental optics coupled with the expansion for the characteristic function have given us the index n and the initial and final path-zenith angles. The correct path (for the assumed coefficients and the resulting $n(r)$) is selected automatically. The path length is also readily evaluated. Direct integration of the θ' equation (Eq. 33) with the now

known $n(r)$ and c give the relation $\theta = \theta(r)$ for $R_t \leq r \leq r_1$. This is the type of expansion suggested by Reference 7. The major remaining tasks are to find a convenient form for the expansion of the characteristic function and to formulate the iteration procedure.

Expansion form for the characteristic function

The geometry is illustrated in Figure 3 on page 14. We seek an expansion to describe the characteristic function, $P(r_0, r)$, for paths originating at r_0 and terminating at an arbitrary position r . The r_0 specifies the transmitter location (R_t, θ_t) . The r values of interest are in the range of R_t to r_1 and $0 \leq \theta \leq \pi/2$. The maximum r is the value r_1 at the receiver position (r_1, θ_1) . Our assumption that $n = n(r)$, independent of θ , gives azimuthal symmetry about the r_0 axis. The expansion represents the characteristic function in the constant-longitude plane of propagation. As a boundary condition, we have $P(r_0, r_0) = 0$. It would also be convenient if the form of the expansion could take advantage of the index of refraction being independent of θ with $n(R_t) = 1$ and $n(r_1) = 1$. The characteristic function also has the limiting value $P = \rho$ as $n(r)$ approaches 1.

Because of the limiting forms, we use an expansion in terms of the (ρ, ζ) coordinates in Figure 3 rather than in terms of (r, θ) . The following expansion has some appropriate properties.

$$P(\rho, \zeta) = \rho + \sum_{i=2}^I \rho^i \sum_{m=0}^M B_{i,m} P_m(\cos \zeta) . \quad (46)$$

At $\rho = 0$ we have $n = 1$ and $P(0, \zeta) = 0$. The symmetry in ζ is automatic, $P(\rho, \zeta) = P(\rho, -\zeta)$. The coefficients $B_{i,m}$ represent the effects of the index of refraction; as $n(\rho, \zeta) \rightarrow 1$ the $B_{i,m}$ become smaller, so $P(\rho, \zeta) \rightarrow \rho$. The expansion represents the optical path length from the transmitter to an arbitrary observation point (ρ, ζ) as an I -th order polynomial. Negative exponents of ρ are omitted to meet the $P(0, \zeta) = 0$ boundary condition. The $P_m(\cos \zeta)$ Legendre polynomials are a complete set of functions on $-1 \leq \cos \zeta \leq 1$. So the expansion has sufficient generality to represent the optical path lengths. In our example where $\partial n / \partial \theta = 0$, the expansion coefficients are the same for paths from each of the transmitters to the

corresponding identical $\theta_i - \theta_t$ difference in angles. A more general case would have coefficients that vary with each choice of transmitter position.

An index of refraction close to 1 makes the ρ term dominant in the expansion. The other terms are expected to be small corrections. The components of the gradient are:

$$\partial P / \partial \rho = 1 + \sum_{i=2}^I i \rho^{i-1} \sum_{m=0}^M B_{i,m} P_m(\cos \zeta) , \quad (47)$$

$$\frac{1}{\rho} \frac{\partial P}{\partial \zeta} = \sum_{i=2}^I \rho^{i-1} \sum_{m=0}^M B_{i,m} \frac{\partial}{\partial \zeta} P_m(\cos \zeta) . \quad (48)$$

The Legendre Polynomials obey the differential equation

$$\frac{1}{\sin \zeta} \frac{\partial}{\partial \zeta} \left[\sin \zeta \frac{\partial P_m}{\partial \zeta} \right] = -m(m+1) P_m(\cos \zeta) . \quad (49)$$

This equation is customarily expressed in terms of $x = \cos \zeta$ instead of ζ itself. It has the form

$$\frac{d}{dx} \left[(1-x^2) \frac{\partial P_m}{\partial x} \right] = -m(m+1) P_m(x) . \quad (50)$$

The first few polynomials are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2-1)/2$, $P_3(x) = (5x^3-3x)/2$, $P_4(x) = (35x^4-30x^2+3)/8$. The polynomials also have the following convenient properties: $P_m(1) = 1$, $P_{2m+1}(0) = 0$,

$$\int_{-1}^1 P_m'(x) P_m(x) dx = \frac{2}{2m+1} \delta_{mm'} , \quad (51)$$

$$(x^2-1) \frac{dP_m}{dx} = mxP_m - mP_{m-1} . \quad (52)$$

Examples of characteristic functions.

The expansion chosen for the characteristic function is most useful if a few terms give an adequate representation. Here we examine a specific example to determine if limitations to the representation are easily discovered. We chose a single transmitter problem illustrated in Figure 4. The transmitter is located at $\theta_t = 57^\circ N$ latitude. Two paths are traced toward

satellites at 1000-km altitude, the first launched toward a satellite directly overhead and a second launched toward a satellite at 59°N latitude. The ionosphere is treated as a Chapman layer:

$$n_e = n_{\max} \exp[(1 - z' - e^{-z'})/2], \quad (53)$$

where $n_{\max} = 10^{12}$ electrons/m³, $z' = (h - 350 \text{ km})/75 \text{ km}$, and where h is the altitude. This electron density is illustrated in Figure 5a. The peak density occurs at 350 km. The corresponding index of refraction is taken from Eq. 5 with frequency at 10 MHz, and illustrated in Figure 5b. Data for the other frequencies in Figure 5b suggest that frequencies from 10 to 100 MHz span a range of curved paths that may improve vertical resolution in CT reconstruction of the ionosphere.

Figure 4. Path geometry for two paths through a Chapman layer (frequency = 10 MHz)

Figure 5. Chapman-layer model of (a) the electron density and (b) the corresponding index of refraction

In Figure 4, the launch directly overhead gives a straight-line path directly to the satellite receiver. The path directed toward the second receiver is deviated as it reaches the gradient in index of refraction and misses the "intended" receiver.

Numerical integration of Eq. 32 gives the optical path length, P , as a function of the corresponding ρ (from Eq. 17). P is illustrated in Figure 6 for the second path in Figure 4. The dashed curve in Figure 6 is the path length evaluated by numerical integration of Eq. 9, with θ' given by Eq. 33.

Figure 6. Optical path length vs ρ for path 2 example in Figure 4. The dashed curve is the path length, S .

To aid judgement about the adequacy of Eq. 46 to represent the optical path length, Figure 7 illustrates the difference between the optical path length and ρ . The few inflection points encountered suggest that a rather low-order polynomial in Eq. 46 will adequately represent the optical path length. A more involved structure in Figure 5 as well as addition of n_e variations with latitude will doubtless add to the complexity. However, the need for more than 100 unknowns is not anticipated. (In Part I, the number of unknowns to be determined is typically >100 and at times $>>100$.) In Figure 6, for example, a third- or perhaps a fifth-order polynomial should suffice.

Figure 7. Difference between ρ and the optical path length for the path 2 example in Figure 4

Expected form of expansion

There are several possible starting points that can provide an expansion for the optical path length. With help from Eq. 28, for example, Equation 32 can be expanded to form

$$P = \int_{R_t}^r \frac{n \, dr}{(1 - c^2/n^2 r^2)^{1/2}} = \int_{R_t}^r \frac{n \, dr}{(1 - \sin^2 \phi)^{1/2}} \quad (54)$$

$$P = \int_{R_t}^r n \left[1 + \frac{1}{2} \sin^2 \phi + \frac{3}{8} \sin^4 \phi + \frac{5}{16} \sin^6 \phi + \frac{35}{128} \sin^8 \phi + \dots \right] dr . \quad (55)$$

This form is convenient for treatment of the $n(h)$ dependence (because $h = r - R_e$ and because the integration is over r). The form is also simple, with odd powers of $\sin \phi$ not contributing. However, the $\phi(r)$ dependence is not simple, nor is the leading term equal to ρ , as needed for direct comparison with Eq. 46.

For direct comparison with Equation 46, as a second alternative we chose an expansion in terms of the ρ, ζ coordinates.

$$P = \int_0^{\rho} n [1 + \rho^2 (d\zeta/d\rho)^2]^{1/2} d\rho . \quad (56)$$

Each separate path through the ionosphere has its unique $\rho \, d\zeta/d\rho$ variation with ρ . The expansion in a binomial series then follows directly. With $y = \rho^2 (d\zeta/d\rho)^2$,

$$P = \int_0^{\rho} n [1 + y/2 - y^2/8 + y^3/16 - 5 y^4/128 + \dots] d\rho . \quad (57)$$

Because of the anticipated small variation of ζ with ρ , y should be small and the expansion reasonable. At the transmitter, $\rho = 0$ and $n(\rho, \zeta) = 1$. Thus, P has the leading term nearly equal to ρ (for n near 1), consistent with Equation 46. Figure 8 illustrates the $\rho \, d\zeta/d\rho$ variation for path 2 in Figure 4. The corresponding maximum value of y is about $(0.28)^2 = 0.078$. This small value indicates that the expansion is appropriate. When the wave frequency is increased from 10 MHz to 15 MHz, the maximum y is reduced to 0.0023. With proper adjustment of the frequency, y may be treated as a small parameter in the expansion for any realistic ionosphere. As the frequency gets larger, the y approaches zero, the paths become straight lines, and we lose any added resolution available from curved paths.

Figure 8. $\rho \, d\zeta/d\rho$ variation along path 2 shown in Figure 4.

Equation 57 and Equation 46 both represent the characteristic function at an arbitrary point (ρ, ζ) for the path originating at the transmitter and terminating at (ρ, ζ) . The two equations must be equivalent. If coefficients could be matched, it would be possible to make more informed initial guesses for the coefficients $B_{i,m}$ in Equation 46. The improved guesses should then speed convergence to values that best match the experimental data. Such coefficient matching is difficult. In the next two paragraphs we describe a procedure that illustrates the difficulty and then offer a more promising alternative.

From Equation 5,

$$n = (1 - \omega_p^2/\omega^2)^{1/2} \quad (58)$$

At large frequencies, expansion gives

$$n = 1 - \frac{1}{2} \frac{\omega_p^2}{\omega^2} - \frac{1}{8} \frac{\omega_p^4}{\omega^4} - \frac{1}{16} \frac{\omega_p^6}{\omega^6} - \frac{5}{128} \frac{\omega_p^8}{\omega^8} - \dots \quad (59)$$

The plasma frequency is given by

$$\omega_p^2 = \frac{n_e e^2}{\epsilon_0 m} \quad , \quad (60)$$

and the electron density is represented by Eq. 1 with its unknown coefficients to be determined. The peak n_e expected is about 2×10^{12} electrons/m³, corresponding to a $\omega_p(\text{max}) = 7.978 \times 10^7$ radians/s, or to a maximum plasma frequency of 12.70 MHz. Smaller values are much more common. The expansion for n in Equation 59 can be inserted into Equation 57. The altitude dependence of the electron density in Eq. 1 can be converted to a ρ, ζ dependence by using the Law of Cosines and solving for h (see Figure 3, page 14).

$$r^2 = (R_e + h)^2 = R_t^2 + \rho^2 + 2 R_t \rho \cos \zeta \quad . \quad (61)$$

$$h = (R_t^2 + \rho^2 + 2 R_t \rho \cos \zeta)^{1/2} - R_e \quad . \quad (62)$$

The variation of y (in Equation 57) with ρ, ζ must also be included to match the expansion coefficients with those in Equation 46. From Figure 3,

$$\zeta = \beta + \theta - \theta_t . \quad (63)$$

With $\mathbf{e}_s$ representing a unit vector tangent to the path at position (r, θ) that is equally well described as (ρ, ζ) , we have

$$\mathbf{e}_s = \frac{dr}{ds} \mathbf{e}_r + r \frac{d\theta}{ds} \mathbf{e}_\theta , \quad (64)$$

$$\mathbf{e}_s = \frac{d\rho}{ds} \mathbf{e}_\rho + \rho \frac{d\zeta}{ds} \mathbf{e}_\zeta . \quad (65)$$

From the geometry illustrated in Figure 3,

$$\cos\phi = \mathbf{e}_s \cdot \mathbf{e}_r = dr/ds \quad (66)$$

$$\sin\phi = \mathbf{e}_s \cdot \mathbf{e}_\theta = r \, d\theta/ds \quad (67)$$

$$\tan\phi = r \, d\theta/dr \quad (68)$$

$$\cos(\beta - \phi) = \mathbf{e}_s \cdot \mathbf{e}_\rho = d\rho/ds \quad (69)$$

$$\sin(\beta - \phi) = \mathbf{e}_s \cdot \mathbf{e}_\zeta = \rho \, d\zeta/ds \quad (70)$$

$$\tan(\beta - \phi) = \rho \, d\zeta/d\rho = y^{1/2} \quad (71)$$

$$\mathbf{e}_\rho = \mathbf{e}_r \cos\beta + \mathbf{e}_\theta \sin\beta \quad (72)$$

$$\mathbf{e}_\zeta = -\mathbf{e}_r \sin\beta + \mathbf{e}_\theta \cos\beta \quad (73)$$

Expansion of the tangent of the difference between two angles, coupled with Equation 63 converts Equation 71 to the form

$$y^{1/2} = \frac{\tan(\zeta - \theta + \theta_t) - \tan\phi}{1 + \tan\phi \tan(\zeta - \theta + \theta_t)} \quad (74)$$

and

$$\tan(\zeta - \theta + \theta_t) = \frac{\tan\zeta - \tan(\theta - \theta_t)}{1 + \tan\zeta \tan(\theta - \theta_t)} . \quad (75)$$

The Law of Sines leads to the form

$$\sin\zeta = \frac{r \sin(\theta - \theta_t)}{\rho} . \quad (76)$$

Thus, for a given ρ, ζ and for a given transmitter location (R_t, θ_t) , r and h are calculated from Eqs. 61-62; θ is available from Eq. 76; β from Eq. 63; ϕ from Eq. 8 and Eq. 33; finally, y is calculated in terms of the given ρ, ζ by using Eq. 74. Our goal is to use y variation with ρ to expand Eq. 57 and match coefficients with Eq. 46. For this procedure to be followed, we

also need estimates of the n^2 and c because of our use of Eq. 33. With all these substitutions it is possible to expand the characteristic function in Eq. 57 in terms of ρ and ζ . The result is not expected to be enlightening and has not been pursued. The main difficulty results from the $n(\rho, \zeta)$ along the path S not remaining at 1 for the leading term in the expansion.

As an alternate route to estimating coefficients in Eq. 46, Eq. 44 can be used,

$$\text{grad } P(r_0, r_1) = n \mathbf{s}_1 \quad .$$

At the receiver position (r_1, θ_1) , the unit vector tangent to the path is $\mathbf{s}_1$. There, $n = 1$ and

$$\mathbf{s}_1 = \frac{\partial P}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial P}{\partial \zeta} \mathbf{e}_\zeta \quad . \quad (77)$$

Because the transmitter and receiver positions are known, $\mathbf{e}_\rho$ and $\mathbf{e}_\zeta$ are available at these positions. The β for use in Eqs. 72-73 is again available from Eq. 63. From Eqs. 47, 48, 69, 70, and 77 at the receiver position, the components of $\mathbf{s}_1$ are

$$\frac{\partial P}{\partial \rho} = \cos(\beta - \phi) = 1 + \sum_{i=2}^I \rho^{i-1} \sum_{m=0}^M B_{i,m} P_m(\cos \zeta) \quad , \quad (78)$$

$$\frac{1}{\rho} \frac{\partial P}{\partial \zeta} = \sin(\beta - \phi) = \sum_{i=2}^I \rho^{i-1} \sum_{m=0}^M B_{i,m} \frac{\partial}{\partial \zeta} P_m(\cos \zeta) \quad . \quad (79)$$

Equations 78-79 can be applied to an arbitrary number of receiver positions ρ_k, ζ_k , all chosen at altitudes such that $n = 1$. At each position, β and ζ are unique and available. The ϕ can be estimated at each position. If ϕ were actually known at the receiver, then c is available and direct ray tracing could occur with iterations on coefficients a_k in Equation 1 to improve the match to the data. With a sufficient number of such chosen positions, all the $(I-1)*(M+1)$ unknown coefficients $B_{i,m}$ can be estimated. Subsequent iteration of the coefficients is expected to improve these estimates and give the best match to the experimental data.

As we found earlier, once the coefficients are known, the actual ϕ_1 , c , and n are directly available. Equation 44 gives the direction $\mathbf{s}_1$ at the receiver. Equation 66 gives the corresponding ϕ_1 . Applying Eq. 27 at the receiver

gives $c = r_1 \sin \phi_1$. The eikonical Eq. 43 gives n . The ray tracing can proceed for comparison with the experimental data.

Another alternative is available to provide initial guesses for the coefficients. The previous method relates the coefficients to the optical path-length integral. The eikonical Eq. 44 gives

$$n^2 = (\partial P / \partial \rho)^2 + \frac{1}{\rho^2} (\partial P / \partial \zeta)^2 . \quad (80)$$

From Equations 78-79,

$$n^2 = \left[1 + \sum_{i=2}^I i \rho^{i-1} \sum_{m=0}^M B_{i,m} P_m(\cos \zeta) \right]^2 + \left[\sum_{i=2}^I \rho^{i-1} \sum_{m=0}^M B_{i,m} \frac{\partial}{\partial \zeta} P_m(\cos \zeta) \right]^2 . \quad (81)$$

Equation 81 is valid at any point along the path as well as at the receiver. Expansion of Eq. 81 gives

$$n^2 = 1 + 2 \sum_{i=2}^I i \rho^{i-1} \sum_{m=0}^M B_{i,m} P_m(\cos \zeta) + \left[\sum_{i=2}^I i \rho^{i-1} \sum_{m=0}^M B_{i,m} P_m(\cos \zeta) \right]^2 + \left[\sum_{i=2}^I \rho^{i-1} \sum_{m=0}^M B_{i,m} \frac{\partial}{\partial \zeta} P_m(\cos \zeta) \right]^2 . \quad (82)$$

In our example, n^2 is independent of θ , but not independent of ρ or ζ . $n = n(h) = n((R_t^2 + \rho^2 + 2R_t \rho \cos \zeta)^{1/2} - R_e)$. In Eq. 82, the squared terms are higher order and can be neglected when making initial guesses for the coefficients. Combining Eqs. 1, 5, and 82 gives the electron density,

$$- \frac{2 \epsilon_0 m \omega^2}{e^2} \sum_{i=2}^I i \rho^{i-1} \sum_{m=0}^M B_{i,m} P_m(\cos \zeta) = \sum_{k=1}^{\infty} \alpha_k H_k(z) . \quad (83)$$

Initial guesses for α_k can take advantage of any previous knowledge of the ionosphere. Then the choice of a sufficiently large number of positions ρ, ζ and use of Eq. 83 will give a series of equations that can be solved for the $(I-1) \cdot (M+1)$ unknown values of $B_{i,m}$. These initial guesses for $B_{i,m}$ can then be inserted into the iteration scheme to improve the match

between (1) the experimental data and (2) the prediction evaluated from the coefficients.

Once the $B_{i,m}$ are determined from the iteration procedure, Eq. 83 can be applied at $\zeta = 0$,

$$\sum_{k=1}^{\kappa} \alpha_k H_k(z) = - \frac{2\epsilon_0 m \omega^2}{e^2} \sum_{i=2}^I i z^{i-1} \sum_{m=0}^M B_{i,m} . \quad (84)$$

Multiplication by H_k' and integration over $0 \leq z \leq 1000$ km gives

$$\alpha_k = - \frac{2\epsilon_0 m \omega^2}{e^2} \sum_{i=2}^I i B_i \int_0^{1000 \text{ km}} z^{i-1} H_k(z) dz , \quad (85)$$

where

$$B_i = \sum_{m=0}^M B_{i,m} . \quad (86)$$

Applying a numerical integration for each k in the interval $1 \leq k \leq \kappa$ yields the required coefficients to specify the electron density within the ionosphere.

One perturbation to this last alternative is to iterate on the α_k values rather than on the coefficients $B_{i,m}$. With a given set α_k for $1 \leq k \leq \kappa$, Eq. 83 can be applied to enough points in the distribution to solve for all the coefficients $B_{i,m}$. Back substitution into Equation 82 furnishes a check on the validity of omitting the higher-order terms when using Eq. 83.

Iteration Procedure

We formulate the problem as a matrix unfolding problem, just as in Part I.¹

$$Rf = T$$

T is the column of TEC measurements for each path. f is the column of unknown coefficients in Equation 1. R is the tomography matrix. The following steps outline the iteration procedure.

Step 1. Choose values of I , M , and ω for use in Eq. 83. The κ is fixed at $\kappa = 7$ for the seven basis functions in Figure 1.

Step 2. Guess at the coefficients α_k , using any knowledge of ionospheres from past studies.

Step 3. Evaluate the $(I-1)*(M+1)$ values of $B_{i,m}$ in Equation 83 by picking a series of positions (ρ, ζ) with their corresponding z values and solving the series of linear equations. Positions are chosen at $M+1$ or more differing ζ values on the interval $0 < z < p/2$, and at $I-1$ or more differing ρ values so all the $B_{i,m}$ are determined. The procedure for finding the $B_{i,m}$ is analogous to the basic unfolding problem, $R_B f_B = T_B$. Here the elements of f_B are the individual $(I-1)*(M+1)$ values of $B_{i,m}$. Each choice of (ρ, ζ) fixes the element of

$$T_B = \sum_{k=1}^{\kappa} \alpha_k H_k(z)$$

and fixes the set of coefficients of the $B_{i,m}$. If enough points are chosen, UFO³ generates a least-square solution for the $B_{i,m}$. Care is needed in the choice of ζ values.

Step 4. (Optional) After the initial values of $B_{i,m}$ are found in Step 3, the values may be refined by using the full Equation 82, with the initial values used in the higher-order terms (to keep the problem linear). The series of positions can be used to generate a refined set of linear equations--where the previous estimates for $B_{i,m}$ are used to evaluate the bracketed terms that are squared. The $B_{i,m}$ in the first summation are evaluated from the refined set of linear equations. This iteration process can be repeated until convergence of the $B_{i,m}$ values. In some problems, the bracketed and squared terms are small, and the first-order estimate in Eq. 83 will suffice.

Step 5. (Optional) In the ρ, ζ space above the transmitter, compare the n_e evaluated from Equation 1 with the representation by the coefficients $B_{i,m}$ in Equation 83. If the representation is inadequate, increase the I or M to improve resolution and return to Step 1.

Step 6. From Eqs. 44, 47, and 48, find e_{s1} at receiver 1.

Step 7. Using Eq. 66, find ϕ_1 at receiver 1, and find $c = n_1 r_1 \sin \phi_1$. The n_1 is available from Eqs. 1 and 5.

Step 8. Numerically integrate $\int (n^2 - 1) ds$ along the path from the transmitter to the receiver.

$$\int_0^S (n^2 - 1) ds = - \frac{e^2}{\epsilon_0 m \omega^2} \int_{R_t}^{r_1} \frac{n_e dr}{(1 - c^2/n^2 r^2)^{1/2}}$$

$$\int_0^S (n^2 - 1) ds = - \frac{e^2}{\epsilon_0 m \omega^2} \sum_{k=1}^{\kappa} \alpha_k \int_{R_t}^{r_1} \frac{H_k(z) dr}{(1 - c^2/n^2 r^2)^{1/2}} .$$

For path 1, the path index $j = 1$ and

$$TEC = \int_0^S n_e ds$$

$$TEC = - \frac{\epsilon_0 m \omega^2}{e^2} \int_0^S (n^2 - 1) ds = \sum_{k=1}^{\kappa} R_{jk} \alpha_k .$$

$$R_{jk} = \int_{R_t}^{r_j} \frac{H_k(z) dr}{(1 - c^2/n^2 r^2)^{1/2}} .$$

In this step, the ray tracing requires the present guess of the $n(r)$ variation. The n^2 might be evaluated from the α_k , from the coefficients $B_{i,m}$ and Eq. 82, or from the first two terms in Eq. 82. The last option is consistent with the estimates of ϕ_1 and c ; giving assurance that the ray tracing will ultimately reach the receiver position. However, here we are merely integrating over dr , not ray tracing where (r, θ) values would be tabulated along the path. If optional steps 4 and 5 give favorable comparisons, the choice of n^2 evaluation scheme should make no difference. Using the α_k and Eq. 1 requires the least amount of calculation.

Step 9. Repeat Steps 6, 7, 8 for paths from the transmitter to each remaining receiver position where data are taken. This generates a value of each element in the R_f vector.

Step 10. Form the residual vector, make corrections to the α_k coefficients. The residual vector has components s_i . Each component is the difference between the data element T_i for path i and its current estimate,

$$s_i = T_i - \langle R_i T_f \rangle .$$

R_i^T is the row vector ($R_{i1} \ R_{i2} \ \dots \ R_{i\kappa}$) that forms the i th row of R .

The coefficients α_k are composed of a current estimate, α_{kc} , plus a small correction term, δ_k .

$$\alpha_k = \alpha_{kc} + \delta_k$$

The electron density becomes

$$n_e(z) = \sum_{k=1}^{\kappa} \alpha_{kc} H_k(z) + \sum_{k=1}^{\kappa} \delta_k H_k(z) .$$

Integrating over the path length gives

$$TEC = \int_0^S n_e(z) \, ds = \sum_{k=1}^{\kappa} \alpha_{kc} \int_0^S H_k(z) \, ds + \sum_{k=1}^{\kappa} \delta_k \int_0^S H_k(z) \, ds .$$

$$TEC = \sum_{k=1}^{\kappa} R_{jk} \alpha_{kc} + \sum_{k=1}^{\kappa} \delta_k \int_0^S H_k(z) \, ds .$$

The first summation term is the current estimate of the TEC for path j .

$$TEC - TEC_{est} = \sum_{k=1}^{\kappa} \delta_k \int_0^S H_k(z) \, ds$$

Each path gives an equation relating the κ values of δ_k . Each set of κ independent paths gives a unique determination of all the δ_k . Solve for average values of each δ_k from such a solution. Add the corrections to the current estimates of α_k , repeat the iteration.

Step 11. If converged, present results for n_e . If not converged, take the new α_k and report to Step 3.

Summary and Conclusions

We have presented a method for 2-D tomographic reconstruction of the electron density in the ionosphere. The method takes advantage of curved wave propagation paths and benefits from the choice of an orthonormal set of basis functions for expansion of the vertical electron-density profile. A direct solution assumes an electron-density profile and iterates on launch angle to find the propagation path that reaches the receiver position where TEC data are recorded. An outer iteration over the assumed electron density profile gives the best fit to the TEC data. A method has been presented that avoids such nested iterations by taking advantage of the eikonal equation that relates the index of refraction to characteristics of the propagation path. A single iteration over coefficients that fix the electron-density profile gives the best fit to the TEC data.

In order to implement the tomographic reconstruction suggested here and judge its advantages over earlier methods, several tasks remain. The method must be translated into a computer code. Practical examples must be presented, with comparisons of resource requirements and examination of limits to the vertical resolution. Should the results justify further study, the representation of electron density should be expanded into a second set of orthonormal basis functions representing the horizontal variations. Favorable results from these tasks would provide a significant improvement to present methods of ionospheric reconstruction over large areas.

With the ability to treat curved paths, possibilities are created for the transmitters to launch signals at a few frequencies. At the receivers, signals at each frequency will measure a different TEC because of the differing paths. The added measurements (with no addition of transmitter or receiver locations) give a more thorough probing of the ionosphere and improve resolution of the electron density in the ionosphere.

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