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Dynamical Impurity Problems

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1. INTRODUCTION

In the past few years there has been a resurgence of interest in dynamical impurity problems, as a result of developments in the theory of correlated electron systems. The general dynamical impurity problem is a set of conduction electrons interacting with an impurity which has internal degrees of freedom. The simplest and earliest example, the Kondo problem, [1] (a spin-half magnetic impurity in a metal) has attracted interest since the mid-sixties not only because of its physical importance but also as an example of a model displaying logarithmic divergences order by order in perturbation theory. It provided one of the earliest applications of the renormalization group method, [2]-[5] which is designed to deal with just such a situation. As we shall see, the antiferromagnetic Kondo model is controlled by a strong-coupling fixed point, and the essence of the renormalization group solution is to carry out the global renormalization numerically starting from the original (weak-coupling) Hamiltonian. [5] In these lectures, we shall describe an alternative route in which we identify an exactly solvable model which renormalizes to the same fixed point as the original dynamical impurity problem. This approach is akin to determining the critical behavior at a second order phase transition point by solving any model in a given universality class.

The general multichannel Kondo model, which was introduced by Nozières and Blandin [6] in 1980, is even richer in behavior. In this generalization, the impurity is allowed to have an arbitrary spin S , and the conduction electrons are assumed to have n degenerate orbital channels or flavors in addition to their usual half-integral spin. The coupling between the impurity and the conduction electrons is a Heisenberg exchange $J\vec{\tau} \cdot \sum_i \vec{S}_i(0)$, where $\vec{\tau}$ is the impurity spin and $\vec{S}_i(0)$ is the conduction electron spin density for flavor i at the impurity site. As it turns out, this generalization provides us with a rare example of a model which is asymptotically free but has a finite-coupling fixed point.

These lectures will be concerned with the one- and two-channel Kondo problems, which are the most interesting from a physical point of view. At the present time, it seems that the most likely physical realizations of the two-channel model involve impurities with internal degrees of freedom other than spin. A number of years ago, it was pointed

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out[7] that the inelastic scattering of conduction electrons from a nonmagnetic two-level impurity is equivalent to a single-channel Kondo problem. In this mapping, the two levels are represented by an impurity pseudospin, but the conduction electrons do not have a corresponding degree of freedom, and indeed true spin plays no role. Subsequently it was emphasized by Zawadowski,[8] in the context of the scattering of conduction electrons from two-level systems in metallic glasses, that inelastic scattering may change the parity of the angular momentum state of the conduction electrons, thereby giving them an effective pseudospin degree of freedom coupled to that of the impurity. In that case the spin of the conduction electrons plays the role of the channel or flavor index, so the number of degrees of freedom is the same as in the two-channel Kondo problem. Other potential applications in a similar spirit are to heavy fermion systems [9, 10] and high temperature superconductors.[11, 12] A particular reason for the interest in multi-channel impurity models is that they provide a tractable way of arriving at non-Fermi liquid behavior of the conduction electrons in the neighborhood of the impurity, [6],[14]-[22] and this is what appears to be required to explain the normal state properties of high temperature superconductors.[24] This impression is strengthened by the fact that, for two-channels and spin-half there is a substantial enhancement of fluctuations of a novel kind of electron pairing,[25] which is odd under time reversal. Fluctuations of this kind may account for the unusual behavior of the normal state properties of high temperature superconductors, and especially their insensitivity to phonons and impurities.[25, 26] It is natural to attribute high temperature superconductivity itself to the same effects, although the ordered state is not necessarily odd under time reversal.[26]

Models of correlated electron systems may also be mapped into dynamical impurity problems in the limit of infinite dimension. Metzner and Vollhardt[27] pointed out that this limit provides a systematic way of obtaining a mean-field theory for correlated electron systems. Starting from the observation of Müller-Hartmann that the electron self-energy is diagonal in real space for the Hubbard model in infinite dimension, Georges and Kotliar[28] showed that the problem is equivalent to a single-impurity Anderson model, together with a self-consistency condition.

Although for many physical applications the internal degrees of freedom of the impurity are characterized by a pseudospin variable rather than a true spin, we shall use the Kondo language of "spin" and "flavor" in these lectures. It is straightforward to transcribe the results for other physical applications, although some care must be exercised in identifying the variables.

There has been a great deal of work on dynamical impurity problems, taking advantage of the fact that they are essentially one-dimensional and may be solved by the Bethe Ansatz,[13-16] or critical, which allows the use of conformal invariance.[18]-[22] Moreover there have been a number of studies using either analytical or numerical renormalization group methods.[3]-[6],[17] As a result of all of this work we now have a rather detailed understanding of the generalized Kondo problem and a wealth of exact results. In these lectures we shall describe another way of solving the $S = \frac{1}{2}$ one- and two-channel Kondo problems, which are the most interesting from a physical point of view. It will be shown that there are solvable free-fermion models which renormalize to the same fixed point as the original Kondo problems. This alternative approach is analytic and deductive, and it allows a very explicit evaluation of correlation functions, while giving a new perspective

on the peculiar properties of the two-channel problem. The method does have some limitations: single-particle properties cannot be evaluated exactly, and it does not work for $n > 2$. On the other hand it may be generalized to solve a one-dimensional array of impurities, [26] which is a crucial first step for applications to heavy-fermion materials and high temperature superconductors.

The layout of the lectures is as follows. The general strategy of dynamical impurity problems will be described in Section 2, and the essential technical background (field theory limit and bosonization) will be given in Sections 3 and 4. Section 5 will explain the construction of the exactly solvable models, and Sections 6 and 7 will evaluate impurity and conduction electron properties. In particular a variety of exactly solvable limits will be used to evaluate the Wilson ratio, a fundamental property of the conduction electrons, defined in Section 2.

2. STRATEGY OF DYNAMICAL IMPURITY PROBLEMS

The essence of a dynamical impurity problem may be appreciated by thinking of it as a problem in statistical mechanics in one dimension, and considering the renormalization group strategy. If the conduction electron variables are integrated out, the remaining Hamiltonian for the impurity spin is a zero-dimensional quantum model or, equivalently, a one-dimensional classical model, in which the imaginary time is regarded as the space dimension.

In following this route for the single-channel problem, Anderson, Yuval, and Hamann[3] generalized the model to an anisotropic exchange coupling with J equal to J_z in the direction parallel to an applied magnetic field and equal to $J_\perp$ in the transverse direction. Then they expanded the partition function in powers of $J_\perp$ to obtain a product of two partition functions; one for the conduction electrons and one for the impurity dynamics, expressed as a one-dimensional gas of "charged" particles with fugacity $J_\perp$ and a logarithmic interaction. As we shall see, the derivation, which made use the long-time approximation of Nozières and de Dominicis[29], is equivalent to introducing a continuum field theory for the conduction electrons (Section 2) and, as such, it is correct in for small $J_\perp$. The details of the perturbation expansion may be found in reference[3]. They are not given here because an alternative but fully equivalent formulation is described in detail in subsequent sections. For now, we only consider the renormalization group equations for the effective one-dimensional model, which are given by:[3]

$$\frac{d\epsilon}{dl} = (2 - \epsilon)(J_\perp \rho)^2 \quad (1)$$

$$\frac{dJ_\perp}{dl} = \frac{1}{2}\epsilon J_\perp \quad (2)$$

Here ρ is the density of states of the conduction electrons and $l \equiv -\ln(e/W)$, where e and W are respectively the renormalized and bare bandwidths, with the boundary condition that the coupling constants are equal to their bare values when $e = W$. Also

$$\epsilon = 8 \frac{\delta}{\pi} - 8 \left(\frac{\delta}{\pi}\right)^2 \quad (3)$$

where the phase shift δ is given by:

$$\tan \delta = \frac{\pi J_z \rho}{4} \quad (4)$$

These equations are expansions in $J_\perp$ but are correct to all orders in J_z . They are very familiar in statistical mechanics as the renormalization group equations of the two-dimensional Coulomb gas, which were in fact derived by Kosterlitz [30] following the procedure devised earlier for the Kondo problem.[3]

Equations (1) and (2) indicate that there is a line of fixed points $J_\perp = 0$, which is stable for ferromagnetic coupling $J_z < 0$. Consequently the renormalization group equations are sufficient to solve the ferromagnetic case, provided the bare coupling $J_\perp$ is weak. However, in the interesting case of antiferromagnetic coupling $J_z > 0$, the line of fixed points is unstable and the system renormalizes to an infinitely strong coupling fixed point. This region corresponds to the high temperature phase of the logarithmic gas for which the fixed point is at infinite temperature and fugacity. Evidently, as fugacity expansions, Eqs. (1) and (2) are inadequate in this phase, and some other method must be used to solve the problem.

The solution proposed by Anderson, Yuval, and Hamann[3] relied on the observation that for $\epsilon = 1$, the expansion in $J_\perp$ coincides with an equivalent perturbation series for a solvable model of free spinless fermions. This equivalence had been pointed out earlier by Toulouse[31], and the point $\epsilon = 1$ for the single-channel problem is known as the Toulouse limit. An alternative derivation will be given in Section 4 and it will be made clear that the solvable limit is not merely a mathematical identity: the free fermions are in fact spin-density waves coupled to "drone" fermions representing the impurity spin. For the present, we merely point out that the free-fermion problem has the generally-expected characteristics of the single-channel problem in the antiferromagnetic region. In particular, the impurity spin is fully compensated by forming a singlet with a resonant state of the conduction electron spin density in the neighborhood of the impurity. Anderson, Yuval, and Hamann[3] argued that the renormalization group equations (1) and (2) scale the original weak-coupling Hamiltonian *onto* the Toulouse limit, and thereby solve the problem. They regarded the solution as *qualitative* because the Toulouse limit is attained for $J_\perp \rho \approx 1$, and this is beyond the range of validity of the approximation.

This approach was regarded as incomplete for two reasons. First of all it appeared to give the wrong value for the Wilson ratio R_W :

$$R_W = \frac{\chi_{imp}/C_{imp}}{\chi_{bulk}/C_{bulk}}, \quad (5)$$

where χ_{imp} , C_{imp} , χ_{bulk} , C_{bulk} are respectively the impurity and bulk contributions to the magnetic susceptibility χ and specific heat C . This quantity is regarded as a fundamental property of dynamical impurity problems because, according to the renormalization group approach, physical quantities should depend on a single energy scale, in this case the Kondo scale Γ . It was expected that $C_{imp} \sim T$ and then, dimensionally $C_{imp} \sim T/\Gamma$ and $\chi \sim \Gamma^{-1}$. Consequently Γ should cancel on the right of Eq. (5) and R_W should be a parameter-independent number. The Toulouse limit gave $R_W = 4$, whereas Wilson found via a numerical renormalization group calculation that $R_W = 2$ and is universal

provided Γ is much smaller than the bandwidth W . This difficulty was resolved by Wiegmann and Finkel'shtein,[32] who pointed out that the value $R_W = 4$ was obtained in the Toulouse limit if the external magnetic field acted only on the impurity. (The field on the conduction electrons was often ignored in practice because it had proved to be unimportant in calculations at weak coupling.) Wiegmann and Finkel'shtein,[32] showed that, even in the Toulouse limit, $R_W = 2$ if the magnetic field acted on the conduction electrons as well as the impurity and both had the same Landé g -factors. Otherwise R_W is not universal although a difference in g -factors is numerically unimportant at weak coupling. These results will be derived via abelian bosonization in Section 6.

A second question concerned the scaling from the weak-coupling problem onto the Toulouse limit. Equations (1) and (2) describe the scaling of the particular exchange couplings which appear in the Hamiltonian. However renormalization also generates other interactions involving higher powers of conduction electron operators. There is no reason to believe that these couplings are unimportant when ϵ reaches the Toulouse value, and consequently the weak-coupling Kondo problem does not truly scale to a free fermion model. In our view this objection is completely overcome by a reinterpretation of the implications of the renormalization group equations. The point is that the Hamiltonian renormalizes to the same strong-coupling fixed point whether the starting point is weak coupling or $\epsilon = 1$. Details of the initial Hamiltonian should only be important for non-universal quantities such as the Kondo scale. As in the theory of critical phenomena, any solvable limit may be used to determine the universal behavior. In order to complete the proof, it is necessary to show that exchange anisotropy is an irrelevant variable, since the Kondo Hamiltonian is anisotropic when $\epsilon = 1$ and $J_{\perp}\rho$ is small. Modulo this last point, the solution of the equivalent free fermion model in fact provides a *quantitative* solution of the single-channel Kondo problem.

The point is underscored by a similar solution for the two-channel model, [25] which is a richer proving ground for proposed solutions. As we shall see, all known universal properties of the two-channel Kondo problem may be obtained by identifying and solving an equivalent free fermion Hamiltonian. The major difference is that the fixed point of the two-channel problem corresponds to finite coupling or, in the one-dimensional analog, to a finite temperature critical point. In other words, the renormalization takes place within a critical surface. The latter is unstable to an interaction which breaks channel degeneracy and diverts the renormalization group flows to the infinite coupling fixed point of the single-channel model.

The influence of the channel variable may be understood by an argument given originally by Nozières and Blandin.[6] The idea is to solve the problem at strong coupling by switching off the kinetic energy. In the single-channel case the exchange coupling produces a fully-compensated state in which the impurity forms a singlet ($[\downarrow] \uparrow - [\uparrow] \downarrow$) with a conduction electron. (Here the direction of the impurity spin is indicated by $[\downarrow]$ or $[\uparrow]$ and the other arrows refer to conduction electrons.) For two-channels *two* conduction electrons, one of each flavor, are localized at the impurity site. The Hamiltonian is $J\tilde{\tau}(\tilde{S}_1(0) + \tilde{S}_2(0))$, and the ground state is $(2[\uparrow] \downarrow\downarrow - [\downarrow] \uparrow\uparrow - [\downarrow] \uparrow\downarrow)$. Now the impurity spin is overcompensated because the three-spin complex has a net spin pointing in the opposite direction to the average spin of the impurity. The interaction between the conduction electrons and the three-spin complex may be evaluated by perturbation theory

in the kinetic energy, and it is just an antiferromagnetic superexchange.[6] Thus the strong coupling fixed point is unstable and the renormalization group trajectories flow to a finite-coupling fixed point. The interesting properties of the spin-half two-channel Kondo problem stem from the facts that the two-fold degeneracy of the impurity is not fully removed and the fixed point is critical.

3. ONE-DIMENSIONAL FIELD THEORY

There are two fundamental reasons why it is possible to solve dynamical impurity problems. In the first place, they are essentially one dimensional: usually the interaction with the conduction electrons involves only a few angular momentum states, and the problem may be reduced to a study of radial motion. This allows us to bring to bear the various techniques that have been used to solve the many-body problem in one dimension. Secondly, it is possible to effect a considerable simplification of the problem and to focus on the long-distance behavior of a correlation functions by taking the continuum or field theory limit (lattice spacing $\rightarrow 0$). The long-distance behavior is the most interesting because it determines the low energy physics. Furthermore a field theory may be constructed at an unstable fixed point of a renormalization group transformation. The fixed-point Hamiltonian is independent of many of the details of a given system, and often is simpler and free of irrelevant transient contributions. Thus a fixed-point theory has the desirable feature that the predictions are very robust and are more reliable than those of a more specific theory. Before showing how this reduction may be accomplished it is instructive to construct the field theory for free Fermions in one dimension.

3.1. Continuum Limit for Fermions in One Dimension

The hopping Hamiltonian for free spinless fermions, given by

$$\mathcal{H}_0 = -t \sum_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) \quad (6)$$

is solvable by Fourier transformation

$$c_n = \frac{1}{\sqrt{N}} \sum_k e^{ikna} c(k) \quad (7)$$

to obtain:

$$\mathcal{H}_0 = -2t \sum_k \cos(ka) c^\dagger(k) c(k) \quad (8)$$

Now, using Eq. (7), it is straightforward to evaluate the correlation function

$$C(n) = \frac{1}{N} \sum_m \langle c_{n+m}^\dagger c_m \rangle \quad (9)$$

at zero temperature and to introduce $x = na$ to find $C(n) \rightarrow C'(x)$ where:

$$C'(x) = \frac{a \sin(k_F x)}{\pi x} \quad (10)$$

where k_F is the Fermi momentum (equal to $\pi/2a$ for a half-filled band) The lattice spacing a is introduced solely in order to construct a field theory, and evidently it must disappear from all physical quantities for a lattice model. Equation (10) gives a hint that the one-dimensional electron gas may be amenable to a field-theory treatment, at least at zero temperature, because the explicit dependence on the lattice spacing a may be removed from the correlation function. The prefactor a , which must be present on dimensional grounds, may be divided out and, as we shall see, the oscillating factor may be transformed away.

In general, a cannot be factored out so easily because correlation functions are not power laws but depend on a coherence length ξ . In that case, $C(n/\xi)$ is rewritten as $C(x/\xi a)$ and, since ξa remains finite, ξ must $\rightarrow \infty$, which implies that the system must approach a critical point. At the same time, $x = na$ remains finite, so n must $\rightarrow \infty$ and only the asymptotic behavior of the lattice correlation function $C(n)$ is important. Thus, although it appears to be quite artificial to introduce a length scale for a lattice model, taking the continuum limit focuses attention on the interesting asymptotic behavior and considerably simplifies the theory by removing unwanted short distance transients.

It is simpler to take the continuum limit in the Hamiltonian itself, before evaluating the correlation functions. First consider the free-fermion Hamiltonian $\mathcal{H}_0$ (Eq.(8)) and linearize the energy spectrum $e(k) = -2t\cos(ka)$ in the neighborhood of the Fermi points $k = \pm k_F$:

$$e(k) = e(\pm k_F) + v_F \cdot (k \mp k_F) \quad (11)$$

Now the oscillating factor $\sin(k_F x)$ on the right of Eq. (10) may be removed by introducing right and left-going fields $\psi_+(x)$ and $\psi_-(x)$ and taking the Fourier transform of $c(k)$ with k in the ranges $k_F - k_c < k < k_F + k_c$ and $-k_F - k_c < k < -k_F + k_c$ respectively:

$$\psi_+(x) = \frac{1}{\sqrt{2\pi}} \int_{-k_c}^{k_c} dk e^{ikx} c(k + k_F) \quad (12)$$

$$\psi_-(x) = \frac{1}{\sqrt{2\pi}} \int_{-k_c}^{k_c} dk e^{ikx} c(k - k_F) \quad (13)$$

This procedure is equivalent to making a Galilean transformation to translate $k = \pm k_F$ to the origin $k = 0$, which is possible because right and left going fermions are treated separately. The cutoff lattice spacing has been replaced by the cutoff $-k_c < k < k_c$, which is symmetric about $k = 0$ because we have tacitly assumed a half-filled band $k_c = \pi/2a$. This simplifies the notation without affecting the final results. In the continuum limit, $k_c \rightarrow \infty$, the Fermi sea of occupied states is $-\infty < k < 0$ for right-going fermions and $0 < k < \infty$ for left-going fermions. Now it is straightforward to verify that the $\psi_{\pm}(x)$ satisfy the equations of motion:

$$\frac{\partial \psi_{\pm}}{\partial t} = \pm v_F \frac{\partial \psi_{\pm}}{\partial x} \quad (14)$$

where $v_F = 2at$. Thus we must take $t \rightarrow \infty$ as $a \rightarrow 0$ in order to keep v_F finite. In the limit $k_c \rightarrow \infty$, the fields ψ_{α} (with α representing $\pm$) satisfy:

$$[\psi_{\alpha}(x), \psi_{\beta}^{\dagger}(x')]_{+} = \delta_{\alpha,\beta} \delta(x - x') \quad (15)$$

Following this procedure, $\mathcal{H}_0$ becomes the Dirac Hamiltonian:

$$\mathcal{H}_0 \rightarrow iv_F \int dx \left[\psi_+^\dagger \frac{\partial \psi_+}{\partial x} - \psi_-^\dagger \frac{\partial \psi_-}{\partial x} \right] \quad (16)$$

In to order consider the other parts of the Hamiltonian, it is necessary to obtain the continuum form of the density operator. First note that, from Eqs. (12) and (13), the total field operator is given by:

$$\psi(x) = \psi_+(x)e^{ik_F x} + \psi_-(x)e^{-ik_F x} \quad (17)$$

Then the density operator $\rho(x) = \psi^\dagger(x)\psi(x)$ becomes:

$$\rho(x) = \rho_+(x) + \rho_-(x) + \psi_+^\dagger(x)\psi_-(x)e^{2ik_F x} + \psi_-^\dagger(x)\psi_+(x)e^{-2ik_F x} \quad (18)$$

where

$$\rho_\pm(x) = \psi_\pm^\dagger(x)\psi_\pm(x) \quad (19)$$

By construction, the fields $\psi_\pm(x)$ pick out long-distance or short-wavelength effects; consequently $\rho(x)$ focusses on wave vectors zero and $2k_F$ which are relevant for processes near to the Fermi points. Moreover the various contributions to $\rho(x)$ should appear in the Hamiltonian only in combinations for which the explicit oscillating factors multiplying the field operators cancel.

3.2. Dynamical Impurity Problems

Now we may show how reduce dynamical impurity problems may be reduced to effective one-dimensional models. The idea has been known for a long time, but we shall follow the prescription of Affleck and Ludwig[21] which is quite convenient for our purposes. For the moment, we suppress spin and flavor labels and show how the argument works for s-states, which are all that occur in the Kondo problem. In three dimensions, the fermion fields $c(\vec{k})$ are defined in terms of the vector momentum $\vec{k}$, and $c(\vec{k})$ may be projected onto s-states via:

$$c_0(k) = \frac{k}{\sqrt{4\pi}} \int d\hat{k} c(\vec{k}) \quad (20)$$

where $\hat{k}$ is the unit vector along $\vec{k}$. Then, if we retain only the s-state component of $c(\vec{k})$ in defining the field operator in real space via:

$$\psi(r) = \frac{1}{(2\pi)^{3/2}} \int d\vec{k} c(\vec{k}) e^{i\vec{k}r} \quad (21)$$

and limit the integral with respect to k to a range $(k_F - k_c, k_F + k_c)$ in the neighborhood of the Fermi surface, as above, we obtain:

$$\psi(r) = \frac{1}{2\sqrt{\pi i r}} \left[\psi_+(r)e^{ik_F r} - \psi_-(r)e^{-ik_F r} \right] \quad (22)$$

where $\psi_{\pm}(r)$ are defined in Eqs. (12) and (13), with $c(k)$ replaced by $c_0(k)$. Physically, the angular variables have been integrated out, leaving a one-dimensional radial problem in which ingoing and outgoing waves correspond to the left-going and right-going waves defined above. The major difference is that the radial problem is confined to the half-space $r > 0$. However, an equivalent representation, which is more convenient in practice, is to define $\psi_-(-r) \equiv \psi_+(r)$ and then to express the problem in terms of left-going fields ψ_- over all space: $-\infty < r < +\infty$. According to Eq. (22), there is no discontinuity at the origin since $\psi(0)$ is finite and hence $\psi_+(0) = \psi_-(0)$.

The one-dimensional form of the Hamiltonian may now be obtained by substituting this representation into the kinetic energy and the Heisenberg exchange $J\vec{\tau} \cdot \sum_i \vec{S}_i(0)$, where $\vec{\tau}$ is the impurity spin and $\vec{S}_i(0)$ is the conduction electron spin density for flavor i at the impurity site:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 \quad (23)$$

where, retaining excitations near the Fermi points,

$$\mathcal{H}_0 = i v_F \sum_{i, \alpha=1}^2 \int_{-\infty}^{\infty} dx \psi_{i\alpha}^{\dagger}(x) \frac{\partial \psi_{i\alpha}(x)}{\partial x} \quad (24)$$

is the kinetic energy of the conduction electrons and $\psi_{i\alpha}(x)$ annihilates a left-going fermion with pseudospin α and flavor i at position x . (Since there are only left-going fermions the left/right labels $\pm$ will be omitted.) The impurity part of $\mathcal{H}$ is given by:

$$\mathcal{H}_{1z} = \frac{J_z^R}{2} \tau^z \sum_{i, \alpha=1}^2 \sigma_{\alpha\alpha}^z \psi_{i\alpha}^{\dagger}(0) \psi_{i\alpha}(0) \quad (25)$$

$$\mathcal{H}_{1xy} = \frac{1}{2} \sum_{i, \alpha, \beta=1}^2 \sum_{\lambda=y,z} J_{\lambda} \tau^{\lambda} \sigma_{\alpha\beta}^{\lambda} \psi_{i\alpha}^{\dagger}(0) \psi_{i\beta}(0) \quad (26)$$

Here τ^{λ} are the three components of the impurity pseudospin operator, σ^{λ} are Pauli matrices, and J_z^R is the renormalized value of J_z . In the naive continuum limit, in which operators in $\mathcal{H}$ are simply replaced by their continuum forms, $J_z^R = J_z$. More accurately, J_z^R is a pseudopotential for scattering from the impurity for $J_z = J_y = 0$, where the impurity is non-dynamical:[3]

$$J_z^R = 8v_F \delta \quad (27)$$

where the phase shift δ is given by Eq.(4) The naive continuum limit is obtained by expanding the right hand side of Eq. (4) to first order in J_z and taking ρ to be the density of states in one dimension. Later it will be seen that $J_x = J_y = J_{\perp}$ also is renormalized and that it is necessary to take $J_{\perp} \rightarrow 0$ as $a \rightarrow 0$ in order to obtain a finite result in the continuum limit. The interaction with an external magnetic field is:

$$\mathcal{H}_2 = H \left[\frac{g_e}{2} \sum_{i, \alpha, \beta=1}^2 \int_{-\infty}^{\infty} dx \sigma_{\alpha\beta}^z \psi_{i\alpha}^{\dagger}(x) \psi_{i\beta}(x) + g_i \tau^z \right] \quad (28)$$

where g_e and g_i are the Landé g -factors for the conduction electrons and the impurity respectively H is the magnetic field. It is assumed that the impurity is at the origin, $x = 0$.

4. BOSONIZATION

The use of a boson representation of fermion operators was introduced in the context of condensed matter physics by Mattis and Lieb[33] in solving the Luttinger model, a simplified model of interacting fermions in one dimension. A few years later, Schotte and Schotte [34] introduced the representation of a fermion field in order to evaluate transition rates in the x-ray threshold problem, and the technique was generalized for application to the single-channel Kondo problem by Blume, Emery and Luther.[36] For applications in condensed matter physics, it was customary at first to express the boson representation of fermion operators in momentum space. However, we shall use the rather more transparent real space representation, in the form introduced by Mandelstam:[37]

$$\psi_{\pm} = \frac{e^{-i\Phi_{\pm}}}{\sqrt{2a}} \quad (29)$$

where

$$\Phi_{\pm}(x) = \sqrt{\pi} \left[\int_{-\infty}^x dx' \hat{\Pi}(x') \mp \phi(x) \right] \quad (30)$$

and $\phi(x)$ and $\hat{\Pi}(x)$ are respectively a Bose field and its conjugate momentum, satisfying the commutation relation:

$$[\phi(x), \hat{\Pi}(x')] = i\delta(x - x') \quad (31)$$

The representation (29)-(30) is useful because it is rather easy to work with exponentials of linear forms of boson operators, and it must be *non-local*, otherwise it would be impossible for a function of commuting variables to satisfy anticommutation relations. The integral $\int_{-\infty}^x dx' \hat{\Pi}(x')$ in $\Phi_{\pm}(x, t)$ makes $\psi_{\pm}$ a displacement operator for the $\phi(x)$ variables. In this sense, $\psi_{\pm}$ corresponds physically to a "backflow" operator.

In these lectures the representation (29) will mainly be used to transform back and forth between fermions and bosons, carrying out canonical transformations on the way. Consequently there will not be too much need for the detailed machinery of bosonization, and we shall simply list the essential formulae and indicate how they may be derived. A more detailed discussion is given in reference [35]. It is straightforward to show from Eq. (31) that $\Phi_{\pm}(x)$ satisfy the commutation relations:

$$[\Phi_{\pm}(x), \Phi_{\pm}(y)] = \pm i\pi \operatorname{sgn}(x - y) \quad (32)$$

and

$$[\Phi_{\pm}(x), \Phi_{\mp}(y)] = \pm i\pi \quad (33)$$

By using these relations together with the identity:

$$e^A e^B = e^B e^A e^{[A, B]} \quad (34)$$

which is true if $[A, B]$ is a c-number, it may be verified that

$$\psi_+(x)\psi_{\pm}(x') = -\psi_{\pm}(x')\psi_+(x) \quad (35)$$

as required. This much would have been true if the boson operators had been defined on a lattice, but the anticommutation relations (15) of $\psi_{\pm}$ with $\psi_{\pm}^{\dagger}$ require that we take the continuum limit. The derivation involves the use of Eq. (34) to arrange $\psi_{\pm}$ $\psi_{\pm}^{\dagger}$ and its Hermitian conjugate into normal order (boson annihilation operators stand to the right of boson creation operators), and is described in reference [35].

We also need the boson representations of the Hamiltonian and the density operators $\rho_{\pm}(x)$:

$$\rho_{\pm}(x) = \mp \frac{1}{2\pi} \left[\frac{\partial \Phi_{\pm}}{\partial x} \right] + \frac{1}{2a} \quad (36)$$

and

$$\mathcal{H}_0 = \frac{v_F}{2} \int dx \left[\hat{\Pi}^2 + \left(\frac{\partial \phi}{\partial x} \right)^2 \right] \quad (37)$$

In Eq. (36), the factors $1/2a$ give an infinite uniform "background" contribution to the density in the limit $a \rightarrow 0$, reflecting the fact that the uniform average density on the lattice becomes infinite in the continuum limit. It is straightforward to verify that $\rho_{\pm}(x)$ and $\mathcal{H}_0$ satisfy the same commutation relations with $\psi_{\pm}(y)$ in the fermion or boson forms. An alternative, but more complicated, way of obtaining Eqs. (36) and (37) is to substitute Eqs (29) and (30) into Eqs (16) and (19) and to arrange the resulting expressions into normal order. (i.e. Introduce boson creation and annihilation operators in k -space, as in Eqs. (39) and (40) below, and move all annihilation operators to the right.) It should be noted that an alternative form of the boson representation may be obtained by making the canonical transformation $(\phi(x), \hat{\Pi}(x)) \rightarrow (-\phi(x), -\hat{\Pi}(x))$, which implies $\Phi(x) \rightarrow -\Phi(x)$. In fermion language, this is equivalent to a particle-hole transformation: $\psi_{\pm}(x) \rightarrow \psi_{\pm}^{\dagger}(x)$.

We shall also need the density-density correlation function $C(t)$ defined as:

$$C(t) = \langle \rho_{-}(0, t) \rho_{-}(0, 0) \rangle \quad (38)$$

where $\langle \hat{O} \rangle$ is the thermal average of an operator $\hat{O}$ in the free boson Hamiltonian $\mathcal{H}_0$. In order to evaluate $C(t)$ we need the Fourier transform of $\rho_{-}(0, t)$: [35]

$$\rho_{-}(0, t) = i \sum_{k < 0} \sqrt{\frac{|k|}{2\pi L}} b_k^{\dagger} e^{i\omega_k t} + H.c., \quad (39)$$

where L is the length of the system, $\omega_k = v_F |k|$, and $b_k^{\dagger}$, b_k are creation and annihilation operators, defined in the usual way in from the Fourier transform of the field operator; they satisfy the commutation relations:

$$[b_k, b_{k'}^{\dagger}] = \delta_{k, k'} \quad (40)$$

Substituting Eq. (39) into Eq. (38), it follows that:

$$C(t) = - \sum_{k < 0} \frac{|k|}{2\pi L} \left[e^{i\omega_k t} \langle b_k^{\dagger} b_k \rangle + e^{-i\omega_k t} \langle b_k b_k^{\dagger} \rangle \right] \quad (41)$$

The thermal averages are given by $\langle b_k^{\dagger} b_k \rangle = n_k$ and $\langle b_k b_k^{\dagger} \rangle = n_k + 1$, where n_k is the Bose distribution function:

$$n_k = \frac{1}{e^{\omega_k/k_B T} + 1} \quad (42)$$

Then, in the limit $L \rightarrow \infty$, the sum in Eq. (41) may be replaced by an integral which can be evaluated in closed form to give:

$$C(t) = \left(\frac{T/2v_F}{\sinh(\pi T t)} \right)^2 \quad (43)$$

This expression is valid for $t > a/v_F$, since the short-time cutoff was lost in taking the continuum limit of the lattice theory.

5. EXACTLY SOLVABLE LIMITS

As we have seen in Section 2, the low-energy behavior of a Kondo problem is dominated by a single angular momentum state of the conduction electrons, and this is equivalent to a one-dimensional problem involving left-going fermions only. The critical behavior may be obtained by taking the continuum limit, in which the kinetic energy of the conduction electrons is given by a Dirac equation involving right- and left-going particles. The Hamiltonian is given in Eqs. (24), (25), (26) and (28). It will now be shown how the problem may be solved by use of the boson representation introduced in Section 3.

The boson representation must be introduced for each of the fields $\psi_{i\alpha}(x)$, using Eqs. (29), (30), (36) and (37). [40] Then the three parts of the Hamiltonian may be written in terms of Bose fields as:

$$\mathcal{H}_0 = \frac{1}{2} v_F \sum_{i, \alpha=1}^2 \int_{-\infty}^{\infty} dx \left[\hat{\Pi}_{i\alpha}^2 + \left(\frac{\partial \phi_{i\alpha}}{\partial x} \right)^2 \right] \quad (44)$$

and

$$\mathcal{H}_1 = \mathcal{H}_{1z} + \mathcal{H}_{1xy} \quad (45)$$

where

$$\mathcal{H}_{1z} = \frac{J_z^R}{4\pi} \tau^z \sum_{i, \alpha=1}^2 \sigma_{\alpha\alpha}^z \left(\frac{\partial \Phi_{i\alpha}}{\partial x} \right)_{x=0} \quad (46)$$

and

$$\mathcal{H}_{1xy} = \frac{1}{4\pi a} \sum_{i, \alpha, \beta=1}^2 \sum_{\lambda=y,z} J_\lambda \tau^\lambda \sigma_{\alpha\beta}^\lambda \exp \left\{ i\Phi_{i\alpha}(0) - i\Phi_{i\beta}(0) \right\}. \quad (47)$$

Also

$$\mathcal{H}_2 = H \left[\frac{g_e}{4\pi} \sum_{i, \alpha=1}^2 \sigma_{\alpha\alpha}^z \int_{-\infty}^{\infty} dx \frac{\partial \Phi_{i\alpha}}{\partial x} + g_i \tau_z \right] \quad (48)$$

Here, for convenience, we have symmetrized the boson form of the kinetic energy by including both right- and left-going fermions, even though only left-going fermions occur in the interaction terms. This makes no difference to the results.

5.1. Single-Channel Problem

The single-channel problem is obtained by omitting the flavor label i . Then the form of $\mathcal{H}_1$ may be simplified by introducing Bose fields corresponding to collective modes for charge $\phi_c(x)$, spin $\phi_s(x)$:

$$\phi_c = \frac{1}{\sqrt{2}} \sum_{\alpha=1}^2 \phi_{\alpha} \quad (49)$$

$$\phi_s = \frac{1}{\sqrt{2}} \sum_{\alpha=1}^2 \sigma_{\alpha\alpha}^z \phi_{\alpha} \quad (50)$$

This is a canonical transformation, and $\hat{\Pi}_c$, $\hat{\Pi}_s$, Φ_c , Φ_s are defined by the same set of linear equations. In terms of the new variables, $\mathcal{H}_0$ remains a diagonal quadratic form, corresponding to free bosons and

$$\mathcal{H}_{1z} = \frac{J_z^R}{2\pi\sqrt{2}} \tau^z \left(\frac{\partial \Phi_s}{\partial x} \right)_{x=0} \quad (51)$$

$$\mathcal{H}_{1xy} = \frac{1}{2\pi a} \left\{ J_x \tau^x \cos(\sqrt{2}\Phi_s(0)) + J_y \tau^y \sin(\sqrt{2}\Phi_s(0)) \right\} \quad (52)$$

$$\mathcal{H}_2 = H \left[\frac{g_e}{2\pi\sqrt{2}} \int_{-\infty}^{\infty} dx \frac{\partial \Phi_s}{\partial x} + g_i \tau^z \right] \quad (53)$$

After this change of variables, the impurity is coupled to the spin density field $\phi_s(x)$ in $\mathcal{H}_1$ and $\mathcal{H}_2$, but it is independent of the charge density field $\phi_c(x)$. At this point, the discussion will be restricted to the antiferromagnetic xxz model ($J_x = J_y = J_{\perp}$, $J_z > 0$). It may be shown that the same critical behavior is obtained for general anisotropic coupling, whatever the sign of J_z . [25] Then it is possible to transform to an exactly solvable problem by carrying out a rotation about the z -axis in τ -spin space via a unitary transformation $U_1 \mathcal{H} U_1^{-1}$ with $U_1 = \exp[i\lambda \tau^z \Phi_s(0)]$. This transforms $\mathcal{H}_0$, $\mathcal{H}_{1z}$, and $\mathcal{H}_2$ to: $\mathcal{H}'_0$, $\mathcal{H}'_{1z}$, and $\mathcal{H}'_2$ where:

$$\mathcal{H}'_0 = \mathcal{H}_0 - \lambda v_F \tau^z \left(\frac{\partial \Phi_s}{\partial x} \right)_{(x=0)}, \quad (54)$$

$$\mathcal{H}'_{1z} = \mathcal{H}_{1z} + \text{constant}, \quad (55)$$

and

$$\mathcal{H}'_2 = H \left[\frac{g_e}{2\pi\sqrt{2}} \int_{-\infty}^{\infty} dx \frac{\partial \Phi_s}{\partial x} + g'_i \tau^z \right], \quad (56)$$

where $g'_i = g_i - \lambda g_e / \sqrt{2}$. This last result will be of great importance for evaluating the Wilson ratio R_W : the impurity sees an effective magnetic field $H g'_i / g_i$ which represents the polarisation of the conduction electrons by the impurity.

From Eqs. (51), (54) and (55) it can be seen that the terms proportional to J_z^R in $\mathcal{H}'_0$ and $\mathcal{H}'_{1z}$ cancel if we choose $J_z^R = 2\sqrt{2}\pi\lambda v_F$. We shall mainly be concerned with this special value J_z^R for which the coupling between the impurity and the conduction electrons is contained entirely in $\mathcal{H}_{1xy}$. Then there are ways of obtaining a solvable problem.

5.1.1. Decoupling Limit

Take $\lambda = \sqrt{2}$. Then $\Phi_s(0)$ is eliminated from $\mathcal{H}'_{1xy} \equiv U\mathcal{H}_{1xy}U^\dagger$:

$$\mathcal{H}'_{1xy} = \frac{J_\perp}{2\pi a} \tau^x \quad (57)$$

and the impurity is decoupled from the conduction electrons. This result may be obtained more directly by recognizing that, according to Eq. (27), the special value of the coupling constant $J_z^R = 4\pi v_F$ corresponds to $J_z = \infty$. In this limit, one conduction electron is localized at the impurity site with its spin opposite to that of the impurity, and the other conduction electrons are decoupled from the impurity. The only remaining problem is to resolve the degeneracy of the two directions of the impurity spin, which is precisely the role $\mathcal{H}'_{1xy}$ in Eq. (57). In other words, we have an anisotropic form of the strong-coupling fixed point which may be solved without the reduction to one dimension and without taking the continuum limit. However the latter are useful for carrying out perturbation theory about the decoupling point, as will be seen in the next section.

5.1.2. Toulouse Limit

The other solvable point is obtained by taking $\lambda = \sqrt{2} - 1$, which removes the factor of $\sqrt{2}$ multiplying $\Phi_s(0)$ in $\mathcal{H}_{1xy}$. Then the full transformed Hamiltonian becomes:

$$\mathcal{H}' = \mathcal{H}_0 + \frac{J_\perp}{2\pi a} \left[\tau^+ e^{-\Phi_s(0)} + \tau^- e^{\Phi_s(0)} \right] + \mathcal{H}'_2 \quad (58)$$

where $\mathcal{H}'_2$ is given in Eq. (56). For this value of λ , which is known as the Toulouse limit,^[31] it is possible to invert the fermion $\rightarrow$ boson transformation (29) and to rewrite the Hamiltonian in terms of spin wave fermions corresponding to Φ_s . Then Eq. (58) is equivalent to a sum of free fermion and free boson Hamiltonians, which is exactly solvable.

The kinetic energy part $\mathcal{H}_0$ of (58) has contributions from ϕ_c which separate because they do not appear in $\mathcal{H}_1$. Therefore we shall focus on the part involving ϕ_s , which may be written in terms of fermion fields by again using a representation similar to (29), but in reverse:

$$\psi_s(x) = \frac{e^{-i\Phi_s(x)}}{\sqrt{2\pi a}} \quad (59)$$

In order to have the $J_\perp$ term of (58) an even function of fermion variables, τ -operators must be rewritten in terms of a drone fermion d to give the various contributions to the Hamiltonian for the spin degrees of freedom:

$$\mathcal{H}'_2 = iv_F \int_{-\infty}^{\infty} dx \psi_s^\dagger(x) \frac{\partial \psi_s(x)}{\partial x} + \frac{J_\perp}{\sqrt{2\pi a}} \left(\psi_s^\dagger(0)d + \psi_s(0)d^\dagger \right) + \mathcal{H}_2^1 \quad (60)$$

Now $\mathcal{H}_2^1$ is a quadratic form in fermion variables and therefore is exactly solvable. The consequences will be worked out in Sections 5 and 6.

If $J_z^R \neq 2\sqrt{2}\pi\lambda v_F$, the term proportional to $(\frac{\partial \Phi_s}{\partial x})_{x=0}$ in $\mathcal{H}$ will not be cancelled. Then there is an additional contribution $(J_z^R - 2\sqrt{2}\pi\lambda v_F)(d^\dagger d - \frac{1}{2})\psi_s^\dagger(0)\psi_s(0)$ to the Hamiltonian. This more general Hamiltonian is fully equivalent the anisotropic Kondo model and is known as the resonant level model. Departures from exactly solvable Toulouse limit may be considered using perturbation theory in the new term, which is not quadratic in fermion operators.

5.2. Two-Channel Problem

In this case we retain the channel label i and there are four collective modes: charge $\phi_c(x)$, spin $\phi_s(x)$, flavor $\phi_f(x)$, and spin-flavor $\phi_{sf}(x)$:

$$\phi_c = \frac{1}{2} \sum_{i, \alpha=1}^2 \phi_{i\alpha} \quad (61)$$

$$\phi_s = \frac{1}{2} \sum_{i, \alpha=1}^2 \sigma_{\alpha\alpha}^z \phi_{i\alpha} \quad (62)$$

$$\phi_f = \frac{1}{2} \sum_{i, \alpha=1}^2 \sigma_{ii}^z \phi_{i\alpha} \quad (63)$$

$$\phi_{sf} = \frac{1}{2} \sum_{i, \alpha=1}^2 \sigma_{ii}^z \sigma_{\alpha\alpha}^z \phi_{i\alpha} \quad (64)$$

The conjugate momenta $\hat{\Pi}_c, \hat{\Pi}_s$, etc., are defined by the same set of linear equations. Once again, in terms of the new variables, $\mathcal{H}_0$ remains a diagonal quadratic form, corresponding to free bosons and

$$\mathcal{H}_{1z} = \frac{J_z^R}{2\pi} \tau^z \left(\frac{\partial \Phi_s}{\partial x} \right)_{x=0} \quad (65)$$

$$\mathcal{H}_{1xy} = \frac{1}{\pi a} \left\{ J_\perp \tau^x \cos \Phi_s(0) + J_y \tau^y \sin \Phi_s(0) \right\} \cos \Phi_{sf}(0), \quad (66)$$

and

$$\mathcal{H}_2 = H \left[\frac{g_e}{2\pi} \int_{-\infty}^{\infty} dx \frac{\partial \Phi_s}{\partial x} + g_i \tau^z \right] \quad (67)$$

Note that the charge- and spin-flavor fields, $\phi_c(x)$ and $\phi_{sf}(x)$ do not enter into $\mathcal{H}_1$.

Once again, the discussion will be restricted to the antiferromagnetic xxz model ($J_x = J_y = J_\perp, J_z > 0$). Then $\Phi_s(0)$ may be eliminated from the J_x, J_y part of $\mathcal{H}_1$ in Eq. (66) by carrying out a rotation in τ -spin space about the z -axis via a unitary transformation $U\mathcal{H}U^{-1}$ with $U = \exp[i\tau^z \Phi_s(0)]$. The kinetic energy $\mathcal{H}_0$ is transformed to:

$$U\mathcal{H}_0U^{-1} = \mathcal{H}_0 - v_F \tau^z \left(\frac{\partial \Phi_s}{\partial x} \right)_{(x=0)} \quad (68)$$

Again we shall mainly be concerned with a special value of the z -component of the exchange, in this case $J_z^R = 2\pi v_F$, for which the Hamiltonian turns out to be particularly simple: later we shall consider the general case. With this assumption, the τ^z terms in $U\mathcal{H}_0U^{-1}$ and $U\mathcal{H}_1U^{-1}$ cancel, and the full transformed Hamiltonian becomes

$$U\mathcal{H}U^{-1} = \mathcal{H}_0 + \frac{J_\perp}{\pi a} \tau^x \cos \Phi_{sf}(0) + \mathcal{H}_2' \quad (69)$$

$$\mathcal{H}_2' = H \left[\frac{g_e}{2\pi} \int_{-\infty}^{\infty} dx \frac{\partial \Phi_s}{\partial x} + (g_i - g_e) \tau_z \right] \quad (70)$$

It is important to note that $\mathcal{H}'_2$ does not depend on the impurity variable τ^z if $g_e = g_i$. This must be true on general grounds since, in that special case, $\mathcal{H}_2$ is proportional to the total spin of the conduction electrons and the impurity, which is conserved. After the canonical transformation Φ_s commutes with the rest of the Hamiltonian but τ^z does not. Consequently $\mathcal{H}'_2$ can only depend on Φ_s . This fact will be important for determining the Wilson ratio for the two-channel Kondo problem.

Again Eq. (69) is equivalent to a sum of free fermion and free boson Hamiltonians, which is exactly solvable. The kinetic energy part of (69), $\mathcal{H}_0$, has contributions from ϕ_c , ϕ_s , and ϕ_f , which separate because they do not appear in $\mathcal{H}_1$. The part involving ϕ_{sf} may be written in terms of fermion fields by again using a representation similar to (29), but in reverse:

$$\psi_{sf}(x) = \frac{e^{-i\Phi_{sf}(x)}}{\sqrt{2\pi a}} \quad (71)$$

Rewriting the τ -operators in terms of the drone fermion d the Hamiltonian for the spin-flavor degrees of freedom becomes:

$$\mathcal{H}_{sf} = iv_F \int_{-\infty}^{\infty} dx \psi_{sf}^\dagger(x) \frac{\partial \psi_{sf}(x)}{\partial x} + \frac{J_\perp}{\sqrt{2\pi a}} \left(\psi_{sf}^\dagger(0) + \psi_{sf}(0) \right) (d^\dagger - d) + \mathcal{H}'_2 \quad (72)$$

This form of $\mathcal{H}_{sf}$ was first derived in reference [25]. In comparison with the single-channel problem, it is analogous to the decoupling limit because Φ_s is completely eliminated, and to the Toulouse limit because $\mathcal{H}_{sf}$ is a free-fermion Hamiltonian. However the differences between the coupling terms, proportional to $J_\perp$, has profound consequences for the correlation functions and thermodynamic properties of the model. This is evident if we transform to Majorana (real) fermions

$$\hat{a} = \frac{d^\dagger + d}{\sqrt{2}} \quad (73)$$

and

$$\hat{b} = \frac{d^\dagger - d}{i\sqrt{2}} \quad (74)$$

Then the two-channel Hamiltonian $\mathcal{H}_{sf}$ does not depend on $\hat{a}$ or $(\psi_{sf}^\dagger(0) - \psi_{sf}(0))$. In a sense only half of the impurity and conduction electron spin-flavor variables at the impurity site are coupled.

Eqs. (60) and (72) are the central results of this section. Since $\mathcal{H}_s$ and $\mathcal{H}_{sf}$ are free-fermion Hamiltonians, it is possible to work out all impurity Greens functions and correlation functions, and correlation functions of the conduction electrons may be evaluated whenever they are expressible as functions of the fermion field ψ_{sf} and the boson fields ϕ_c , ϕ_s , and ϕ_f .

If $J_z^R/2\pi v_F \neq 1$, the term proportional to $(\frac{\partial \Phi_s}{\partial x})_{x=0}$ in $\mathcal{H}$ will not be cancelled. Then the spin-wave part of the Hamiltonian may be rewritten in terms of a fermion field $\psi_s(x)$ (defined by Eq. (71) with Φ_{sf} replaced by Φ_s) as

$$\mathcal{H}_s = \mathcal{H}_{s0} + \mathcal{H}_{s1} + \mathcal{H}'_2 \quad (75)$$

where

$$\mathcal{H}_{s0} = i v_F \int_{-\infty}^{\infty} dx \psi_s^\dagger(x) \frac{\partial \psi_s(x)}{\partial x} \quad (76)$$

and

$$\mathcal{H}_{s1} = 2(J_z^R - 2\pi v_F) (d^\dagger d - \frac{1}{2}) \psi_s^\dagger(0) \psi_s(0) \quad (77)$$

The coupling between the conduction electrons and the impurity is now fully described by $\mathcal{H}_s + \mathcal{H}_{sf}$ which is the two-channel version of the resonant-level model. Departures from exactly solvable limit may be considered by using perturbation theory in $\mathcal{H}_{s1}$.

6. IMPURITY PROPERTIES

It is straightforward to obtain the correlation functions and thermodynamic properties of the impurity from the neighborhood of the decoupling limit of the single-channel problem and the free fermion Hamiltonians (60) and (72). Once again the single channel and two channel cases will be considered in turn.

6.1. Single-Channel Problem

6.1.1. Decoupling Limit

The decoupling limit $J_z^R = 4\pi v_F$ or $J_z = \infty$ is special inasmuch as a conduction electron forms a bound state rather than a resonance with the impurity. What is more, $\mathcal{H}'_2$ does not depend on the impurity variable τ^z if $g_e = g_i = g$. This must be true on general grounds since, for $g_e = g_i$, $\mathcal{H}_2$ is proportional to the total spin of the conduction electrons and the impurity, which is conserved. After the canonical transformation $\int_{-\infty}^{\infty} dx \frac{\partial \Phi_s}{\partial x}$ commutes with the rest of the Hamiltonian but τ^z does not. Consequently $\mathcal{H}'_2$ can only depend on Φ_s . This fact will be important for determining the Wilson ratio which, in fact, has no meaning in this limit, since $\chi_{imp} = 0$ and C_{imp} does not have a term proportional to T .

However, more characteristic behavior, and particularly the Wilson ratio may be obtained by evaluating χ_{imp}/C_{imp} by taking the limit $J_z \rightarrow \infty$, instead of setting $J_z = \infty$. This amounts to working to second order in $\Delta J_z^R \equiv J_z^R - 4\pi v_F$. To start with it will be assumed that $g_e = g_i = g$. The easiest way to obtain χ_{imp} is to choose λ to eliminate J_z^R from $\mathcal{H}'_0 + \mathcal{H}'_{1z}$ and then $\mathcal{H}'_2$ contains an impurity contribution $(-Hg\Delta J_z^R \tau^z / 4\pi v_F)$. Now, although Φ_s is not precisely eliminated from $\mathcal{H}'_{1xy}$, Eq. (57) is sufficient to obtain χ_{imp} to second order in ΔJ_z^R :

$$\chi_{imp} = \frac{g^2 a (\Delta J_z^R)^2}{16\pi v_F^2 J_\perp} \quad (78)$$

which diverges at the decoupling point, as it should. It is convenient to use a slightly different procedure to obtain C_{imp} . Choose λ to eliminate Φ_s from $\mathcal{H}'_{1xy}$, leaving a contribution $\Delta \mathcal{H}_z$ given by the right hand side of Eq. (51) with J_z^R replaced by ΔJ_z^R . This term gives a second order contribution to the free energy:

$$F_2 = -\frac{1}{2} \int_0^\beta dt \langle \Delta \mathcal{H}_z(t) \Delta \mathcal{H}_z(0) \rangle \quad (79)$$

where the (imaginary) time evolution of $\Delta\mathcal{H}_z$ and the thermal average (written $\langle \dots \rangle$) are evaluated using the free boson Hamiltonian for Φ , and $\mathcal{H}'_{1xy}$ for τ^z . F_2 is most easily calculated from the expression (36) for $\rho_s = \frac{1}{2\pi} \frac{\partial \Phi_s}{\partial x}$ and carrying integrals in k -space. To lowest order in T :

$$C_{imp} = \frac{\pi T a (\Delta J_z^R)^2}{24 v_F^2 J_\perp} \quad (80)$$

In order to obtain the Wilson ratio, we also need C_{bulk} and χ_{bulk} . The bulk specific heat may be obtained by regarding each degree of freedom either as an ideal Bose gas (for $k < 0$) or an ideal Fermi gas of particles ($k < 0$) and holes ($k > 0$), all with energy spectrum $v_F |k|$. Since there are two degrees of freedom:

$$\frac{C_{bulk}}{N} = \frac{\pi T}{3 v_F a} \quad (81)$$

for $H = 0$ and to lowest order in T . The bulk susceptibility may be obtained by using the fermion form of $\mathcal{H}_s^1$ and recognizing that the magnetic field is equivalent to a chemical potential $\mu = g_e / \sqrt{2}$:

$$\frac{\chi_{bulk}}{N} = \frac{(g_e)^2}{4 \pi v_F a} \quad (82)$$

for $T = 0$.

Combining Eqs. (78), (80), (81), and (82), the Wilson ratio for $g_e = g_i$ is:

$$R_W = 2 \quad (83)$$

which is the same as the value obtained by a numerical renormalization group calculation for weak-coupling.[5] Note that, as discussed in Section 1, a sensible result would not have been obtained if g_e had been set equal to zero at the outset. Then χ_{imp} would have been finite at the decoupling point but there would not have been a T -linear contribution to C_{imp} . The same situation will be encountered for the two-channel problem.

6.1.2. Toulouse Limit

The impurity Green's function is given by:

$$\hat{G}_{d1}(t) = -i \langle T d(t) d^\dagger \rangle \quad (84)$$

where T is the time ordering operator and $\langle \dots \rangle$ denotes thermal average. Then it is straightforward to solve the equations of motion and evaluate the Fourier transform $G_{d1}(\omega)$ of $\hat{G}_{d1}(t)$ to find

$$G_{d1}(\omega) = [\omega - g'_i H \pm i \Gamma_1]^{-1} \quad (85)$$

where $\Gamma_1 = J_\perp^2 / (\pi v_F a)$ and the upper or lower sign corresponds to ω in the upper or lower half plane respectively. The corresponding impurity spectral function is

$$A_1(\omega) = \frac{2\Gamma_1}{(\omega - g'_i H)^2 + \Gamma_1^2} \quad (86)$$

This expression shows the characteristic impurity resonance of the Kondo problem.

The impurity contribution to the free energy F may be calculated in the usual way by evaluating the thermodynamic average of $(\psi_s^\dagger(0)d + \psi_s(0)d^\dagger)$ using Eq. (85), and integrating over the coupling constant. The result is

$$F = F_0 + \frac{1}{\pi} \int_{-v_F/a}^{v_F/a} d\omega f(\omega) \tan^{-1} \left(\frac{\Gamma_1}{\omega - g_i' H} \right) \quad (87)$$

where F_0 is the free energy of the impurity with $J_\perp = 0$, and $f(\omega) = (e^{\beta\omega} + 1)^{-1}$. It is necessary to retain the short-distance cutoff a here in order to obtain a finite expression for F , but a may be set equal to zero in evaluating universal quantities obtained by differentiating F .

Now it is straightforward to show from Eq. (87) that, for $g_e = g_i$, the impurity contributions the magnetic susceptibility and specific heat are given by $\chi_{imp} = g^2/2\pi\Gamma$ and $C_{imp} = \pi T/3\Gamma$. Then, as before, $R_W = 2$. The fact that R_W has the same value at three points along the antiferromagnetic side of the line of fixed points $J_\perp = 0$ strongly suggests that it is universal for $\Gamma_1 \ll v_F a$, i.e. that it is independent of $J_\perp$ as well as the lattice spacing a . Indeed, this was proved by Wiegmann and Finkel'shtein.[32] Once this is accepted, it follows that, for arbitrary values of g_i/g_e , R_W is given by:

$$R_W = \frac{2 \left(\frac{g_i}{g_e} - J_z^R/4\pi v_F \right)^2}{\left(1 - J_z^R/4\pi v_F \right)^2} \quad (88)$$

where J_z^R is expressed in terms of J_z in Eq. (27). This result makes explicit the absence of universality of R_W for arbitrary values of g_e and g_i . As mentioned in Section 1, $R_W = 4$ if $g_e = 0$, or to $2\pi^2/3$ if $g_e = g_i$. All of these results, except for the discussion of the decoupling point, were obtained by Wiegmann and Finkel'shtein,[32] using perturbation theory in $J_\perp$, rather than bosonization.

6.2. Two-Channel Problem

In the two-channel problem the total number of fermions is not conserved, so there are anomalous Green's functions, as in the theory of superconductivity, and it is convenient to use the Nambu notation[41]

$$\Delta = \begin{pmatrix} d \\ d^\dagger \end{pmatrix} \quad (89)$$

and obtain the impurity Green's function as a 2×2 matrix

$$\hat{G}_d(t) = -i \langle T \Delta(t) \Delta^\dagger \rangle \quad (90)$$

where T is the time ordering operator and $\langle \dots \rangle$ denotes thermal average. Then it is straightforward to solve the equations of motion and evaluate the Fourier transform $G_d(\omega)$ of $\hat{G}_d(t)$ to find

$$G_d(\omega) = \left[\omega + 2g_i' H \tau^z \pm \frac{i\Gamma}{2} (1 - 2\tau^x) \right]^{-1} \quad (91)$$

where $\Gamma = J_{\perp}^2/(\pi v_F a)$ and the upper or lower sign corresponds to ω in the upper or lower half plane respectively. For $H = 0$, the corresponding impurity spectral function is

$$A(\omega) = \pi \delta(\omega) (1 + 2\tau^z) + \frac{\Gamma}{\omega^2 + \Gamma^2} (1 - 2\tau^z) \quad (92)$$

This expression brings out the essential feature of the two-channel version of the Kondo problem. Half of the spectral weight of the impurity is decoupled from the electron gas (the $\delta(\omega)$ term in Eq. (92)) and half has the usual Lorentzian form, with the width Γ , which is another way of showing that only half of the impurity degrees of freedom are coupled to the conduction electrons. It will be seen that the interplay between the two contributions to $A(\omega)$ is responsible for the characteristic behavior of the correlation functions at low temperature.

Once more the impurity contribution to the free energy F may be calculated by evaluating the thermodynamic average of $(\psi_{sf}^\dagger(0) + \psi_{sf}(0)) (d^\dagger - d)$, using Eq. (91), and integrating over the coupling constant. The result is

$$F = F_0 + \frac{1}{2\pi} \int_{-v_F/a}^{v_F/a} d\omega f(\omega) \tan^{-1} \left(\frac{\Gamma\omega}{\omega^2 - (g'_i H)^2} \right) \quad (93)$$

where F_0 is the free energy of the impurity with $J_{\perp} = 0$, and $f(\omega) = (e^{\beta\omega} + 1)^{-1}$. From Eq. (93) the impurity contribution to the entropy $(-\frac{\partial F}{\partial T})$ may easily be evaluated, to find

$$\lim_{T \rightarrow 0} \lim_{H \rightarrow 0} S = \frac{1}{2} \ln 2 \quad (94)$$

a result obtained previously.[14-16] There are two contributions to Eq. (94). For $J_{\perp} = 0$, $S_0 \equiv -\frac{\partial F_0}{\partial T} = \ln 2$, because an isolated impurity has a two-fold degenerate ground state. But the integral in Eq. (93) contributes $-\frac{1}{2} \ln 2$ to S ; once again only half of the impurity is coupled to the conduction electrons. If the order of limits in Eq. (94) is reversed, the value of the entropy is zero, which agrees with the conclusion of Tsvetick[18] and Sacramento and Schlottmann.[16] Note that the same argument gives zero for the ground state entropy of the single channel problem, (compare Eqs. (87) and (93)) as it should be for a fully saturated impurity.

For $g_e = g_i$, the singular contributions to the susceptibility and specific heat vanish at the solvable value of J_z^R , just as they do at the decoupling point of the single-channel problem. Since $\int_{-\infty}^{\infty} dx \frac{\partial \mathcal{H}}{\partial g_e}$ commutes with the Hamiltonian at this point, it follows as before that τ^z does not couple to the magnetic field in the transformed Hamiltonian and χ_{imp} vanishes. Similarly it is known[14-16],[18], [21] that, at low temperatures, C_{imp} has a contribution which varies as $T \ln T$. However, using Eq. (93), we find that the leading term in C_I is $\pi T/6\Gamma$. Once again, it is necessary to use second order perturbation theory to find the singular terms and obtain the Wilson ratio as a limit. The procedure has already been described for the decoupling point of the single-channel problem. The susceptibility may be evaluated in the same way: choose λ to eliminate J_z^R from $\mathcal{H}'_0$ and $\mathcal{H}'_{1z}$, and then $\mathcal{H}'_2$ contains a contribution $g'_i H \tau^z$, where $g'_i = -g \Delta J_z^R / 2\pi v_F$ with $\delta J_z^R = J_z^R - 2\pi v_F$. Substituting in Eq. (93), it is found that χ_{imp} has a logarithmic divergence:

$$\chi_{imp} = \frac{1}{\pi^3 \Gamma} \left(\frac{\Delta J_z^R}{2v_F} \right)^2 \ln \left(\frac{\Gamma}{T} \right) \quad (95)$$

In evaluating the specific heat it is easier to use the real-time expression for the second order contribution to the free energy:

$$F_2 = i(\Delta J_z^R)^2 \int_0^\infty dt C(t) g(t) \quad (96)$$

where the conduction electron spin density correlation function $C(t)$ is given in Eq. (43), and the τ^z correlation function must be evaluated using the two-channel Hamiltonian $\mathcal{H}_{sf}$ to obtain:

$$g(t) = -i \frac{T/2\Gamma}{\sinh(\pi T t)} \quad (97)$$

for $t > \Gamma^{-1}$, and is roughly constant otherwise. Then cutting off the integral at Γ^{-1} for small t , it is straightforward to extract the contribution of order T^2 , from which:

$$C_{imp}/T = \frac{1}{2\pi\Gamma} \left(\frac{\Delta J_z^R}{2v_F} \right)^2 \ln \left(\frac{\Gamma}{T} \right) \quad (98)$$

The bulk ratio χ_{bulk}/C_{bulk} is the same as in the single-channel case: the numerator is twice as big because the effective field on the conduction electron spin density is multiplied by $\sqrt{2}$ (compare Eqs. (56) and (70)), and the denominator is twice as big because there are four degrees of freedom instead of two. Thus we obtain

$$R_W = \frac{8}{3} \quad (99)$$

in agreement with the value obtained by Affleck and Ludwig[21] using conformal invariance. This result was anticipated in reference[26] and also has been obtained independently using the method described here by Sengupta and Georges.[42]

If $g_e = 0$, the field on the impurity does not vanish at $J_z^R = 2\pi v_F$ and χ_{imp} has a logarithmic divergence, as above:

$$\chi_{imp} = \frac{g_i^2}{\pi\Gamma} \ln(u) \quad (100)$$

where $u = \Gamma/T$ for $H = 0$ and $u = (\Gamma/H)^2$ for $T = 0$. The coefficient of the logarithm in Eq. (100) agrees with that obtained from the Bethe Ansatz solution[14–16] if we identify $\Gamma = 2\pi k_B T_K$, where T_K is the Kondo temperature. Then the coefficient of $\ln H$ agrees exactly with the result of Tsvelick and Wiegmann, [15] and the coefficient of $(\ln T)/T_K$ is $(2\pi^2)^{-1}$, whereas Sacramento and Schlottmann[16] obtain 0.05 numerically. Note that H and T have different scale dimensions $T \sim H^2$.

The most interesting impurity correlation function for our purposes is the longitudinal spin susceptibility, which may be expressed in terms of the fermion variables as

$$\chi(\omega, T) = -i \int_{-\infty}^0 dt e^{-i\omega t} \langle [\hat{n}, \hat{n}(t)] \rangle \quad (101)$$

where $\hat{n} \equiv d^\dagger d$. Since the fermions are noninteracting, the thermal average in Eq. (101) factors into averages of pairs of $d^\dagger, d$, which may then be obtained from the matrix elements of $A(\omega)$. The result of these manipulations is

$$\chi(\omega, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_1 \frac{\tanh(\frac{1}{2}\beta\omega_1)}{\omega_1 - \omega} \frac{\Gamma}{\omega_1^2 + \Gamma^2} \quad (102)$$

This expression involves one frequency integral instead of the usual two because of the $\delta(\omega)$ contribution to $A(\omega)$ in Eq. (92). For $\omega = 0$, Eq. (102) agrees with Eq. (95) at low temperatures. For $T = 0$, the real part of χ is given by Eq. (95) with T replaced by ω . The imaginary part of $\chi(\omega, T)$, for any ω, T is given by

$$\text{Im } \chi(\omega, T) = \frac{1}{2} \tanh\left(\frac{1}{2}\beta\omega\right) \frac{\Gamma}{\omega^2 + \Gamma^2} \quad (103)$$

For small ω , this agrees with the result obtained by Tsvetick,[18] apart from a factor of 4.

7. CONDUCTION ELECTRON PROPERTIES

In this section, we shall evaluate conduction electron correlation functions of the two-channel Kondo problem, especially those of interest for high temperature and heavy-fermion superconductors. The procedure has been set out in the preceding sections:

- (a) introduce the boson representation (29)
- (b) change to collective variables via Eqs. (61)-(64).
- (c) carry out the rotation in τ -spin space generated by U
- (d) change back to the fermion field $\psi_{sf}(x)$

Then any correlation functions which are expressible in terms of free boson or free fermion variables may be evaluated exactly. This procedure is straightforward for two-particle operators but single-particle operators involve (formally) $\sqrt{\psi_{sf}(x)}$, which is difficult to work with.

The z -component of the conduction electron spin density is given by

$$S^z(x) = \frac{1}{2} \sum_{i,\alpha,\beta} \sigma_{\alpha\beta}^z \psi_{i\alpha}^\dagger(x) \psi_{i\beta}(x), \quad (104)$$

and following the steps outlined above we arrive at

$$U S^z(x) U^{-1} = \frac{1}{\pi} \frac{\partial \Phi_s}{\partial x} + \tau^z \delta(x) \quad (105)$$

Here, only the second term on the right involves the impurity because the spin density is decoupled from the Hamiltonian $\mathcal{H}_{sf}$. Since $\tau^z = \hat{n} - \frac{1}{2}$, the impurity contribution to the correlation function for the total spin density is given by $\chi(\omega, T)$, defined by Eq. (101) and evaluated in Eq. (102) and (103). This result, which shows that the time-evolution of the impurity spin feeds back into conduction electron properties, is of interest for calculating the optical conductivity in a particular realization of the two-channel Kondo problem, proposed in the context of high-temperature superconductivity.[11] In that realization, the flavor variables correspond to the spin of the conduction electrons and the spin is an internal degree of freedom of a local collective excitation represented by the "impurity." Since there are four collective degrees of freedom; it is possible to construct four current-density operators for the one-dimensional version of the model. They are the charge-, spin-, flavor- and spin-flavor densities of the left-going fermions. In the boson representation,[39] they are proportional to $\frac{\partial \Phi_s}{\partial x}$, $\frac{\partial \Phi_f}{\partial x}$, $\frac{\partial \Phi_\sigma}{\partial x}$, and $\frac{\partial \Phi_{sf}}{\partial x}$. Of these, only the last two

couple to the impurity: $\frac{\partial \Phi_{sf}}{\partial \sigma}$ (because $\Phi_{sf}(0)$ occurs in Eq. (69)) and $\frac{\partial \Phi_{sf}}{\partial \sigma}$ (because the canonical transformation generated by U gives it a contribution $\tau^z \delta(x)$, as in Eq. (105)). Then the corresponding current-current correlation functions give a Drude contribution to the spin-flavor conductivity, but the impurity contribution to the spin conductivity is proportional to $\chi(\omega, T)$ whose real part is given by:[25]

$$\text{Re } \sigma_s(\omega, T) \sim \frac{\tanh(\frac{1}{2}\beta\omega)}{\omega} \frac{\Gamma}{\omega^2 + \Gamma^2} \quad (106)$$

This quantity is proportional to ω^{-1} for $T \ll \omega \ll \Gamma$ and to T^{-1} for $\omega \ll T \ll \Gamma$ which is the behavior characteristic of the normal state of high-temperature superconductors.[43] We have shown that, in an orbital Kondo model,[26] that the z-component of the impurity spin has the same transformation properties as a current operator (i.e., it is odd under parity and time-reversal) and that the optical conductivity maps into the spin conductivity of the equivalent two-channel Kondo problem.

Another issue of importance for applications is the behavior of the conduction electron pairing susceptibility at the impurity site. In particular, since the low-temperature state is a pairing resonance, rather than a single-particle resonance as in the single-channel Kondo problem, it is conceivable that pairing correlations might be enhanced. The correlation functions of the relevant pairing operators $\psi_{i\alpha}(0) \psi_{j\beta}(0)$ may be evaluated by the method outlined above and it is found that they vary as t^{-2} for long times.[35] This behavior is the same as for free fermions, and there is no enhancement of ordinary pairing at the impurity site.

But this is not the whole story, for the pairing resonance also involves the impurity spin. Consequently we consider the "composite pairing" operator, first introduced in ref. [25]:

$$P_{\pm} = \left(\frac{1}{2} \pm \tau^z\right) (\psi_{11} \psi_{22} + \psi_{21} \psi_{12}) \quad (107)$$

which is a flavor singlet, spin triplet $S = 0$ and projects onto impurity spin $\pm \frac{1}{2}$. Expressed in terms of the collective modes and the d -fermions

$$P_+ = \frac{1}{2\pi a} d^\dagger d (\psi_{sf}^\dagger + \psi_{sf}) \psi_c, \quad (108)$$

where ψ_c is defined by Eq. (71) with Φ_{sf} replaced by Φ_c . Using Eqs. (73) and (74), $d^\dagger, d$ may be replaced by the Majorana fermions $\hat{a}, \hat{b}$, to give

$$P_+ = (1 - i\hat{a}\hat{b}) (\psi_{sf}^\dagger + \psi_{sf}) \psi_c \quad (109)$$

Now, when $H = 0$, $\hat{a}$ is a constant of the motion with $\hat{a}^2 = \frac{1}{2}$. Also, since $\hat{b}$ appears in $\mathcal{H}_{sf}$, $\langle b(\psi_{sf}^\dagger + \psi_{sf}) \rangle$ is finite. Then there is a contribution to the P_+ susceptibility equal to a constant multiplied by the ψ_c susceptibility, which in turn is proportional to t^{-1} . This is a slower falloff than for free fermions and hence the frequency-dependent susceptibility is substantially enhanced. The same behavior is obtained for P_- .

It also was shown in ref. [25], that the composite pairing operator, introduced in Eq. (107), may be written as a singlet, odd-time pairing operator. This may be seen as follows: Introduce the Heisenberg operators $\psi_{i\alpha}(t)$ and define the flavor singlet operator:

$$\hat{O}(t) = \psi_{11}(t)\psi_{22}(0) - \psi_{21}(t)\psi_{12}(0) \quad (110)$$

Then, using Eqs. (23)-(25), it is straightforward to show that all odd time derivatives $\frac{d^n \hat{O}(t)}{dt^n}$, evaluated at $t=0$, contain a contribution from the J_z term of the Hamiltonian which is proportional to $P_+ - P_-$, whose correlation function is enhanced at the impurity site, as shown above. It follows that the susceptibility of any odd time derivative of $\hat{O}(t)$, evaluated at $t = 0$, is similarly enhanced, but not the susceptibilities of even time derivatives. It follows that the order parameter corresponding to $\hat{O}(t)$, given by $\langle \hat{T} \hat{O} \rangle$ (where $\hat{T}$ is the time ordering operator), is odd in time and even in space, and it is a spin-singlet as well as a flavor singlet. This behavior is strikingly different from conventional superconductivity, for which the order parameter is even in time. An ordered superconducting state will emerge from a collection of two-channel Kondo impurities when the fluctuations associated with different impurity sites become coherent. We have shown [26] that the nature of such an ordered state depends on the coupling between the different impurities and that it need not have an parameter that is odd in time.

There is by now a growing literature concerned with odd-time, or equivalently odd-frequency pairing. The idea was considered first for spin-triplet pairs by Berezinskii [44] and for spin-singlet pairs by Balatsky and Abrahams.[45] They pointed out that Fermi statistics require that $\langle \hat{T} \hat{O} \rangle$ be odd in (space $\times$ spin $\times$ time), whereas the usual assumption in the BCS theory of superconductivity is that the order parameter is even in time and therefore odd in (space $\times$ spin). We introduced the concept of composite pairing in the context of the two-channel Kondo problem [25] and showed that it is equivalent to odd-frequency pairing. Coleman *et al* [46] found a triplet odd-frequency paired state in a mean-field theory of a single channel Kondo lattice. However, to our knowledge, the two-channel Kondo problem is the only one for which the enhancement of odd-frequency pairing has been *proved*. [25]

These results provide a new perspective on the evaluation of pairing correlation functions in the two-channel Kondo problem, using conformal invariance. Ludwig and Affleck[23] showed that only the spin-singlet flavor-singlet channel has a diverging susceptibility, and they concluded that "by Fermi statistics the susceptibility of the pairing $\psi_{\alpha i} \psi_{\beta j}$ field itself does *not* diverge." However, in light of the above discussion, it is clear that conformal invariance *does* in fact allow a diverging susceptibility for spin-singlet flavor-singlet pairing, provided it is odd in frequency.

The discussion of dynamical impurity problems given in these lectures illustrates some of the features of modern many-body theory and, in particular, the way it makes use of concepts borrowed from the theory of critical phenomena. In particular it shows that the idea of obtaining the universal low-energy behavior of a many-body system by studying a simple fixed point Hamiltonian is very powerful, especially when combined with the techniques of quantum field theory.

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