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FREE VIBRATION ANALYSIS OF A TANK
CONTAINING TWO LIQUIDS

by

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May 1993

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FREE VIBRATION ANALYSIS OF A TANK CONTAINING TWO LIQUIDS

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Yu Tang and Y. W. Chang

ABSTRACT

A study of the dynamic characteristics of rigidly supported upright circular cylindrical tanks containing two different liquids is presented. The governing differential equations for the tank-two liquid system are obtained by application of the Rayleigh-Ritz procedure in combination with Lagrange's equation. The response functions examined include the fundamental natural frequency, the associated mode of vibration and hydrodynamic pressure exerted against the tank wall. Unlike the cases of tanks containing one liquid in which the dynamic response is controlled by four parameters, the dynamic response of a tank that contains two liquids is controlled by six parameters. The numerical results are presented in tabular and graphic forms, and are compared with those of the identical tank filled with one liquid. Also, a simple approximate equation for evaluating the fundamental natural frequency for preliminary design is proposed.

I. INTRODUCTION

The dynamic response of liquid-storage tanks subjected to earthquake ground motions has been a subject of numerous studies in the past thirty years. An excellent review of this topic can be found in Ref. 1. Most of the previous studies were focused on the tank containing only one liquid. However, there are cases in which the density of the tank content is not uniform. For example, in the paper by Burris, et al. [2], a new concept of reprocessing reactor spent fuel has been developed. This fuel recycling process is carried out primarily in a tank which contains two liquids with different densities. Also, a large number of high level waste storage tanks at various Department of Energy (DOE) facilities contain liquid with nonuniform density, see Ref. 3. For such cases, the adequacy of the design procedure based on the solutions obtained by assuming that tanks contain only one liquid becomes questionable. A study on the sloshing motion in a tank containing two liquids as reported in Refs. 4 and 5 shows that the sloshing wave height may be increased significantly as a result of the interaction of the two liquids. Therefore, it is necessary to understand how the two liquid interaction can affect the induced hydrodynamic pressure. In response to this need, a comprehensive study on the dynamic response of upright circular cylindrical liquid-storage tanks subjected to a horizontal ground motion has been studied by Tang [6,7,8]. However, in those papers only the rigid tank solutions were considered. Therefore, the effect of the tank wall flexibility on the response still needs to be answered. As the first step toward this goal, the dynamic characteristics of the tank filled with two liquids is studied, and the results are presented herein.

The objectives of this paper are: (1) to understand the effect of two-liquid interaction on the dynamic characteristics of the liquid storage tanks; (2) to present an analytical method of analysis for the computation of the natural frequencies of tanks containing two different liquids; and (3) to propose a simple approximate equation for the evaluation of the fundamental frequency of the tank-two liquid system.

The method of analysis presented herein is similar to that used in Ref. 9, and is basically the same as that used by Yang [10] and Tang [11], except that in the evaluation of the added mass of liquid, two liquids are involved in the present case. This approach is based on the Rayleigh-Ritz energy procedure in which the deflections of the tank wall are assumed to be linear combinations of the natural vibration modes of a uniform cantilever beam or the derivatives of

those modes, and the liquid region is solved analytically. The governing differential equations of the tank-two fluid system are obtained by the application of Lagrange's equation.

Like the previous studies of Housner [12] and in Ref. 1, the hydrodynamic pressure is divided into two components: the impulsive component and the convective component. For the definitions of these two components, the reader is referred to Refs. 1 and 12. It has been pointed out in Ref. 1 and by Veletsos and Yang [13] and confirmed by Housner and Haroun [14] that the effect of the tank flexibility on the convective component is unimportant and can be neglected. Therefore, only the impulsive component is examined in this paper. Note that the convective component can be evaluated by considering the tank to be rigid.

II. SYSTEM DESCRIPTION

The tank-two liquid system investigated is shown in Fig. 1. It is a ground-supported upright circular cylindrical tank of radius R filled with two liquids to a total height of H . The lower portion of liquid, identified as Liquid I, has a heavier mass density, ρ_1 , and the upper portion of liquid, identify as Liquid II, has a lighter mass density, ρ_2 . The height of Liquid I and II are H_1 and H_2 , respectively. Both liquids are considered to be incompressible and inviscid. The tank wall is assumed to be uniform thickness, h , and clamped to a rigid base. The mass density of the tank wall is denoted by ρ , and the modulus of elasticity and Poisson's ratio of the tank wall are denoted by E and ν , respectively. Points on the tank wall are specified by the cylindrical coordinate system, r, θ, z . The origin of the coordinate system is taken at the center of the base. For convenience, two local cylindrical coordinates are used for the two liquids. Let r, θ, z_1 denote the radial, circumferential, and vertical axial coordinates of a point in the Liquid I, and let r, θ, z_2 be the corresponding coordinates for a point in Liquid II as shown in Fig. 1. The coordinate z is related to z_1 and z_2 by the equation

$$z = \begin{cases} z_1 & \text{for } 0 \leq z \leq H_1 \\ z_2 + H_1 & \text{for } H_1 \leq z \leq H \end{cases} \quad (1)$$

III. METHOD OF ANALYSIS

The analysis is carried out by the application of the Rayleigh-Ritz method in combination with Lagrange's equation to obtain the governing differential equations.

The axial, tangential and radial components of the deflections of a point on the tank wall, u , v and w respectively, are expressed in terms of generalized coordinate functions $\chi_i(z)$ and $\psi_i(z)$, i.e.,

$$u = \sum_{i=1}^{N1} U_i(t) \chi_i(z) \cos \theta \quad (2a)$$

$$v = \sum_{i=1}^{N2} V_i(t) \psi_i(z) \sin \theta \quad (2b)$$

$$w = \sum_{i=1}^{N3} W_i(t) \psi_i(z) \cos \theta \quad (2c)$$

where $U_i(t)$, $V_i(t)$ and $W_i(t)$ are time-dependent displacement amplitudes of the i th coordinate functions $\chi_i(z)$ and $\psi_i(z)$, and $N1$, $N2$ and $N3$ are the number of coordinate functions used to approximate the displacements. A suitable choice of the generalized function $\psi_i(z)$ for a tank with free boundary conditions at top and fixed boundary condition at bottom would be the natural modes of vibration of a uniform cantilever beam. The second generalized coordinate function $\chi_i(z)$ is the first derivative of $\psi_i(z)$. These sets of generalized functions have been used previously in related studies [9,10,11]. Note that these generalized functions are geometrically admissible since they satisfy the boundary conditions at the tank base, i.e., $u=0$, $v=0$, $w=0$ and $dw/dz=0$.

The governing differential equations are obtained by the application of Lagrange's equation given as

$$\frac{d}{dt} \left(\frac{\partial(T_s + T_l)}{\partial \dot{q}_i} \right) - \frac{\partial(T_s + T_l)}{\partial q_i} + \frac{\partial S}{\partial q_i} = 0 \quad (3)$$

for the i th equation in which T_s is the kinetic energy of the tank and T_l is the kinetic energy of the two liquids; S is the strain energy of the tank wall, and q_i is the generalized coordinates. In this study $q_i = U_i$ or V_i or W_i . Obviously, if $T_l=0$, one would obtain the governing differential equations for the empty tank.

A. Kinetic Energy of Tank, T_s

The kinetic energy of the tank is given by

$$T_s = \frac{\rho h}{2} \int_0^{2\pi} \int_0^H (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) R dz d\theta \quad (4)$$

in which a dot superscript denotes the differentiation with respect to time.

B. Strain Energy of Tank, S

The strain energy of the tank is computed by making use the expression given in Bleich and Dimaggio [15]. It is

$$\begin{aligned}
S = & \frac{E}{2(1-\nu^2)} \frac{h}{R} \int_0^{2\pi} \int_0^H \left[R^2 u_z^2 + (v_\theta + w)^2 \right. \\
& + 2\nu R u_z (v_\theta + w) + \frac{1-\nu}{2} (u_\theta + R v_z)^2 \left. \right] dz d\theta \\
& + \frac{E}{24(1-\nu^2)} \frac{h^3}{R^3} \int_0^{2\pi} \int_0^H \left[R^4 w_{zz}^2 + (w_{\theta\theta} + w)^2 \right. \\
& + \frac{1-\nu}{2} (R w_{z\theta} - u_\theta)^2 + \frac{3(1-\nu)}{2} R^2 (v_z + w_{z\theta})^2 \\
& \left. + 2\nu R^2 w_{zz} (w_{\theta\theta} - v_\theta) - 2R^3 u_z w_{zz} \right] dz d\theta
\end{aligned} \tag{5}$$

in which the terms in the first bracket represent the extensional energy and the terms in the second bracket represent the bending energy. The subscripts z and θ in the expression represent the derivatives of the quantity considered with respect to z and θ . Note that the expression presented in Eq. (5) differs a sign for w from that given in Ref. 15. This is due to the different sign convention used for w in this paper.

Substituting the expressions for T_i and S into Eq. (3), omitting T_i from the equation, and after repeating application of Eq. (3), one obtains a set of governing equations for the free vibration analysis of empty tanks. This work has been done by Yang [10] where the resultant equations are presented in a matrix form given by

$$\begin{bmatrix} [A] & 0 & 0 \\ 0 & [B] & 0 \\ 0 & 0 & [C] \end{bmatrix} \begin{Bmatrix} \{\ddot{U}\} \\ \{\ddot{V}\} \\ \{\ddot{W}\} \end{Bmatrix} + \begin{bmatrix} [D] & [E] & [F] \\ [E] & [G] & [H] \\ [F] & [H] & [I] \end{bmatrix} \begin{Bmatrix} \{U\} \\ \{V\} \\ \{W\} \end{Bmatrix} = \{0\} \tag{6}$$

where $[A]$ through $[I]$ are square submatrices, and $\{U\}$, $\{V\}$ and $\{W\}$ are the column submatrices whose elements are U_i , $i = 1, 2, \dots, N1$; V_i , $i = 1, 2, \dots, N2$; and W_i , $i = 1, 2, \dots, N3$, respectively. The elements of matrices $[A]$ through $[I]$ and the details of derivation of Eq. (6) can be found in Ref. 10. For the sake of presentation, Eq. (6) is written in a more compact form as

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\} \quad (7)$$

with the column matrix $\{q\}$ defined by

$$\{q\} = \{U_1(t), U_2(t), \dots, U_{N1}(t), V_1(t), V_2(t), \dots, V_{N2}(t), \\ W_1(t), W_2(t), \dots, W_{N3}(t)\}^T \quad (8)$$

where the superscript T denotes the transpose of the row matrix.

It has been shown by Yang [10] that the governing equations for a tank filled with liquid can be obtained by including the so-called liquid added mass to the mass matrix of the empty tank, and this liquid added mass is added to the elements associated with the radial displacement only; the presence of the liquid has no effect on the stiffness matrix, $[K]$. Specifically, if c_{ij} denotes the element of the submatrix $[C]$ in Eq. (6), c_{ij} needs to be replaced by c_{ij}^* where c_{ij}^* is related to c_{ij} by

$$c_{ij}^* = c_{ij} + m_{ij}^t \quad (9)$$

The quantity m_{ij}^t in Eq. (9) is the so-called liquid added mass. The same procedure is employed herein to obtain the governing equations for tanks containing two liquids. The equation for computing the corresponding liquid added mass is presented as follows.

C. The Governing Equations and Boundary Conditions for Liquids I and II

Given the conditions that the liquids are incompressible and inviscid, the velocity potentials for Liquids I and II, denoted by ϕ_1 and ϕ_2 respectively, must satisfy the Laplace equations

$$\nabla^2 \phi_1 = 0 \quad (10a)$$

in the region $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$, and $0 \leq z_1 \leq H_1$, and

$$\nabla^2 \phi_2 = 0 \quad (10b)$$

in the region $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$, and $0 \leq z_2 \leq H_2$.

The liquid velocity at an arbitrary point along x-direction, v_x , is given by

$$v_x = - \frac{\partial \phi_1}{\partial x} \quad (11)$$

for points in Liquid I, and

$$v_x = - \frac{\partial \phi_2}{\partial x} \quad (12)$$

for points in Liquid II.

The boundary conditions for Liquid I are:

- (a) The vertical velocity of Liquid I at the tank base must equal zero, i.e.,

$$\left. \frac{\partial \phi_1}{\partial z_1} \right|_{z_1=0} = 0 \quad (13)$$

- (b) The radial velocity of Liquid I adjacent to the tank wall must equal the velocity of the tank wall, i.e.,

$$-\left. \frac{\partial \phi_1}{\partial r} \right|_{r=R} = \dot{w} = \sum_{i=1}^{N3} \dot{W}_i(t) \psi_i(z) \cos \theta, \text{ and} \quad (14)$$

- (c) The value of ϕ_1 at $r=0$ is finite

The boundary conditions for Liquid II are:

- (a) The radial velocity along the tank wall is given by

$$-\left. \frac{\partial \phi_2}{\partial r} \right|_{r=R} = \dot{w} = \sum_{i=1}^{N3} \dot{W}_i(t) \psi_i(z) \cos \theta, \text{ and} \quad (15)$$

- (b) At the free surface, the pressure is equal to zero, i.e.,

$$\left. \frac{\partial \phi_2}{\partial t} \right|_{z_2=H_1} = 0, \text{ and} \quad (16)$$

(c) The value of ϕ_2 at $r=0$ is finite.

The boundary conditions at the interface of two liquids are:

(a) Continuity of hydrodynamic pressure, i.e.,

$$\rho_1 \frac{\partial \phi_1}{\partial t} \Big|_{z_1=H_1} = \rho_2 \frac{\partial \phi_2}{\partial t} \Big|_{z_2=0} \quad (17)$$

and

(b) Continuity of vertical velocity, i.e.,

$$\frac{\partial \phi_1}{\partial z_1} \Big|_{z_1=H_1} = \frac{\partial \phi_2}{\partial z_2} \Big|_{z_2=0} \quad (18)$$

The method of separation of variables is used to solve the differential equations (10a) and (10b). The boundary conditions, Eqs. (13), (16), (17) and (18), are used to determine the eigenfunctions and eigenvalues, and the boundary conditions, Eqs (14) and (15), are used to determine the integration constants. The n th eigenfunction, denoted by $\Phi_n(z)$, is found to be

$$\Phi_n(z) = \begin{cases} \alpha \cos\left(\lambda_n \frac{z_1}{H}\right) & 0 \leq z \leq H_1 \\ \cos\beta_{1n} \cos\left(\lambda_n \frac{z_2}{H}\right) - \alpha \sin\beta_{1n} \sin\left(\lambda_n \frac{z_2}{H}\right) & H_1 \leq z \leq H \end{cases} \quad (19)$$

where λ_n is the n th eigenvalue which is the n th root of the characteristic equation given by

$$\cos\beta_{1n} \cos\beta_{2n} - \alpha \sin\beta_{1n} \sin\beta_{2n} = 0 \quad (20)$$

Note that Eq. (20) has been presented in the paper by Tang [6] dealing with the hydrodynamic pressure in a rigid tank filled with two liquids. The notations used in Eqs (19) and (20) are explained as follows:

$$\alpha = \frac{\rho_2}{\rho_1} \quad (21)$$

$$\beta_{1n} = \lambda_n \frac{H_1}{H} \quad (22)$$

and

$$\beta_{2n} = \lambda_n \frac{H_2}{H} \quad (23)$$

Introducing a notation $\phi(r, \theta, z)$ denoting the velocity potential at any point, (r, θ, z) , in the liquid field, the solutions for the velocity potentials, ϕ_1 and ϕ_2 , can be cast into an expressed given by

$$\begin{aligned} \phi(r, \theta, z) &= \begin{cases} \phi_1(r, \theta, z_1) & \text{for } 0 \leq z \leq H_1 \\ \phi_2(r, \theta, z_2) & \text{for } H_1 \leq z \leq H_2 \end{cases} \\ &= \left[\sum_{i=1}^{N3} \dot{W}_i F_i(r, z) \right] R \cos\theta \end{aligned} \quad (24)$$

Function $F_i(r, z)$ in Eq. (24) is given by

$$F_i(r, z) = - \frac{H}{R} \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{I_1\left(\epsilon_n \frac{r}{R}\right)}{I'_1(\epsilon_n)} \frac{d_{in}}{\Delta_n} \Phi_n(z) \quad (25)$$

in which I_1 is the modified Bessel function of the first kind and I'_1 is its derivative, and

$$\epsilon_n = \lambda_n \frac{R}{H} \quad (26)$$

$$\begin{aligned} d_{in} = & \alpha \int_0^{\frac{H_1}{H}} \psi_i\left(\frac{z}{H}\right) \cos\left(\lambda_n \frac{z_1}{H}\right) d\left(\frac{z_1}{H}\right) \\ & + \cos\beta_{1n} \int_0^{\frac{H_2}{H}} \psi_i\left(\frac{z}{H}\right) \cos\left(\lambda_n \frac{z_2}{H}\right) d\left(\frac{z_2}{H}\right) \\ & - \alpha \sin\beta_{1n} \int_0^{\frac{H_1}{H}} \psi_i\left(\frac{z}{H}\right) \sin\left(\lambda_n \frac{z_2}{H}\right) d\left(\frac{z_2}{H}\right) \end{aligned} \quad (27)$$

$$\Delta_n = \alpha \frac{H_1}{H} + \frac{H_2}{H} \left(\alpha^2 \sin^2 \beta_{1n} + \cos^2 \beta_{1n} \right) \quad (28)$$

Consequently, the corresponding hydrodynamic pressure, denoted by $p(r, \theta, z)$, can be obtained from the equation given by

$$p(r, \theta, z) = \begin{cases} \rho_1 \frac{\partial \phi_1}{\partial t} & \text{for } 0 \leq z \leq H_1 \\ \rho_2 \frac{\partial \phi_2}{\partial t} & \text{for } H_1 \leq z \leq H \end{cases} \quad (29)$$

D. Kinetic Energy of Liquids, T_i

For the incompressible and nonviscous fluids considered herein, it is shown by Currie [12] that the kinetic energy of the two liquids can be evaluated by the equation

$$T_i = \frac{\rho_1}{2} \iint_{s_1} \phi_1 \frac{\partial \phi_1}{\partial n} ds_1 + \frac{\rho_2}{2} \iint_{s_2} \phi_2 \frac{\partial \phi_2}{\partial n} ds_2 \quad (30)$$

in which $\frac{\partial \phi_1}{\partial n}$ and $\frac{\partial \phi_2}{\partial n}$ are the derivatives of ϕ_1 and ϕ_2 in the direction normal to the boundaries

of each liquid, respectively, and the symbols s_1 and s_2 are the surfaces which enclose Liquids I and II, respectively. The only non-zero part of the surface integral in Eq. (30) is the one associated with the tank wall; the value of $\frac{\partial \phi_i}{\partial n}$, $i = 1, 2$, in this case is equal to $\frac{\partial \phi_i}{\partial r}$, $i = 1, 2$,

respectively. The integrals involving the tank bottom and free surface equal to zero are the results from the boundary conditions Eqs. (13) and (16), and the integrals at the interface of two liquids canceled each other are the result of the boundary conditions Eqs. (17) and (18). After eliminating those zero surface integrals, Eq. (30) can be rewritten as

$$\begin{aligned}
T_t = & \frac{\rho_1}{2} \int_0^{2\pi} \int_0^{H_1} \left[\phi_1 \frac{\partial \phi_1}{\partial r} \right] \bigg|_{r=R} R d\theta dz_1 \\
& + \frac{\rho_2}{2} \int_0^{2\pi} \int_0^{H_2} \left[\phi_2 \frac{\partial \phi_2}{\partial r} \right] \bigg|_{r=R} R d\theta dz_2
\end{aligned} \tag{31}$$

Substituting Eqs. (14), (15) and the resultant of Eq. (24) evaluated at $r=R$ into Eq. (31) and performing the indicated integration, one obtains

$$T_t = \frac{1}{2} \sum_{i=1}^{N3} \sum_{j=1}^{N3} m_{ij}^t \dot{W}_i \dot{W}_j \tag{32}$$

where

$$m_{ij}^t = m_{t2} \left[\frac{H}{R} \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{I_1(\epsilon_n)}{I_1'(\epsilon_n)} \frac{d_{in} d_{jn}}{\Delta_n} \right] \tag{33}$$

in which $m_{t2} = \pi H R^2 \rho_2$, and d_{jn} is obtained by replacing ψ_i in Eq. (27) by ψ_j . The quantity m_{ij}^t is the liquid added mass mentioned in Eq. (9).

E. Eigenvalue Problem

Replacing elements, c_{ij} by c_{ij}^* in the mass matrix, $[M]$, and assuming that the time function of the vibration is a sinusoidal function, Eq. (7) leads to the standard form of an eigenvalue problem

$$[K]\{q\} = \omega^2[M]\{q\} \quad (34)$$

where $\{q\}$ = the eigenvector which is the vector of the displacement amplitudes of vibration, and ω = the eigenvalue which is the natural circular frequency of the tank-two liquid system.

IV. NUMERICAL RESULTS

Unlike the case of a tank containing one liquid where the response is controlled by four parameters, H/R , h/R , ρ_2/ρ and ν , the response of tanks containing two liquids is controlled by six parameters. These six parameters are H/R , h/R , ρ_2/ρ , ν , H_2/H_1 and α .

The numerical results presented in this paper are for steel tanks where ν is taken to be 0.3. The lighter density liquid, Liquid II, is assumed to be water and the value for ρ_2/ρ is taken to be 0.127.

A. Fundamental Natural Frequency

The fundamental natural frequency of the tank-two liquid system may conveniently be expressed in the form of

$$f = \frac{1}{2\pi} \frac{c_f}{H} \sqrt{\frac{E}{\rho}} \quad (35)$$

in which f has the unit of Hz (cycles per second), and c_f is a nondimensional coefficient.

Values of c_f for steel tanks with $H/R = 0.5, 1, 2$ and 3 were obtained by using different combinations of N_1, N_2 and N_3 , and the results are listed in Table I. As expected, the value of c_f decreases monotonously as the number of terms, N_1, N_2 and N_3 increase. In Table I, the combination of N_1, N_2 and N_3 stops at $N_1=9, N_2=9$ and $N_3=9$. This is due to the limitation of the arithmetic precision imposed by the digital computer. In the computation of the integral defined in Eq. (27) for a tank containing two liquids, of the vibration modes of a uniform cantilever beam are evaluated at intermediate points and end points. It has been found that even

with the double precision words of the digital computer (IBM 3270), these vibration modes can be calculated accurately only up to the ninth mode. To assess the accuracy of the numerical results obtained by taking $N_1=9$, $N_2=9$ and $N_3=9$, the data reported in Ref. 11 for the case of one liquid ($\alpha=1$) are also listed in Table I for comparison. These data were obtained by taking $N_1=10$, $N_2=15$ and $N_3=20$. Note that for the case of $\alpha = 1$, only the two end points of the vibration modes need to be calculated, and the exact values for these two end points are available in Ref. 17. From Table I one can see clearly that the difference of the two results is less than 1.5 percent, which is within engineering accuracy. Thus all solutions presented herein are obtained by using the combination of $N_1=9$, $N_2=9$ and $N_3=9$.

To study the effect of the liquid densities on the fundamental frequency, the values of c_f are presented graphically in Fig. 2 for two different values of H_2/H_1 , 0.5 and 2, as a function of α . Four different values of H/R are considered; they are 0.5, 1, 2 and 3. From Fig. 2 it is noted that the value of c_f decreases with decreasing value of α , and the decrease is more rapid for small values than for large values of H_2/H_1 . This indicates that due to the interaction of the two liquids the natural frequency of the tank-two liquid system may be quite different from that of the identical tank filled with one liquid. For example, for a steel tank with the value of H/R equal to 1, the value of c_f is 0.0877 if the tank is filled with water; whereas the value of c_f is 0.0757 if the tank is filled with two liquids with $\alpha = 0.5$ and $H_2/H_1 = 1$, a difference of 15 percent is noted.

B. Fundamental Mode of Vibration

If $\{\hat{q}\}$ denotes the fundamental eigenvector obtained by solving the eigenvalue problem defined by Eq. (34), and the elements of $\{\hat{q}\}$ are denoted by $\hat{U}_i$, $i=1$ to N_1 , $\hat{V}_i$, $i=1$ to N_2 and $\hat{W}_i$, $i=1$ to N_3 , the distributions of the axial, circumferential, and radial displacements, denoted by $\hat{u}(z)$, $\hat{v}(z)$ and $\hat{w}(z)$ respectively, can be computed from the equations

$$\hat{u}(z) = \sum_{i=1}^{N_1} \hat{U}_i \chi_i(z) \quad (36a)$$

$$\hat{v}(z) = \sum_{i=1}^{N2} \hat{V}_i \psi_i(z) \quad (36b)$$

and

$$\hat{w}(z) = \sum_{i=1}^{N3} \hat{W}_i \psi_i(z) \quad (36c)$$

These equations are deduced from Eqs. (2a), (2b) and (2c).

The distribution functions, $\hat{u}(z)$, $\hat{v}(z)$ and $\hat{w}(z)$, for $\alpha = 0.5$ and 1, $H_2/H_1 = 0.5$ and 2 are shown in Figs. 3, 4 and 5, respectively, for a broad tank, $H/R = 0.5$ and a tall tank, $H/R = 3$. These distributions are normalized with respect to the maximum value of the three functions. Note that from these figures the displacements $\hat{u}$ and $\hat{v}$ have an opposite sign to that of $\hat{w}$, i.e., when $\hat{w}$ is positive, both $\hat{u}$ and $\hat{v}$ are negative. One also notes that for the broad tank, the magnitudes of $\hat{u}$ and $\hat{v}$ are much less than that of $\hat{w}$, while for the tall tank, $\hat{v}$ and $\hat{w}$ are comparable and $\hat{u}$ is still small. Generally speaking, the distributions of $\hat{u}$, $\hat{v}$ and $\hat{w}$ have the same shape for two values of α considered except for $\hat{w}$ in the case of the broad tank where a more clear change in the distribution is observed, see Fig. 5 (a) for $H/R = 0.5$. Inspection of Figs. 3, 4 and 5 also show that for tall tanks the mode shapes of $\hat{v}$ and $\hat{w}$ are very close to that of the cantilever beam, and the mode shape of $\hat{u}$ is close to that of the shear beam.

C. Fundamental Mode of Hydrodynamic Pressure

The fundamental mode of the hydrodynamic pressure exerted at the tank wall can be obtained by replacing $\dot{W}_i$ in Eq. (24) by $\hat{W}_i$, then, substituting the resultant equation into Eq. (29), and evaluating the resultant equation at $r = R$. As can be seen, the circumferential distribution of the pressure is proportional to $\cos\theta$ which is the same as that for the case of tanks filled with one liquid.

The distributions of the fundamental mode of the hydrodynamic pressure exerted at the tank wall, denoted by $\hat{p}(z)$, are presented in Fig. 6 for the same parameters considered in Figs. 3, 4 and 5. Note that unlike the displacement distributions shown in Figs. 3, 4 and 5 where the effect of the parameter α is appreciable only in the radial displacement of the broad tank, the effect of α is noticeable for both the broad and tall tanks. It can be seen that with the presence of two liquids ($\alpha = 0.5$) the distribution has a slope change at the interface of the two liquids, and the center of gravity of the area enclosed by the curve for $\alpha = 0.5$ is lower than that enclosed by the curve for $\alpha = 1$. This indicates that the base moment arm is smaller for the tank filled with two liquids than that of an identical tank filled with only one liquid.

V. SIMPLE EQUATION FOR EVALUATING FUNDAMENTAL NATURAL FREQUENCY

The simple procedure for evaluating the impulsive response for a broad tank ($H/R \leq 1$) containing only one liquid proposed by Veletsos and Yang [13] is very practical and useful for the preliminary design. It requires that the input ground acceleration, denoted by $\ddot{x}(t)$, in the rigid tank solutions be replaced by the pseudoacceleration function corresponding to the fundamental frequency of the tank-liquid system. It is believed that the proposed simple procedure can be extended for the preliminary design of the tanks containing two liquids. However, to implement that procedure for tanks containing two liquids, the fundamental frequency of the tank-two liquids system must be estimated. An equation for estimating the fundamental frequency of the tank-two liquid system is proposed herein. This equation is based on two assumptions: (1) the effective mass of the tank-two liquids system that participates in the first mode of vibration is equal to the impulsive component of the liquid added mass in the rigid tank, and (2) the mass of the tank wall itself is negligible in comparison to that of the liquid. The first assumption has been shown by Veletsos and Yang [13] to be a very good approximation for tanks filled with one liquid and with values of H/R in the range between 0.25 and 1.0. The second assumption is a condition that normally can be justified in practice. Therefore, the equation proposed herein for estimating the fundamental frequency of the tank-two liquids system is also limited for tanks with values of H/R in the range between 0.25 and 1.0. Note that this range is the practically important range.

Conceptually, the original two liquids are replaced by an equivalent liquid, and the mass density of this equivalent liquid is determined by the criterion that the tank filled with the equivalent liquid yields the same base shear (considering only the impulsive component) as that in the same tank filled with the original two liquids on the assumption that the tank is rigid. Presented in a paper by Tang [7], the impulsive component of the base shear, denoted by Q_i for rigid tanks containing two liquids can be expressed as

$$Q_i = \gamma_0 \rho_1 \pi R^2 H \ddot{x}(t) \quad (37)$$

where γ_0 is a dimensionless coefficient depending on the values of the parameters, H/R , H_2/H_1 , and α . If the ratio of two liquids is α_0 , then it is easy to show that the mass density of the equivalent liquid, denoted by ρ^* , is given by

$$\rho^* = \frac{(\gamma_0)_{\alpha=\alpha_0}}{(\gamma_0)_{\alpha=1}} \rho_1 \quad (38)$$

After the mass density for the equivalent liquid is obtained, the fundamental frequency of the tank-two liquid system can be estimated from the data for the fundamental natural frequency of the tank filled with water using the equation given in Ref. 1. Specifically, if the fundamental natural frequency for tanks filled with water is expressed in the same form defined by Eq. (35), and the frequency coefficient c_f for this case is denoted by c_w , then the frequency coefficient for a tank filled with two liquids with the value of $\alpha = \alpha_0$, denoted by c^* , can be computed from the equation

$$c^* = c_w \sqrt{\frac{\rho_w}{\rho^*}} \quad (39)$$

where ρ_w is the density of water. Extensive numerical data for the value of c_w are available in Refs. 18, 1 and 11.

The frequency coefficient computed by Eq. (39) for three values of α , 0.25, 0.5 and 0.75, two values of H_2/H_1 , 0.5 and 2, and values of H/R in the range between 0.3 and 1 are listed in Table II. These data are referred to as approximate solutions, and also listed in Table II are the corresponding data, referred to as the exact solution, obtained by the analytical methods presented in the paper for the purpose of comparison. It can be seen that the proposed equation yields satisfactory results.

VI. CONCLUSIONS

An analytical method of analysis for the study of the dynamic characteristics of tanks containing two liquids is presented. The method is based on the Rayleigh-Ritz procedure in combination with Lagrange's equation. The liquid region is solved analytically, and the displacements of the tank wall are expanded by the vibration modes of the cantilever beam. The method is easy to implement and may be expanded to include seismic excitations. The study presented herein shows that unlike the case of tanks containing one liquid in which the response is controlled by four parameters, the response of tanks containing two liquids is controlled by six parameters. Also, the dynamic characteristics of the system, the fundamental natural frequency, the associated mode of vibration and the hydrodynamic pressure, are affected by the interaction of the two liquids. Therefore, it is important to consider the presence of the two liquids in the design.

The fundamental frequency for tanks with values of H/R between 0.25 and 1 can be estimated with good accuracy by the simple equation provided in this paper. This simple equation utilizes the rigid tank solution for a tank containing two liquids and the available data for tanks filled with water.

ACKNOWLEDGMENT

The authors wish to thank Mr. Ralph Seidensticker of Argonne National Laboratory for helpful discussions during the performance of this work. This work was performed in the

Engineering Mechanics Program of the Reactor Engineering Division of Argonne National Laboratory under the auspices of the U.S. Department of Energy, Contract No. W-31-109-Eng-38.

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Table I. Convergence Table of Frequency Coefficient, c_i (steel tanks with $h/R = 0.001$, $\nu = 0.3$, $\rho_2/\rho = 0.127$)

No. of Terms N1-N2-N3	Frequency Coefficient, c_i			
	H/R = 0.5	H/R = 1	H/R = 2	H/R = 3
	$\alpha = 0.5$		$H_2/H_1 = 1$	
1-1-1	0.111	0.131	0.199	0.0967
3-3-3	0.0670	0.0820	0.0843	0.0756
3-3-5	0.0639	0.0795	0.0835	0.0736
5-5-9	0.0619	0.0765	0.0809	0.0733
7-7-9	0.0617	0.0760	0.0801	0.0727
9-9-9	0.0617	0.0757	0.0796	0.0724
	$\alpha = 1$		$H_2/H_1 = 1$	
1-1-1	0.116	0.139	0.127	0.102
3-3-3	0.0769	0.0943	0.0948	0.0828
3-3-5	0.0744	0.0923	0.0942	0.0824
5-5-9	0.0726	0.0896	0.0917	0.0807
7-7-9	0.0724	0.0890	0.0910	0.0801
9-9-9	0.0723	0.0887	0.0906	0.0799
10-15-20*	0.0719	0.0875	0.0896	0.0792

*Taken from Ref. 8.

Table II. Comparison of Exact and Approximate Fundamental Natural Frequency for Steel Tank with $h/R = 0.001$, $\nu = 0.3$ and $\rho_2/\rho = 0.127$

	Frequency Coefficient, c^*					
	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
H/R	Exact	Approx.	Exact	Approx.	Exact	Approx.
	$H_2/H_1 = 0.5$					
0.3	0.0369	0.0387	0.0482	0.0492	0.0552	0.0555
0.4	0.0408	0.0430	0.0535	0.0545	0.0613	0.0616
0.5	0.0442	0.0462	0.0578	0.0586	0.0663	0.0663
0.6	0.0471	0.0488	0.0615	0.0620	0.0704	0.0702
0.7	0.0495	0.0507	0.0646	0.0646	0.0739	0.0734
0.8	0.0516	0.0522	0.0672	0.0667	0.0768	0.0759
0.9	0.0535	0.0533	0.0694	0.0684	0.0793	0.0780
1.0	0.0550	0.0542	0.0712	0.0697	0.0812	0.0798
	$H_2/H_1 = 2$					
0.3	0.0490	0.0498	0.0552	0.0552	0.0581	0.0580
0.4	0.0537	0.0553	0.0610	0.0612	0.0645	0.0644
0.5	0.0579	0.0594	0.0659	0.0658	0.0696	0.0693
0.6	0.0616	0.0628	0.0699	0.0696	0.0740	0.0734
0.7	0.0649	0.0652	0.0734	0.0725	0.0776	0.0767
0.8	0.0677	0.0672	0.0763	0.0749	0.0807	0.0794
0.9	0.0700	0.0686	0.0787	0.0768	0.0832	0.0817
1.0	0.0721	0.0697	0.0806	0.0783	0.0852	0.0835

Fig. 1. Tank-Two Liquid System Considered

Fig. 2. Effect of Mass Densities on Fundamental Natural Frequency for Steel Tanks with $h/R = 0.001$, $\nu = 0.3$ and $\rho_2/\rho = 0.127$

(a) $H/R = 0.5$ (b) $H/R = 3$

Fig. 3. Distributions of Axial Displacement of Fundamental Mode of Vibration for Steel Tanks with $h/R = 0.001$, $\nu = 0.3$ and $\rho_2/\rho = 0.127$

(a) $H/R = 0.5$ (b) $H/R = 3$

Fig. 4. Distributions of Circumferential Displacement of Fundamental Mode of Vibration for Steel Tanks with $h/R = 0.001$, $\nu = 0.3$ and $\rho_2/\rho = 0.1$

(a) $H/R = 0.5$ (b) $H/R = 3$

08021

Fig. 5. Distributions of Radial Displacement of Fundamental Mode of Vibration for Steel Tanks with $h/R = 0.001$, $\nu = 0.3$ and $\rho_2/\rho = 0.127$

(a) $H/R = 0.5$ (b) $H/R = 3$

08022

Fig. 6. Distributions of Fundamental Model Hydrodynamic Pressure of Steel Tank with $h/R = 0.001$, $\nu = 0.3$ and $\rho_2/\rho = 0.127$

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