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**CONSTRUCTION Of The COURANT-SNYDER INVARIANTS For The
NON-LINEAR EQUATIONS Of MOTION UNDER PRESENCE
Of The LINEAR COUPLING**

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Abstract

A single particle subjected to an arbitrary static magnetic field, which includes skew-quadrupole and solenoidal fields, is considered. The Courant-Snyder invariants are constructed, and the long-term stability of the beam in a storage ring is formulated in terms of the normal modes. The generalized Courant-Snyder invariants become Lyapunov functions when the new generalized B-functions admit non-zero lower and finite upper bounds.

1. Introduction.

In the paper we construct the Courant-Snyder invariants for the full, linearly coupled equations of motion, and we formulate the long-term stability criterion for the beam in a storage ring. Hence, the paper can be viewed as an addition to our first paper on the subject in which the linear coupling was ignored.¹

2. The Model

We consider an almost linear, two-dimensional, dynamical system in which the linear part is generated by the Hamiltonian

$$H = \frac{1}{2} (p_x^2 + p_y^2) + L(p_x y - p_y x) + \frac{1}{2} Fx^2 + Kxy + \frac{1}{2} Gy^2, \quad (2.1)$$

where x and y are orthogonal coordinates, p_x and p_y are their conjugate momenta, and L , F , K and G are the periodic in s with a period C . This form of the Hamiltonian may describe small betatron oscillations in a synchrotron in which case K and L would arise from skew quadrupole and solenoid fields, respectively. The system is characterized by the following equations of motion

$$\begin{aligned} x' &= p_x + Ly + \varphi_x, \\ p_x' &= Lp_y - y - Fx + \varphi_p, \\ y' &= p_y - Lx + \varphi_y, \\ p_y' &= -Lp_x - Kx - Gy + \varphi_q, \end{aligned} \quad (2.2)$$

where the nonlinearities, coming from the higher-order multipoles or from relativistic factors, and other sources, are represented by the functions φ_x , φ_p , φ_y , and φ_q . In short, the equations of motion are

$$z' = A(s)z + \varphi(z,s), \quad (2.3)$$

where

$$z = \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix}, \quad (2.4)$$

and

$$\dot{z} = \begin{bmatrix} 0 & 1 & L & 0 \\ -F & 0 & -K & L \\ -L & 0 & 0 & 1 \\ -K & -L & -G & 0 \end{bmatrix} z \quad (2.5)$$

We do not require the full equations to be Hamiltonian, only their linear part are assumed to be such.

It is assumed further that the reference trajectory: $z = 0$, is a solution of the equations of motion i.e.,

$$\varphi(0,s) = 0. \quad (2.6)$$

We are interested in the stability of the reference trajectory, that is we ask whether nearby curves stay that way for all times. Following Poincare's method we consider first the linearized equations of motion

$$\dot{z} = A(s)z, \quad (2.7)$$

which can be solved by the transfer matrix method

$$z(s) = T(s,s_0)z(s_0), \quad (2.8)$$

$$T = \begin{bmatrix} M & n \\ m & N \end{bmatrix}. \quad (2.9)$$

The fundamental matrix T is symplectic and periodic, and is of the coupled form which prevents direct application of our earlier results. The matrix can, be however, brought to the quasidiagonal, decoupled form, by a linear change of the variables ^{2,3,4} (i.e.,

by passing to the normal modes),

$$z = R w , \quad (2.10)$$

where

$$w = \begin{bmatrix} u \\ p_u \\ v \\ p_v \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} , \quad (2.11)$$

and R is a symplectic matrix (cf. Appendix for details). The new linearized equations are

$$w' = P(s)w , \quad (2.12)$$

where the matrix $P(s)$ is of the form

$$P = R^{-1}AR - R^{-1}R' = \begin{bmatrix} P_u & O \\ O & P_v \end{bmatrix} , \quad (2.13)$$

and

$$P_{u,v} = \begin{bmatrix} p & q \\ -k & -p \end{bmatrix}_{1,2} , \quad (2.14)$$

where p, q, k are periodic functions of s which can be expressed by the coefficients of the initial Hamiltonian and the matrix R , (cf. Appendix).

The transfer matrix T can be written in the form

$$T = R U R^{-1} , \quad (2.15)$$

where the symplectic matrix U becomes quasidiagonal

$$U = \begin{bmatrix} A & O \\ O & B \end{bmatrix} , \quad (2.16)$$

while the 2×2 submatrices A, B are symplectic and thus admit the familiar Courant-Snyder-Twiss (CST) parametrizations. In the case of a single-turn transfer matrix T one has for A

and B the following expressions in terms of the new tunes- μ and the new CST parameters

$$A = \cos \mu_1 \mathbf{1} + \sin \mu_1 J_1 , \quad (2.17)$$

and

$$B = \cos \mu_2 \mathbf{1} + \sin \mu_2 J_2 , \quad (2.18)$$

where

$$J_{1,2} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}_{1,2} , \quad \gamma\beta = 1 + \alpha^2 , \quad (2.19)$$

are symplectic and periodic. The new CST parameters $(\beta, \alpha, \gamma)_{1,2}$ satisfy the following differential equations

$$\begin{aligned} \beta' &= 2p - 2q\alpha , \\ \alpha' &= k\beta - q\gamma , \\ \gamma' &= -2p\gamma + 2k\alpha , \end{aligned} \quad (2.20)$$

and, as the result the new $\beta_{1,2}$ functions satisfy the equations.

$$2\beta\beta'' - (\beta')^2 + 2 \frac{q'}{q} \beta\beta' + 4(qk + p \frac{q'}{q} - p^2 - p')\beta^2 - 4q^2 = 0. \quad (2.21)$$

The equation for γ follows from the two previous ones if the constraint (2.19) holds.

Following Courant-Snyder one may construct two invariants of the linearized equation of motion as follows

$$\begin{aligned} \epsilon_1 &= - (\sin \mu_1)^{-1} \tilde{U} \sigma A U = - \tilde{U} \sigma J_1 U \\ &= \tilde{U} \begin{bmatrix} \gamma_1 & \alpha_1 \\ \alpha_1 & \beta_1 \end{bmatrix} U \\ &= \gamma_1 u^2 + 2\alpha_1 u p_u + \beta_1 p_u^2 \\ &= \frac{1}{\beta_1} [u^2 + (\alpha_1 u + \beta_1 p_u)^2] \geq 0 \end{aligned} \quad (2.22)$$

and similar for ϵ_2

$$\epsilon_2 = - (\sin \mu_2)^{-1} \tilde{V} \sigma B V = - \tilde{V} \sigma J_2 V . \quad (2.23)$$

It is clear that our earlier results apply here and the long-term stability of the linearized motions is guaranteed when the new β -functions admit finite lower and upper bounds, {cf. (3.47), Ref. 1]. This in turn imposes restrictions on the functions $(k,p,q)_{1,2}$ occurring in the equation (2.12) and, further, on the functions K,L,F,G from the original equations (2.7), and (2.2). This is equivalent to the usual stability criterion in terms of the traces of the transfer matrices.

$$\text{tr}A = 2 \cos \mu_1 \leq 2, \quad \text{tr}B = 2 \cos \mu_2 \leq 2. \quad (2.24)$$

This criterion, however, is not portable to the non-linear case.

We will assume in the sequel that the stability conditions are satisfied on the level of the linearized equations of motion. Furthermore, we will demand that the functions K,L,F , and G are such that the following characteristic determinant is non-negative around a ring, viz.,

$$|m + n| \geq 0. \quad (2.25)$$

This will ensure stability of the betatron oscillations when the tunes are close to one another.

The new tunes $-\mu_{1,2}$ are connected with the new β -function as follows:

$$\mu_{1,2} = \int_0^C \left[\frac{q}{\beta} \right]_{1,2} ds \quad (2.26)$$

where $q_{1,2}$ are given in terms of the solenoidal coupling coefficient L , (cf. Appendix, A.6), while $\beta_{1,2}$ can be found from the formulae (2.17), and (2.18), respectively.

The matrices A and B are given in terms of the submatrices of the single-turn transfer matrix T as follows^{3,7}

$$A = M + (t + \delta)^{-1} (\bar{m} + n) m, \quad (2.27)$$

and

$$B = N + (t + \delta)^{-1} (m + \bar{n}) n , \quad (2.28)$$

where

$$t = \frac{1}{2} \text{tr} (M - N) , \quad (2.29)$$

and

$$\delta = \frac{1}{2} \text{tr} (A - B) = (t^2 + |\bar{m} + n|)^{\frac{1}{2}} . \quad (2.30)$$

3. Generalization of the Courant-Snyder Invariants to the Non-Linear Case and the Stability Criterion.

Performing the substitution (2.10) in the original equations of motion (2.3) we get the new equations of motion

$$w' = P(s)w + \varphi(w, s) , \quad (3.1)$$

where the new function φ is

$$\varphi_{new}(w, s) \equiv R^{-1} \varphi_{old}(Rw, s) \equiv \begin{bmatrix} \varphi_u \\ \varphi_p \\ \varphi_v \\ \varphi_q \end{bmatrix} , \quad (3.2)$$

and P is given by (2.13), (2.14). In components these equations are of the form

$$\begin{aligned} u' &= p_1 u + q_1 p_u + \varphi_u , \\ p_u' &= -k_1 u - p_1 p_u + \varphi_p , \\ v' &= p_2 v + q_2 p_v + \varphi_v , \\ p_v' &= -k_2 v - p_2 p_v + \varphi_q \end{aligned}$$

Following our approach we define the "quadratic" form

$$E_1 = \tilde{U} \begin{bmatrix} r_1 & A_1 \\ A_1 & B_1 \end{bmatrix} U , \quad (3.4)$$

$$= \Gamma_1 u^2 + 2A_1 u P_u + B_1 P_u^2 , \quad (3.5)$$

and similar for the V-normal mode, where the generalized CST parameters B , A , Γ are assumed to be periodic functions of s satisfying the conditions similar to those from the linearized case (we omit the subscript 1, for clarity),

$$\Gamma B - A^2 = 1 , \quad (3.6)$$

$$\frac{dB}{ds} = 2PB - 2QA , \quad (3.7)$$

$$\frac{dA}{dS} = KB - Q\Gamma , \quad (3.8)$$

$$\frac{d\Gamma}{ds} = -2P\Gamma + 2KA , \quad (3.9)$$

where the functions K, P, Q are determined by the requirement that E is a constant of motion of the full equations (3.)

$$\frac{dE}{dS} \equiv 0 . \quad (3.10)$$

As the result we get for the quantities K , Q , and P the expressions

$$K = k + \Delta K , \quad (3.11)$$

where

$$\Delta k = - \frac{1}{Au^2 + Bup_u} [(\Gamma u + Ap_u) \varphi_u + (Au + Bp_u) \varphi_p] , \quad (3.12)$$

and

$$P = p \quad , \quad (3.13)$$

and

$$Q = q \quad . \quad (3.14)$$

The generalized B-function satisfies the equation which follows from the equations

$$\begin{aligned} B' &= 2pB - 2qA \quad , \\ A' &= kB - q\Gamma + B\Delta k \quad , \\ \Gamma' &= -2p\Gamma + 2kA + 2A\Delta k \quad , \end{aligned} \quad (3.15)$$

and, as the result

$$2BB'' - (B')^2 + 2\frac{q'}{q}BB' + 4(qK + p\frac{q'}{q} - p')B^2 - 4q^2 = 0 \quad . \quad (3.16)$$

The B-function has properties similar to those of the original β -function. It is positive, periodic function which is continuous, together with its first derivative. Due to the condition (3.6) the generalized Courant-Snyder invariant is positive definite and becomes Lyapunov function when the B-function admits non-zero lower, and finite upper bounds. The long-term stability of the beam in a storage ring is then guaranteed. This in turn imposes restrictions on the nonlinear terms $\varphi(\omega, s)$ under which the motion is stable. Since, after the decoupling, the theory becomes quite analogous to the previously treated case, there is no need to repeat it here. The formal solutions of the equations (3.1) and (3.15) can be found in the same way as it is done in our first paper on the subject.¹

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Appendix

a. The Edwards-Teng Decomposition.

Any symplectic 4x4 single-turn transfer matrix T can be decomposed as follows:

$$T = \begin{bmatrix} M & n \\ m & N \end{bmatrix} = R \begin{bmatrix} A & O \\ O & B \end{bmatrix} R^{-1} , \quad (\text{A.1})$$

where the symplectic matrix R can be parametrized in the following way³

$$T = \begin{bmatrix} 1 \cos \phi & D^{-1} \sin \phi \\ -D \sin \phi & 1 \cos \phi \end{bmatrix} , \quad (\text{A.2})$$

and the 2x2 symplectic submatrix D is

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} , \quad (\text{A.3})$$

and

$$| D | = ad - bc = 1 . \quad (\text{A.4})$$

It has been demonstrated by Edwards and Teng³ that the parameters ϕ , a , b , c , and d , which are periodic in s , in case of a ring, satisfy the following set of the differential equations:

$$\begin{aligned} \phi' &= \frac{1}{2}(a + d)L - \frac{1}{2}bK , \\ a' &= c + bF - \{[a(a + d) - 2]L - abK\} \cot 2\phi , \\ b' &= d - a - \{b(a + d)L - b^2K\} \cot 2\phi , \\ c' &= -aG + dF - \{c(a + d)L - (ad + 1)K\} \cot 2\phi , \\ d' &= -c - bG - \{[d(a + d) - 2]L - bdK\} \cot 2\phi , \end{aligned} \quad (\text{A.5})$$

where K , L , F , and G are coefficient functions occurring in the initial Hamiltonian.

Appendix (continued).

b. The Decoupled Equations of Motion.

The coefficient $(p, q, k)_{1,2}$ occurring in (3.3) are given by the expressions:

$$\begin{aligned} p_1 &= -\frac{1}{2} [(a - d)L + bK] \tan \phi , \\ q_1 &= 1 - bL \tan \phi , \end{aligned} \quad (A.6)$$

and

$$\begin{aligned} k_1 &= F + (cL - aK) \tan \phi , \\ p_2 &= \frac{1}{2} [(a - d)L - bK] \tan \phi , \\ q_2 &= 1 + bL \tan \phi , \\ k_2 &= G - (cL - dK) \tan \phi . \end{aligned}$$

Strictly speaking the Edwards-Teng parametrization holds at these regions in a ring where the determinant $|m + \bar{n}|$ is positive. The parameters a, b, c, d , and ϕ are then real, and

the angle ϕ is contained in the interval $[\frac{\pi}{4}, \frac{\pi}{4}]$. If, however, the determinant is negative the parameters become purely imaginary

$$\begin{aligned} \phi &= i\tilde{\phi} , \\ a &= -i\tilde{a} , \quad \text{etc.} \end{aligned} \quad (A.7)$$

and

$$D \sin \phi = iD \operatorname{sh} \tilde{\phi} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} \operatorname{sh} \tilde{\phi} , \quad (A.8)$$

where now

$$\tilde{a} \tilde{d} - \tilde{b} \tilde{c} = -1 . \quad (A.9)$$

and the parameter $\tilde{\phi}$ can now assume any real value. We exclude this case from the considerations assuming that the determinant $|m + \bar{n}|$ is positive around a ring, cf. (2.25).

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