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Particle Motion in Crystalline Beams

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*Abstract **

We study the possibility of storing a highly intense charged-particle beam in a storage ring. High beam density is obtained by cooling and freezing the beam to a very low temperature. Particles will eventually form an ordered configuration. In this report we derive the equations of motion of a particle moving under strong space-charge influence.

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1 Introduction

Studying the possibility of storing a low emittance (or “cooled”) beam of charged particles in a storage ring, we are faced with the effect of space charge by which particles are repelled and influence each others’ motion. The correct evaluation of the space-charge effects is important to determine the attainment and properties of Crystalline Beams, a phase transition which intense beams of ions can undergo when cooling is applied.

In this report we derive the equations of motion of a particle moving under the action of external restoring forces generated by the magnets of the storage ring, and of the electromagnetic fields generated by the other particles. The motion in every direction is investigated: in the longitudinal, as well as vertical and horizontal direction. The external forces are assumed to be linear with the particle displacement from the reference orbit. The space-charge forces are comparable in magnitude to the external focussing forces.

The equations of motion so derived are then used to determine confinement and stability conditions for the attainment of Crystalline Beams, using transfer matrices.

2 The Hamiltonian

In the relativistic relationship between the particle momentum $\mathbf{p}$, its total energy E , and its rest mass m_0 , viz.:

$$E^2 = m_0^2 c^4 + \mathbf{p}^2 c^2, \quad (1)$$

we substitute

$$\begin{aligned} E &\rightarrow H - Qe\phi, \\ \mathbf{p} &\rightarrow \mathbf{P} - \frac{Qe}{c}\mathbf{A}. \end{aligned} \quad (2)$$

H is the Hamiltonian of a single particle with charge Qe moving with generalized momentum $\mathbf{P}$ in electric and magnetic fields given by the scalar potential ϕ and vector potential $\mathbf{A}$. The Hamiltonian of such a system in an orthogonal Cartesian coordinate system is [1]:

$$(H - Qe\phi)^2 = m_0^2 c^4 + c^2 \left(\mathbf{P} - \frac{Qe}{c} \mathbf{A} \right)^2. \quad (3)$$

To study the single-particle dynamics around a closed, *planar* reference orbit, we introduce a new curvilinear coordinate system (x, s, y) . In this system, the three spatial coordinates are the arc or path length s along the reference orbit, and the displacement x in the radial and y in the vertical direction [2]. The reference trajectory is described by the curvature function $h(s)$. The curvature function is a piecewise constant with a period equal to the total arc length of one traversal of the reference trajectory. $h(s)$ is zero in a drift and quadrupole magnet, and equal to the inverse of the radius of curvature in the bending magnet.

In this new curvilinear coordinate system Hamiltonian Eq. (3) becomes:

$$\begin{aligned} (H - Qe\phi)^2 = & m_0^2 c^4 + c^2 \frac{(P_s - \frac{Qe}{c} A_s)^2}{[1 + h(s)x]^2} \\ & + c^2 (P_x - \frac{Qe}{c} A_x)^2 + c^2 (P_y - \frac{Qe}{c} A_y)^2 \end{aligned} \quad (4)$$

In order to study the dynamics around the reference orbit it is more convenient to use s instead of time t as the independent variable. Consequently our new Hamiltonian becomes [2]:

$$\mathcal{G} = -P_s(x, P_x, y, P_y, H, t; s), \quad (5)$$

where the new canonically conjugate variables are

$$x, P_x \quad y, P_y \quad \text{and} \quad H, t.$$

From Eq. (4) we obtain:

$$\begin{aligned} \mathcal{G} = & -\frac{Qe}{c}A_s - [1 + h(s)x]\left\{\frac{(H - Qe\phi)^2}{c^2}\right. \\ & \left. - m_0^2c^2 - (P_x - \frac{Qe\phi}{c}A_x)^2 - (P_y - \frac{Qe\phi}{c}A_y)^2\right\}^{\frac{1}{2}} \end{aligned} \quad (6)$$

The potentials ϕ and $\mathbf{A}$ in the Hamiltonian are made up of ϕ_{ext} and $\mathbf{A}_{ext}$, the external fields needed to confine the beam, as well as interparticle or space-charge contributions, ϕ_{sc} and $\mathbf{A}_{sc}$. We assume that $\phi_{ext} = 0$, so that:

$$\begin{aligned} \phi &= \phi_{sc}, \\ \mathbf{A} &= \mathbf{A}_{sc} + \mathbf{A}_{ext}. \end{aligned} \quad (7)$$

3 The External Potentials

Our aim is to retain only contributions up to (and including) quadratic in the displacements x and y and momentum error δ (to be defined later). So for the external field we take:

$$\mathbf{A}_{ext} \equiv (0, A_{ext}, 0), \quad (8)$$

i.e., $A_{x_{ext}} = A_{y_{ext}} = 0$, with

$$A_{ext} = \mu_1 x + \mu_2 x^2 + \mu_3 y^2. \quad (9)$$

We also have, since we consider static potentials in a current source free region of space:

$$\nabla^2 \mathbf{A} = 0 \quad \text{and} \quad \nabla \mathbf{A} = 0. \quad (10)$$

In a drift:

$$\mu_i = 0; \quad i = 1 \dots 3. \quad (11)$$

In a pure quadrupole (or magnet edge):

$$\mu_1 = 0 \quad \text{and} \quad \mu_2 + \mu_3 = 0, \quad (12)$$

and μ_2, μ_3 are related to the quadrupole gradient $G' = B'$. In a pure dipole:

$$\mu_2 = \mu_3 = 0 \quad (13)$$

and μ_1 is related to the bending field B . In a combined function magnet with curvature h :

$$2(\mu_2 + \mu_3) = h\mu_1, \quad (14)$$

where the field index n enters. The actual expressions for μ_i will be given later on.

The storage ring consists of drifts, regular quadrupoles, and bending magnets with a field index n . The bending magnets may have nonzero entrance and exit angles, in which case the edge effects are described by thin-lens quadrupoles.

The variation of A_{ext} from element to element implies that $A_{x_{ext}}$ and $A_{y_{ext}}$ are nonvanishing at the edge of each element. Nevertheless, their contribution is of higher order than quadratic and will therefore be neglected.

4 The Space-Charge Potential

The space-charge potential acting on the test particle is the sum of the single-particle potentials, viz.:

$$\phi_{sc} = \sum_i \frac{Qe}{s_i}, \quad (15)$$

with

$$s_i^2 = \mathbf{r}^2 - (\mathbf{r}_i \times \boldsymbol{\beta}_i)^2. \quad (16)$$

$\mathbf{r}_i$ is the position vector of the i th particle with respect to the test particle, and $\boldsymbol{\beta}_i$ its velocity normalized with the speed of light. All

these quantities are to be evaluated at the same instant [3, 4, 5].

Also:

$$\mathbf{A}_{sc} = \sum_i \frac{Q_i^e}{s_i} \beta_i. \quad (17)$$

Because we are dealing with a well-directed particle beam, the transverse components of the velocity are considerably smaller than the longitudinal one. Within the limit of our approximation:

$$A_{x_{sc}} = A_{y_{sc}} = 0, \quad (18)$$

and

$$A_{sc} \equiv A_{s_{sc}} = \beta \phi_{sc}. \quad (19)$$

β is the value of the reference velocity. Consequently, the space-charge contribution to the single-particle Hamiltonian Eq. (6) is expressed completely by its scalar part ϕ_{sc} . For the same reason as outlined for the external forces, the space-charge potential contains at least a linear term in x , due to the curvature of the trajectory:

$$\phi_{sc} = \nu_1 x + \phi_{sc}(x, y, \sigma). \quad (20)$$

The coefficient of the first term on the right-hand side is proportional to the (local) curvature of the reference trajectory, and consequently is zero in a drift, quadrupole, or at the edge of a magnet. The potential depends also on the displacement σ along the forward direction of the motion of the test particle; it describes the change in the potential as the particle travels in the forward direction. The coordinate σ will be defined more elaborately later on. The actual derivation of the space-charge potential is given in sections 8 – 10.

5 Quadratic Hamiltonian

We obtain the linearized equations of motion by expanding the right-hand side of the Hamiltonian $\mathcal{G}$, Eq. (6). Only terms up to

quadratic order are retained, and within this approximation $A_x = A_y = 0$. Hence, we replace $P_{x,y}$ with the actual momentum $p_{x,y}$. Observe that:

$$\left(\frac{H}{c}\right)^2 - m_0^2 c^2 = p^2 = p_0^2(1 + \delta)^2, \quad (21)$$

from which we obtain in particular:

$$H_0^2 = m_0^2 c^4 + p_0^2 c^2. \quad (22)$$

p_0 is the momentum of a particle on the reference trajectory and δ is the (relative) momentum error of the test particle with respect to p_0 . Consequently, the Hamiltonian for the motion around the reference orbit is:

$$\begin{aligned} \mathcal{G} = & -\frac{Qe}{c} A_s - p_0[1 + h(s)x][(1 + \delta)^2 - \\ & - 2\frac{QeH}{(p_0c)^2}\phi_{sc} + \frac{(Qe)^2}{(p_0c)^2}\phi_{sc}^2 - \frac{p_x^2 + p_y^2}{p_0^2}]^{\frac{1}{2}}, \end{aligned} \quad (23)$$

with (by virtue of Eq. (19)):

$$A_s = A_{ext} + \beta\phi_{sc}. \quad (24)$$

We note that:

$$\begin{aligned} hx &<< 1, \\ \delta &<< 1, \\ p_x^2 + p_y^2 &<< p_0^2. \end{aligned} \quad (25)$$

In this paper we consider cooled particle beams with the kinetic energy dominating the potential energy owing to the space charge. We call this energy regime “dominated by the kinetic energy”. The opposite case of being “dominated by the potential energy”, as in the case of a plasma or ion trap [6, 7], though interesting, is not considered in this paper.

Consequently, we demand that:

$$Qe\phi_{sc} << cp_0. \quad (26)$$

Since nearest neighbors contributions dominate the space charge, Eq. (26) sets a lower limit on the interparticle distance:

$$\lambda \gg \frac{(Qe)^2}{p_0 c}. \quad (27)$$

We now proceed by expanding the square root on the right-hand side of Eq. (23), taking conditions Eq. (25) and Eq. (26) into account, using the Taylor-series expansion:

$$f(\delta, \epsilon) = \{(1 + \delta)^2 + \epsilon\}^{\frac{1}{2}}; \quad (28)$$

$$\begin{aligned} f(\delta, \epsilon) &\approx f(0, 0) + \frac{\partial f}{\partial \delta}|_0 \delta + \frac{\partial f}{\partial \epsilon}|_0 \epsilon + \\ &+ \frac{1}{2} \frac{\partial^2 f}{\partial \delta^2}|_0 \delta^2 + \frac{1}{2} \frac{\partial^2 f}{\partial \epsilon^2}|_0 \epsilon^2 + \frac{\partial^2 f}{\partial \delta \partial \epsilon}|_0 \epsilon \delta + \dots \\ &= 1 + \delta + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} - \frac{\epsilon \delta}{2} + \dots \end{aligned} \quad (29)$$

Note that $(\partial^2 f)/(\partial \delta^2)|_0 = 0$. Consequently we obtain:

$$\begin{aligned} \mathcal{G} &= -\frac{Qe}{c} A_s - p_0 [1 + h(s)x] \left\{ 1 + \delta - \frac{p_x^2 + p_y^2}{2 p_0^2} + \right. \\ &- \frac{Qe H_0}{(p_0 c)^2} \phi_{sc} + \frac{(Qe)^2}{2(p_0 c)^2} \phi_{sc}^2 + \\ &- \left. \frac{(Qe H_0)^2}{2(p_0 c)^4} \phi_{sc}^2 + \frac{Qe H_0}{(\gamma p_0 c)^2} \phi_{sc} \delta + \dots \right\}. \end{aligned} \quad (30)$$

We have retained all the terms that can give contributions up to quadratic order in x, y, σ and δ .

Since p_0 is a constant of the motion, we scale both sides of Eq. (30) with p_0 , replacing $\frac{\mathcal{G}}{p_0}$ by $\mathcal{G}$, and defining:

$$x' = \frac{p_x}{p_0}, \quad (31)$$

$$y' = \frac{p_y}{p_0}. \quad (32)$$

Introducing the curvature contribution inside the graph parenthesis, replacing A_s by Eq. (24) and keeping up to and including

quadratic order terms gives:

$$\begin{aligned} \mathcal{G} = & \frac{(x')^2 + (y')^2}{2} - \delta - h(s)x - h(s)x\delta - \frac{Qe}{p_0c}A_{ext} + \\ & + \frac{Qr_0}{eA\beta^2\gamma} \left[\frac{1}{\gamma^2} + h(s)x - \frac{\delta}{\gamma^2} \right] \phi_{sc} + \\ & + \frac{(Qr_0)^2}{2(eA\beta^2\gamma^2)^2} \phi_{sc}^2, \end{aligned} \quad (33)$$

where $\gamma^2 = 1/(1 - \beta^2)$, $r_0 = e^2/(m_{proton}c^2)$ is the classical proton radius and A is the mass number of the particle.

The Hamiltonian Eq. (33) consists of two parts. The first part, on the first line of the right-hand side of Eq. (33), is the Hamiltonian when space-charge forces are negligible, i.e., for a dilute beam. The solution of the resulting equations of motion has been studied extensively [2]. In the following we require that the equations of motion we obtain from the "full Hamiltonian" $\mathcal{G}$ Eq. (33) in the presence of a space charge reduce to the (well-known) equations of motion for the dilute beam, when we set $\phi_{sc} = 0$.

The second part of the Hamiltonian is a result of the presence of the strong space-charge forces in the Crystalline Beam. In the spirit of the foregoing discussion, the last term in Eq. (33) will contribute to $\mathcal{G}$ only when the linear term of the space-charge potential ϕ_{sc} , proportional to ν_1 (see Eq. (20)), is not zero. This is the case along the curved parts of the reference orbit (e.g., in bending magnets), and the last term is proportional to $(\nu_1 x)^2$.

Introducing the explicit form of the space-charge potential Eq. (20), the Hamiltonian is rewritten in a more convenient form:

$$\begin{aligned} \mathcal{G} = & \frac{(x')^2 + (y')^2}{2} - \delta - h(s)x f_- - h(s)x\delta f_+ - \frac{Qe}{p_0c}A_{ext} + \\ & + \frac{Qr_0}{eA\beta^2\gamma^3} \phi_{sc}, \end{aligned} \quad (34)$$

with

$$f_{\pm} \equiv 1 \pm \frac{Qr_0}{eA\beta^2\gamma^3 h(s)} \nu_1 \quad (35)$$

Note that $f_{\pm} = 1$ in the absence of curvature. Furthermore, by inserting typical parameters it is easily seen that $f_{\pm} \approx 1$.

6 The Equations of Motion

In this section we derive the equations of motion. We start with the easiest one, the motion in the vertical direction (i.e., motion out of the plane of the reference orbit), viz:

$$\frac{d^2 y}{ds^2} \equiv y'' = -\frac{\partial \mathcal{G}}{\partial y}. \quad (36)$$

Note that there are no curvature effects here. Inserting the expressions for the external vector potential A_{ext} , Eq. (9) we find:

$$y'' + K_v(s) y + \frac{Qr_0}{eA\beta^2\gamma^3} \frac{\partial \phi_{sc}}{\partial y} = 0, \quad (37)$$

where

$$K_v = 2\frac{Qe}{p_0c}\mu_3 \quad (38)$$

is the vertical focussing function.

The radial equation of motion is:

$$\begin{aligned} x'' &= -\frac{\partial \mathcal{G}}{\partial x} \\ &= h(s)f_- + \frac{Qe}{p_0c}\mu_1 + h(s)f_+\delta + \\ &\quad - K_h(s)x - \frac{Qr_0}{eA\beta^2\gamma^3} \frac{\partial \phi_{sc}}{\partial x}, \end{aligned} \quad (39)$$

where

$$K_h = 2\frac{Qe}{p_0c}\mu_2. \quad (40)$$

is the horizontal focussing function.

In the absence of the space charge, the Lorentz force is balanced by the centrifugal force along the reference orbit:

$$h(s) + \frac{Qe}{p_0c}\mu_1 = 0, \quad (41)$$

that is with $\mu_1 = -B_0$ and $\rho = 1/h(s)$:

$$QeB_0\rho = p_0c. \quad (42)$$

The space charge adds a small contribution due to the repulsion between the particles in the beam. It is easy to see that such a term exists along the curved part of the reference orbit only, viz.:

$$h(s) + \frac{Qe}{p_0c}\mu_1 - \frac{Qr_0}{eA\beta^2\gamma^3}\nu_1 = 0. \quad (43)$$

There is a similar, and small, contribution to the dispersive term in δ . The most prominent effect of the space charge is a defocusing effect.

We now derive the equation of motion for the last set of canonically conjugate variables, namely H and t , viz.:

$$\frac{dt}{ds} = -p_0 \frac{\partial \mathcal{G}}{\partial H}, \quad (44)$$

$$\frac{dH}{ds} = p_0 \frac{\partial \mathcal{G}}{\partial t}. \quad (45)$$

In the above equations we have taken care of the scaling of $\mathcal{G}$ with p_0 ; this explains the appearance of p_0 in front of the partial differentials.

In order to evaluate Eq. (44), we note that:

$$\begin{aligned} \frac{\partial \mathcal{G}}{\partial H} &= \frac{dH^2}{dH} \frac{\partial \mathcal{G}}{\partial H^2} \\ &= 2H \frac{\partial \delta}{\partial H^2} \frac{\partial \mathcal{G}}{\partial \delta} \end{aligned} \quad (46)$$

We then use Eq. (21) to evaluate $\frac{\partial H^2}{\partial \delta}$. Hence, we obtain:

$$\begin{aligned} \frac{dt}{ds} &= -\frac{H_0(1+\beta^2\delta)}{p_0c^2(1+\delta)} \frac{\partial \mathcal{G}}{\partial \delta} \\ &= -\frac{1}{\beta c} \left(1 - \frac{\delta}{\gamma^2}\right) \frac{\partial \mathcal{G}}{\partial \delta} \end{aligned} \quad (47)$$

After linearization this equation becomes:

$$\frac{dt}{ds} = \frac{1}{\beta c} \left(1 - \frac{\delta}{\gamma^2} + h(s)f_{+x} \right) \quad (48)$$

For the reference particle $\delta = x = 0$ and, as expected, dt/ds reduces to the inverse of the velocity. Again, rather than working with the time t , we prefer to work with the distance. More specifically, we introduce the “path length difference” as a canonical variable:

$$d\sigma = \beta c d\tau, \quad (49)$$

with $\tau = t - t_0$ the time delay between the test and reference particle to arrive at a given location, described by the curvilinear coordinate s . Note that for the reference particle, $\sigma = 0$. The equation of motion of σ then is:

$$\frac{d\sigma}{ds} = h(s)f_{+x} - \frac{\delta}{\gamma^2}. \quad (50)$$

The momentum variable canonically conjugate to the path length difference σ is the momentum error δ itself which replaces H . We replace H using Eq. (46), noting that the explicit time dependence enters only in the space-charge terms:

$$\begin{aligned} \frac{d\delta}{ds} &= \frac{\partial \mathcal{G}}{\partial \sigma} \\ &= \frac{Qr_0}{eA\beta^2\gamma^3} \frac{\partial \phi_{sc}}{\partial \sigma}. \end{aligned} \quad (51)$$

7 Equilibrium, Confinement, and Stability

In the foregoing we determined the equations of motion of a test particle around a closed, planar reference orbit moving in a “mean field” of the Crystalline Beam. In subsequent sections we will discuss specific models of Crystalline Beams, namely *strings*, *zig-zags*

and *helices*, and we shall derive the corresponding space-charge potentials.

In order to determine whether the lattice of a storage ring will support a stable Crystalline Beam we proceed as follows: After having defined a reference configuration (e.g., *string*, *zig-zag*, *helix*, ...), we determine if it is an equilibrium configuration, and if additional external fields are needed to confine the reference configuration (i.e., the confinement conditions).

As the last step we determine whether the motion of a test particle perturbed slightly from its equilibrium position is stable. In other words, we check to see if the particle is locally in a potential well.

Given the complex nature of the lattice of a storage ring, and since we are dealing with linearized equations of motion, the easiest way to propagate the motion around the storage ring is to use transfer matrices. It will be shown in subsequent notes that both the confinement and the stability can be investigated with the transfer matrices. In particular, the effect of curvature and of the inherent momentum spread in the beam will be transparent.

We need to determine realistic models for the space-charge potential ϕ_{sc} . Molecular Dynamics (MD) simulations [8], confirmed by experiments with atoms in Paul traps [6, 7], have given some insight into the most probable equilibrium structures. Both experiments and simulations suggest that possible equilibrium configurations are indeed made of *strings*, *zig-zags*, and *helices*. In the following we shall take these models one at a time and follow the approach outlined above.

8 General Estimate of the Space-Charge Potential

The equations of motion can be correctly derived from the Hamiltonian Eq. (33). The last term in ϕ_{sc}^2 has been introduced to determine the contribution of the space charge in bending magnets where the trajectory has a finite curvature. This term is usually small, and can be neglected. The term linear in ϕ_{sc} is the term proper to the space charge, and in principle contains both the contribution of the equilibrium configuration as well as the potential introduced by a small perturbation. Here we derive the complete expression of ϕ_{sc} to be used in conjunction with Eq. (33) and the equations of motion Eq. (37), Eq. (39), Eq. (50) and Eq. (51). We consider Crystalline Beams made of a number n_s of parallel substrings, properly located, symmetrically around the common axis and shifted relative to each other.

The ions in each substring are separated by a lattice constant λ , which is the same for all substrings involved. Thus if there are N ions in the storage ring with circumference $2\pi R$:

$$\lambda = \frac{2\pi R}{N} n_s. \quad (52)$$

At equilibrium all the particles on a substring are located at the same fixed location x_n, y_n in the transverse plane. The longitudinal coordinate of the i th particle is $i\lambda - \lambda_n$, where λ_n varies from string to string ($n = 1, 2, \dots, n_s$), and is the spatial phase shift of the substrings relative to each other. The location x_n, y_n , as well as the longitudinal position λ_n , of the n th substring may vary slowly around the circumference to allow for the variation of transverse focusing and the curvature of the trajectory.

Let us consider a test particle on the n th substring. We can estimate the scalar potential acting in the proximity of its location

as the sum of the scalar potentials induced by all the substrings involved:

$$\phi_{sc} = \sum_{m=1}^{n_s} \phi_{nm}, \quad (53)$$

where ϕ_{nm} is the scalar potential produced by the m th substring on the test particle on the n th substring. Aside from the slow variation around the ring, and a negligible effect of the curvature of the trajectory in bending magnets, the potential is the same for all particles on the same substring.

Let u, v , and σ denote small displacements of the test particle around its reference position in the radial, vertical, and longitudinal direction, respectively. In the approximation of small displacements with respect to λ and to the transverse separation between substrings:

$$\phi_{nm} = \bar{\phi}_{nm}(x_n, y_n, \lambda_n) + \nu_2^{nm} u^2 + \nu_3^{nm} v^2 + \nu_4^{nm} \sigma^2, \quad (54)$$

where the first term is the potential at the location of the unperturbed position, used to determine the equilibrium configuration and the confinement conditions. The remaining terms are the potentials induced by the perturbation of the motion of the test particle, used to determine the stability of the motion. The contribution from all the particles on a different substring is

$$\bar{\phi}_{nm} = \sum_i \frac{Qe\gamma}{[(x_n - x_m)^2 + (y_n - y_m)^2 + \gamma^2(\lambda_i - \lambda_{nm})^2]^{\frac{1}{2}}} \quad (55)$$

with

$$\lambda_{nm} = \lambda_m - \lambda_n \quad (56)$$

and

$$\nu_2^{nm} = \frac{1}{2} Qe\gamma \sum_i \frac{2(x_n - x_m)^2 - (y_n - y_m)^2 - \gamma^2(\lambda_i - \lambda_{nm})^2}{[(x_n - x_m)^2 + (y_n - y_m)^2 + \gamma^2(\lambda_i - \lambda_{nm})^2]^{5/2}} \quad (57)$$

$$\nu_3^{nm} = \frac{1}{2} Qe\gamma \sum_i \frac{2(y_n - y_m)^2 - (x_n - x_m)^2 - \gamma^2(\lambda i - \lambda_{nm})^2}{[(x_n - x_m)^2 + (y_n - y_m)^2 + \gamma^2(\lambda i - \lambda_{nm})^2]^{5/2}} \quad (58)$$

$$\nu_4^{nm} = -\gamma^2(\nu_2^{nm} + \nu_3^{nm}). \quad (59)$$

Eqs. (57, 58) have been derived by adding a small perturbation u, v to x_n and y_n of Eq. (55), and retaining only the quadratic-order terms in the expansion. Eq. (59) follows from Laplace's equation:

$$\frac{\partial^2 \phi_{sc}}{\partial u^2} + \frac{\partial^2 \phi_{sc}}{\partial v^2} + (1 - \beta^2) \frac{\partial^2 \phi_{sc}}{\partial \sigma^2} = 0 \quad (60)$$

which applies to the perturbative term. Expressions (57 – 59) can be simplified using the following notation:

$$r_{nm}^2 = (x_n - x_m)^2 + (y_n - y_m)^2 \quad (61)$$

$$\alpha_{nm} = \frac{r_{nm}}{\gamma\lambda} \quad (62)$$

$$\tau_{nm} = \frac{\lambda_{nm}}{\lambda} \quad (63)$$

and defining

$$f_n(\alpha, \tau) = \sum_i [\alpha^2 + (i - \tau)^2]^{-\frac{n}{2}} \quad (64)$$

We then derive

$$\nu_2^{nm} = \frac{1}{2} \frac{Qe}{\lambda^3 \gamma^2} \left[3u_{nm}^2 f_5(\alpha_{nm}, \tau_{nm}) - f_3(\alpha_{nm}, \tau_{nm}) \right] \quad (65)$$

$$\nu_3^{nm} = \frac{1}{2} \frac{Qe}{\lambda^3 \gamma^2} \left[3v_{nm}^2 f_5(\alpha_{nm}, \tau_{nm}) - f_3(\alpha_{nm}, \tau_{nm}) \right], \quad (66)$$

where

$$u_{nm} = \frac{x_n - x_m}{\gamma\lambda}, \quad (67)$$

$$v_{nm} = \frac{y_n - y_m}{\gamma\lambda}; \quad (68)$$

$$\alpha_{nm}^2 = u_{nm}^2 + v_{nm}^2. \quad (69)$$

By adding Eq. (65) and Eq. (66) we also obtain:

$$\nu_4^{nm} = \frac{Qe}{\lambda^3} [f_3(\alpha_{nm}, \tau_{nm}) - \frac{3}{2} \alpha_{nm}^2 f_5(\alpha_{nm}, \tau_{nm})]. \quad (70)$$

If we let

$$\nu_0 = \frac{Qe g_0}{\gamma^2 \lambda^3} \quad (71)$$

with

$$g_0 = \frac{1}{2} f_3(0, 0) \approx 1.2 \dots, \quad (72)$$

we can write formally:

$$\nu_2 = \sum_m \nu_2^{nm} = -\nu_0 \eta_h, \quad (73)$$

$$\nu_3 = \sum_m \nu_3^{nm} = -\nu_0 \eta_v. \quad (74)$$

where

$$\eta_h = \frac{1}{2g_0} \sum_m [f_3(\alpha_{nm}, \tau_{nm}) - 3u_{nm}^2 f_5(\alpha_{nm}, \tau_{nm})] \quad (75)$$

$$\eta_v = \frac{1}{2g_0} \sum_m [f_3(\alpha_{nm}, \tau_{nm}) - 3v_{nm}^2 f_5(\alpha_{nm}, \tau_{nm})]. \quad (76)$$

Similarly

$$\nu_4 = \sum_m \nu_4^{nm} = 2\gamma^2 \nu_0 \eta_e \quad (77)$$

with

$$\eta_e = \frac{1}{2} (\eta_h + \eta_v). \quad (78)$$

9 The Potential for Confinement and the Shear Potential

Using the definitions introduced in the previous section, the potential Eq. (55), which defines the equilibrium configuration and the confinement conditions, can be formally expressed as follows:

$$\bar{\phi}_{nm} = \frac{Qe}{\lambda} f_1(\alpha_{nm}, \tau_{nm}). \quad (79)$$

The forces acting on the test particle easily follow from Eq. (79), viz.:

$$F_h^{nm} = \frac{\partial \bar{\phi}_{nm}}{\partial \bar{x}_n}, \quad (80)$$

$$= -\frac{\nu_0}{g_0} f_3(\alpha_{nm}, \tau_{nm})(\bar{x}_n - \bar{x}_m); \quad (81)$$

$$F_v^{nm} = \frac{\partial \bar{\phi}_{nm}}{\partial \bar{y}_n}, \quad (82)$$

$$= -\frac{\nu_0}{g_0} f_3(\alpha_{nm}, \tau_{nm})(\bar{y}_n - \bar{y}_m). \quad (83)$$

In locations where the trajectory is curved (e.g., in bending magnets) the motion of particles on different substrings have a shear with respect to each other. The reason is that in order for the equilibrium configuration to have the same periodicity as the storage ring (and therefore to close after each period), particles on different substrings are forced to shift longitudinally with respect to each other. This introduces a (local) longitudinal component of the space-charge forces. This component is derived from Eq. (55) by letting the shift λ_m vary from string to string, viz.:

$$\lambda_m = \bar{\lambda}_n + \sigma_n. \quad (84)$$

$\bar{\lambda}_n$ is the shift in absence of curvature, and σ_n is the displacement due to the shear common to all particles in the m th substring. After inserting Eq. (84) in Eq. (55) we expand for small σ_n .

The substrings making up the Crystalline Beam are to be arranged in such a way that no net longitudinal force is introduced. Therefore, the relative shifts among the substrings are such that the first-order term in the expansion of ϕ_{sc} vanishes, viz.:

$$\sum_m \sum_i \frac{i - \tau_{nm}}{[\alpha_{nm}^2 + (i - \tau_{nm})^2]^{\frac{3}{2}}} = 0. \quad (85)$$

Consequently, we avoid having to use external fields for longitudinal confinement. The shear then contributes only quadratic order terms

to the space-charge potential:

$$\phi_{shear}^{nm} = \nu_4^{nm} (\sigma_m - \sigma_n)^2 \quad (86)$$

where ν_4^{nm} is given by Eq. (70).

10 Application to Simple Crystalline Structures

The expressions derived in sections 8 and 9 can be applied easily to a few simple Crystalline Beam configurations, namely the string, the vertical and horizontal zigzag, and the helix with $n_s = 4$ substrings.

10.1 The String

In this case, $n_s = 1$, and $\alpha_{11} = \tau_{11} = 0$. Therefore:

$$\eta_h = \eta_v = \eta_e = 1. \quad (87)$$

Condition Eq. (85) is satisfied. By definition, $F_h = F_v = 0$, and we have no confinement conditions.

10.2 The Vertical Zigzag

The vertical zigzag consists of two substrings, in the vertical plane, symmetric with respect to the horizontal plane, and shifted with respect to each other by $\lambda/2$. Again, Eq. (85) is satisfied. Since $x_1 = x_2 = 0$, there is no net horizontal force and $F_h = 0$. On the other hand, $y_1 = -y_2 = b$, and we have a net vertical force F_v acting on the test particle:

$$F_v = -4 \frac{\nu_0}{g_0} f_3(\alpha, 1/2) b \quad (88)$$

where the (vertical) aspect ratio α is:

$$\alpha = \frac{2b}{\gamma\lambda}. \quad (89)$$

The form factors for determining the stability are:

$$\eta_v = 1 + \frac{f_3(\alpha, 1/2) - 3\alpha^2 f_5(\alpha, 1/2)}{2 g_0}; \quad (90)$$

$$\eta_h = 1 + \frac{f_3(\alpha, 1/2)}{2 g_0}. \quad (91)$$

10.3 The Horizontal Zigzag

The two substrings are side-by-side in the horizontal plane and a distance $2a$ apart. They are shifted relative to each other by $\lambda/2$, and Eq. (85) is satisfied. The (horizontal) aspect ratio is similar to the vertical one, with height $2b$ replaced by width $2a$ in Eq. (89). Instead of a net vertical force, we have a net horizontal force. F_h is obtained by replacing $2b$ by $2a$ in the right-hand side of Eq. (88). By making similar transformations to Eqs. (90) and (91) we obtain the form factors for the stability equations, noting that the role of vertical and horizontal is interchanged.

The horizontal zigzag is the simplest structure for which shear forces appear. For the two substrings we have $\sigma_1 = -\sigma_2 = \sigma$. That is, they drift in opposite direction with respect to each other. The shear force is then the same for both:

$$\frac{\partial \phi_{shear}}{\partial \sigma} = 8\nu_0 \gamma^2 \eta_e \sigma \quad (92)$$

10.4 The Helix

The simplest three-dimensional structure is a helix with $n_s = 4$ substrings, shifted by τ_{nm} with respect to each other. The substrings are located at four symmetric locations:

$$\bar{x} = \pm a, \quad (93)$$

$$\bar{y} = \pm b. \quad (94)$$

There is no net longitudinal force, but confinement is needed in both vertical and radial direction, where the net forces are not zero. We

have:

$$F_h = -2 \frac{\nu_0}{g_0} [f_3(\alpha_h, \tau_{12}) + f_3(\alpha, \tau_{13})] a, \quad (95)$$

$$F_v = -2 \frac{\nu_0}{g_0} [f_3(\alpha_v, \tau_{14}) + f_3(\alpha, \tau_{13})] b. \quad (96)$$

where

$$\alpha_h = \frac{2a}{\gamma\lambda}, \quad (97)$$

$$\alpha_v = \frac{2b}{\gamma\lambda}, \quad (98)$$

$$\alpha^2 = \alpha_v^2 + \alpha_h^2. \quad (99)$$

The magnitude of the force is the same for all particles; there is only one set of (coupled) envelope equations for a and b , describing the shape of the helix as it travels around the storage ring.

The same applies to the shear force. All particles suffer the same amount of shear, though it may be in opposite directions. The shear force is given by Eq. (92).

The form factors are now:

$$\begin{aligned} \eta_h = & 1 + \frac{1}{2g_0} f_3(\alpha_v, \tau_{14}) + \\ & + \frac{f_3(\alpha_h, \tau_{12}) - 3\alpha_h^2 f_5(\alpha_h, \tau_{12})}{2g_0} + \\ & + \frac{f_3(\alpha, \tau_{13}) - 3\alpha_h^2 f_5(\alpha, \tau_{13})}{2g_0} \end{aligned} \quad (100)$$

$$\begin{aligned} \eta_v = & 1 + \frac{1}{2g_0} f_3(\alpha_h, \tau_{12}) + \\ & + \frac{f_3(\alpha_v, \tau_{14}) - 3\alpha_v^2 f_5(\alpha_v, \tau_{14})}{2g_0} + \\ & + \frac{f_3(\alpha, \tau_{13}) - 3\alpha_v^2 f_5(\alpha, \tau_{13})}{2g_0} \end{aligned} \quad (101)$$

The form factors and the expressions we have just defined to represent the space-charge potential satisfy some interesting properties, for example:

$$f_n(\alpha, 1 - \tau) = f_n(\alpha, \tau), \quad (102)$$

and

$$\sum_m f_3(0, \tau_{mn}) = n_s^3. \quad (103)$$

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