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May 4, 1994

Department of Advanced Technology

Informal Report

Brookhaven National Laboratory
Associated Universities, Inc.
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Crystalline Beams: The Vertical Zigzag

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*Abstract **

This note is the continuation of our comprehensive investigation of *Crystalline Beams*. After having determined the equations of motion [1], and the conditions for the formation of the simplest configuration, i.e. the *string* [2], we study the possibility of storing an intense beam of charged particles in a storage ring where they form a *vertical zigzag*. We define the equilibrium configuration, and examine the confinement conditions. Subsequently, we derive the transfer matrix for motion through various elements of the storage ring. Finally we investigate the stability conditions for such a beam.

*Work performed under the auspices of the U.S Department of Energy.

1 Introduction

We continue our investigation of *Crystalline Beams*. At high enough ion density, the string configuration we have studied previously [1, 2] separates into two *substrings* that make up the vertical zigzag, provided that the string becomes unstable first in the vertical direction. We think of the zigzag as two strings “on top” of each other. In this section we study the onset of the vertical zigzag and its confinement and stability condition. We follow the same method as outlined for the string [2], using transfer matrices. There were no confinement conditions to be satisfied for the string, but only stability conditions. The vertical zigzag is the first example where confinement is to be studied in its simplest form: vertical confinement.

The notation used in this paper is the same as introduced in [1, 2]. The reader should familiarize him/herself with these references, since they contain the derivation of the equations used in this paper.

2 Equilibrium Configuration

The equilibrium configuration for this geometry is given by two identical strings placed at equal distances from the midplane $y = 0$. The particles on the string on the top are located on the plane $y = +b$ and those on the string at the bottom on the plane $y = -b$, so that the total vertical separation is $2b$. The longitudinal spacing or lattice constant in both substrings is λ (see Fig. 1), as in the case of the string. All particles have the same momentum (i.e., $\delta = 0$) and are located at $x = 0$.

To determine the equilibrium configuration, we look for minima of the single-particle potential. Intuitively, two configurations come to mind: either the two “substrings” are right above each other, or

they are shifted relative to each other by $\lambda/2$. In both of these cases there is no net force in the longitudinal direction. There is a net force in the transversal direction, and the confinement of the string in the presence of these net forces will be discussed in the next section. In Fig. 1 we show the zigzag for arbitrary shift Δ . To determine which configuration corresponds to a potential minimum, Δ is treated as a free parameter. The potential consists of two parts, ϕ_1 and ϕ_2 . ϕ_1 is the interaction of the test particle with the ions in its own substring; ϕ_2 is due to the ions in the other substring, viz.:

$$\phi_1 = 2 \sum_{i=1}^{\infty} \frac{Qe}{\gamma i \lambda}, \quad (1)$$

$$\begin{aligned} \phi_2 = & \sum_{i=0}^{\infty} \frac{Qe}{[4b^2 + \gamma^2(\Delta + i\lambda)^2]^{\frac{1}{2}}} \\ & + \sum_{i=1}^{\infty} \frac{Qe}{[4b^2 + \gamma^2(\Delta - i\lambda)^2]^{\frac{1}{2}}}. \end{aligned} \quad (2)$$

We find it convenient to take λ outside the expression, and to study the asymptotic properties of the potential Eq. (2) as a function of the aspect ratio α of the separation between the substrings $2b$ and the lattice constant λ :

$$\alpha = \frac{2b}{\gamma\lambda}. \quad (3)$$

After we rescale the potential with $Qe/\gamma\lambda$, we obtain:

$$\begin{aligned} \tilde{\phi}_2 = & \frac{1}{\alpha^2 + \bar{\Delta}^2} \\ & + \sum_{i=1}^{\infty} \left\{ \frac{1}{[\alpha^2 + (\bar{\Delta} + i)^2]^{\frac{1}{2}}} + \frac{1}{[\alpha^2 + (\bar{\Delta} - i)^2]^{\frac{1}{2}}} \right\}, \end{aligned} \quad (4)$$

where $\bar{\Delta} = \Delta/\lambda$. Clearly $0 \leq \bar{\Delta} \leq 0.5$.

When $\alpha \ll 1$ – that is, when the separation between the substrings is less than the lattice constant – we can ignore α in Eq. (4), and we obtain the series

$$\tilde{\phi}_2 = \frac{1}{\bar{\Delta}} + \sum_{i=1}^{\infty} \frac{1}{|i + \bar{\Delta}|} + \frac{1}{|i - \bar{\Delta}|}. \quad (5)$$

Obviously, the minimum corresponds to $\bar{\Delta} = 1/2$, i.e., $\Delta = \lambda/2$.

In the continuum limit $\alpha \gg 1$ the potential Eq. (4) depends weakly on $\bar{\Delta}$, and all possible relative shifts may be energetically allowed. This is understandable, since the two substrings are so far apart, that they cannot "resolve" each others' single ions.

In Fig. 2 we plot the potential Eq. (4) for $\bar{\Delta} = 0$ and $\bar{\Delta} = 0.5$. It is remarkable that the potential is quite independent of $\bar{\Delta}$ for $\alpha \approx 1$.

We conclude that for $\alpha \ll 1$, the equilibrium configuration of the vertical zigzag corresponds to two substrings shifted half a lattice constant relative to each other. For larger α (for $\alpha \gtrsim 1$ and upwards), the relative shift has negligible influence on the particle potential, and all relative shifts are allowed. In the following we simply set $\bar{\Delta} = 0.5$, since this corresponds to the global minimum in the potential energy.

Figure 1: The vertical zigzag

Figure 2: The potential $\tilde{\phi}_2$ as a function of α , at $\bar{\Delta} = 0$ and $\bar{\Delta} = 0.5$.

3 Confinement for the Vertical Zigzag

3.1 The Effect of Curvature

It is seen that there are no net forces in the radial direction corresponding to the x coordinate, except when the trajectory is bent in the arcs, as we have already seen in the case of the string [2]. Therefore, the scalar potential due to the space charge [1] contains at least a radial component, viz.:

$$\phi_{sc} = \nu_1 x. \quad (6)$$

The radial force has the same magnitude for all particles, viz.:

$$\begin{aligned} \nu_1 = & -\frac{Qe}{2\rho^2\gamma^2} \sum_{i=1}^{N/2} \frac{1}{[1 - \beta^2 \sin^2(\frac{\theta_i}{2})]^{3/2} \sin(\frac{\theta_i}{2})} \\ & -\frac{Qe}{2\rho^2\gamma^2} \sum_{i=1}^{N/2} \frac{\sin^2(\frac{\theta_i}{2})}{\{\frac{b^2}{2\rho^2\gamma^2} + [1 - \beta^2 \sin^2(\frac{\theta_i}{2})] \sin^2(\frac{\theta_i}{2})\}^{3/2}}. \end{aligned} \quad (7)$$

The first summation includes the contribution from the particles on the same string as the test particle, the second summation is the contribution from all particles on the other string. Both summations are taken only over half the beam ahead of the test particle. Denoting with $\Delta = \lambda h$ the angular separation, we have $\theta_i = i\Delta$ in the first summation and $\theta_i = (i - 1/2)\Delta$ in the second summation.

For nonrelativistic beams we ignore the term in β . The largest contribution is from the neighboring particles. Therefore, we replace the sin- function with its argument. Using the functions

$$f_n(\alpha) = \sum_{i=1}^{\infty} \frac{1}{[\alpha^2 + (i - \frac{1}{2})^2]^{n/2}} \quad (8)$$

introduced in [1], we can rewrite ν_1 as:

$$\nu_1 = -\frac{Qe}{\gamma^2\lambda\rho} [f_3(0) + f_3(\alpha)]. \quad (9)$$

Recall that α is the aspect ratio introduced earlier (see Eq. (3)). Since typically $N \gg \alpha$, in the limit that $\Delta \ll 1$ we can also approximate the summation with an integral, which gives

$$f_3(0) + f_3(\alpha) \approx \ln\left(\frac{\pi N}{1 + \sqrt{1 + 4\alpha^2}}\right). \quad (10)$$

For $\alpha = 0$ this reduces to the $\ln$ previously derived for the string [2]. Clearly, like in the case of the string, in the straight sections $h = 0$, and therefore $\nu_1 = 0$.

We conclude that the corresponding equations for the horizontal and longitudinal motion are identical to Eqs (9–13) of Ref. [2], with ν_1 replaced by the expression appropriate for the vertical zigzag, and given above.

The equilibrium configuration of the vertical zigzag, $x = \delta = 0$ and $\sigma = \text{constant}$, satisfies the equations of motion, provided that the generalized Lorentz condition [1, 2] is satisfied. Of course, all the considerations already made for the string still apply. In particular the term in ν_1 owing to the curvature is generally very small and will be ignored.

3.2 Confinement in the Vertical Direction

For the vertical zigzag in its equilibrium configuration, the interparticle repulsion needs to be balanced by the focussing of the storage ring. The vertical component of the space-charge force is given by the first-order derivative of the space-charge potential ϕ_{sc} with respect to the vertical coordinate y , evaluated at equilibrium [1]. Ignoring the curvature in the bending magnets, we find:

$$\frac{\partial \phi_{sc}}{\partial y} \Big|_{eq.} = -2Qe\gamma \sum_{i=1}^{\infty} \frac{2b}{[4b^2 + \gamma^2 \lambda^2 (i - \frac{1}{2})^2]^{3/2}}. \quad (11)$$

Therefore, the equation for the vertical motion is

$$y'' + K_v(s)y - \frac{4Q^2 r_0 f_3(\alpha)}{A\beta^2 \gamma^5 \lambda^3} y = 0. \quad (12)$$

This equation can be easily solved by taking a constant-gradient circular magnet, like a betatron. Denoting with n the field index, the motion of a single particle is stable provided that n takes any positive value. For the vertical confinement of a vertical zigzag the following condition has to be satisfied:

$$n = \frac{4Q^2 r_0 f_3(\alpha)}{A\beta^2 \gamma^5 \lambda^3 h^2}. \quad (13)$$

By increasing the gradient nh^2 of the magnet, the height of the zigzag $2b$ reduces. As we shall see later, nonetheless, this configuration is unstable against single-particle perturbations. We have, therefore, to consider the case of storage rings with alternating-gradient magnets.

3.3 Vertical Confinement with Alternating-Gradient Focusing

In this case one has to allow the height of the zigzag to vary around the storage ring. The variation will be small compared to b and λ , and will take place over lengths comparable to the length of the magnets. One then derives the following envelope equation [1]:

$$b'' + K_v(s)b - K_{sc} \frac{2f_3(\alpha)}{g_0} b = 0 \quad (14)$$

where

$$K_{sc} = 2 \frac{Q^2 r_0 g_0}{A\beta^2 \gamma^5 \lambda^3} = h_{sc}^2 \quad (15)$$

The exact solution is nonlinear because of the nonlinear dependence of $f_3(\alpha)$ on b . However, simple and intuitive physical arguments provide an approximate solution. We are interested in a nontrivial ($b \neq 0$) solution which has the same fundamental periodicity as the storage ring. That is, its period is a harmonic of that of the storage ring:

$$\nu_y = \frac{m}{2} N_p,$$

which is satisfied only when the space-charge force is large enough to shift the dilute beam tune to the nearest stopband. Treating the space-charge force as a perturbation, the tune shift is given by

$$\Delta\nu_v = \frac{Q^2 r_0}{\pi \beta^2 \gamma^5 \lambda^3 A} \oint \beta_v f_3(\alpha) ds. \quad (16)$$

The integral is taken over the storage-ring circumference. Calculations done with SYNCH and GETBETAS [3] show that this formula is correct even for large values of the space-charge force. In fact, it is a good approximation until a limit of stability – or stopband – is encountered.

We require that the tune shift brings the betatron tune down to the next integer stopband value. This occurs when

$$\Delta\nu_v = \delta\nu_v. \quad (17)$$

$\delta\nu_v$ is half of the distance of the original tune value from the center of the stopband [2]. Using this we derive

$$F(\alpha) \equiv \frac{1}{f_3(\alpha)} = \frac{2Q^2 r_0 R}{A \beta^2 \gamma^5 \lambda^3 \delta\nu_v} \beta_v, \quad (18)$$

which can be solved to obtain the displacement b as a function of λ and β_v . The function $F(\alpha)$ is plotted in Fig. 3. There is a minimum corresponding to $\alpha = 0$, that is $b = 0$,

$$F(0) = \frac{1}{7g_0}, \quad (19)$$

where

$$g_0 = \sum_{i=1}^{\infty} i^{-3} = 1.2 \dots \quad (20)$$

From Eq. (18) we determine an *upper limit* on λ in order for the zigzag to exist:

$$\lambda < \lambda_1 \quad (21)$$

$$\equiv \left[\frac{2Q^2 r_0 R \beta_v}{A \beta^2 \gamma^5 F(0) \delta\nu_v} \right]^{1/3}. \quad (22)$$

When the stopband is first encountered, with $b = 0$, λ is equal to λ_1 in Eq. (22). Eq. (18) can be solved to estimate the actual value of b around the ring. β_v varies around the ring, and so does the threshold value of λ_1 . In order to obtain a zigzag throughout the ring, Eq. (22) has to be satisfied everywhere, especially where β_v is the smallest.

We now show that the vertical zigzag arises quite naturally out of the string. For a dilute beam – that is, for large values of λ – the space-charge force is small, and can be neglected or treated as a small perturbation. The motion is then completely described by the storage ring focussing parameter K_v , from which the original tune ν_v and amplitude lattice function can be derived. The only acceptable solution to Eq. (14) is the trivial solution $b = 0$, and consequently the Crystalline Beam is a string. We now decrease λ further until the limit of stability of the string is reached [2]:

$$\lambda_{string} = \lambda_c \left(\frac{1}{\nu_v \delta_v} \right)^{\frac{1}{3}} = \lambda_c c_v, \quad (23)$$

and

$$\lambda_c = \left(\frac{Q^2 r_0 g_0 R^2}{A \beta^2 \gamma^5} \right)^{1/3} \quad (24)$$

is the critical lattice constant. Therefore, the value of the lattice constant for which the zigzag first appears, given by λ_1 , and the critical lattice constant of the string λ_{string} for which the vertical motion of the string becomes unstable are related:

$$\lambda_1 = 2.410 \dots \lambda_{string}. \quad (25)$$

In other words, the particle spacing λ of the two substrings forming the vertical zigzag is roughly double the value of that of the string from which the zigzag originates. Furthermore, the substrings are shifted relative to each other by $\frac{\lambda}{2}$, and therefore this does indeed correspond to the minimum-energy configuration discussed in section 2. Therefore, we regard the creation of the zigzag

as the byproduct of a *pitchfork bifurcation* of the reference orbit: for a critical value of the lattice spacing the reference orbit becomes unstable, and gives rise to two stable orbits of the same period. Simultaneously, the string splits into two substrings, which have roughly twice the lattice constant as the original string.

We conclude this section with two more remarks: For $\alpha \gtrsim 1$, $F(\alpha) \approx \alpha^2$ and one obtains:

$$b^2 = \frac{\epsilon_v \beta_v}{\pi}, \quad (26)$$

with

$$\epsilon_v = \frac{\pi Q^2 r_0 R}{2A\beta^2 \gamma^3 \lambda \delta \nu_v} \quad (27)$$

the equivalent betatron emittance. Thus, the height of the zigzag closely follows the lattice amplitude function β_v of the storage ring.

From the above discussion it follows that:

$$\lambda^3 F(\alpha) = \text{constant.} = F(0) \lambda_1^3. \quad (28)$$

As λ decreases, the spatial separation b has to increase at a faster rate to keep the betatron tune near that of the stopband.

3.4 Using the Transfer Matrix to Determine the Vertical Confinement

In order to determine confinement conditions and configurations we need to confine our analysis only to the vertical motion and, to the corresponding vertical zigzag matrices.

A more exact estimate of the equilibrium configuration of the vertical zigzag can be found on the computer using the matrix formulation. For this purpose we still assume that the storage ring is made of drifts, quadrupoles and bending magnets. Each of these magnet components can be subdivided in a sufficiently large number of sections where the zigzag dimension b can be approximated

as constant. For each of these sections we can associate a 2×2 transfer matrix, where the elements are in general nonlinear functions of b . The transfer matrix for each magnetic section can then be estimated as done in [2].

In [2] we derived the transfer matrices for motion through a drift, dipole, and quadrupole in the presence of strong space-charge forces in the form of a string. For the space charge forces in the form of a vertical zigzag, the transfer matrices have a analogous form, if we replace $K_{sc} = h_{sc}^2$ with

$$K_{sc} \frac{2f_3(\alpha)}{g_0} = \frac{4Q^2 r_0 f_3(\alpha)}{A\beta^2 \gamma^5 \lambda^3}. \quad (29)$$

everywhere.

To determine a closed periodic solution, we usually choose a symmetry point as the start of the superperiod so that $b' = 0$. We take an initial guess for b and propagate the vector

$$\mathbf{B} = (b, b') \quad (30)$$

by multiplication with the sequence of the transfer matrices of the sections as they appear in the storage ring. Before each multiplication, we replace the value of b in the elements of the matrix with the one obtained at the end of the previous multiplication. This way, we obtain the total transfer matrix M for a superperiod which is a function of the original value of b . If $\mathbf{B}_1$ and $\mathbf{B}_2$ denote the two vectors at the beginning and at the end of the superperiod respectively,

$$\mathbf{B}_2 = M(\mathbf{B}_1)\mathbf{B}_1. \quad (31)$$

In order for the orbit to close $\mathbf{B}_2 = \mathbf{B}_1$.

There are many iteration methods to find the desired solution. For instance we can plot the first diagonal element m_{11} of M as a function of b (initial). By inspection, the value of b which corresponds to

$$m_{11} = 1 \quad (32)$$

is the desired initial value. All other values of $\mathbf{B}$ at different locations can then be calculated by matrix multiplication. The solution of the equation above can also be found by the Newton method. Another method requires inspection of the eigenvalues of the total transfer matrix M . Only solutions with $b > 0$ are acceptable.

4 Stability of the Test-Particle Motion

4.1 Stability of the Vertical Motion

To estimate the stability of the vertical zigzag we proceed as follows. We derive first the equations of motion in the vertical plane.

$$y'' + K_v(s)y + \frac{Qr_0}{eA\beta^2\gamma^3} \frac{\partial\phi_{sc}}{\partial y} = 0. \quad (33)$$

We ignore the curvature in the bending magnets, as discussed previously in subsection 3.1. The potential is evaluated with all the particles in their equilibrium (ground state) position. The position of the test particle is perturbed by an amount small compared to b and λ . The space-charge potential for the vertical zigzag is derived in [1]. By inserting it in the equation of vertical motion and by letting $y = b + v$, we obtain two equations. The one for the equilibrium b of the zigzag we have already studied. The other equation is for the perturbation v and it is used to determine the stability of motion, that is

$$v'' + K_v(s)v - K_{sc}\eta_v v = 0, \quad (34)$$

where

$$\eta_v = 1 + \frac{f_3 - 3\alpha^2 f_5}{g_0}. \quad (35)$$

Apart from the form factor η_v , this equation is exactly the same as the one derived for the string. η_v , which is a function of the aspect ratio $\alpha = 2b/\gamma\lambda$, is plotted in Fig. 4.

When the string splits into the two substrings forming the zigzag, $\alpha = 0$, and $\eta_v = 8$ (because $f_3(0) = 7g_0$). In Eq. (34), λ refers to the spacing of the substring, which is twice the spacing of the string giving rise to the zigzag. In other words, Eq. (34) reduces to the equation of the vertical motion of the string, if we replace λ_{zigzag} by

$$\lambda_{string} = \frac{\lambda_{zigzag}}{2}. \quad (36)$$

Therefore, we again conclude that when the string becomes unstable in the vertical direction, this gives rise to a vertical zigzag, consisting of two substrings with twice the lattice constant of the original string.

The stability of the vertical motion of the vertical zigzag can be determined exactly from the eigenvalues of the transfer matrix [2]. The matrices corresponding to the different elements making up the storage ring can be obtained from those of the string by multiplying the space charge parameter K_{sc} by η_v .

Before estimating the stability one needs to determine the periodic solution of the confinement equation as it has been explained before.

It is also possible to estimate the threshold of stability of the vertical zigzag analytically, using the same method as outlined for the string. The motion is stable as long as the spacing λ is larger than the critical value λ_2 , viz.:

$$\lambda > \lambda_2, \quad (37)$$

where

$$\lambda_2 = \left(\frac{Q^2 r_0 g_0 R \bar{\beta}_v}{A \beta^2 \gamma^5 \delta \nu_v} \eta_v \right)^{1/3} \quad (38)$$

$$= \lambda_c c_v \eta_v^{1/3}. \quad (39)$$

Since η_v itself is a function of λ through α , taking into account Eq. (18), the same stability condition can also be expressed as follows

$$F(\alpha)\eta_v(\alpha) \leq \frac{2}{g_0}. \quad (40)$$

The equality sign holds exactly when the zigzag becomes unstable in the vertical direction. The resulting equation is easily solved numerically, and yields the critical value of the aspect ratio α_v :

$$\alpha_v \approx 1.58. \quad (41)$$

We note that this value is *independent* of the storage-ring characteristics, and is in that sense *universal*.

4.2 Stability of the Radial – Longitudinal Motion

The equations of motion for the radial and longitudinal direction of motion can be derived as in the case of the string, viz.:

$$x'' + (K_h - K_{sc}\eta_h)x = h\delta, \quad (42)$$

$$\sigma' = hx - \frac{\delta}{\gamma^2}, \quad (43)$$

$$\delta' = 2\gamma^2 K_{sc}\eta_e \sigma. \quad (44)$$

with the horizontal and longitudinal form factors given by

$$\eta_h = 1 + \frac{f_3}{g_0}, \quad (45)$$

$$\eta_e = 1 + \frac{f_3 - 1.5\alpha^2 f_5}{g_0}. \quad (46)$$

η_h and η_e are derived in ref. [1], and plotted in Fig. 4. To determine the stability of the planar motion we follow the same procedure outlined in [1] for the string, and derive similar stability conditions, viz.:

$$\lambda > \lambda_c c_i \eta_i^{1/3}, \quad (47)$$

with c_i and η_i replaced by the following pairs:

$$c_h = \left(\frac{1}{\nu_h \delta \nu_h} \right)^{1/3}, \quad \text{and} \quad \eta_h; \quad (48)$$

$$c_s = \left(\frac{4}{N_p} \right)^{2/3}, \quad \text{and} \quad \eta_e; \quad (49)$$

$$c_c = \left(\frac{\eta_h}{\nu_h q \eta_e} \right)^{2/3}, \quad \text{and} \quad \eta_r. \quad (50)$$

q is the smallest positive value of

$$q = 1 \pm \sqrt{1 + \frac{\eta_h}{\eta_e} \frac{\nu_h - m_+ N_p}{\nu_h}}. \quad (51)$$

N_p is the storage-ring periodicity, and m_+ an integer. Each of these three constants (four, including c_v) are inversely proportional to the distance between the dilute beam tune and the nearest integral stopband responsible for the destruction of the zigzag stability. These are the vertical, radial, and longitudinal stopband, and the sum-coupling resonance between the radial and longitudinal motion. The largest value of these four constants will determine whether the zigzag breaks up in the vertical direction to form four staggered zigzags, or in the horizontal plane to form a *helix*.

Using the fact that $\lambda^3 F_3(\alpha)$ remains constant, as determined from Eq. (18), the stability condition Eq. (47) is rewritten as:

$$F(\alpha) \eta_i(\alpha) \leq \frac{2}{g_0} \left(\frac{c_v}{c_i} \right)^3 \quad (52)$$

η_i and c_i are replaced by either one of the three pairs listed in Eqs (48 – 50), or indeed η_v and c_v . In the latter case, we recover Eq. (40). The equality sign holds when the zigzag becomes unstable, and Eq. (52) can be used to estimate the threshold value of the aspect ratio α for which the stability of the vertical zigzag is lost. The smallest value of α_i which satisfies Eq. (52) determines which component of the motion becomes unstable first.

We obviously demand, that the lattice constant λ_1 , for which the vertical zigzag is formed, is larger than the stability threshold λ_2 . Combining Eq. (23), Eq. (25) and Eq. (47) yields the following condition:

$$c_i < g_0 c_v, \quad i = h, s, c. \quad (53)$$

This is equivalent to requiring that the vertical stopband is encountered first, before the other ones, as the linear particle density of the string is increased. Therefore, the right-hand side of Eq. (52) is roughly equal to or larger than unity.

The functions $F(\alpha)\eta_i(\alpha)$ that determine the stability threshold are plotted in Fig.5. The vertical zigzag becomes unstable in the radial - longitudinal plane, when

$$\frac{2}{g_0} \left(\frac{c_v}{c_h} \right) < F(\alpha_v)\eta_h(\alpha_v) \approx 3.36 \quad (54)$$

or

$$\frac{2}{g_0} \left(\frac{c_v}{c_{s,c}} \right) < F(\alpha_v)\eta_e(\alpha_v) \approx 2.52, \quad (55)$$

since $\alpha_{h,s,c} < \alpha_v$, and the vertical zigzag goes over into a *helix*. The conditions Eq. (54) and Eq. (55) correspond to a tuning range of the storage ring.

Figure 3: Plot of the $f_3(\alpha)$ and $F(\alpha)$ function versus α

Figure 4: Plot of the $\eta_v(\alpha)$, $\eta_h(\alpha)$ and $\eta_e(\alpha)$ functions versus the aspect ratio α

5 The Betatron Magnet as a Storage Ring

The stability condition for the vertical motion in a Betatron Magnet is derived from Eq. (34) by letting $K_v = nh^2$, viz.:

$$n > \xi^2 \eta_v > 0, \quad (56)$$

where $\xi = h_{sc}/h$.

The equations of motion on the plane are given by Eqs (42 – 44) where all parameters are constant. In particular we have $K_h = h^2(1 - n)$. Solving for the eigenvalues we find, like in the case of the string, that they satisfy the following equation:

$$\alpha^4 + \Gamma^2 \alpha^2 - \Upsilon^2 = 0, \quad (57)$$

where Γ and Υ are given by:

$$\Gamma^2 = h^2(1 - n + \xi^2 \eta_v), \quad (58)$$

$$\Upsilon^2 = 2h^4 \xi^2 \eta_e (\gamma^2 + n - 1 + \xi^2 \eta_h). \quad (59)$$

For the motion to be stable the eigenvalues all need to be purely imaginary. This means, that the following quantity has to be real and negative:

$$2\alpha^2 = -\Gamma^2 \pm \sqrt{\Gamma^4 + 4\Upsilon^2}. \quad (60)$$

This in turn requires that

$$\Gamma^4 + 4\Upsilon^2 > 0, \quad (61)$$

and this condition is always satisfied. The other condition

$$\Gamma^2 > 0 \quad \text{with} \quad \Upsilon^2 < 0 \quad (62)$$

needs to be satisfied simultaneously. That is

$$n < 1 - \gamma^2 - \xi^2 \eta_h < 0. \quad (63)$$

Clearly, the two stability conditions – Eq. (56) for the vertical motion and Eq. (63) for the planar motion – cannot be satisfied simultaneously. Therefore, in a betatron the vertical zigzag is unstable against small perturbations.

6 Application of the Results

We have previously studied a few examples [1,2] of storage-ring lattices. The string in the CRYSTAL8 lattice [3], became unstable in the vertical plane. In our view, the string therefore “bifurcates” into the vertical zigzag. So, we bring the insights developed above to this case.

The periodicity of the CRYSTAL8 storage ring is $N_p = 8$ with the circumference $2\pi R = 64.82$ m. The original betatron tunes are $\nu_h = 1.9$ and $\nu_v = 1.09$. Taking a beam of Au ions with $Q = 51$ at a magnetic rigidity of $B\rho = 1.2$ Tm, the critical ion separation is $\lambda_c = 63.4$ μm . From the stability arguments presented in [2] we conclude that the string loses stability at $\lambda = 75$ μm . On the other hand, using the confinement condition for the vertical zigzag Eq. (18), we conclude that a vertical zigzag already appears at $\lambda = 90$ μm . That is, the actual critical lattice spacing at which the zigzag appears is in fact 20% higher than the critical lattice spacing at which the string loses its stability, in complete agreement with the discussion at the end of subsection 3.3. According to our analytical estimates:

$$c_v = 1.19$$

$$c_h = 0.82$$

$$c_s = 0.63$$

$$c_c = 0.36$$

and the stability conditions yield the threshold values

$$\alpha_v = 1.58$$

$$\alpha_h = 2.25$$

$$\alpha_s = 3.35$$

$$\alpha_c = 7.72.$$

The lowest threshold is given by α_v , so that the vertical zigzag is stable for $\alpha < 1.58$ which also corresponds to $\lambda < 75 \mu\text{m}$. At the limit of stability the vertical zigzag bifurcates in its turn in a double vertical zigzag, which is made of four substrings on top of each other.

One can conjecture that the stability is generally lost when the height of the zigzag is comparable to the longitudinal particle separation on each substring. Also, the beam becomes unstable once the separation λ on each substring is reduced to the original limit value for the string, which in good approximation is given by the critical distance λ_c .

Our analytical predictions are in good agreement with the more accurate numerical search for stability, by investigating the eigenvalues of the total transfer matrix per period. In Fig. 6 we show the relation between the aspect ratio α and the lattice spacing λ in the CRYSTAL8 storage ring. We subsequently use this to determine the limit of stability of the vertical zigzag using the transfer matrices. We find that the zigzag becomes unstable in the vertical direction for $\lambda = 66 \mu\text{m}$, with the critical lattice spacing $\alpha_v = 1.53$, in good agreement with the analytical results.

Figure 5: Plot of the stability threshold functions $F(\alpha)\eta_{v,h,e}$ versus the aspect ratio α .

Figure 6: Particle spacing λ for the Vertical Zigzag in the CRYSTAL8 storage ring as a function of the aspect ratio α .

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