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REFRIGERATION

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REGENERATOR OPTIMIZATION FOR STIRLING CYCLE REFRIGERATION

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Abstract

A cryogenic regenerator for a Stirling cycle is designed using a fractional loss or entropy gain as the criterion of performance. The gas losses are treated separately from heat storage medium losses. For the optimum channel flow nonturbulent design, the maximization of heat transfer from the gas to the wall and the minimization of entropy production by friction leads to a gas flow velocity criterion of sound speed times loss fraction. This velocity with a given frequency leads to a Stirling cycle dead volume ratio and consequently a channel of specified length and width. The thermal properties of the channel wall must then accommodate this cyclic heat flow without substantially increasing the loss fraction. It is found that stainless steel or plastics are adequate for 30 to 300 K, but that lower temperature stages 4 to 30 K require either a special lead alloy of moderate conductivity or a segmented anisotropic construction of alternate highly conducting lead layers and alternate insulating glass or epoxy fiber glass spacers. An overall efficiency of $\cong 50\%$ of Carnot is predicted at a frequency of 30 Hz and a pressure of one atmosphere.

I. Introduction

Regenerators are used in Stirling refrigeration cycles to store and return the heat of the working fluid, i.e. gas, with small entropy gain during each of two phases of the cycle. If the refrigerator operates across a small temperature difference, relative to absolute temperature, as in domestic heat pumps and refrigerators, the regenerator is simpler to design because the thermal properties of the materials and of the gas remain nearly constant throughout the regenerator. In cryogenic refrigerators where the temperature ratio is large, the regenerator is more complex and the efficiency more sensitive to its properties.

We will discuss cryogenic regenerators and show how one arrives at a design that may reach 50% of Carnot efficiency at 4 K. Our objective is to comprehend the design

space which involves very many variables. Because of this complexity, whenever the effect of a variable on the loss fraction is weak, we replace it with an approximate number and display a simplified but approximate relationship. Later when the design direction is understood, computer optimization can take place using a more accurate description.

Sullivan, Radebaugh, Daney, and Zimmerman (1982) have considered a similar design, but did not include the entropy gain from the frictional heat of the gas flow. Here we believe we have considered all thermally important constraints. The dead volume of the regenerator is specifically included as one constraint because it affects strongly the useful refrigeration.

In this paper we address primarily the problem of the efficiency of the regenerator in a Stirling cycle. There are, however, several important additional components to a cryocooler that will affect its efficiency. We assume that our regenerator is bounded at each end by isothermal compression and expansion volumes. These volumes will have their own losses although we expect that the use of special metal bellows (Colgate 1984) can keep these losses small.

Similarly, radiation and conduction to the cold end should be intercepted at as high a temperature as possible so that the entropy gain will be a minimum. A regenerator is a passive element in the sense that it can perform no work and hence, can generate no refrigeration. On the other hand, the interception of heat losses along the length of the regenerator is an advantage compared to intercepting the loss at the cold end. This is because the entropy generated by a given loss Q at an intermediate temperature T_i is Q_i/T_i . This will be considerably smaller than intercepting the loss at T_c if $T_i \gg T_c$. A refrigerator transfers entropy from the cold to the hot reservoir so that Q_c/T_c and hence, all parasitic losses should be intercepted at as high a temperature as possible.

Therefore from the standpoint of intermediate heat losses, heat can be transferred to and from the gas stream in the regenerator without a penalty greater than the entropy flow. However, there are reasons to divide the regenerator into stages by separate compression and expansion volumes.

The advantage of staging is that the working fluid or its pressure can be different in different stages. Since the ultimate goal of a cryocooler is to achieve liquid helium temperature efficiently, the working fluid at the lowest temperature must also be helium.

Classically the bottom stage of most helium cryocoolers to 4K are Joule Thompson because of the liquid phase transition at one atmosphere pressure. Here, however, we have the possibility of doing refrigeration work at subatmospheric pressure in the lowest temperature stage. The highest temperature stages may operate at a minimum cycle pressure P_O of 4 atmospheres, depending upon the compression and expansion volume limitations, but as one goes to lower temperatures and the working fluid mass is conserved, the working pressure can be reduced while the working volumes remain constant. Thus the lowest stage may have $P_O \cong 1/3$ to $1/2$ atm. Hence, we give the regenerator design in terms of the minimum working pressure, P_O , of the cycle in atmospheres and expect the final design to take advantage of reduced pressure in the lower temperature stages.

In Section II we will discuss the geometry of the regenerator with particular emphasis on the question of why we choose a smooth wall channel of nearly constant width instead of the frequently used geometries of wire mesh or spheres of heat mass material. In addition, we will consider dividing the regenerator into uniform sections in each of which the temperature difference from one end to the other is small enough that the working fluid and material properties can be approximated as constant.

In Section III we will enumerate all the losses using as a non-dimensional loss the fractional addition of entropy per cycle. We will assume laminar flow and diffusive heat transport and show that these assumptions are consistent with the results. The working fluid losses will then define a relatively simple optimum mean gas velocity as a fraction of sound speed. This fraction of sound speed turns out to be conveniently the above nondimensional loss or fractional addition of entropy per cycle. Using this velocity criterion, the dead volume ratio of the Stirling cycle due to the regenerator volume is derived in Section IV. This allows one to relate frequency, channel length, temperature, dead volume ratio, and nondimensional loss. This relationship in Section V defines the length and width of the regenerator channel as a function of temperature resulting in an overall gas efficiency and dead volume. At the lowest temperature there will be some turbulence, but the overall design remains valid.

We then ask in Section VI whether there are channel wall materials with heat capacity and thermal conductivity suitable for storing the working fluid heat as required by the above separately optimized velocity, channel width, and length without too large additional losses. In other words we first optimize the working fluid gas parameters and hope to find wall materials that satisfy the working fluid gas requirements.

The thermal skin depth defines the optimum wall thickness. The longitudinal thermal conduction in this thickness must not short circuit the cycle thermally. At low temperatures, 4-16 K, and at the relatively high frequency of 30 Hz, lead, alloyed with small amounts of cesium or bismuth, may prove adequate, but at a lower frequency like 15 Hz, which is necessary for good gas efficiency, a thermally anisotropic channel wall will be required. A multilayer, laminated material of pure lead and epoxy fiber glass sheet, for example, allows heat to penetrate parallel to the laminations (perpendicular to the gas flow) and access a large heat mass of lead, whereas the very low conductivity of the epoxy fiber glass prevents longitudinal thermal conduction. With such a material for the low temperature sections (4 to 30 K) and with plastic or stainless steel foil or laminated copper-epoxy fiber glass for the upper sections, a complete regenerator design can be made. The overall efficiency of the regenerator will be calculated to be of the order of 50% of Carnot. This efficiency and the power density is discussed in the summary of Section VII.

II. Channel Geometry

Classically, regenerators in Stirling cycle heat pumps are heterogeneous, typically made of wire mesh, spheres, or random wires (see Walker 1980). Recently, Radebaugh and Louie (1984) and independently Colgate (1986) have pointed out that the smooth wall channel is a far better geometry for efficient heat transfer. This is based upon the extensive measurements of transient heat flow by Coopage and London (1956). A quantitative comparison is complex and given in the above references. The additional dead volume penalty for a random matrix regenerator giving the same heat transfer and friction loss compared to a uniform width channel is $\cong 13$ -fold. It is sufficient that the uniform channel flow is better to proceed with such a design. The following simple argument may help in understanding why the uniform channel geometry is optimum. Consider a channel of widths W_1 and W_2 alternating in a repeat length ΔL . All matrix regenerators can be approximated this way. Later we will derive an optimum channel width W . If in our alternating design, we choose $W_1 = W$, then W_2 will not be optimum by definition. In general if W_2 is less than W , the velocity will be too high and the friction loss will dominate. On the other hand, if W_2 is greater than W , increased dead volume will cause an unnecessary cycle loss. One questions whether turbulence induced by the irregularities will be an advantage. If eddy motion decayed only by friction with the walls, then the effect of turbulence might possibly be beneficial. However, the Prandtl number (ratio of thermal to viscous diffusivity) is reduced from 1.66 for a monatomic gas, i.e. helium, to 1.0 for

turbulence, i.e. an eddy transports heat and momentum macroscopically and thus equally. Thus the particular advantage of larger thermal diffusivity is lost. We therefore choose channel flow as the optimum regenerator configuration.

Regenerator Section Independence

For practical reasons of material properties and fabrication, it will be difficult to produce a regenerator with monotonically varying properties as a function of temperature. Instead we imagine sections of one given wall material and possibly constant channel width. We would like to know if the lack of perfection or inefficiency of one section adversely affects others. Here we find that the overall efficiency is greater than the product of the efficiencies of sections.

Let Γ be the refrigerator coefficient of performance (COP) that is

$$\Gamma_i = Q_i/W_i \quad , \quad (1)$$

where Q_i is the heat removed from the low temperature side of the i^{th} stage or section and w_i is the work done in the i^{th} stage or section. Here a stage or section is any unit of refrigeration where a sequence, i , is connected in series. We use the term stage hereafter in a general sense that can mean either regions of regenerator of uniform properties, or refrigerators using different compression and expansion volumes. Index $i = 1$ refers to the lowest temperature of the entire refrigerator. Figure (1) shows the transfer of heat Q_i , stage by stage by performing an amount of work w_i per stage. By conservation of energy,

$$Q_j = Q_1 + \sum_{i=1}^{j-1} w_i \quad (2)$$

so that for the second of two stages,

$$\Gamma_2 = Q_2/w_2 = (Q_1 + w_1)/w_2 \quad . \quad (3)$$

while for the two stages combined

$$\Gamma_{12} = Q_1/(w_1 + w_2) \quad (4)$$

Together, these equations give

$$\Gamma_{12} = \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2 + 1} . \quad (5)$$

We define the efficiency, η , by

$$\Gamma = \eta \Gamma_C ,$$

where the Carnot COP is $\Gamma_C = T_1/(T_2 - T_1)$. Then the combined efficiency of two stages become

$$\eta_{12} = \frac{\eta_1 \eta_2 \Gamma_{C1} \Gamma_{C2}}{(\eta_1 \Gamma_{C1} + \eta_2 \Gamma_{C2} + 1) \Gamma_{C12}} . \quad (6)$$

For example, if each stage temperature ratio is 2:1, then, conveniently,

$$\Gamma_C = T_j/(T_{j+1} - T_j) = 1 , \quad (7)$$

and the combined efficiency of two stages with an overall temperature ratio of 4 and Carnot COP of 1/3 becomes

$$\eta_{12} = \frac{3\eta_1 \eta_2}{\eta_1 + \eta_2 + 1} . \quad (8)$$

For three such stages the result is

$$\eta_{13} = 7/3 \frac{\eta_{12} \eta_3}{(\eta_{12}/3) + \eta_3 + 1} = \frac{7\eta_1 \eta_2 \eta_3}{1 + \eta_1 + \eta_2 + \eta_3 + \eta_1 \eta_2 + \eta_1 \eta_3 + \eta_2 \eta_3} \quad (9)$$

with more complicated expressions for higher stages. We note that each combination of efficiencies is greater than the simple product, which is more favorable than one might have thought, and that if all $\eta_i = 1$, then the combined efficiency is also equal to 1 as it should be. In Appendix II the continuum solution of the problem shows a similar result.

III. Working Fluid Losses

Measure of Refrigeration

The refrigeration work required to transfer an amount of heat Q from T to $T+dT$ is QdT/T , proportional to Q and inversely proportional to the temperature past which it is transferred. Thus, a measure of refrigeration performance, or heat transfer difficulty is Q/T or the entropy transferred per cycle or unit of time. The coefficient of performance is the ratio of heat transferred, Q , to the PdV work required to transfer it. The Carnot COP is T/dT , while the actual COP is Q/PdV so that the local efficiency is $QdT/(TPdV)$. This efficiency is a measure of the extra PdV work required to transfer a given entropy Q/T through dT . Losses (inefficiencies) are due to the production of entropy by friction, conduction, and mixing. Our purpose is to minimize the entropy gain in each stage in the spirit of Minta and Smith (1984).

Regenerator Function

A regenerator must conduct a gas from a hot region to a cold region, store the thermal energy of the gas, and then, when the flow reverses, restore the energy to the gas with as little entropy generation as possible. The regenerator must also not conduct heat from the hot reservoir to the cold reservoir. The flow is driven by a large volume change at the high temperature end. The transfer of a given mass of gas to the cold end results in a small volume at the cold end. The regenerator also must transmit more gas in a cycle time than it contains in dead volume; otherwise the cycle efficiency becomes too small. These requirements are limited by the following losses.

1. The failure of ideal heat exchange between the working fluid gas and the stationary regenerator.
2. The viscous heat generation leading to heat that must be removed and to a pressure drop along the length of the regenerator.
3. Cycle loss due to the dead volume within the regenerator. This limits the cycle compression ratio.
4. Losses due to limited heat mass of the wall.

5. Entropy generation by heat flow in finite temperature gradients perpendicular to the gas flow in the regenerator material. These gradients are required for the storage of heat during the cycle.
6. Thermal conduction in the direction of the primary heat flow, i.e. in the axial direction along the regenerator. (The axial thermal conduction in the working fluid, helium, is neglected as being too small to constitute a meaningful constraint.)

Relating these six losses in a fashion that leads to a sensible design is a difficult task; many are conflicting. For example, in order to minimize thermal conduction, we would like the heat storage mass to be a minimum so that there is not a great deal of material to conduct heat longitudinally. On the other hand, we want large heat storage mass to store the thermal energy during the cycle. In order to have the storage mass small, we would like the frequency high so that the heat of the storage mass is used many times per second. On the other hand, we would like much more gas to flow through the regenerator from the hot end to the cold end and back each cycle than is stored within the regenerator itself; i.e. we want a large, active volume as compared to the dead volume. Otherwise the compression ratio and hence, the cycle efficiency, becomes too small. This in turn means that for high frequency we would like high gas velocity in order to transfer as large a volume of gas as possible. But high gas velocity leads to large viscous loss, pressure drop, waste work, and heat that short circuits the desired cooling. This combination of concepts is an outline of the conflict among the various losses and gains in the system.

Heat Exchange-Working Fluid to Wall

The primary purpose of a regenerator is to exchange heat reversibly between a moving working fluid, i.e. gas for the Stirling cycle, and a stationary heat mass. The lack of perfect heat exchange leads to a mean temperature difference between the gas and the wall. The gradient of this temperature difference both drives the heat flow from the gas to the wall and also leads to an irreversible loss, that is an entropy gain, or equivalently extra PdV work per cycle, Fig. 2.

We now define a mean temperature difference as an appropriate average to describe these losses. Since this temperature difference, δT , must reverse with the direction of heat flow, the fractional loss per cycle will be $2\delta T/T$. In order that this loss be small, the

distance through which the gas must move in the z direction parallel to the wall to exchange heat with the regenerator must be small. This distance we call the relaxation length. It is the distance in which, for uniform conditions, i.e. no temperature gradient in the z direction, the temperature contrast between gas and regenerator would be reduced by a factor e . In particular, this distance must be much less than the distance in the z direction in which the actual temperature changes appreciably.

The definition of thermal relaxation length, L_t , just given is $d\delta T/dz = -\delta T/L_t$. Adding to this equation the effect of a longitudinal gradient in regenerator temperature $\Delta T/L$ where ΔT is the temperature difference across a regenerator section of length L gives

$$\frac{d\delta T}{dz} = \frac{\delta T}{L_t} + \frac{\Delta T}{L} . \quad (10)$$

Assuming δT constant then gives $\delta T = L_t \Delta T/L = \Delta T/n_t$ where n_t is the number of thermal relaxation lengths in the length L . Thus the fractional loss per cycle is of order $(2\Delta T/T)(1/n_t)$. In Appendix II the relationship of thermal relaxation time to gas properties, transverse flow profile and channel width W is derived. The result is that if D is the thermal diffusion coefficient of the gas, i.e. κ/C_v , then the thermal relaxation time, t_t , is defined by the skin depth relationship

$$W/3 = \sqrt{Dt_t} . \quad (11)$$

The factor of $1/3$ is plausible in that the heat can diffuse to either wall and not all the gas is half way from either wall. This approximation and the corresponding entropy generation are also discussed in Appendix III. Then the average relaxation length must be the half cycle average mean velocity v_a times t_t . Here $v_a/(f/2)$ is the volume displacement per unit area in the time of one half cycle. The number of such relaxation lengths is then

$$n_t = L/(v_a t_t) = 9LD/(W^2 v_a) = 9L P_r \nu / (W^2 \rho v_a) . \quad (12)$$

Here ν is the viscosity, independent of density and $P_r(\nu/\rho)$ is the thermal diffusion coefficient. Hence, for a given gas fractional thermal loss, we have a complete specification of the channel parameters in terms of the gas thermal diffusivity and mean velocity.

Gas Friction Loss

The velocity, however, is constrained by friction with the wall. We have to pay for this friction in two ways: (1) frictional heat requiring additional refrigeration and (2) extra work performed against the resulting pressure drop. Since the removal of the friction heat, Q_f , requires a refrigeration work, $Q_f(T_h - T_c)/(\eta T)$, and the extra PdV work is just Q_f , the frictional heat is most important at low temperature where the regenerator design is most crucial. The work required to make up for a frictional heat Q_f is then the sum or $Q_f((T_h - T_c)/\eta T + 1)$. At low temperature where $(T_h - T_c)/T \gg 1$, the additional PdV work or the 1 in the above expression can be neglected. At higher temperatures it will be included.

The result of gas friction can be described in the same terms as thermal relaxation. In the case of friction, the quantity relaxed is momentum resulting in a dissipation of the kinetic energy of flow, $\sim \rho v_a^2/2$, in a relaxation length. The diffusion of heat and momentum are closely related because they both depend on molecular collisions. The ratio of heat diffusion to momentum diffusion is the Prandtl number P_r and hence, the thermal diffusivity D of Eqs. (11) and (12) is just $P_r \mu$ where μ is the kinematic viscosity. The number of viscous diffusion lengths will then be η/P_r . The actual kinetic energy per unit volume of the flow is larger than $\rho v_a^2/2$ because of the flow profile in the channel $\langle v^2 \rangle / \langle v \rangle^2 = (8/15)/(2/3)^2 = 6/5$ and the harmonic average $= \langle \sin^2 \theta \rangle / \langle \sin \theta \rangle^2 = \pi^2/8$. This results in a mean kinetic energy of $1.48 \rho(v_a^2/2)$ dissipated in each viscous diffusion length. Here v_a is the mass average velocity in the channel per half cycle. This relaxation approximation is verified in Appendix II.

The viscous loss occurs primarily in the expanded half cycle because the volume and hence, velocity is less by the inverse of the compression ratio, $1/C_R$, in the compressed half cycle. Hence, the friction loss is less by $(1/C_R)^2$ in the compressed half cycle. The thermal loss on the other hand is the same for each half cycle since ΔQ_i is a constant. The mechanical cycle can be so arranged that the gas transfer through the regenerator occurs in a shorter time for the compressed half cycle compared to the expanded one. In this case a slightly different optimization occurs for viscous and thermal losses, because the viscous loss would decrease and the thermal increase. We nevertheless take the transfer times the same each half cycle because of dead volume considerations and neglect the small gain ($\cong 20\%$) associated with this optimization. Then the total viscous heating loss per unit volume becomes the kinetic energy times the number of times it is dissipated or:

$$\text{viscous loss} \equiv 1.48(1 + C_R^{-2})(\rho_0 v_a^2/2)n_t/P_r \text{ ergs cm}^{-3} \text{ displaced per cycle.} \quad (13)$$

We define the fractional viscous loss, $1/n_f$, as the viscous loss per useful refrigeration work. In the Stirling cycle the work performed is approximately $(P_0)\ln C_R$ per unit volume displaced where P_0 is the pressure in the expanded state and C_R is the compression ratio, Appendix III. (This is just the work performed in an isothermal compression of volume ratio C_R .) In addition we have $P_0 = (\rho_0/\gamma)C_s^2$ where ρ_0 is the density in the expanded state, C_s is the sound speed, and γ the ratio of specific heats. Hence, the fractional viscous loss becomes

$$1/n_f = \frac{1.48(1+C_R^{-2})\rho_0 (v_a^2/2) n_t/P_r}{(\rho_0/\gamma) C_s^2 (\ln C_R)} \quad (14)$$

Then rearranging gives

$$v_a = \left(\frac{P_r \ln C_R}{0.74\gamma(1+C_R^{-2})} \right)^{1/2} C_s \left(\frac{1}{n_t n_f} \right)^{1/2} \quad (15)$$

Minimization of Friction and Conduction Loss

We desire the gas velocity to be large in the constant volume regenerator so that the dead volume fraction is not so large that the compression ratio, C_R , approaches unity and thus performs zero useful work. We define a total fractional gas loss, $1/n_g$, that is composed of the sum of the fractional friction loss, $1/n_f$, and the fractional thermal loss. The fractional thermal loss will be $(2\Delta T/T)/n_t$ from Eq (12). Since for reasonably constant properties of the construction of each regenerator stage, we have chosen $\Delta T = T_1 = 1/2 T_2$, then $\Delta T / \langle T \rangle = 1/\sqrt{2}$. Then equal losses implies $n_t = n_f/\sqrt{2}$, and so $(n_t n_f) = ((1+\sqrt{2})^2/\sqrt{2}) (n_g)^2$. If we use this in Eq. (15), then we obtain the expression for the average velocity in terms of sound speed and the fractional gas loss

$$v_a = (1/2)((1+\sqrt{2})^2/\sqrt{2})^{1/2} \left(\frac{P_r \ln C_R}{0.74\gamma(1+C_R^{-2})} \right)^{1/2} C_s n_g \quad (15)$$

Later we will justify a useful compression ratio of $CR \cong 2$, which enters the expression only weakly as $(1 + CR^{-2})$ and as the square root of a logarithm, and so using this value and $\gamma = 5/3$ and $P_r = 5/3$ for helium, we obtain the expression for the optimum gas velocity.

$$v_a = 0.55 C_s/n_g . \quad (16)$$

On the other hand dead volume in the cycle has the effect that the actual displacement of working fluid is less than the ideal. This is a second order effect which we crudely estimate as actual displaced volume = $V_0(1 - (1/CR)^2)$ and hence, the gas loss will be less by this factor, or the ideal v_a larger. The mean velocity then becomes

$$v_a = 0.7 C_s/n_g . \quad (17)$$

Dead Volume

With this mean velocity the dead volume ratio can be determined as a function of frequency and channel length. The volume of gas inside a regenerator reduces the useful work per cycle. To the extent that the same volume of gas is transported through the regenerator, the cycle efficiency is reduced. This is because the volume of gas in the regenerator does not participate in the active compression or expansion of the working fluid and so limits the effective compression ratio. A relatively significant loss of effectiveness occurs if the "dead volume" of the regenerator is greater than the displacement volume. Here dead volume in the regenerator must be scaled by $1/T$ since the gas occupies a larger volume at the high temperature end. The work done in an isothermal cycle is discussed in Appendix IV.

$$PdV \cong P_0 V_0 (\ln CR) \quad (18)$$

where V_0 is the total volume at the high temperature end and the compression ratio is.

$$CR = (V_{dead} + V_{replacement})/V_{dead} . \quad (19)$$

As the dead volume becomes large compared to the displacement volume, the compression ratio approaches unity, and $\ln CR$ approaches zero. The useful work also approaches zero. The gas is then transferred back and forth and no useful work is

performed but the friction and thermal losses remain unaffected. On the other hand, a large compression ratio can be achieved only at low frequency in order that the displaced volume is large compared to the dead volume. A low frequency requires a larger regenerator heat mass, which adds its own losses, as well as reducing the refrigeration power. Here we choose a practical compression ratio of $C_R = 2.0$, ($\ln 2 = 0.69$), where we expect, Appendix IV, that the useful refrigeration is optimized.

In addition it is likely that the expansion and compression volumes will be driven either harmonically by a crankshaft with a given phase difference, or by a cycle with smaller effective dead volume (than the harmonic cycle), but a higher velocity transfer, e.g. the Ross Yoke or Rhombic drive. Here we use harmonically driven compression and expansion volumes where a phase shift shuttles the gas and compresses it as:

$$V = V_0(2 + \sin[\omega t] + \sin [\omega t + \theta]) . \quad (20)$$

This supposes no dead volume. The gas is shuttled back and forth between hot and cold volumes, and the compression ratio is just the ratio of minimum and maximum of V . Actually the cold volume will be smaller proportional to $1/T$, but the density of the gas will be larger by T so that the mass is independent of T . Therefore, the volume ratio is the same as assuming that the temperature of both ends is the same.

The compression ratio then becomes

$$C_R = (1 + \cos\theta/2)/(1 - \cos\theta/2) . \quad (21)$$

It is expected that the cycle efficiency will optimize close to where the phase shift is 90° because in this case the gas transfer time is equal to the compression and expansion time. In this case the compression ratio for zero dead volume becomes:

$$C_R 90^\circ = (1 + \cos 45^\circ)/(1 - \cos 45^\circ) = 5.8 . \quad (22)$$

The equivalent dead volume of the phase difference of 90° becomes

$$V_{\text{dead}}/V_{\text{displacement}} = 1/C_R - 1 = 0.207 , \quad (23)$$

and that for $C_R = 2.0$ is

$$V_{\text{dead}}/V_{\text{displacement}} = 1.0 .$$

Therefore the dead volume ratio permissible in the regenerator, which we assume is the total dead volume, is the difference or

$$R_d \equiv V_{\text{dead}}/V_{\text{displacement}} = 0.79 . \quad (24)$$

We will use this value of regenerator dead volume ratio as the maximum permissible. On the other hand, the dead volume ratio of local regions of regenerator, dz , are additive because they depend upon a local flux and local volume, and is therefore independent of temperature. Since we know the gas velocity as a function of temperature, the gas displacement volume per unit flow area is determined by the frequency. The dead volume ratio is then the regenerator volume WL divided by the displacement in the time of $1/2f$ or

$$R_d = WL/(Wv_a/(2f)) = 2fL/v_a . \quad (25)$$

Here it is convenient to assign finite values to the differentials so that in a differential length $dL \equiv l$ there is a small dead volume, $dR_d \equiv r_d$, so that

$$r_d = 2fl/v_a , \quad (26)$$

and

$$R_d = \int_0^L (2f/v_a)dz , \quad l = dz \quad (27)$$

where L is the total length of the regenerator. In this discretization of the regenerator, we think of many exchange lengths, thermal or viscous (n_t and n_f) within a finite section of length L and finite temperature ratio T_1/T_2 . This is because in the construction with small channel width and wall thickness, one is naturally led to use foils of constant thickness at constant width for a finite length L . The sensitivity of the specified spacing W to T and later wall thickness, d to T , then become the measure of the accuracy of the approximation of sections of uniform properties within length L of a given temperature ratio.

Working Fluid Design Criterion

If we parameterize the sound speed for helium as

$$C_s = 5800 \text{ T cm s}^{-1} , \quad (28)$$

then using Eq. (17) for v_a in Eq. (26) we obtain the design criterion of:

$$n_g = 2000 r_d T^{1/2} / (f l) . \quad (29)$$

This is the primary result of optimizing the working fluid, or gas efficiency of the regenerator. The cycle effectiveness demands a small dead volume, but the efficiency of the working fluid in the regenerator improves with a larger one. However, high frequency and long length reduce the gas efficiency. On the other hand we desire the frequency to be large in order that the useful work be large and that the regenerator heat mass be reduced and accordingly the regenerator heat mass conduction less. Each section must be long enough so that both the accessible stationary heat mass be large and the longitudinal thermal conduction loss is small.

We consider the dead volume ratio and the gas efficiency as the primary constraints because they affect the useful work linearly. Then we can choose the frequency and then the length of each section such as to satisfy a minimum efficiency. We note that as the temperature becomes larger, the constraint of dead volume ratio is easier to meet so that we look for a design that assigns the resource of dead volume ratio preferentially to the lowest temperature. The penalty to efficiency of dead volume in a series of sections, i.e. Eq. (26), is only additive, but the gas efficiency, $1/n_g$, on the other hand leads approximately to a product, and every single section that is inefficient will proportionately adversely effect the whole regenerator. We therefore look to a design where r_d changes as a function of temperature (as $1/T$) and n_g is nearly constant.

Total Dead Volume

We choose $l = l_0 (T_c/T)^{1/2}$, with n_g a constant and of course f is independent of T . Then r_d will decrease as $1/T$ and its sum over k sections, the total dead volume of the regenerator must be less than a maximum, 0.79 from Eq. (25). Then

$$r_d = r_{do}(T_c/T) ,$$

and

$$R_d = r_{do} \ln(T/T_c) ,$$

or

$$r_{do} \leq 0.79/\ln(T/T_c) . \quad (30)$$

Total Gas Efficiency

We now need to choose a design goal gas efficiency, $1/n_g$. We are concerned with liquid helium temperatures, $T_c = 4\text{K}$ and the lowest temperature stage from 4K to 32K becomes the most critical, and so by Eq. (30), $r_{do} \leq 0.38$. Above a temperature of 32 K different design criteria become important. Thus for a desired value of efficiency of $\cong 70\%$ so that the losses are not too large an effect on the upper stages, we calculate $n_g = 6.53$ from Eq. (10) of Appendix I and $n_g = 1/(m-1)$ for $x = T/T_c = 8$. Then Eq. (28) and (30) give

$$l_o = 306 r_{do} T_c^{1/2}/f \leq 116 T_c^{1/2}/f \text{ cm}$$

and the total length

$$L = \int_{T_c}^T \left(\frac{dT}{T_c} \right) = 233(T^{1/2} - T_c^{1/2})/f . \quad (31)$$

We desire a maximum frequency and by trial and error select a design with $f = 30 \text{ Hz}$ suitable for efficient electric motors, and presumably the maximum feasible and reasonable power density. Then $l_o = 7.8/f_{30} \text{ cm}$ where f_{30} is frequency in units of 30 Hz. For example, a stage from 4 to 32 K is $28/f_{30} \text{ cm}$ long.

Channel Width and Reynolds Number

From Eq. (10), (11), and (17) the channel width becomes

$$\begin{aligned}
 W &= 3(ID/v_{an_t})^{1/2} \\
 &= (6.4 ID/C_S)^{1/2} \text{ cm} .
 \end{aligned} \tag{32}$$

For $l = 15.6 T^{-1/2} f_{30}^{-1}$, $C_S = 5800 T^{1/2}$, and $D = \kappa/C_V = 3.3 \times 10^{-4} T^{3/2} P_O^{-1} \text{ cm}^2 \text{ s}^{-1}$ for helium, we then obtain

$$W = 2.4 \times 10^{-3} T^{1/4} P_O^{-1/2} f_{30}^{-1/2} \text{ cm} . \tag{33}$$

Here P_O is the pressure in atmospheres.

Since n_g , l and W from Eqs. 29, 31, and 33 are proportional to T^0 , $T^{-1/2}$ and $T^{1/4}$ respectively, a centered value of T in each section of temperature ratio of $\times 2$ introduces an error of the order of 10% and hence, assuming uniform properties results in an error that is adequately small for this analysis.

Reynolds Number and Turbulence

One can calculate the Reynolds number for channel flow in terms of the equivalent "hydraulic" radius of a pipe or $W/4$ for a channel such that the Reynolds number becomes

$$\text{Rey}_{W/4} = (W/4) v_a P_r / D , \tag{34}$$

where we have substituted D/P_r for μ . Then using Eq. (33) for W

$$\text{Rey}_{(W/4)} = 1900 T^{-3/4} P_O^{1/2} f_{30}^{-1/2} . \tag{35}$$

For the lowest temperature, $T = 4 \text{ K}$ and $P_O = 1$, $\text{Rey}_{(W/4)} = 670$. For channel flow in Appendix III, we show that the coefficient of friction $C_f = 3/\text{Rey}_{(W/4)}$ or $C_f \text{Rey}_{(W/4)} = 3$. For steady-state pipe flow $C_f \text{Rey} = 4.5$ at $\text{Rey} = 670$ (Handbook of Physics). Hence, we would accept a correction of 1.5 increase in wall friction and heat transfer due to turbulence. Hence l can be correspondingly reduced with a corresponding relaxation of r_d . At the high temperature end of the first stage, $T = 32\text{K}$, or an increase in temperature by a factor of 8, $\text{Rey}_{(W/4)} = 140$. Therefore the upper temperature stages will be laminar.

Higher Temperature Stage

The design of the regenerator divides naturally into two stages, each a factor of 8 in temperature because of both dead volume, and as we consider later, the need for special wall materials. Here we consider the reduced constraint of dead volume. The higher temperature upper stage from 32 K to 300 K is not constrained as much by dead volume as the lower stage, since from Eq. (30) $r_d \propto T^{-1}$. The upper stage (32 to 300 K) will have a dead volume ratio smaller by a factor of 8, or $r_d = 0.049$ at $T = 32$ K.

If we keep this ratio a constant for $T > 32$ K for the upper stage, then the total dead volume ratio becomes

$$R_d \equiv 2r_d((T_h/T_c)^{1/2} - 1) = 0.18 . \quad (36)$$

This is acceptably low for good cycle efficiency, i.e. half of the maximum of Eq. (24). This means that the dead volume ratio is not a significant design constraint for the upper stage. Similarly, we choose l a constant equal to that at $T = 32$ K or from Eq. (31)

$$l(T = 32) = 2.8 f_{30}^{-1} = l = l_0 \text{ cm for the upper stage.} \quad (37)$$

Therefore

$$L = l(T_h/T_c - 1) = 22 \text{ cm} .$$

Therefore the advantage of the constant value of the differential length, l , and longer length, L , implied by Eq. (31) will be two-fold. The gas efficiency improves in the upper stages, Eq. 29, and the longer length allows a better distribution of regenerator heat mass discussed in the next section. Then the gas efficiency, n_g , increases as $T^{1/2}$ from Eq. (29) ($n_g = 1.1 T^{1/2}$). This value of n_g should be corrected for the extra PdV work neglected in the friction optimization of Eq. (12), which doubles the fractional friction loss at the higher temperature. Then the value of n_g becomes approximately a constant 6.5. The channel width from Eq. (32) then becomes

$$W = 1.01 \times 10^{-3} T^{1/2} P_0^{-1/2} f_{30}^{-1/2} \text{ cm} , \quad 32 \text{ K} \leq T \leq 300 \text{ K} . \quad (38)$$

The Lateral Extent of Channel (perpendicular to W)

The mass flow through the regenerator is the same for each section so that the cross section area of gas flow, A , must be such as to result in a constant mass flux. Since the channel width is determined, this leads to a channel extent, Y . (Extent is measured in the Y direction perpendicular to X , the width, and Z , the flow direction, Fig. 1.) Then the mass flux becomes

$$\rho v_a A_g = v_a WY(\rho_0 T_h/T) = \text{constant, g s}^{-1}, \quad (39)$$

where $A_g = WY$, and $\rho = \rho_0 T_h/T$. Here again ρ_0 is the gas density in the expanded state at the hot end and T_h is its temperature. The mass flux can be related to a more useful quantity, the PdV work performed at the hot end H_h , which is

$$H_h = (1/2) v_a (P_0/10) A_g (\ln C_R) (T_h/T) \text{ watts} . \quad (40)$$

where $P_0/10$ is the pressure in joules cm^{-3} for P_0 in atmospheres. The lateral extent of the regenerator then becomes

$$Y = A_g/W = 20 H_h T / (W v_a P_0 T_h \ln C_R) ,$$

or

$$Y = 19.3 T^{1/4} P_0^{-1/2} f_{30}^{1/2} H_h T_h^{-1} \quad \text{cm/watt lower stage,} \quad (41)$$

$$Y = 46.5 P_0^{-1/2} f_{30}^{1/2} H_h T_h^{-1} \quad \text{cm/watt upper stage.}$$

VI. Regenerator Stationary Heat Mass

The stationary wall heat mass that bounds the channel leads to two losses:

1. The entropy gain due to the cyclic temperature difference in x needed to access a finite heat mass.
2. The enhancement of all losses due to the cyclic distorted temperature distribution in z resulting from too small a heat mass exclusive of (1).

3. The longitudinal thermal conduction loss.

The wall heat mass accessible to the gas is limited by either its thickness d , or a finite skin depth $\delta = [\kappa/(C_M \pi f)]^{1/2}$ where κ is now the thermal conduction of the heat mass and C_M its specific heat per unit volume. Since the skin depth is the characteristic length in which the temperature fluctuations die out exponentially as a function of depth (in the x direction, Fig. 1) into the material, it is intuitively evident that material at a depth greater than δ will not contribute effectively to the heat mass in thermal contact with the gas. This is because within the effective cycle time, $1/(\pi f)$, heat is deposited or extracted roughly only from the depth, δ . Hence, depths $d > \delta$ do not significantly improve the thermal efficiency of the cyclic heat storage. On the other hand the longitudinal conduction loss depends only on the longitudinal average temperature gradient as opposed to the oscillating fraction, and so increases linearly with thickness d . Hence, we will look for a regenerator design where $d \leq \delta$.

Heat Access Loss

If heat cyclically enters and is extracted from a material, entropy is produced. We call this loss of refrigeration heat access loss. In Appendix IV the oscillating temperature heat access loss is derived in some detail with the effective result that if $d \leq \delta/2$, then the fractional loss is less than $(1/12)(\delta T/T)$ due to heat flow in and out of the material alone. Here δT is the temperature variation of the wall material. Since $\delta T < T$, the access loss can be neglected for $d < \delta/2$. This is then the maximum useful thickness.

We will later find that the optimum thickness will be just this half skin depth, $\delta/2$, for the optimum material. However, we must next understand the cycle heat loss due to finite heat mass.

Finite Heat Mass

If a thin layer of heat mass is used such that $d \ll \delta/2$, so that the heat access loss can be neglected, and provided in addition the longitudinal conduction loss can be neglected, then the wall material will tend to follow the gas in temperature each half cycle. Since the gas in either the compression or expansion volume is assumed isothermal by contact with a large heat mass, the small heat mass of our thin wall will follow either the

input or output temperature conditions. When the gas enters from an isothermal volume, this means that a constant temperature region will extend into the regenerator. On the reverse half cycle this uniform or constant region will be transformed to the gradient distribution of the remaining regenerator length. The oscillation in temperature will be just the ratio of heat masses (gas/wall) times temperature difference. This is illustrated in Fig. 2. Again we emphasize that the access heat loss is assumed small here, so that the only effect of the finite heat mass is the large distortion in the temperature profile depicted in Fig. 2. This distortion effectively reduces the length over which the temperature gradient exists by just twice the ratio of heat masses, i.e. $2 \times (\text{transferred gas heat mass})/(\text{wall heat mass})$. This shortened length has the effect of increasing all the losses, gas as well as conduction, because now the effective length of a series of sections is $(L_1 - \Delta L) + L_2 + (L_3 - \Delta L)$ where ΔL is the length whose heat mass equals that of the transferred gas. The gas loss due to conduction to the wall, Eq. 11, as well as the regenerator material conduction and the heat access loss all increase as $1/L$. (The friction loss is not changed.) These increased losses occur alternately in the higher or lower temperature section. This increase means that the effective loss in these two sections will be larger by $L/(L - \Delta L) 1/n_t + L/(L - \Delta L) 1/n_{\text{conduction}} + L/(L - \Delta L) 1/n_{\text{access}}$. This introduces a complicated nonlinear coupling among the losses. It will become important when all the losses are large or n is small. We can estimate that our design will lead to $1/n_{\text{conduction}}$ and $1/n_{\text{access}}$ to be much less than $1/n_t$. Then in order that $1/n_t$ not be increased by more than a small fraction, $\Delta L/L$ must also be small. The finite heat mass loss and conduction loss compete for the same resource, d , the wall thickness. The heat mass loss is proportional to $1/d$ and the conduction loss proportional to d . Hence, a minimum total loss will correspond to equal losses. The fractional finite heat mass loss will be smaller than $\Delta L/L$ and larger than $(\Delta L/L) 1/n_t$ because of the compounding of losses of Appendix I as well as the contribution of the access and conduction losses. We view the gas loss, $1/n_t$, as most critical and so investigate a design where the finite heat mass loss is overestimated as equal to $\Delta L/L = (\text{gas heat mass}/\text{wall heat mass})$, and see if materials and construction exist that permits both the finite heat mass loss and conduction loss to be negligibly small jointly.

Finite Heat Mass Loss

The fractional cyclic heat storage loss is defined as $(1/n_{\text{Material Heat Loss}} \equiv 1/n_{\text{MH}})$. Since it is not a direct loss but instead multiplies other losses, it will be approximately half the temperature variation of the heat mass, δT , cycle averaged, divided by the absolute temperature or

$$n_{MH} = T/\delta T \quad (42)$$

The heat stored each half cycle

$$Q_M = 2(\delta T) dC_M \quad \text{per cm} \quad (43)$$

where C_M is the heat mass per unit volume of the regenerator material and the factor 2 is for two surfaces.

However, the heat of the gas to be stored each half cycle is

$$Q_g = (\rho C_p)(dT/dz) v_a W P_O / 2f \quad (44)$$

where $dT/dz = T/l$ and (ρC_p) is the specific heat of helium per unit volume at atmospheric pressure = $0.062 T^{-1} \text{ cal deg}^{-1} \text{ cm}^{-3}$. Hence, Eq. (31) for $DL/dT = (dT/dz)^{-1}$ and from Eqs. (33) and (38) for W and Eqs. (16) and (17) for v_a with our choice of $n_g = 6.53$ we have

$$Q_g = 1.0 \times 10^{-4} T^{5/4} P_O^{1/2} f_{30}^{-1/2} \text{ cal cm}^{-1} \text{ for the lower temperature stage,}$$

and

$$Q_g = 2.30 \times 10^{-4} T P_O^{1/2} f_{30}^{-1/2} \text{ cal cm}^{-1} \text{ for the upper temperature stage.} \quad (45)$$

Then from Eq. (42) and (43) and letting $Q_M = Q_g$ we then have

$$n_{MH} = 2.0 \times 10^4 (dC_M) T^{-1/4} P_O^{-1/2} f_{30}^{1/2} \text{ for the lower temperature stage,}$$

and

$$n_{MH} = 8.7 \times 10^3 (dC_M) P_O^{-1/2} f_{30}^{1/2} \text{ for the upper temperature stage.} \quad (46)$$

Conduction Loss

The conduction loss is just $2\kappa d \, dT/dz$ for the two walls and so the fractional loss, $1/n_{MC}$, results in

$$n_{MC} = \frac{\int P dV}{2\kappa d \, dT/dz} = \frac{(v_a/2) P_o W(\ln C_R)}{2\kappa d \, T/l} \quad (47)$$

or

$$n_{MC} = 0.10 \, T^{-3/4} P_o^{1/2} (\kappa d)^{-1} f_{30}^{3/2} \quad \text{for the lower stage,}$$

and

$$n_{MC} = 7.3 \times 10^{-3} \, T^{-3/4} P_o^{1/2} (\kappa d)^{-1} f_{30}^{3/2} \quad \text{for the upper stage,} \quad (48)$$

We note that these two losses depend inversely upon d , the thickness of regenerator material. Hence, we look for an optimum when $n_{MH} = n_{MC}$ resulting in

$$d = 2.2 \times 10^{-3} (\kappa C_M)^{-1/2} T^{1/4} P_o f_{30}^{12} \, \text{cm} \quad \text{for the lower stage,}$$

and

$$d = 92.2 \times 10^{-4} (\kappa C_M)^{1/2} P_o^{1/2} f_{30}^1 \, \text{cm} \quad \text{for the upper stage.} \quad (49)$$

With this value of d we can calculate an optimum material loss factor. Here we assume the actual loss due to finite heat mass, $1/n_{MH}(1/n_t + 1/n_{access} + 1/n_{MC})$, will be substantially smaller than $1/n_{CM}$. Therefore a total material loss factor, $n_M = n_{MC}$, becomes

$$n_M = 44 (C_M/\kappa)^{1/2} T^{-1/2} f_{30}^{1/2} \quad \text{for the lower stage,}$$

and

$$n_M = 8.0 (C_M/\kappa)^{1/2} f_{30}^{1/2} \quad \text{for the lower stage} \quad (50)$$

Thus for high efficiency we desire materials of large specific heat and low conductivity. This combination, however, means a small thermal skin depth that prevents

material at greater depth from participating as accessible heat mass to the fluctuating temperature of the gas. In other words, the heat mass of the wall must be accessible within the half cycle, or $d \leq \delta/2 = 1/2[(\kappa/C_M)(1/\pi f)]^{1/2}$. Applying this condition to Eq. (49) requires that

$$\kappa \geq 4.27 \times 10^{-2} T^{-1/4} P_o^{1/2} f_{30}^{1/2} \text{ cal K}^{-1} \text{ cm}^{-2} \text{ s}^{-1}. \text{ for the lower stage,}$$

and

$$\kappa \geq 1.79 \times 10^{-2} P_o^{1/2} f_{30}^{1/2} \text{ cal K}^{-1} \text{ cm}^{-2} \text{ s}^{-1}. \text{ for the upper stage. (51)}$$

These conductivities at low temperature restrict wall materials to metal alloys. Once this condition is satisfied, however, we desire the largest heat capacity and lowest conductivity in Eq. (53) to obtain the best efficiency. Thus, if we use the lower bound $d = \delta/2$ in Eq. (50) and retain the conductivity condition, Eq. (51), we obtain the efficiency with the ideal material

$$n_M = 213 C_M^{1/2} T^{-3/8} P_o^{-1/4} f_{30}^{1/4} \text{ for the lower stage,}$$

and

$$n_M = 60 C_M^{1/2} P_o^{-1/4} f_{30}^{1/4} \text{ for the upper stage. (52)}$$

If we do not have ideal materials such that κ is less than that expressed in Eq. (51), then we can use $\delta/2$ for d in the finite heat mass Eq. (46) for n_M , since the conduction can be neglected. Then

$$n_M = n_{MH} = 1.03 \times 10^3 (\kappa C_M)^{1/2} T^{-1/4} P_o^{-1/2} \text{ for the lower stage,}$$

and

$$= 4.5 \times 10^2 (\kappa C_M)^{1/2} P_o^{-1/2} \text{ for the upper stage. (53)}$$

As discussed before this loss due to finite heat mass must be small in order that other losses are not increased. The heat capacity of common metals at low temperature maximizes for lead, a soft material with low Debye temperature. (Gd-Rh is not considered.) The ferrous

alloys all have nearly the same specific heat, significantly larger ($\cong 3$ -fold) than the copper or aluminum alloys. Hence, we explore the material wall efficiency for alloys of lead and steel assuming that for lead we can retain the ideal thermal conductivity of Eq. (51). In this case at low temperature all metals follow the Debye specific heat law. For steels we have

$$\begin{aligned} C_M &\cong 2.5 \times 10^{-5} T^2 \text{ cal K}^{-1} \text{ cm}^{-3} & T < 150 \text{ K} \\ &\cong 4.7 \times 10^{-2} T^{1/2} \text{ cal K}^{-1} \text{ cm}^{-3} & T > 150 \text{ K}, \end{aligned} \quad (54)$$

and for lead

$$\begin{aligned} C_M &\cong 3.6 \times 10^{-4} T^2 \text{ cal K}^{-1} \text{ cm}^{-3} & T < 20 \text{ K} \\ &= 3.1 \times 10^{-2} T^{1/2} & 20 \text{ K} < T < 100 \text{ K} \\ &= .31 & 00 \text{ K} < T < 300 \text{ K} . \end{aligned} \quad (55)$$

Thus above 116 K the ferrous alloys will have a larger specific heat than lead, but for low temperature lead has a major advantage. If we substitute the low temperature behavior of lead into the efficiency condition (52), which assumes the ideal conductivity, we obtain

$$\begin{aligned} n_M &= 4.0 T^{5/8} P_o^{1/4} f_{30}^{1/4} & T < 20 \text{ K} \\ &= 38 T^{-1/8} P_o^{1/2} f_{30}^{1/4} & 20 \text{ K} < T < 32 \text{ K} . \end{aligned} \quad (56)$$

and

$$\begin{aligned} d &= 0.568 T^{-9/8} P_o^{1/4} f_{30}^{3/4} \text{ cm} & T < 20 \text{ K} \\ &= 6.0 \times 10^{-2} T^{-3/8} P_o^{1/4} f_{30}^{3/4} \text{ cm} & 20 \text{ K} < T < 32 \text{ K} . \end{aligned}$$

At the temperature of 4 K the loss factor $1/nM = 10\%$, a satisfactory low value and improves rapidly at higher temperatures provided the conductivity is maintained at the optimum value of Eq. (51).

In the higher temperature sections we desire the lowest conductivity, but must satisfy the skin depth condition dependent upon the conductivity of Eq. (51). The thermal conductivity of the lowest conductivity ferrous alloy with good specific heat is stainless steel with

$$\kappa = 2.1 \times 10^{-4} T \text{ cal K}^{-1} \text{ cm}^{-1} \text{ s}^{-1} \quad T < 150$$

and

$$\kappa = 2.1 \times 10^{-3} T^{1/2} \quad T > 150 \quad (57)$$

With Eq. (54) this gives a half skin depth:

$$\delta/2 = 0.15 T^{-1/2} f_{30}^{1/2} \text{ cm} \quad T < 150 \text{ K} \quad (58)$$

and

$$\delta/2 = 0.244 f_{30}^{1/2} \text{ cm} \quad T > 150 \text{ K} .$$

If we compare this half skin depth with the ideal thickness d of Eq. (49), we find that it is satisfied only for $T > 85 P_0^{1/2}$, which means that the loss factors must be calculated separately for $d = \delta/2$. Then using the heat capacity of Eq. (54) and the conductivity of stainless steel, Eq. (57) the loss factors, Eq. (53) and (48) becomes for the upper stage

$$n_{MH} = 3.26 \times 10^{-2} T^{3/2} P_0^{-1/2} \quad T < 150 \text{ K} , \quad (59)$$

$$C = 232 T^{-1/2} P_0^{1/2} f_{30}^{1/2} \quad T < 150 \text{ K} . \quad (60)$$

At $T = 32 \text{ K}$, $n_{MH} = 5.9$ and improves rapidly at higher temperatures being > 10 at $T \geq 46 \text{ K}$. Conversely, $n_{MC} = 34$ at $T = 46 \text{ K}$ and becomes smaller or a larger loss at higher temperatures. Since a value of $n_{MH} > 10$ satisfactorily avoids the compounding of other losses ($C_M d$) in Eq. (46) it can be made a constant for $T \geq 46 \text{ K}$. Then in Eq. (46) for $n_{MH} = 10$,

$$d C_M P_0^{-1/2} = 1.15 \times 10^{-3} ,$$

or using C_M from Eq. (54) for stainless steel,

$$d = 46 T^{-2} C_M P_o^{1/2} f_{30}^{1/2} \text{cm} \quad 46 \text{ K} < T < 150 \text{ K}$$

and

$$d = 2.45 \times 10^{-2} T^{-1/2} C_M P_o^{1/2} f_{30}^{1/2} \text{cm} \quad T > 150 \text{ K}$$

and

$$d = \delta/2 = 0.244 f_{30}^{1/2} \quad 32 \text{ K} < T < 46 \text{ K} . \quad (61)$$

From Eq. (48)

$$\begin{aligned} n_{MC} &= .756 T f_{30}^{1/2} \quad 46 < T < 150 \text{ K} \\ &= 114 T f_{30}^{1/2} \quad T > 150 \text{ K} . \end{aligned} \quad (62)$$

The loss factor with stainless steel for the upper stage is $< 1\%$, which is satisfactorily small. It is noteworthy that nylon or mylar plastic gives an efficiency factor from Eq. 53 of

$$n_{MH} = 0.18 T^{5/4} p_o^{-1/2} \quad \text{for the lower stage,}$$

and

$$n_{MH} = 0.5 T^{1/2} p^{-1/2} \quad \text{for the upper stage} . \quad (63)$$

Thus nylon or mylar are marginal for the upper temperature stage where $n_{MH} \geq 4.5$ for $T \geq 80 \text{ K}$.

Special Lead Alloy

For the lower sections the conduction penalty for using pure lead would be $\times 100$ in conductivity and an increase in loss of ~ 10 and hence, would reduce the efficiency of the regenerator to less than 30%. Instead it is fortunate that trace impurities modify the thermal

conductivity of pure metals in a major fashion. On the other hand impurities also harden the metal and increase the Debye temperature, reducing the specific heat. It is believed that the soft metals, cesium or bismuth, can be added to pure lead in amounts less than 1% and not significantly reduce the specific heat. The conductivity of the lead with $\cong 1/2\%$ bismuth is known to result in a thermal conductivity in the desired range (Touloukian et al. 1970).

Segmented Regenerator

The development of a special alloy is difficult. Instead a smaller wall loss can be obtained by making the wall material anisotropic such that it has high conductivity transverse to the flow, and low conductivity parallel to the flow. Then the Eqs. (46) and (48) that describe the heat mass loss and longitudinal loss are decoupled. The regenerator wall can be made anisotropic by using alternate layers of high thermal conductivity, pure lead, and a thermal insulator like epoxy fiber glass or foamed ceramic. The thermal insulation of fiber glass epoxy or even glass is roughly 10^{-2} of the optimum conductivity of an isotropic material, Eq. (51), and so gives very good thermal isolation in the longitudinal direction. In the transverse direction, as we have already pointed out, the conductivity of pure lead is 100 times larger than we need, and so the thermal heat mass will be adequate. Therefore, by a segmented construction we can reduce the heat mass loss and thermal conduction loss to a negligible amount. The only remaining loss we call the segmentation loss. We must have enough segments, N_s , in the longitudinal direction per e-fold change in temperature such that the temperature difference segment to segment is small. In addition the insulating gap does not contribute to the regenerator function so that the fractional gap length should be small like $< 10\%$. The dead volume ratio and friction would then be increased by this factor, i.e. $\times 1.10$.

The temperature drop across each gap is $\delta T = T/N_s$. Therefore the fractional loss for N_s sections per e-fold change in temperature from Appendix I becomes

$$\text{Fractional segmentation loss} \cong 1/(N_s) . \quad (64)$$

We therefore will require $N_s \geq 20$ segments or bands of alternate insulation and lead per to maintain a negligible material loss.

Regenerator Heat Transport

We have considered regenerator losses only internal to the regenerator. These losses are manifested as an increased temperature and hence, entropy of the gas, exciting the regenerator enters either the isothermal compression or expansion volumes. However, we expect other heat losses, that is, heat to enter the regenerator along its length. These heat sources may be due, for example, to radiation or conduction as might occur from supports or radiation shields. They will affect the temperature distribution along the length of the regenerator and hence, its efficiency. We would also like to know whether it is advantageous to deposit the same heat loss, as for example, due to a radiation shield at an appropriate temperature point along the regenerator length, or whether the shield heat should be extracted always at the cold expansion end. Entropy is only extracted at the cold end since no work is done within the regenerator. A heat shield, Q_{shield} , may be effective at a temperature, T_{shield} , significantly greater than T_{cold} . The entropy produced by T_{shield} ($Q_{\text{shield}}/T_{\text{shield}}$) connected to the regenerator at $T_{\text{shield}} = T_{\text{regenerator}}$ will be much less than if T_{shield} is connected to T_{cold} , $Q_{\text{shield}}/T_{\text{cold}}$, if $T_{\text{shield}} \gg T_{\text{cold}}$. Therefore, the heat shield should be connected to the regenerator at as high a temperature as possible.

Temperature Distribution

The heat redistribution equation within an ideal regenerator section, defined as having wall dimensions independent of z , can be described for a heat source Q as a function of z and assuming the gas losses are negligible, i.e. $n_g \rightarrow \infty$, as

$$\partial T / \partial t = \left(\frac{\phi_0 C_{gm} e^{i\omega t}}{C_M} + 2d\kappa \right) \frac{\partial T}{\partial z} + Q . \quad (65)$$

Here $\phi_0 e^{i\omega t}$ is the gas mass flux and C_{gm} is the specific heat per unit mass of the gas. We can assume C_{gm} is independent of T , but C_M the wall heat mass per unit volume, and κ the wall conductivity will be temperature dependent. However, for the sake of simplicity, we will assume $\kappa \rightarrow 0$ as in a segmented design and that C_M is a constant. Then we note that for the unique condition $\partial T / \partial z = \text{constant}$, we obtain a uniform temperature distribution that oscillates as depicted in Fig. 2. If we now introduce Q as for example at one point, z_1 , then the heat must be transported in such a fashion that the additional entropy is removed at the cold end provided that $T(z_1) < T(z_{\text{hot}})$. This is because no PdV work is done within the regenerator and so there can be no reduction in entropy by work. The effect of $Q(z_1)$ will be to establish a temperature perturbation relative to the uniform $dT/dz = \text{constant}$ value that

will cyclically move back and forth similar to the exaggerated rhomboid shape of Fig. 2. We surmise that any heat input will alter the constant slope shape thereby transporting entropy to the cold end. However, the entropy generated by a heat input $Q(z = z_1) \equiv Q_1$ will be $s_1 = Q_1/T_1$. This entropy is less than if the same heat were introduced at the cold by the ratio T_{cold}/T_1 . Therefore it is advantageous to dump support driven heat or radiation heat losses at as high a temperature as possible along the length of the regenerator. All the entropy produced by these respective heat loads must be supplied at the cold end. However, the volume of the cold end will be smaller than the hot by T_c/T_h and so the refrigerator work $\propto P_{\text{ovC}} \propto P_{\text{ovN}}(T_c/T_h)$ will be the same. Thus even though the magnitude of heat removal decreases as T_{cold} , the absolute value of the entropy removal is independent of T_{cold} . Thus depending on the magnitude and temperature of the parasitic heat losses, it may be sufficient to distribute these along the length of the regenerator and avoid multiple stages of expansion.

Design Example

Table I shows a regenerator design according to these principles. The equation numbers are given for each variable.

In Table I the efficiency is primarily determined by the gas losses. By Eq. (29) the gas loss is independent of pressure provided the gap, W , is optimized. The gas loss fraction is also proportional $1/f$. For the upper stages where the wall losses are larger, Eq. (59) is also independent of pressure and so the same fractional loss applies. However, the wall skin depth condition will be partially violated and a segmented fourth stage may have to be considered, or possibly a higher conductivity alloy.

Conclusion

We have optimized the design of a cryogenic regenerator for a Stirling cycle that leads to an overall efficiency of $\cong 38\%$ at a pressure of 1 atm helium, 30 Hz frequency and compression ratio 2:1 with a dead volume based upon a harmonic cycle.

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Table I
Regenerator Design for $f = 30$ Hz and $P_O = 1$ atm.

Eq. #	(31)	(33)	(29)	(49)	(56)	$(\sum n^{-1})^{-1}$	(41)	
T	l cm	W cm $\times 10^{-3}$	n_{gas}	d _{cm} lead alloy	n_M	n_{total}	$Y_{\text{cm/W}}$ input	A_t $Y(2d+W)$
4	7.8	3.4	6.53	0.12	9.5	3.9	0.09	0.097
8	5.2	4.0	6.53	0.054	15	4.5	0.11	0.023
16	3.9	4.8	6.53	0.025	23	5.1	0.13	0.012
32	2.8	5.7	6.53	0.016	25	5.2	0.16	0.010
	L = 28 cm		Eff $\cong$ 56%, (Appendix I, Eq. (10))					

Stainless Steel

Eq. #	(37)	(38)	(29)	(58)	(60)	(46)	(41)		
T	l	W	n_{gas}	(61) d	(62) n_{MC}	(59) n_{MH}	n_{total}	Y	A
32	2.8	5.7	6.53	0.24	41	5.9	5.3	0.155	8.9×10^{-3}
64	2.8	8.1	6.53	0.112	48	10	5.4	0.155	4.7×10^{-3}
128	2.8	11.4	6.53	0.0028	97	10	5.8	0.155	2.6×10^{-3}
256	2.8	16.2	6.53	0.0015	114	10	5.9	0.155	3.0×10^{-3}
300	2.8	17.5	6.53	0.0014	114	10	5.9	0.155	4.0×10^{-3}
L = 23.5			Efficiency = 75%, Appendix I. Eq. (10)						
			Joint Efficiency = 38%, Eq. (6)						

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APPENDIX I

The Compounding of Losses

Let $S(T)$ be the rate at which the refrigerator pumps entropy past temperature T , i.e. $\dot{S} = d/dt S(T)$. Then the flow of heat past T is

$$Q(T) = T \dot{S} \quad (1)$$

and the work for a Carnot refrigerator to pump the entropy from T to $(T + dT)$ is

$$W_C(T) dT = Q(T + dT) - Q(T) = \dot{S} dT. \quad (2)$$

Suppose the actual work is $W(T) dT = m W_C(T) dT = m \dot{S} dT$. Then if the work not required to pump the heat is locally deposited (i.e. wall friction) it leads to an increase in entropy flow rate $d\dot{S} = (m-1) \dot{S} dT/T_C$. $W/W_C = m$, so the efficiency, relative to Carnot, locally (i.e. at T) is m^{-1} . Since

$$T d\dot{S} = (m-1) \dot{S} dT, \quad (3)$$

the integration of these losses becomes

$$\ln(\dot{S}/\dot{S}_c) = \int_{T_c}^T [(m-1)dT/T], \quad (4)$$

where $\dot{S}_c$ is the entropy pumped from the cold reservoir at T_c . Then

$$\dot{S} = \dot{S}_c \exp \left[\int_{T_c}^T \{(m-1)/T\} dT \right]. \quad (5)$$

The integral of work performed is

$$\dot{W} = \dot{W}(T) dT = \int_{T_c}^{T_1} m \dot{S} dt = \dot{S}_0 m-1) dT/T \int_{T_c}^{T_1} m \exp \left[\int_{T_c}^T (m-1) dT/T \right] . \quad (6)$$

Let $m = 1$, then $\dot{S} = \dot{S}_c$,

$$W = S_c (T_1 - T_c) = \{(T_1 - T_c)/T_c\} Q_c = W_c \text{ as exxpected,} \quad (7)$$

where the Carnot work is W_c . Let m be a constant not equal to 1. Then

$$\dot{S} = S_c \exp[(\ln T - \ln T_c)(m-1)] = S_c (T/T_c)^{(m-1)} . \quad (8)$$

Also

$$W = m \dot{S}_c \int (T/T_c)^{(m-1)} dT = \dot{S}_c (T_1^m/T_c^{(m-1)} - T_c) , \quad (9)$$

and the desired total efficiency becomes

$$\begin{aligned} \text{Eff}^{-1} &= W/W_c = (\\ &= (T_1^m - T_c^m)/[T_1 - T_c][T_c^{(m-1)}] \\ &= (T_c/[T_1 - T_c]) ([T_1 - T_c]^m - 1) . \end{aligned} \quad (10)$$

If we evaluate this expression for the temperature ratios $T_1/T_c = x = 75$, $x = 8$, and $x = 2$ corresponding to the ratio of room temperature, 300 K to 4 K, 32 K to 4 K and 300 K to 150 K respectively, we obtain the Table.

$W/W_c = (x^{m-1})/(x-1)$			
m	W/W_c (x=75)	W/W_c (x=8)	W/W_c (x=2)
1.1	1.547	1.264	1.144
1.2	2.39	1.589	1.30
1.5	8.76	3.090	1.83
2.0	76.0	9.00	3.00
1.01	1.045	1.024	
$1 + \epsilon$	$1 + 4.38 \epsilon$	$1 + 2.37 \epsilon$	
Infinite	$x^{(m-1)}$		

Approximation for m approaching 1:

$$x^m = x \cdot x^{(m-1)} = x^{(m-1)} \ln x = x(1 + [m-1] \ln x)$$

and so

$$\begin{aligned} (x^{m-1})/(x-1) &= (x-1 + [m-1]x \ln x)/(x-1) \\ &= 1 + (m-1)(x/[x-1]) \ln x, \end{aligned} \tag{11}$$

which is valid if $(m-1) \ln x \ll 1$ and $(m-1) \ll .22$. As $\ln 75 = 4.32$ this gives $(75/74) \ln 75 = 4.38$.

APPENDIX II

Explanation of Thermal Exchange Length in Channel Flow

A comparison is needed of the thermal and viscous exchange length approximation of a channel regenerator to the steady-state solution of heat and momentum transfer.

The diffusive exchange length, Δz , is the distance the gas will move in the channel parallel to the walls in the z direction, in the time required for thermal diffusion in the x direction to reach approximate equilibrium with the walls. The direction perpendicular to the wall is x . The velocity used to calculate the length, $\langle x \rangle$, is a velocity averaged over the x coordinate. The time for diffusive equilibrium from the usual skin depth argument is calculated as follows: The diffusion depth is $\delta = (Dt)^{1/2}$, where D is the diffusivity. For heat $D = \kappa/(C_p \rho)$, κ = thermal conductivity, C_p is the heat capacity of the gas per unit mass at constant pressure, and ρ is the density of the gas. (Hence, $C_p \rho$ is the heat capacity per unit volume at constant pressure.) For momentum $D_\mu = \nu/\rho = \mu$ where ν is the viscosity and μ the kinematic viscosity. The kinematic viscosity is related to the thermal diffusivity $\kappa/(C_p \rho)$ by the Prandtl number $Pr = D_T/D_\mu = 5/3$ for a monatomic gas.

For a channel width W , half-width $W/2$, an average depth of heat diffusion that constitutes approximate equilibrium has been estimated as:

$$\delta = (2/3)(W/2) \quad (1)$$

Confirming this estimate for both heat and momentum is the purpose of this Appendix. Then

$$t = W^2/(9D) \quad (2)$$

and the diffusive exchange length of the fluid with the wall is the distance in the z direction that the fluid moves in this time, or $\Delta z = t \langle v \rangle$.

Thermal Relaxation

The motivation for using the thermal exchange or relaxation length is the following: If the temperature relaxes n_t times in the length L , then the local temperature difference or departure from local equilibrium is (temperature difference over the length of the

regenerator section)/ $n_t = \Delta T/n_t$. This average local temperature difference between the fluid and the wall we call $\langle \delta T \rangle$.

This time and space dependent temperature difference δT leads to an entropy generation and a refrigeration loss. The fractional loss due to this difference will be $\langle Q\delta T \rangle / \langle Q \rangle \Delta T$ and thus is $1/n_t$. To the extent that $\langle \delta T \rangle \ll \Delta T$, i.e. that $n_t \gg 1$, then the temperature is approximately uniform across the channel and $Q(x) \rightarrow C_{pp}v(x)$ and the entropy gain or fractional loss becomes

$$\langle Q\delta T \rangle / \langle Q \rangle \Delta T = 1/n_t \quad (3)$$

Ideal Channel Flow

Ideal channel flow is Poiseuille flow and is discussed in many textbooks on laminar flow. However, we need the derivation in several places and so repeat it here.

Laminar flow in a uniform width channel experiences a shear stress that balances the force per unit extent ("extent" is measured in the y direction perpendicular to x and z , Fig. 1). The pressure gradient in the z direction produces this force. This pressure gradient is independent of x , i.e. normal to the channel walls. By symmetry the shear stress in the midplane of fluid must be zero, and at each wall it becomes maximum. Therefore, the fluid shear stress will be proportional to the integral of the z component of the pressure gradient from the midplane or measuring x as the distance from the midplane, then:

$$(\eta) dv/dx = \int_0^x (dP/dz) dx = P'x$$

and so

$$v = v_{\max}(1-4x^2/W^2) \quad (4)$$

Hence,

$$v_{\max} = P' W^2/(8\eta)$$

where

ν = viscosity, $P' = dP/dz$, and $x = W/2$ at the wall.

The stress of the wall, τ , becomes

$$\tau = \nu \left. \frac{dv}{dx} \right| (x = \frac{W}{2}) = -4 \nu_{\max} \nu/w^2 = -P' w/2 \quad (5)$$

and

$$P' = \tau/(w/2) = 8\nu_{\max} \nu/w^2 . \quad (6)$$

Therefore, the velocity is quadratic from $v = \nu_{\max}$ at the midplane to $v = 0$ at the wall and $\langle v \rangle = 2/3 \nu_{\max}$ and $\langle v^2 \rangle = (8/15)\nu_{\max}^2$ by integrating Eq. 4. The number of exchange lengths in a channel of length L using Eq. (2) for t becomes

$$n_t = L/(\langle t \rangle \langle v \rangle) = (27/2)(DL)/(W^2 \nu_{\max}) = (27/2)(\kappa/C_p \rho)L/(W^2 \nu_{\max}) . \quad (7)$$

Since the flow is nondivergent in steady state, the heat flux convected into any layer must be balanced by conduction to the wall. Therefore,

$$\kappa dT/dx = (-C_p \rho) \int_0^x v T_x dx . \quad (8)$$

Here the convected flux is $(C_p \rho T_z v)$ and T_z is the temperature gradient in the z direction = $\Delta T/L$. We assume that T_z is independent of x because $L/n_T \gg W/2$ so that $(dT/dz) \gg (dT/dx)$. Then from Eq. (4), $(dT/dz) \gg dT/dx$,

$$dT/dx = -(C_p \rho / \kappa)(\Delta T/L) \nu_{\max} (x^4 - x^3/3W^2) , \quad (9)$$

and the temperature difference between the fluid and the wall, δT , becomes

$$\delta T = T_{\max} - T = (C_p \rho / \kappa)(\Delta T/L) \nu_{\max} (x^2/2 - x^4/3W^2) . \quad (10)$$

If $\delta T_{\max}$ is the maximum temperature between the midplane and the wall, then by substituting $w/2$ for x in Eq. (10),

$$\delta T_{\max} = 5/12 (C_p \rho / \kappa) (\Delta T / L) v_{\max} (W/2)^2 . \quad (11)$$

But the velocity averaged temperature difference that leads to entropy gain each half cycle will be $\langle \delta T v \rangle / \langle v \rangle$. By integrating Eqs. (4) and (10), we obtain

$$\begin{aligned} n_t &= \langle \delta T v \rangle / (\langle v \rangle \Delta T) = L (\kappa / C_p \rho) \{63/5\} / (W^2 v_{\max}) , \\ &= 12.6 L (\kappa / C_p \rho) / (W^2 v_{\max}) . \end{aligned} \quad (12)$$

This is close to that derived by thermal relaxation length analysis, Eq. (7) where the factor is 13.5. Since both derivations used several approximations, the agreement is satisfactory and the simplifying concept of thermal and viscous relaxation lengths is substantiated.

Friction Loss

The fractional friction loss, $1/n_f$, will be

$$1/n_f = \int dt \int P' v dx / \int P v dt \quad (13)$$

Here taking P' from Eq. 5 and noting that P' is constant in a length L results in

$$\frac{1}{n_f} = \frac{\int \langle v \rangle (8 v_{\max} v L / W^2) dt}{(2/\pi) P \langle v \rangle \ln C_R} \quad (14)$$

$$= (4\pi/3) v_{\max} v L / (W^2 P \ln C_R) \quad (15)$$

where the half cycle time average of $\langle v_{\max} \rangle$ is $2/\pi$ and that of $\langle v^2_{\max} \rangle = 1/2$. We compare this to the relaxation length argument where the kinetic energy of the gas,

$\rho \langle v^2 \rangle / 2$, is dissipated (n_f/P_r) times per transit of the gas through the length L . Then using Eq. 7 for n_f this loss becomes

$$\frac{1}{n_f} = \frac{27}{2} \frac{(\kappa/C_p \rho) L \rho (8/15) v_{\max}^2 / 2}{W_{v_{\max}}^2 P_r (2\pi) P \langle v \rangle \ln C_R} \quad (16)$$

$$= (6\pi/5) v_{\max} v L / (W^2 P \ln C_R) \quad (17)$$

where $P_r = 3/2(\kappa/C_p \rho)/(v/\rho)$.

When Eq. (17) is compared to (15) we find a factor of 9/10, or sufficient agreement for the use of the relaxation length arguments.

Pressure Drop

It is useful to know the maximum fractional pressure drop $\Delta P_{\max}$ as a function of n_f and n_g . From Eq. (6) the fractional pressure drop becomes

$$\Delta P_{\max}/P = 8 v_{\max} v L / (W^2 P) \quad (19)$$

and using Eq. (16)

$$\Delta P_{\max}/P = (6/\pi)(\ln C_R)/n_f = (3/\pi)(\ln C_R)/n_g . \quad (20)$$

since $n_f = 2 n_g$ by the relations before Eq. (15) of the text.

APPENDIX III

The Stirling Cycle and useful Refrigeration

The pressure volume relation for an ideal Stirling cycle is shown in Fig. 3.

The compression ratio is defined by $V_1/V_2 = C_R$ and each of two volumes are assumed isothermal. In the diagram, the state conditions of just the expansion (refrigerated) volume is shown. Starting at state 0, both volumes are expanded and the pressure is a minimum P_1 . The expansion volume compresses along the isotherm $1 \rightarrow 2$ doing work $P_1 V_1 \ln C_R$. The expansion volume is then static (compressed) during the subsequent compression by the unseen second volume or compression volume. This unseen compression takes the gas from the P_2 to P_3 . No work is done by the expansion volume since it is static. The expansion volume then expands from state 3 to 4 performing work $P_2 V_1 \ln C_R$. Following state 4, the expansion volume is again static and the second volume expands the gas from state 4 to the original state 1. The network in the cycle performed by the expansion volume is $(P_2 - P_1) V_1 \ln C_R$. If each volume holds an equal mass of gas, that is their expanded volumes must be in the ratio of their respective temperatures, then we expect $P_2 = 2P_1$ for zero dead volume. Then of course C_R would be infinite, but the work of the cycle would be $P_1 V_1 \ln C_R$.

Practical considerations alter this. The mechanical strength of the respective volumes limit the value of the maximum pressure, P_3 . Since $P_3 = 2 C_R P_1$, the work becomes $(P_3/2 C_R) \ln C_R$ which maximizes for $C_R = e$. An additional limitation is the drive mechanism. The ideal cycle has periods where each volume is static. This can be created only by a special drive mechanism, i.e. the rombic or the Ross drive. On the other hand, the harmonic drive has several distinct advantages. The drive mechanism is simple to build, the accelerations are less so that mechanical balancing is simpler at high speed. In addition mass transfer occurs during both compression and expansion. This reduces the extremum of P_3 and P_1 . If the pressure is now increased such that the maximum is P_3 , then a cycle shape closer to the inscribed dotted curve results. This shape is the function of the phase angle between the two harmonic drives and the dead volume.

From the standpoint of the regenerator efficiency, one desires a maximum work per unit of displacement through the regenerator, since it is the process of displacement that generates losses. The displacement we define as $V_1 - V_2 = V_0$. If we define $C_R = P_{\max}/P_{\min}$, then the maximum possible work of the cycle is $(P_{\max} - P_{\min}) V_0 = P_{\max}$

$v_0(1 - 1/C_R)$, or if we define $P_{min} = P_0$, $W_{max} = P_0 V_0(C_R - 1)$. The actual work is less than this depending upon the shape of the inscribed loop. In the neighborhood of $C_R = 2$ for the present designs an approximation to the inscribed loop area is $\ln C_R$ from several numerical integrations. We therefore will use $P_0 V_0 \ln C_R$ as an approximate measure of the effective work of the cycle in the cold expansion volume recognizing that P_0 is the minimum pressure of the cycle, V_0 the displaced volume and that $C_R P_0$ the maximum pressure is held constant.

APPENDIX IV

Cycle Heat Energy Storage Loss

When heat is conducted into or out of a medium with finite conductivity there is an irreversible entropy gain depending upon the thermal diffusivity and time of transfer. Since the objective of a regenerator is to store heat reversibly and cyclically, this entropy gain is an effective loss to the system. We call this the thermal access loss. In refrigeration this leads to an inefficiency. This inefficiency will be evaluated as a ratio of (irreversible heat)/(heat stored) per cycle. The irreversible heat is just the (entropy gain)/T, T being the absolute temperature.

Let κ = thermal conductivity

C_V = thermal heat capacity per unit volume

$D = \kappa/C_V$ = thermal diffusivity.

Let the average local temperature be T, and the oscillating temperature T' be

$$T' = T + \theta, \quad \theta = \theta_0 e^{i(kx\omega t)} \quad (1)$$

the heat equation gives

$$\frac{\partial T'}{\partial t} = D \frac{\partial^2 T}{\partial x^2}, \quad (2)$$

$$\frac{\partial \theta'}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2}, \quad (3)$$

or

$$-i\omega\theta = Dk^2\theta, \quad (4)$$

so that

$$k = \sqrt{\frac{i\omega}{D}} = \frac{1+i}{\sqrt{2}} \sqrt{\omega/D} = \frac{1+i}{\delta} , \quad (5)$$

where the thermal skin depth δ is defined as

$$\delta = \sqrt{2D/\omega} = \sqrt{D/\pi f} . \quad (6)$$

We next choose the physically appropriate solution. We know that the temperature fluctuation must decay as one goes far into the material, i.e. as $x \rightarrow \infty$. Then

$$\begin{aligned} \int \theta \, dx &= -\frac{1}{ik} \theta_0 e^{-i\omega t} = \frac{-\delta}{\pm i(1+i)} \theta_0 e^{-i\omega t} \\ &= \frac{-(i+1)\delta}{\pm 2} \theta_0 e^{-i\omega t} , \end{aligned} \quad (7)$$

$$\text{when } x \rightarrow \infty, e^{ikx} \rightarrow 0 \text{ so } \text{Im}k > 0 . \quad (8)$$

Therefore we must use the positive root. Hence,

$$\text{Re} \int_0^\infty \theta \, dx = \delta \theta_0 e^{i\omega t} . \quad (9)$$

We now calculate the entropy fluctuation.

$$\begin{aligned} \delta S &= \int_0^\infty dx \int_0^{2\pi/\omega} dt \frac{C_v \delta\theta/\delta t}{T'} \\ &= \int_0^\infty dx \int_0^{2\pi/\omega} dt \frac{C_v}{T'} (1-\theta/T) (-i\omega\theta) \end{aligned}$$

$$= \frac{i\omega C_v}{T^2} \int_0^\infty dx \int_0^{2\pi/\omega} dt \frac{C_v}{T'} (\text{Re}\theta)^2 . \quad (10)$$

But using k in the definition of θ gives

$$\theta = \theta_0 e^{(ix/\delta - i\omega t)} e^{-x/\delta} \quad (11)$$

so that

$$\text{Re}\theta = \theta_0 \delta^{-\omega t} e^{-x/\delta} . \quad (12)$$

To check the reasonableness of our solution, physically, we know that the entropy loss or gain at $x = 0$ must be zero; that is

$$\int_{\text{cycle}}^? \frac{\delta\theta}{T+\theta} = \ln(T+\theta)_{\text{cycle}} = 0 . \quad (13)$$

For our solution at $x = 0$

$$dQ = \kappa \frac{\partial\theta}{\partial x} = \kappa\theta_0 \frac{d}{dx} [\cos(\frac{x}{\delta} - \omega t) e^{-x/\delta}] , \quad (14)$$

and so,

$$dS = \kappa \frac{dQ}{T'} = \kappa \frac{\theta_0^2}{T^2} \cos(\frac{x}{\delta} - \omega t) e^{-x/\delta} \times Q \frac{d}{dx} [\cos(\frac{x}{\delta} - \omega t) e^{-x/\delta}] . \quad (15)$$

Then

$$\int_{\text{cycle}} dS = \int_{\text{cycle}} \cos \cdot \sin = 0 , \quad (16)$$

since $d/dx \cos x \rightarrow \cos x \sin x$.

Therefore, our solution for S has the appropriate value at $x = 0$. If we integrate S Eq. (15) over time for $x \neq 0$, then

$$\Delta S = \kappa \frac{\theta_o^2}{T^2} \frac{1}{\delta} \frac{t}{2} . \quad (17)$$

For a cycle $t = 2\pi/\omega$. In this time the useful energy exchange is from Eq. 7,

$$\Delta E = C_v Re \int_{\pi/2}^{3\pi/2} \theta dx = C_v \theta_o \delta . \quad (18)$$

The irreversible energy loss is

$$T' \Delta S \equiv T \Delta S , \quad (19)$$

and the useful energy exchange is ΔE . Therefore the fractional loss becomes

$$\frac{T \Delta S}{\Delta E} = \left(\frac{D}{\delta^2} \right) \frac{\pi}{\omega} \left(\frac{\theta_o}{T} \right) = \frac{\theta_o}{T} . \quad (20)$$

Since the fluctuating temperature at the boundary is $\pm \theta_o$ during a cycle, the fractional loss is just twice the (fractional temperature change)/ temperature. As the thickness, d , of the heat storage layer is made thinner than a skin depth δ , a similar analysis shows that the entropy loss decreases as $(d/\delta)^3$ and the useful energy stored decreased as d/δ . Hence, one obtains that the fractional loss decreases as

$$1/3(d/\delta)^2 . \quad (21)$$

This is such a rapid improvement that a useful thickness for a regeneration material is limited to $1/2$ the normal skin depth, in which case the entropy loss due to the phase lag of the heat storage in and out will be small compared to the gas loss from the same effect.

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