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EIGENFUNCTIONS OF THE TRANSFER MATRIX IN THE PRESENCE OF LINEAR COUPLING*

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Abstract

This paper presents an approach to computing the change in the linear orbit parameters, due to a perturbing field that couples the x and y motions, by computing the change in the eigenfunctions of the transfer matrix and then using the relationship between the eigenfunctions and the orbit parameters. This can be compared with the approach [1] that computes the change in the transfer matrix and uses the relationship between the elements of the transfer matrix and the orbit parameters. For the case of coupled motion, the eigenfunction approach appears to be easier to apply than the transfer matrix approach, partly because the relationship between the transfer matrix and the linear orbit parameters is considerably more complicated in this case. Results are found for the change in the four eigenfunctions of the transfer matrix in the presence of linear coupling, and for the relationship between the eigenfunctions and the orbit parameters.

I. THE EIGENFUNCTIONS AND THE LINEAR ORBIT PARAMETERS

The eigenfunctions may be defined in terms of the transfer matrix, $T(s, s_0)$,

$$x(s) = T(s, s_0) x(s_0) \quad (1a)$$

In Eq. (1) $T(s, s_0)$ is a 4×4 matrix, $x(s)$ is a 4×1 column vector

$$x = x, p_x, y, p_y \quad (1b)$$

In the absence of solenoids, $p_x = x'$ and $p_y = y'$. The eigenfunctions are those $x(s)$ that satisfy

$$T(s + L, s) x = \lambda x, \quad (1c)$$

where L is the period of the magnetic guide field.

It can be shown [1] that there are 4 eigenfunctions $x_i(s)$, $i = 1, 2, 3, 4$ with eigenvalues λ_i , and which occur in pairs such that for stable motion,

$$x_2 = x_1^*, \quad x_4 = x_3^*, \quad \lambda_2 = \lambda_1^*, \quad \lambda_4 = \lambda_3^*.$$

It can be shown [2] that the eigenfunctions are solutions of the equations of motions, and that

$$x_1(s) = \exp(i2\pi\nu_1 s/L) f_1(s), \quad (2)$$

$$x_3(s) = \exp(i2\pi\nu_2 s/L) f_3(s).$$

$f_1(s)$, $f_3(s)$ are periodic in s with period L , and ν_1, ν_2 are the normal mode tunes. Note x_i and f_i are both 4×1 column vectors.

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A. The Transfer Matrix in Terms of the Eigenfunctions

Given the four eigenfunctions x_i , $i = 1, 4$ which are normalized such that $\tilde{x}_i^* S x_i = 2i$, (3)

where S is

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

then it will be shown that one can find the transfer matrix $T(s, s_0)$ from $T(s, s_0) = (-1/2i) U(s) \bar{U}(s_0)$

$$\bar{U} = \tilde{S} \tilde{U} S \quad (4)$$

$$U = [x_1 x_2 x_3 x_4].$$

U is a 4×4 matrix and x_i is a 4×1 column vector.

Eq. (4) will be derived for the 2-dimensional case. The generalization to 4 or more dimensions is clear. In two dimensions a solution of the equation of motion can be written as

$$x = a_1 x_1 + a_2 x_2, \quad x_2 = x^*, \quad (5)$$

$$a_1 = \tilde{x}_1^* S x / 2i, \quad a_2 = a_1^* = \tilde{x}_2^* S x / (-2i)$$

Evaluate a_1 and a_2 using $x(s_0)$. Then

$$\begin{aligned} x &= (1/2i) (x_1(s) \tilde{x}_1^*(s_0) - x_2(s) \tilde{x}_2^*(s_0)) S x(s_0) \\ x &= (1/2i) (x_1(s) \tilde{x}_1^*(s_0) - x_1^*(s) \tilde{x}_1(s_0)) S x(s_0) \\ x &= (1/2i) [x_1(s) \quad x_1^*(s)] S \begin{bmatrix} \tilde{x}_1(s_0) \\ \tilde{x}_1^*(s_0) \end{bmatrix} S x(s_0) \\ x &= (-1/2i) U(s) \bar{U}(s_0) x(s_0). \end{aligned} \quad (6)$$

$$\text{Thus} \quad T(s, s_0) = (-1/2i) U(s) \bar{U}(s_0) \quad (7)$$

One may note that $V = (-2i)^{-1/2} U(s)$ is symplectic as $T(s, s) = I$ and $V \bar{V} = I$.

Eq. (7) shows that knowing the eigenfunctions x_i is equivalent to knowing the transfer matrix $T(s, s_0)$. Eq. (7) also shows that $T(s, s_0)$ is symplectic as it is the product of two symplectic matrices, $V(s)$ and $\bar{V}(s_0)$.

B. Relationship Between Eigenfunctions and the Linear Orbit Parameters

The eigenfunctions will now be related to the 10 linear orbit parameters for the coupled motion.

In two dimensions, the eigenfunction is related to the 3 orbit parameters β, α, ψ by

$$x_1 = \begin{pmatrix} \beta^{1/2} \\ \beta^{-1/2}(-\alpha + i) \end{pmatrix} \exp(i\psi) \quad (8)$$

and $x_2 = x_1^*$. x_1 obeys the normalization condition

$$\tilde{x}_1^* S x_1 = 2i \quad (9)$$

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In four dimensions, one can go from the coordinates, x, p_x, y, p_y to an uncoupled set of coordinates v, p_v, u, p_u the normal coordinates, by the transformation [3]

$$x = R v \quad (10)$$

$$R = \begin{pmatrix} I \cos \varphi & \bar{D} \sin \varphi \\ -D \sin \varphi & I \cos \varphi \end{pmatrix} \quad (11)$$

I and R are 2×2 matrices. I is the 2×2 identity matrix. $\bar{D} = D^{-1}$ and $|D| = 1$. R is a symplectic matrix,

$$\bar{R}R = I \quad (12)$$

$$\bar{R} = \bar{S}R S$$

φ and the 3 independent elements of D may be considered as 4 of the orbit parameters. They are periodic in s . The other 6 orbit parameters are the $\beta_1, \alpha_1, \psi_1$ and $\beta_2, \alpha_2, \psi_2$ of the 2 normal modes.

It can be shown that $\tilde{x}^* S x$ is a constant [1] of the motion. Also if x and v are related by a symplectic matrix then

$$\tilde{x}^* S x = \tilde{v}^* S v \quad (13)$$

The transfer matrix for the v coordinates is given by

$$v(s) = U(s, s_0) v(s_0) \quad (14)$$

$$U = \bar{R}(s) T R(s_0)$$

It can then be shown that the eigenfunction of U , v_i , and the eigenfunctions of T are related by

$$x_i = R v_i \quad (15)$$

The v coordinates are uncoupled, so the v_i eigenfunctions can be written down using Eq. (8) as

$$\begin{aligned} v_1 &= \begin{pmatrix} \mu_1 \\ 0 \end{pmatrix} & v_3 &= \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix} \\ \mu_1 &= \begin{pmatrix} \beta_1^{\frac{1}{2}} \\ \beta_1^{-\frac{1}{2}}(-\alpha_1 + i) \end{pmatrix} \exp(i\psi_1), & (16) \\ \mu_2 &= \begin{pmatrix} \beta_2^{\frac{1}{2}} \\ \beta_2^{-\frac{1}{2}}(\alpha_2 + i) \end{pmatrix} \exp(i\psi_2) \\ v_2 &= v_1^*, & v_4 &= v_3^* \end{aligned}$$

one may note that $\tilde{v}_1^* S v_1 = \tilde{v}_3^* S v_3 = 2i$. The x_i can then be written down using $x = R v$ as

$$\begin{aligned} x_1 &= \begin{pmatrix} \mu_1 \cos \varphi \\ -D \mu_1 \sin \varphi \end{pmatrix} \\ x_3 &= \begin{pmatrix} \bar{D} \mu_2 \sin \varphi \\ \mu_2 \cos \varphi \end{pmatrix} \end{aligned} \quad (17)$$

Eq. (17) relates the eigenfunctions x_i to the 10 orbit parameters. Also $\tilde{x}_1^* S x_1 = \tilde{x}_3^* S x_3 = 2i$.

If the eigenfunctions are known, then Eqs. (17) can be inverted to find the 10 orbit parameters. One can use the additional relationships

$$\frac{d\psi_1}{ds} = \frac{1}{\beta_1}, \quad \frac{d\psi_2}{ds} = \frac{1}{\beta_2} \quad (18)$$

$\alpha_1 = -\frac{1}{2}\beta_1' + \beta_1 \tan \varphi d\varphi/ds$, $\alpha_2 = -\frac{1}{2}\beta_2' + \beta_2 \tan \varphi d\varphi/ds$ which are valid in absence of solenoidal fields.

To further illustrate how expressions for the eigenfunctions can be used to compute the linear orbit parameters, consider the problem of finding β_1 , the beta function of the normal mode with ν_1 , assuming that the eigenfunctions are known (see section II).

The eigenfunction x_1 may be written as

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix} \quad (19)$$

From Eq. (17)

$$x_{11} = \beta_1^{\frac{1}{2}} \cos \varphi \exp(i\psi_1) \quad (20)$$

which gives the two relations

$$\psi_1 = ph(x_{11}), \beta_1^{\frac{1}{2}} \cos \varphi = |x_{11}| \quad (21)$$

where $ph(x_{11})$ means the phase of x_{11} .

Assuming x_{11} is known, ψ_1 can be found from Eq. (21) and one can find β_1 from $\beta_1 = d\psi_1/ds$. Once β_1 is known, one can find $\cos \varphi$ from Eq. (21). One may note that the results for the tune ν_1 and ν_2 comes directly out of solving the equations of motion for the eigenfunctions (see section II).

To find the emittance ϵ_1 from the eigenfunctions, one can use the relationship [4]

$$\epsilon_1 = |\tilde{x}_1^* S x_1|^2 \quad (22)$$

II. PERTURBATION SOLUTIONS FOR THE EIGENFUNCTIONS

The skew quadrupole field is described by $a_1(s)$. On the median plane, the field B_x is given by

$$B_x = -B_0 a_1 x, \quad (23)$$

where B_0 is the main dipole field. ρ is the radius of curvature of the main dipole.

The solutions of Eq. (23) were found in two previous papers [5,6] when ν_x, ν_y are near the resonance line $\nu_x = \nu_y + p$. These solutions may be written as

$$\eta_x = A \exp(i\nu_{x,s} \theta_x) \left\{ 1 + \sum_{n \neq -p} f_n \right\} \quad (24)$$

$$f_n = \frac{(\nu_{x,s} - \nu_x) 2\nu_x b_n \exp[-i(n+p)\theta_x]}{\Delta\nu (n - \nu_x - \nu_y)(n+p)}$$

$$\eta_y = B \exp(i\nu_{y,s} \theta_y) \left\{ 1 + \sum_{n \neq p} g_n \right\}$$

$$g_n = \frac{(\nu_{y,s} - \nu_y) 2\nu_y c_n \exp[-i(n-p)\theta_y]}{\Delta\nu^* (n - \nu_x - \nu_y)(n-p)}$$

$$\Delta\nu = (1/4\pi\rho) \int ds (\beta_x \beta_y)^{\frac{1}{2}} a_1 \exp[i(-\nu_{x,s} \theta_x + \nu_{y,s} \theta_y)]$$

$$b_n = \frac{1}{4\pi\rho} \int ds (\beta_x \beta_y)^{\frac{1}{2}} a_1 \exp[i((n - \nu_y) \theta_x + \nu_y \theta_y)]$$

$$c_n = \frac{1}{4\pi\rho} \int ds (\beta_x \beta_y)^{\frac{1}{2}} a_1 \exp[i(\nu_x \theta_x + (n - \nu_x) \theta_y)]$$

$$\theta_x = \psi_x / \nu_x, \quad \theta_y = \psi_y / \nu_y$$

$\nu_{x,s}$ and $\nu_{y,s}$ are the solutions of

$$\nu_{x,s} = \nu_{y,s} + p, \quad (\nu_{x,s} - \nu_x) = |\Delta\nu|^2 \quad (25)$$

There are two solutions of Eq. (25) corresponding to the two normal modes. For the mode for which $\nu_{x,s} \rightarrow \nu_x$ when $a_1 \rightarrow 0$, we will put $\nu_{x,s} = \nu_1$, $\nu_{y,s} = \nu_1 - p$. For the mode for which $\nu_{y,s} \rightarrow \nu_y$ when $a_1 \rightarrow 0$, we will put $\nu_{y,s} = \nu_2$, $\nu_{x,s} = \nu_2 + p$. The A and B coefficients are related by

$$\begin{aligned} B_1 &= \frac{-(\nu_1 - \nu_x)}{\Delta\nu} A_1 \text{ for the } \nu_1 \text{ mode} \\ A_2 &= \frac{-(\nu_2 - \nu_y)}{\Delta\nu^*} B_2 \text{ for the } \nu_2 \text{ mode} \end{aligned} \quad (26)$$

The results for the eigenfunctions, Eq. (24) were found by solving the equations of motion to first order terms in a_1 . It has been assumed that ν_x, ν_y the unperturbed tune, is close to the coupling resonance $\nu_x = \nu_y + p$ and the $\nu_x - \nu_y - p$ can be considered to be small, of the same order as a_1 . This last assumption allows the equations to be simplified and it is the case of most interest to us.

The A and B coefficients in Eq. (24) have now to be chosen so that the eigenfunctions are properly normalized, which means the eigenfunctions can be then expressed in terms of the orbit parameters like $\beta_1, \alpha_1, \psi_1$ and $\beta_2, \alpha_2, \psi_2$. To understand this better consider the 2 dimensional case. If we wish the eigenfunction to be related to β, ψ by

$$x = \beta^{1/2} \exp(i\psi),$$

then

$$p_x = x' = \beta^{-1/2} (-\alpha + i) \exp(i\psi)$$

and the two eigenfunctions are given by x_1, x_1^* where

$$x_1 = \left[\beta^{-\frac{1}{2}} \frac{\beta^{\frac{1}{2}}}{(-\alpha + i)} \right] \exp(i\psi) \quad (27)$$

These eigenfunctions are normalized so that

$$\tilde{x}_1^* S x_1 = 2i \quad (28)$$

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\tilde{x}_1$ is the transpose of x_1 .

The normalization given by Eq. (28) gives the relationship between x_1 and β, α, ψ given by Eq. (27). It is shown in section I, that in the 4 dimensional case the normalization Eq. (28) will allow the eigenfunctions x_1, x_3 to be related to $\beta_1, \alpha_1, \psi_1$ and $\beta_2, \alpha_2, \psi_2$ in a corresponding way. In this case, S is now the corresponding 4×4 matrix. Eq. (28) will be used to determine the coefficients A, B . This gives the relationship, see Ref. 7 where the following relationship is derived,

$$|A|^2 (\nu_{x,s}/\nu_x) + |B|^2 (\nu_{y,s}/\nu_y) = 1. \quad (29)$$

Eq. (29) together with Eq. (26) determine A and B .

For the ν_1 mode

$$\begin{aligned} B_1 &= -\frac{\nu_1 - \nu_x}{\Delta\nu} A_1, \\ |A_1|^2 \left(\frac{\nu_1}{\nu_x} + \frac{\nu_1 - p}{\nu_y} \left| \frac{\nu_1 - \nu_x}{\Delta\nu} \right|^2 \right) &= 1. \end{aligned} \quad (30)$$

For the ν_2 mode

$$\begin{aligned} A_2 &= -\frac{\nu_2 - \nu_y}{\Delta\nu^*} B_2, \\ |B_2|^2 \left(\frac{\nu_2}{\nu_y} + \frac{\nu_2 + p}{\nu_x} \left| \frac{\nu_2 - \nu_y}{\Delta\nu} \right|^2 \right) &= 1. \end{aligned} \quad (31)$$

A case of particular interest is when the linear coupling has been corrected to make $\Delta\nu \simeq 0$. There are then two solutions of interest,

1. $|\Delta\nu| \ll |\nu_x - \nu_y - p|$
2. $|\nu_x - \nu_y - p| \ll |\Delta\nu|$

In case 1., $\Delta\nu$ has been made small enough so that the tune ν_x, ν_y is well outside the width of the difference resonance. This may not always be achieved. ν_x, ν_y may be very close to the difference resonance, and the best setting of the correction system to minimize the tune splitting does not have to correspond to $\Delta\nu = 0$.

If $|\Delta\nu| \ll |\nu_x - \nu_y - p|$, one finds

$$|A_1| = 1, \quad B_1 = 0, \quad (32)$$

$$|B_2| = 1, \quad A_2 = 0.$$

The two modes appear to be decoupled.

If $|\nu_x - \nu_y - p| \ll |\Delta\nu|$, one finds

$$\begin{aligned} |A_1| &= |B_1| = 1/\sqrt{2} \\ |A_2| &= |B_2| = 1/\sqrt{2} \end{aligned} \quad (33)$$

The two modes appear completely coupled.

Examples of using the eigenfunction approach to compute the linear orbit parameters are given in Ref. 4,7.

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