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Tests of a Homogeneous Poisson Process Against Clustering and Other Alternatives

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ABSTRACT

This report presents three closely related tests of the hypothesis that data points come from a homogeneous Poisson process. If there is too much observed variation among the log-transformed between-point distances, the hypothesis is rejected. The tests are more powerful than the standard chi-squared test against the alternative hypothesis of event clustering, but not against the alternative hypothesis of a Poisson process with smoothly varying intensity.

SUMMARY

This report presents three related tests of the hypothesis that data points come from a homogeneous Poisson process, based either on a fixed number of observed events in a random time or on a random number of observed events in a fixed time. An important alternative hypothesis allows clustering of events in time. Such clustering is sometimes seen in failure times at operating nuclear power plants, for example. The tests proposed here could be screening tools during analysis of such data.

The tests are all based on examination of the log-transformed distances between successive points. If there is too much observed variation among these transformed between-point distances, the hypothesis is rejected.

Under the null hypothesis, the distances between successive points are independent exponential(λ), and the logarithms of the distances have an extreme value distribution with location parameter $-\log\lambda$. The density is approximately bell-shaped, suggesting that the sample variance can be used to test whether the log-distances all have the same location parameter. Three sample variances are considered in this report, based on (1) estimating the mean of the log-distance by the sample mean, or (2) estimating the mean by the maximum likelihood estimate, or (3) using the known mean conditional on the number of events in time-censored data or on the final event time in event-censored data. Each sample variance is approximately a multiple of a chi-squared random variable, with multiplier and degrees of freedom tabulated in this report. The approximation can be improved (a second-order refinement) by another table in this report.

These tests were compared to each other, to the Pearson chi-squared test, and to two standard tests for trend in three settings: a Poisson process with increasing intensity function and time-censored data, the same process with event-censored data, and a Markov process that switches between a homogeneous Poisson process with low intensity function and one with high intensity function. The Markov process produces clusters of events separated by widely spaced events. Against the Markov alternative, the tests introduced here were superior to the other tests, with test (3) being slightly better than the other two. Against the increasing Poisson intensity, the tests intro-

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Tests of a Homogeneous Poisson Process Against Clustering and Other Alternatives

1 INTRODUCTION

1.1 Background and Motivation

This report presents three related tests of the hypothesis that data points come from a homogeneous Poisson process (HPP), based either on a fixed number of observed events in a random time (event-censored data) or on a random number of observed events in a fixed time (time-censored data). Such a statistical analysis can be of interest in analyzing failures in time at an operational facility. In studies of failures at nuclear power plants, clustering of failures has sometimes been observed. For example, Atwood (1992) presents a data set, similar to other such data sets, which exhibits this clustering. If it is difficult to obtain detailed information that might reveal the causes of such clustering, it could be useful to test whether the clustering is statistically significant. If it is, this could justify further work to try to investigate the causes of the clustering. A test for clustering is seen as a possible screening tool during data analysis.

The tests are all based on examination of the transformed distances between successive points. If there is too much observed variation among these transformed between-point distances, the hypothesis is rejected.

Some possible alternatives to the HPP model are

- A non-Poisson model, in which the events cluster together. Such clusters can occur if the events are equipment failures: a misdiagnosed failure can lead to a sequence of short times to failure, each followed by a superficial "repair" that does not really correct the problem. Clusters can also occur if the events are discoveries of failed standby equipment: if discovery of a failure sometimes initiates a search for other similarly failed equipment, there will be occasional clusters of failure discoveries.
- A smoothly varying Poisson intensity, which goes up and down as conditions

change. A trend is one example.

- A between-event time X with $\text{var}(X) > E^2(X)$. If the events arise from an HPP, X is exponentially distributed and $\text{var}(X) = E^2(X)$. The alternative considered here has a larger relative variance.

In all the above alternative models, the between-event times show more variation than if they had arisen from an HPP. Therefore, it is natural to consider tests based on the variation in the times between events.

1.2 Summary of Technical Results

Assume that the data are generated by an HPP. For event-censored data the logarithms of the between-event times have a distribution that is known except for a location parameter. The sample variance is approximately a multiple of a $\chi^2(df)$ random variable with appropriate degrees of freedom df . The multiplier and df are easily calculated based on theoretical formulas for the moments. A slight variation in constructing the sample variance is to center not on the customary sample mean but instead on the maximum likelihood estimate (MLE) of the mean; the resulting test is similar.

If instead the data are time censored, considerations of ancillarity suggest that inference should be conditional on the count n . Event-censored data can also be examined conditionally on the final event time, although there is no ancillarity argument to justify it. For either type of data, the events are conditionally uniformly distributed under the HPP hypothesis. The conditional distribution of the between-event times is a completely specified Dirichlet distribution, and the sample variance of the logarithms of the between-event times has a distribution approximated by a multiple of a $\chi^2(df)$ distribution. The multiplier and df/n do not vary much with n , and are given by a short table.

The χ^2 approximations all are found to be rather good in the upper tail of the distribution, especially around the 95th percentile. To improve the approximations, the

empirical distributions are found by simulation, and short tables are given relating the approximate percentiles, using the χ^2 approximations, with the empirically determined percentiles.

Therefore, with only a little computational effort any of the tests can be used to test whether the distribution is uniform, against a variety of alternatives.

The above tests were compared to several standard tests in simulations: Pearson's chi-squared test for nonconstant intensity, and the Mann test and Laplace test for trend. Against alternatives with clustering, they were superior to these tests. Against smoothly varying intensities, they were inferior to the chi-squared test, which in turn was inferior to the Mann and Laplace tests when the intensity was monotone.

2 DISTRIBUTIONAL RESULTS

Section 2.1 considers how to test the null hypothesis

H_0 : the events are generated by a homogeneous Poisson process (HPP).

Denote the unknown constant intensity by λ . Sections 2.2 and 2.3 then give distributional results when H_0 is true.

2.1 Unconditional and Conditional Distributions

Let T_i be the time of the i th event, and let

$$X_1 = T_1, \text{ and } X_i = T_i - T_{i-1} \text{ for } i > 1. \quad (2-1)$$

If the events arise from a homogeneous Poisson process (HPP) with intensity λ , the X_i 's are i.i.d exponential(λ), with $E(X_i) = 1/\lambda$. (X is mnemonic for "exponential.")

The data can be event censored or time censored. Event-censored data arise

when a fixed number, m , of successive events are observed in a random total time, T_m . Time-censored data arise when a random number, N , of events are observed from time 0 to some fixed time limit L .

The facts in this paragraph are presented by Cox and Lewis (1966, Sec. 3.3). Suppose that the data arise from a Poisson process with intensity function $\lambda(t)$. The process is homogeneous (the null hypothesis) when $\lambda(t)$ is constant. It can be useful to condition time-censored data on n , the observed number of events. Conditional on n , the T_i 's form an ordered random sample from a distribution with density proportional to $\lambda(t)$. Under the null hypothesis, T_1 through T_n are an ordered random sample from a uniform(0, L) distribution. Similarly, it can be useful to condition event-censored data on t_m , the observed time of the final event. Then T_1 through T_{m-1} form an ordered random sample from a density proportional to $\lambda(t)$; if the process is homogeneous, the distribution is uniform(0, t_m).

The justification for conditioning is especially strong when the alternative model is a nonhomogeneous Poisson process (NPP) and the data are time censored. Write $\lambda(t) = \lambda_0 g(t; \beta)$, where λ_0 is a normalizing parameter, and β is a possibly multidimensional parameter controlling the shape of the intensity function. Because the null hypothesis states that $g(t; \beta)$ is constant, testing the homogeneity of the process is a kind of inference about β . The process can be reparameterized in terms of β and $\omega \equiv \int \lambda(t) dt$, where the integration is from 0 to L . Together, N and the T_i 's are a sufficient statistic for (ω, β) . N , the random number of events, is Poisson(ω); this distribution does not involve β , so N is ancillary for β . Conditionally on n , the unordered T_i 's have a density proportional to $g(t; \beta)$; this distribution does not involve ω . Thus, inference for β (including any test for homogeneity) should be based on the T_i 's, conditional on n (Cox and Hinkley, 1974, Sections 2.2vii and 2.3iii).

For event-censored data, it can be shown that $\Lambda(T_m)$ has a gamma(m) distribution, where

$$\Lambda(t) = \int_0^t \lambda(u) du . \quad (2-2)$$

Therefore, the distribution of the final failure time, T_m , depends on β , and some

information about β is lost by conditioning on $T_m = t_m$.

In short, for either kind of data it is valid to condition and obtain a uniform distribution of event times. However, a compelling argument for doing this has been given only when the alternative model is NPP and the data are time censored.

2.2 Tests Using the Unconditional Distribution

Suppose now that H_0 is true, and consider the logarithmic transformation on the unconditional random variables. Let $G_i = \log X_i$. (G is mnemonic for "Gumbel," another name for the extreme value distribution.) The distribution of $G_i + \log \lambda$ is standard extreme value (Lawless 1982, p. 18). The scale parameter λ for X corresponds to a location parameter $\log \lambda$ for G , the central moments of G are all known and do not depend on λ , and the density of G is roughly bell shaped. Therefore, the logarithm is an exact variance stabilizing and approximately normalizing transformation. Therefore, a comparison of the sample variance of the G_i 's to the theoretical variance should reveal departures from the HPP assumption.

To carry this out, note that the moment generating function of G is

$$M_G(t) = E[\exp(tG)] = E[\exp(t \log X)] = E[X^t]$$

where X is exponential(λ). When $\lambda = 1$ this is easily shown to equal $\Gamma(1 + t)$. The first two derivatives at $t = 0$ are given by Lawless (1982). The work below requires the following moments, which were evaluated by the symbolic computer program Mathematica (Wolfram 1988):

$$E(G) = -\gamma \equiv -0.57722$$

$$E(G^2) = \pi^2/6 + \gamma^2 = 1.97811$$

$$E(G^3) = -5.44487$$

$$E(G^4) = 23.56147$$

These are the noncentral moments when $\lambda = 1$. Here, γ is Euler's gamma, a quantity that will appear throughout this report. Letting μ_i denote the i th central moment of

the standard extreme value distribution, we therefore have

$$\mu_2 = \pi^2/6 = 1.64493 \quad (2-3)$$

$$\mu_4 = 14.61136 \quad (2-4)$$

Let $\bar{G}$ denote the sample mean of G_1 through G_m , and let S_U denote the sum of squares

$$S_U = \sum_i (G_i - \bar{G})^2 \quad (2-5)$$

The subscript U stands for "unconditional," indicating that, in contrast to the distributions of Section 2.3, the distribution of the G_i 's is not conditional. Clearly, the distribution of S_U does not depend on λ . The first two moments are

$$E(S_U) = (m-1)\mu_2 \quad (2-6)$$

$$\text{var}(S_U) = [m-2 + (1/m)]\mu_4 + (m-1)[-1 + (3/m)]\mu_2^2 \quad (2-7)$$

The formula (2-6) for the mean of S_U is well known, while the derivation of the formula (2-7) is tedious but direct, and sketched in the appendix.

This report will give several test statistics, of which S_U is one. Denote them generically by S . If S were c times a χ^2 random variable with df degrees of freedom, the moments of S would be related to c and df by

$$c = \text{var}(S) / [2E(S)] \quad (2-8)$$

$$df = 2[E(S)]^2 / \text{var}(S) \quad (2-9)$$

Therefore, a natural approximation of the distribution of S_U is $c\chi^2(df)$, with c and df obtained by substituting (2-3) and (2-4) into (2-6) and (2-7), which then are substituted into (2-8) and (2-9). This approximation turns out to be rather good near the 95th percentile, but the accuracy of the approximation, and empirical improvements to it, are discussed in a later section.

It might be objected that the sample mean $\bar{G}$ in (2-5) is not a function of the

sufficient statistic for λ , and that therefore S_U should be defined differently. The MLE of $E(G_i)$ is

$$\hat{E}(G_i) = -\gamma - \log \hat{\lambda} = -\gamma + \log \bar{X}.$$

This leads to the following statistic

$$\begin{aligned} S_0 &= \sum_i [G_i - \hat{E}(G_i)]^2 \\ &= \sum_i [\log X_i + \gamma - \log \bar{X}]^2 \\ &= \sum_i [\log(X_i / \Sigma X_j) + \gamma + \log m]^2 \end{aligned} \tag{2-10}$$

Treatment of S_0 is deferred to Section 2.3, where it will be considered together with the closely related statistic S_C . The reasoning is as follows: The distribution of S_0 clearly does not depend on λ . Because ΣX_i is a complete sufficient statistic for λ , it follows from a theorem of Basu (Bain and Engelhardt 1992, Theorem 10.4.7) that S_0 and ΣX_i are independent. Therefore, the distribution of S_0 is the same as the conditional distribution of S_0 given ΣX_i . Therefore, S_0 can be treated in either Section 2.2 or 2.3, and it is more convenient to consider it with S_U in Section 2.3.

2.3 Tests Using Conditional Distributions

Suppose now that T_1 through T_n are an ordered random sample from a uniform distribution. This can arise as the conditional distribution of time-censored data, conditional on n . Then define $t_{n+1} = L$, the upper end of the observation period. It can also arise as the conditional distribution of event-censored data, conditional on the final event time t_m . Then define $n = m - 1$. In either case, T_1 through T_n are an ordered random sample from a uniform(0, t_{n+1}) distribution. Let the X_i 's and T_i 's be related by (2-1).

Let us now find the conditional distribution of the above X_i 's under the null hypothesis of an HPP. It is easiest to derive if we assume that the conditioning is on t_m with event-censored data. Unconditionally, the X_i 's are the between-event times of (2-1), i.i.d. exponential(λ). Equivalently, $2\lambda X_i$ is $\chi^2(2)$. Let

$$D_i = X_i / \Sigma X_j \quad (2-11)$$

for i and j from 1 to m . Then any $n = m - 1$ of the D_i 's have a Dirichlet distribution (see Johnson and Kotz 1972, Sec. 40.5); the notation D_i is the obvious mnemonic. This is the unconditional distribution, but it is also the distribution conditional on $\Sigma X_i \equiv T_m$, by the same argument as was used for S_0 .

This shows that the uniform conditional distribution of the T_i 's, regardless of how it arose, results in a Dirichlet distribution for any n of the D_i 's. Individually, each D_i has a beta(1, n) distribution. A formula for the mixed moments is given by Johnson and Kotz.

Motivated by Section 2.1, let $Y_i = \log D_i$, for i from 1 to $n + 1 \equiv m$. The moment generating function for any two of the Y_i 's is

$$M_{YY}(s, t) = E[\exp(sY_i + tY_j)] = E[D_i^s D_j^t]$$

This expectation can be evaluated by the formula for mixed moments given by Johnson and Kotz, yielding

$$M_{YY}(s, t) = \Gamma(n + 1)\Gamma(1 + s)\Gamma(1 + t) / \Gamma(n + 1 + s + t)$$

This can be differentiated by Mathematica, and the derivative evaluated at $s = 0$, $t = 0$, for any desired n . In particular, $M_{YY}(s, 0) = \Gamma(n + 1) / [(1 + s) \cdots (n + s)]$, from which it follows that

$$E(Y_i) = - \sum_1^n (1/k) . \quad (2-12)$$

This can be calculated directly, or evaluated as $\psi(1) - \psi(n)$ from Table 6.3 of Abramowitz and Stegun (1964).

The analogue of S_U is

$$S_C = \sum_{i=1}^{n+1} [Y_i - E(Y_i)]^2 \quad (2-13)$$

with the subscript C standing for "conditional." Any n of the Y_i 's determine the remaining Y_i , because the D_i 's sum to 1. Therefore, one of the $n+1$ terms in the sum S_C is redundant. Nevertheless, all $(n+1)$ terms are included in S_C , because we want the sum to be unique, not to depend on which n terms are selected. If, for example, the sum went from 1 to n , S_C would not be invariant with respect to symmetry: if every t_i were replaced by $t'_{n+1-i} = L - t_i$, the pattern of event times would appear the same (except in reverse order), but the value of S_C using the t' values would be different from S_C using the t values.

Both S_C and S_0 are of the form

$$S = \sum (Y_i - a)^2$$

for some constant a , and $Y_i \equiv \log D_i$. Denote the noncentral moments of Y by $E(Y^k) = \mu'_k$, denote the mixed noncentral moments $E(Y_i^h Y_j^k)$ by μ'_{hk} , and abbreviate μ'_1 as μ . The first two moments of a statistic S can be shown by direct calculation to be

$$E(S) = (n+1)(\mu'_2 - 2a\mu + a^2)$$

$$\begin{aligned} \text{var}(S) = (n+1) \{ & \mu'_4 - (n+1)(\mu'_2)^2 + n\mu'_{22} + 4a[-\mu'_3 - n\mu'_{12} + (n+1)\mu'_2\mu] \\ & + 4a^2[\mu'_2 + n\mu'_{11} - (n+1)\mu^2] \} \end{aligned}$$

For S_0 , the constant a equals $-\gamma - \log(n+1)$. For S_C , the constant a equals μ , and the expressions simplify somewhat; two ways to evaluate μ are given by equation (2-12) and the associated text. To approximate either S_0 or S_C as c times a $\chi^2(df)$ random variable, obtain c and df by equations (2-8) and (2-9).

The moments of Y depend on n , and are difficult to evaluate without a computer program such as Mathematica that can symbolically differentiate the moment generating function M_{YY} . Therefore, selected values of c and df/n are given in Table 1. For comparison, the values are also given for S_U , as calculated directly from equations (2-3) through (2-9) with $n = m - 1$.

Table 1. c and df/n , for S_C , S_0 , and S_U .

n	S_C		S_0		S_U	
	c	df/n	c	df/n	c	df/n
1	4.17	0.480	3.13	0.686	2.63	0.625
2	3.95	0.474	3.35	0.574	2.96	0.556
3	3.86	0.470	3.43	0.536	3.13	0.526
4	3.81	0.468	3.47	0.516	3.23	0.510
5	3.77	0.466	3.50	0.504	3.29	0.500
7	3.73	0.463	3.53	0.491	3.37	0.488
10	3.70	0.461	3.56	0.480	3.44	0.478
15	3.67	0.459	3.58	0.472	3.50	0.471
20	3.66	0.458	3.59	0.467	3.52	0.467
30	3.64	0.457	3.60	0.463	3.56	0.463
50	3.63	0.456	3.61	0.460	3.58	0.459
100	3.63	0.455	3.61	0.457	3.60	0.457
∞	3.619	0.4545	3.619	0.4545	3.619	0.4545

2.4 Comparisons with Empirical Distributions

The three test statistics have all been asserted to be approximate multiples of χ^2 random variables. This section assesses how good the approximations are, by comparing the approximate distributions with the empirical distributions found from simulations. The approximations are seen to be rather good near the 95th percentile. A second-order refinement is given to adjust the approximate percentiles by tabulated multipliers and make them equal to the percentiles of the empirical distributions.

The IMSL (1987) random number generator RNUN was used to generate uniform(0, 1) random variables. RNUN is a multiplicative congruential generator with modulus $2^{31} - 1$, and the multiplier 950706376 was used. For S_U , m such values U_i were generated, and S_U was calculated from equation (2-5) with $G_i = \log[-\log(U_i)]$. Based on 100,000 simulated values of S_U , the empirical cumulative distribution function

(e.c.d.f.) was found at selected points. Similarly, RNUN was used to generate $n \equiv m - 1$ uniform(0, 1) random variables T_i , which were then arranged in increasing order. T_{n+1} was defined as 1, and X_1 through X_{n+1} were defined by equation (2-1). Then S_0 was defined by equation (2-10), and S_C was defined by equation (2-13) with $Y_i = \log X_i$. The e.c.d.f. is shown in Table 2 for S_C , S_0 and S_U , for the case $n = 30$. For any of the three statistics, x is the value such that $P[c\chi^2(df) < x]$ equals the nominal value shown in the left column. Then e.c.d.f.(x) is the fraction of cases for which the simulated test statistic was $\leq x$. If the χ^2 approximation were perfect, the e.c.d.f. would equal the nominal probability, except for Monte Carlo variability. Note that the e.c.d.f. is smaller than the nominal c.d.f. near $x = 0$, but it changes from too small to too large near the 95th percentile. This pattern is also seen for other values of n , although not shown here.

For each statistic S it was noticed that $\log[P(S) > x]$ is almost a linear function of x in the right tail of the distribution. Percentiles of interest from the e.c.d.f. were then found by interpolating this nearly straight line. The interpolating routine was IMSL's CSAKM, a cubic spline interpolator using an algorithm by Akima designed to minimize oscillations. Visual inspection in selected cases showed that it seemed to be interpolating smoothly. Table 3 shows the ratio [empirical percentile]/[percentile of $c\chi^2(df)$]. This ratio refines the χ^2 approximation: multiply the approximate percentile found from the χ^2 distribution by the ratio to obtain the corresponding percentile of the empirical distribution.

For example when $n = 30$, Table 2 shows that the empirical 95th percentile of S_C is a little larger than the nominal value of 84.94. Table 3 quantifies this: the 95th percentile is calculated as $c\chi^2_{0.95}(df)$ with Table 1 supplying $c = 3.64$ and $df = 30 \times 0.457$. By Table 3, this should be multiplied by 1.013 to obtain the true 95th percentile of S_C . The assertion that the χ^2 approximations are rather good near the 95th percentile is now stated more formally: for the sample sizes shown in Table 3, the ratio of the true 95th percentile to the approximate 95th percentile is never less than 0.945 or more than 1.013.

Note in Table 3, the approximation becomes better (that is, the ratio becomes closer to 1.0) as n increases. For very large n , the Central Limit Theorem is driving

this, making both the true distribution of S and the χ^2 approximation approximately normal, and therefore approximately equal to each other.

Table 2. Empirical cumulative distribution function (e.c.d.f.) at selected x , with $n = 30$.

Nominal c.d.f.(x)	S_C		S_0		S_U	
	x	e.c.d.f.(x)	x	e.c.d.f.(x)	x	e.c.d.f.(x)
0.010	16.40	0.000 (± 0.000) ^a	16.55	0.000 (± 0.000)	16.33	0.002 (± 0.000)
0.050	23.23	0.015 (± 0.001)	23.38	0.016 (± 0.001)	23.07	0.027 (± 0.001)
0.100	27.60	0.060 (± 0.001)	27.74	0.060 (± 0.001)	27.37	0.073 (± 0.002)
0.300	38.50	0.302 (± 0.003)	38.59	0.301 (± 0.003)	38.09	0.297 (± 0.003)
0.500	47.57	0.535 (± 0.003)	47.61	0.534 (± 0.003)	47.00	0.522 (± 0.003)
0.700	57.97	0.734 (± 0.003)	57.94	0.733 (± 0.003)	57.20	0.729 (± 0.003)
0.800	64.93	0.822 (± 0.002)	64.86	0.821 (± 0.002)	64.04	0.820 (± 0.002)
0.900	75.46	0.905 (± 0.002)	75.30	0.904 (± 0.002)	74.36	0.906 (± 0.002)
0.950	84.94	0.947 (± 0.001)	84.71	0.947 (± 0.001)	83.64	0.949 (± 0.001)
0.980	96.48	0.974 (± 0.001)	96.14	0.973 (± 0.001)	94.95	0.976 (± 0.001)
0.990	104.7	0.984 (± 0.001)	104.3	0.984 (± 0.001)	103.0	0.985 (± 0.001)
0.995	112.5	0.990 (± 0.001)	112.1	0.990 (± 0.001)	110.7	0.991 (± 0.001)
0.999	130.0	0.996 (± 0.000)	129.4	0.996 (± 0.000)	127.8	0.996 (± 0.000)

a. Number in parentheses gives approximate 95% confidence limits on the probability.

Table 3. Ratio of [empirical percentile]/[percentile of $c\chi^2(df)$]

	S_C	S_0	S_U
<u>c.d.f.</u>	<u>$n = 5$</u>		
0.800	0.877 (± 0.007) ^a	0.873 (± 0.006)	0.913 (± 0.006)
0.900	0.910 (± 0.008)	0.901 (± 0.008)	0.922 (± 0.007)
0.950	0.956 (± 0.010)	0.961 (± 0.010)	0.945 (± 0.009)
0.975	1.016 (± 0.013)	1.018 (± 0.014)	1.003 (± 0.012)
0.990	1.098 (± 0.021)	1.103 (± 0.019)	1.071 (± 0.019)
0.995	1.174 (± 0.029)	1.164 (± 0.027)	1.130 (± 0.026)
	<u>$n = 7$</u>		
0.800	0.902 (± 0.006)	0.899 (± 0.006)	0.926 (± 0.005)
0.900	0.927 (± 0.007)	0.928 (± 0.007)	0.941 (± 0.007)
0.950	0.976 (± 0.009)	0.978 (± 0.009)	0.973 (± 0.009)
0.975	1.027 (± 0.012)	1.037 (± 0.013)	1.016 (± 0.012)
0.990	1.113 (± 0.019)	1.121 (± 0.019)	1.080 (± 0.018)
0.995	1.161 (± 0.025)	1.193 (± 0.025)	1.140 (± 0.024)
	<u>$n = 10$</u>		
0.800	0.920 (± 0.005)	0.920 (± 0.005)	0.938 (± 0.005)
0.900	0.947 (± 0.006)	0.947 (± 0.006)	0.953 (± 0.006)
0.950	0.984 (± 0.008)	0.991 (± 0.008)	0.986 (± 0.008)
0.975	1.026 (± 0.010)	1.039 (± 0.011)	1.023 (± 0.011)
0.990	1.105 (± 0.017)	1.114 (± 0.017)	1.079 (± 0.015)
0.995	1.168 (± 0.023)	1.184 (± 0.024)	1.140 (± 0.021)
	<u>$n = 15$</u>		
0.800	0.941 (± 0.004)	0.941 (± 0.004)	0.955 (± 0.004)
0.900	0.964 (± 0.005)	0.967 (± 0.005)	0.969 (± 0.005)
0.950	0.996 (± 0.007)	1.005 (± 0.007)	0.997 (± 0.007)
0.975	1.029 (± 0.009)	1.043 (± 0.010)	1.026 (± 0.009)
0.990	1.096 (± 0.014)	1.106 (± 0.014)	1.084 (± 0.014)
0.995	1.141 (± 0.017)	1.158 (± 0.017)	1.122 (± 0.018)
	<u>$n = 20$</u>		
0.800	0.955 (± 0.004)	0.954 (± 0.004)	0.962 (± 0.004)
0.900	0.977 (± 0.005)	0.977 (± 0.005)	0.977 (± 0.005)
0.950	1.004 (± 0.006)	1.010 (± 0.006)	1.002 (± 0.006)
0.975	1.035 (± 0.009)	1.037 (± 0.008)	1.027 (± 0.008)
0.990	1.096 (± 0.013)	1.087 (± 0.012)	1.069 (± 0.012)
0.995	1.136 (± 0.017)	1.132 (± 0.017)	1.102 (± 0.015)

Table 3. (Continued)

	S_C	S_0	S_U
c.d.f.	$n = 30$		
0.800	0.969 (± 0.003)	0.970 (± 0.003)	0.973 (± 0.003)
0.900	0.988 (± 0.004)	0.991 (± 0.004)	0.988 (± 0.004)
0.950	1.013 (± 0.005)	1.013 (± 0.005)	1.005 (± 0.005)
0.975	1.039 (± 0.007)	1.039 (± 0.007)	1.026 (± 0.007)
0.990	1.084 (± 0.011)	1.078 (± 0.010)	1.062 (± 0.010)
0.995	1.128 (± 0.016)	1.111 (± 0.014)	1.101 (± 0.015)
	$n = 50$		
0.800	0.981 (± 0.003)	0.983 (± 0.003)	0.986 (± 0.003)
0.900	0.996 (± 0.003)	0.996 (± 0.003)	0.995 (± 0.003)
0.950	1.012 (± 0.004)	1.013 (± 0.004)	1.010 (± 0.004)
0.975	1.031 (± 0.006)	1.035 (± 0.005)	1.026 (± 0.005)
0.990	1.062 (± 0.008)	1.063 (± 0.008)	1.049 (± 0.007)
0.995	1.087 (± 0.011)	1.086 (± 0.011)	1.071 (± 0.011)
	$n = 100$		
0.800	0.991 (± 0.002)	0.991 (± 0.002)	0.993 (± 0.002)
0.900	0.999 (± 0.002)	1.000 (± 0.002)	0.997 (± 0.002)
0.950	1.010 (± 0.003)	1.011 (± 0.003)	1.006 (± 0.003)
0.975	1.021 (± 0.004)	1.024 (± 0.004)	1.016 (± 0.004)
0.990	1.036 (± 0.005)	1.046 (± 0.006)	1.033 (± 0.005)
0.995	1.048 (± 0.007)	1.061 (± 0.008)	1.044 (± 0.007)
	$n = 500$		
0.800	0.998 (± 0.001)	0.998 (± 0.001)	0.999 (± 0.001)
0.900	1.000 (± 0.001)	1.000 (± 0.001)	1.000 (± 0.001)
0.950	1.003 (± 0.001)	1.004 (± 0.001)	1.003 (± 0.001)
0.975	1.006 (± 0.002)	1.007 (± 0.002)	1.006 (± 0.002)
0.990	1.008 (± 0.002)	1.010 (± 0.002)	1.008 (± 0.002)
0.995	1.011 (± 0.003)	1.013 (± 0.003)	1.012 (± 0.003)

a. Number in parentheses gives approximate 95% confidence limits on the ratio.

3 POWER

3.1 Simulated Models

To explain the ideas, consider the simplest alternative hypothesis, that the data are generated from a Poisson process with nonconstant $\lambda(t)$. The example chosen for simulations was

$$\lambda(t) = \lambda_0 \exp(\beta t) \quad . \quad (3-1)$$

Consider time-censored data first, with upper limit L arbitrarily set to 1. A sample of size n was drawn from a population with density proportional to $\lambda(t)$, namely

$$\begin{aligned} f(t) &= \beta \exp(\beta t) / [\exp(\beta L) - 1] && \text{if } 0 \leq t \leq L \text{ and } \beta \neq 0 \\ f(t) &= 1 && \text{if } 0 \leq t \leq L \text{ and } \beta = 0 \quad . \end{aligned}$$

This is the distribution conditional on n , and it does not depend on λ_0 . It is possible to do this for any β , positive or negative, but a negative β yields a density which is the mirror image of the density corresponding to $|\beta|$.

For event-censored data, the distribution of X_i , conditional on t_{i-1} , satisfies

$$\Lambda(t_{i-1} + X_i) - \Lambda(t_{i-1}) \sim \text{exponential}(1) \quad ,$$

where $\Lambda(t)$ is defined by (2-2), and equals $(\lambda_0/\beta)[\exp(\beta t) - 1]$ in the present example. To carry out the simulation of m events, both λ_0 and β must be specified, and the relative spacing of the simulated events depends on *both* parameters. If λ_0 is very large, the m events will occur in a short time, and the Poisson intensity (3-1) will be almost constant in the observed range $[0, t_m]$. If instead λ_0 is very small, the final event time will be large, and the intensity (3-1) will vary greatly in the observed range $[0, t_m]$. To make the time-censored and event-censored simulations comparable, a different λ_0 was chosen for each β and m , to satisfy

$$\Lambda(L) = E[\Lambda(T_m)] \equiv m \quad .$$

Here L is the upper limit for time-censored data with $n = m - 1$ events, and the value of the expectation follows from the gamma(m) distribution of $\Lambda(T_m)$. It is interesting to note that event-censored data with intensity (3-1) can only be simulated when $\beta \geq 0$. When $\beta < 0$, there is a nonzero probability that no more events will occur.

The above discussion shows that there are differences between time-censored data conditioned on n and event-censored data conditioned on the final event time, even when the underlying model is simple, a Poisson process. Consider now a non-Poisson process.

A Markov model was chosen for simulation. The events were thought of as failures of a system, and two states were assumed, G and B ("good" and "bad"). When the system is in state G , the most recent failure was correctly diagnosed and repaired, and the time until the next failure is exponential(λ_G). When the system is in state B , the most recent failure was misdiagnosed and therefore the time until the next failure is exponential(λ_B), with $\lambda_B > \lambda_G$. A potential change of state occurs at the time of each failure. For the simulations, the transition probability P_{BG} was always set to 0.5. This corresponds to an average run of 2 failures in the bad state. Simulations were performed for various sample sizes, values of the ratio λ_B/λ_G , and values of the transition probability P_{GB} .

With such a model, it is natural to simulate a fixed number of events in a random time period (event-censored data). It is not obvious how to simulate time-censored data with a predetermined number of observations. Therefore, only event-censored data were simulated.

3.2 Tests Considered

The three test statistics introduced above were S_U , S_0 , and S_C . All three statistics were used in the power investigations, no matter how the data were generated. S_U was used with time-censored data by setting $m = n + 1$ and $t_m = L$. S_C was used with event-censored data by setting $n = m - 1$ and $L = t_m$, treated as fixed.

Three other standard tests were also used, for comparison. The Pearson chi-squared test was constructed by dividing $[0, L]$ or $[0, t_m]$ into $n/5$ cells of equal length. Then under H_0 , the expected count in each cell was 5. The values for n were 10, 20, and 50, leading to 2, 4, and 10 cells, respectively.

The Mann (1945) test for trend, described by Ascher and Feingold (1984, p. 80), is constructed as follows. A reverse arrangement is said to exist when $X_i < X_j$ for $i < j$. There are $nm/2$ comparisons made. Under H_0 , the number of reversals R has mean and variance

$$E(R) = nm/4$$

$$\text{var}(R) = (2m^3 + 3m^2 - 5m)/72 .$$

For $m \geq 10$,

$$Z = \frac{R + 0.5 - E[R]}{[\text{var}(R)]^{1/2}}$$

is approximately standard normal. The value 0.5 is a continuity correction. A two-sided test was used, rejecting H_0 if $|Z| > 1.96$.

Finally the Laplace test for trend is described by Ascher and Feingold (1984, p. 79) and by Cox and Lewis (1966, Sec. 3.3). Let $\bar{T}$ be the sample mean of the n event times in the interior of the observation period. Under H_0 the T_i 's are uniformly distributed, and therefore $\bar{T}$ has mean $L/2$ and standard deviation $L/\sqrt{12n}$, and is approximately normal even for quite small n . The two-sided test was used, rejecting H_0 if

$$|(\bar{T} - L/2)\sqrt{12n}/L| > 1.96 .$$

Cox and Lewis derive the Laplace statistic as the derivative of the conditional loglikelihood under model (3-1). It follows (Cox and Hinkley, 1974, Sec. 4.8.iii) that the one-sided Laplace tests are locally most powerful against one-sided alternatives of the form

(3-1) with $\beta > 0$ or with $\beta < 0$.

3.3 Results of Simulations

The results are shown in Tables 4-6. The simulated power, or $P(\text{reject } H_0)$, is shown for each of the six tests. Each tabulated power is the observed fraction of times when the test statistic was greater than the critical value that corresponds to 5% Type I error. The standard error of any tabulated power p is $[p(1 - p)/100000]^{1/2}$.

As can be seen, the tests introduced in this report are better than any of the others against the Markov model with clustering. They are better than the chi-squared test, which is the standard test that one would consider against such an alternative model. The test based on S_C is slightly better than the other two, although it is not clear whether this is a result of small miscalibration of the tests. It is surprising that the Laplace test for trend succeeds in detecting the departure from the null model at all; its power is low, and not very dependent on the sample size, but greater than the probability of Type I error. Apparently, the Markov model sometimes produces data sequences that seem to show a trend, and the Laplace test catches these sequences.

Against a nonhomogeneous Poisson process with monotone failure rate, the Laplace test is the best, just as would be expected, and the Mann test is second best. This result holds whether the data are time censored or event censored. The chi-squared test is next, and the three tests introduced in this report have much lower power than the chi-squared test. The chi-square test uses data pooled into bins, while the tests introduced in this report use the exact event times. Apparently, the pooling of events with similar λ gives the chi-square test an advantage that more than compensates for the loss of the exact event times.

Table 4. Power of tests for event-censored Markov process.

λ_B/λ_G	P_{GB}	S_U	S_0	S_C	X^2	Mann	Laplace
$n = 10$							
5.0	0.00	0.049	0.049	0.050	0.020	0.049	0.047
5.0	0.01	0.053	0.053	0.055	0.023	0.053	0.054
5.0	0.05	0.066	0.068	0.070	0.033	0.061	0.073
5.0	0.10	0.082	0.087	0.090	0.042	0.065	0.090
5.0	0.20	0.099	0.108	0.112	0.056	0.066	0.115
5.0	0.50	0.121	0.140	0.145	0.076	0.049	0.132
5.0	0.90	0.115	0.140	0.145	0.080	0.035	0.126
10.0	0.00	0.048	0.048	0.049	0.020	0.049	0.046
10.0	0.01	0.059	0.061	0.062	0.023	0.054	0.056
10.0	0.05	0.095	0.102	0.105	0.036	0.067	0.086
10.0	0.10	0.133	0.146	0.150	0.053	0.077	0.113
10.0	0.20	0.189	0.215	0.221	0.076	0.076	0.150
10.0	0.50	0.248	0.303	0.312	0.113	0.050	0.189
10.0	0.90	0.241	0.317	0.327	0.123	0.027	0.187
20.0	0.00	0.049	0.048	0.050	0.021	0.050	0.049
20.0	0.01	0.071	0.072	0.074	0.025	0.055	0.059
20.0	0.05	0.153	0.162	0.166	0.041	0.073	0.094
20.0	0.10	0.236	0.256	0.261	0.059	0.083	0.127
20.0	0.20	0.356	0.397	0.404	0.088	0.082	0.174
20.0	0.50	0.492	0.583	0.594	0.136	0.050	0.227
20.0	0.90	0.478	0.609	0.620	0.158	0.023	0.237
50.0	0.00	0.050	0.049	0.050	0.021	0.049	0.046
50.0	0.01	0.093	0.095	0.097	0.026	0.055	0.059
50.0	0.05	0.255	0.261	0.265	0.044	0.075	0.099
50.0	0.10	0.419	0.433	0.437	0.062	0.087	0.136
50.0	0.20	0.636	0.661	0.667	0.097	0.087	0.190
50.0	0.50	0.837	0.894	0.899	0.154	0.048	0.254
50.0	0.90	0.814	0.904	0.910	0.185	0.019	0.281

Table 4. (continued)

λ_B/λ_G	P_{GB}	S_U	S_0	S_C	X^2	Mann	Laplace
<hr/>							
$n = 20$							
5.0	0.00	0.048	0.048	0.049	0.040	0.051	0.049
5.0	0.01	0.055	0.055	0.056	0.048	0.055	0.054
5.0	0.05	0.077	0.079	0.081	0.074	0.068	0.076
5.0	0.10	0.099	0.104	0.107	0.100	0.074	0.095
5.0	0.20	0.129	0.142	0.145	0.142	0.074	0.120
5.0	0.50	0.161	0.190	0.195	0.192	0.051	0.143
5.0	0.90	0.152	0.186	0.192	0.196	0.034	0.139
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10.0	0.00	0.049	0.049	0.050	0.041	0.051	0.048
10.0	0.01	0.063	0.064	0.065	0.049	0.057	0.057
10.0	0.05	0.123	0.130	0.133	0.086	0.076	0.084
10.0	0.10	0.191	0.207	0.212	0.127	0.086	0.114
10.0	0.20	0.292	0.326	0.333	0.188	0.084	0.150
10.0	0.50	0.394	0.474	0.483	0.288	0.051	0.194
10.0	0.90	0.367	0.476	0.487	0.318	0.027	0.202
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20.0	0.00	0.049	0.049	0.050	0.041	0.051	0.049
20.0	0.01	0.083	0.084	0.085	0.051	0.059	0.057
20.0	0.05	0.215	0.224	0.227	0.094	0.082	0.092
20.0	0.10	0.363	0.382	0.388	0.143	0.095	0.123
20.0	0.20	0.568	0.609	0.617	0.220	0.092	0.170
20.0	0.50	0.751	0.833	0.840	0.347	0.049	0.225
20.0	0.90	0.708	0.832	0.839	0.402	0.021	0.245
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50.0	0.00	0.048	0.047	0.048	0.040	0.052	0.049
50.0	0.01	0.119	0.119	0.121	0.052	0.060	0.059
50.0	0.05	0.366	0.370	0.373	0.102	0.087	0.096
50.0	0.10	0.601	0.606	0.610	0.151	0.101	0.132
50.0	0.20	0.855	0.863	0.866	0.239	0.100	0.182
50.0	0.50	0.983	0.993	0.993	0.385	0.052	0.249
50.0	0.90	0.966	0.990	0.990	0.460	0.018	0.278

Table 4. (continued)

λ_B/λ_G	P_{GB}	S_U	S_0	S_C	X^2	Mann	Laplace
<hr/>							
$n = 50$							
5.0	0.00	0.049	0.049	0.050	0.045	0.048	0.049
5.0	0.01	0.058	0.058	0.060	0.060	0.054	0.054
5.0	0.05	0.099	0.103	0.106	0.114	0.066	0.074
5.0	0.10	0.148	0.157	0.161	0.174	0.074	0.096
5.0	0.20	0.219	0.240	0.246	0.265	0.074	0.121
5.0	0.50	0.288	0.335	0.342	0.372	0.049	0.148
5.0	0.90	0.260	0.316	0.324	0.378	0.031	0.147
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10.0	0.00	0.049	0.048	0.050	0.044	0.048	0.048
10.0	0.01	0.073	0.074	0.076	0.066	0.055	0.057
10.0	0.05	0.192	0.200	0.204	0.144	0.077	0.084
10.0	0.10	0.345	0.365	0.371	0.233	0.088	0.112
10.0	0.20	0.566	0.609	0.616	0.372	0.084	0.148
10.0	0.50	0.733	0.814	0.821	0.565	0.048	0.198
10.0	0.90	0.681	0.795	0.801	0.618	0.023	0.207
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20.0	0.00	0.047	0.047	0.048	0.046	0.048	0.049
20.0	0.01	0.103	0.104	0.106	0.067	0.058	0.058
20.0	0.05	0.361	0.369	0.375	0.160	0.084	0.090
20.0	0.10	0.632	0.647	0.653	0.268	0.097	0.121
20.0	0.20	0.895	0.910	0.913	0.434	0.095	0.164
20.0	0.50	0.985	0.995	0.995	0.658	0.049	0.227
20.0	0.90	0.968	0.991	0.992	0.745	0.018	0.252
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50.0	0.00	0.049	0.049	0.051	0.045	0.050	0.049
50.0	0.01	0.163	0.164	0.166	0.070	0.059	0.057
50.0	0.05	0.583	0.585	0.589	0.170	0.087	0.091
50.0	0.10	0.867	0.869	0.872	0.290	0.102	0.125
50.0	0.20	0.992	0.992	0.992	0.467	0.101	0.177
50.0	0.50	1.000	1.000	1.000	0.717	0.049	0.245
50.0	0.90	1.000	1.000	1.000	0.812	0.014	0.278

Table 5. Power of tests for time-censored Poisson process with $\lambda(t) = \lambda_0 \exp(\beta t)$.

β	Test					
	S_U	S_0	S_C	X^2	Mann	Laplace
$n = 10$						
0.00	0.050	0.050	0.051	0.023	0.050	0.050
0.50	0.051	0.052	0.053	0.032	0.056	0.072
1.00	0.056	0.058	0.059	0.063	0.089	0.141
1.50	0.065	0.068	0.070	0.119	0.144	0.257
2.00	0.075	0.082	0.084	0.204	0.211	0.407
2.50	0.088	0.101	0.104	0.311	0.282	0.568
3.00	0.104	0.124	0.128	0.433	0.349	0.715
$n = 20$						
0.00	0.049	0.049	0.050	0.041	0.051	0.049
0.50	0.052	0.051	0.053	0.061	0.081	0.096
1.00	0.057	0.059	0.060	0.130	0.171	0.241
1.50	0.068	0.072	0.074	0.268	0.307	0.467
2.00	0.084	0.093	0.096	0.460	0.459	0.700
2.50	0.105	0.121	0.125	0.662	0.589	0.869
3.00	0.129	0.159	0.164	0.821	0.687	0.955
$n = 50$						
0.00	0.050	0.050	0.051	0.047	0.050	0.050
0.50	0.054	0.053	0.055	0.082	0.135	0.174
1.00	0.065	0.067	0.068	0.218	0.384	0.523
1.50	0.085	0.090	0.092	0.485	0.674	0.853
2.00	0.116	0.129	0.133	0.774	0.864	0.978
2.50	0.159	0.188	0.193	0.941	0.947	0.999
3.00	0.212	0.267	0.273	0.991	0.976	1.000

Table 6. Power of tests for event-censored Poisson process with $\lambda(t) = \lambda_0 \exp(\beta t)$.

β	λ_0	Test					
		S_U	S_0	S_C	X^2	Mann	Laplace
$n = 10$							
0.00	11.000	0.051	0.051	0.052	0.022	0.051	0.048
0.50	8.478	0.052	0.053	0.054	0.031	0.057	0.072
1.00	6.402	0.057	0.058	0.060	0.063	0.090	0.142
1.50	4.739	0.065	0.068	0.069	0.119	0.142	0.256
2.00	3.443	0.074	0.081	0.083	0.201	0.207	0.400
2.50	2.459	0.087	0.098	0.101	0.305	0.277	0.556
3.00	1.729	0.101	0.121	0.125	0.422	0.341	0.700
$n = 20$							
0.00	21.000	0.050	0.049	0.051	0.041	0.052	0.048
0.50	16.186	0.052	0.052	0.053	0.061	0.082	0.097
1.00	12.222	0.058	0.059	0.061	0.133	0.169	0.243
1.50	9.047	0.069	0.073	0.075	0.268	0.305	0.463
2.00	6.574	0.085	0.093	0.096	0.454	0.453	0.688
2.50	4.695	0.106	0.122	0.126	0.651	0.583	0.854
3.00	3.301	0.129	0.157	0.162	0.810	0.679	0.947
$n = 50$							
0.00	51.000	0.049	0.049	0.050	0.047	0.050	0.049
0.50	39.308	0.053	0.053	0.054	0.080	0.135	0.173
1.00	29.681	0.064	0.065	0.067	0.215	0.380	0.521
1.50	21.972	0.085	0.090	0.092	0.481	0.666	0.844
2.00	15.965	0.115	0.129	0.132	0.769	0.857	0.974
2.50	11.402	0.158	0.187	0.192	0.935	0.943	0.998
3.00	8.017	0.211	0.266	0.272	0.989	0.975	1.000

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APPENDIX: TECHNICAL DETAILS

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The variance of

$$S_U = \sum_i (G_i - \bar{G})^2$$

is given by Equation (2-7). The derivation of that equation is given here. Without loss of generality, the location parameter $\log \lambda$ can be chosen such that $E(G_i) = 0$. Then the k th central moment μ_k equals the k th moment about zero, $E(G_i^k)$. Define $Y_i = G_i - \bar{G}$. Then S_U equals $\sum_i Y_i^2$ and $\text{var}(S_U)$ equals $E[(\sum_i Y_i^2)^2] - E^2(S_U)$. The second term in the variance, $E^2(S_U)$, can be found from Equation (2-6). Therefore, consider the term $(\sum_i Y_i^2)^2$. This expression breaks into two parts

$$(\sum_i Y_i^2)^2 = \sum_i Y_i^4 + \sum_i \sum_{j \neq i} Y_i^2 Y_j^2 \quad (\text{A-1})$$

which we consider separately.

The first term on the right in Equation (A-1) is

$$\begin{aligned} \sum_i Y_i^4 &= \sum_i (G_i - \bar{G})^4 \\ &= \sum_i [G_i^4 - 4G_i^3 \bar{G} + 6G_i^2 \bar{G}^2 - 4G_i \bar{G}^3 + \bar{G}^4] \\ &= \sum_i G_i^4 - 4 \sum_i (G_i^3 + J_1)/n + 6 \sum_i G_i^2 (\bar{G}^2 + \sum_{j \neq i} G_j^2 + J_2)/n^2 - 4n \bar{G}^4 + n \bar{G}^4 \end{aligned}$$

where the J terms consist of sums of products involving G_j to the first power, so that $E(J_1) = E(J_2) = 0$. It follows that

$$E(\sum_i Y_i^4) = n\mu_4 - 4\mu_4 + 6\mu_4/n + 6\mu_2^2(n-1)/n - 3nE(\bar{G}^4) \quad (\text{A-2})$$

The second term on the right in Equation (A-1) is

$$\begin{aligned} \sum_i \sum_{j \neq i} Y_i^2 Y_j^2 &= \sum_i \sum_{j \neq i} (G_i^2 - 2G_i \bar{G} + \bar{G}^2)(G_j^2 - 2G_j \bar{G} + \bar{G}^2) \\ &= \sum_i \sum_{j \neq i} (G_i^2 G_j^2 - 4G_i G_j^2 \bar{G} + 4G_i G_j \bar{G}^2 + 2G_i^2 \bar{G}^2 - 4G_i \bar{G}^3 + \bar{G}^4) \\ &= \sum_i \sum_{j \neq i} (G_i^2 G_j^2 - 4G_i^2 G_j^2/n + 8G_i^2 G_j^2/n^2) \end{aligned}$$

$$+ 2(n-1)\Sigma_i G_i^2 (G_i^2 + \Sigma_{k \neq i} G_k^2)/n^2 - 3n(n-1)\bar{G}^4 + J$$

where J is a term with zero mean. The expectation of this expression is

$$\begin{aligned} E(\Sigma_i \Sigma_{j \neq i} Y_i^2 Y_j^2) \\ = n(n-1)(1 - 4/n + 8/n^2)\mu_2^2 \\ + 2(n-1)[\mu_4 + (n-1)\mu_2^2]/n - 3n(n-1)E(\bar{G}^4). \end{aligned} \quad (A-3)$$

Equations (A-2) and (A-3) both involve the fourth moment of $\bar{G}$. This is evaluated by noting that

$$\bar{G}^4 = \Sigma_i G_i^4/n^4 + 3\Sigma_i \Sigma_{j \neq i} G_i^2 G_j^2/n^4 + J$$

where $E(J) = 0$. Therefore, we have

$$E(\bar{G}^4) = \mu_4/n^3 + 3(n-1)\mu_2^2/n^3. \quad (A-4)$$

Combining Equations (A-2) and (A-3) shows that the expectation of Expression (A-1) is

$$\mu_4(n-2+4/n) + (n-1)(n-2+12/n)\mu_2^2 - 3n^2 E(\bar{G}^4).$$

Substitution of Equation (A-4) shows that the expectation of Expression (A-1) is

$$\mu_4(n-2+1/n) + (n-1)\mu_2^2(n-2+3/n).$$

Equation (2-7) follows from combining this result with Equation (2-6).

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