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STABILITY ANALYSIS FOR THE FLOW OF GRANULAR MATERIALS DOWN AN INCLINED PLANE USING KINETIC MODEL

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SUMMARY

The flow of granular materials down an inclined plane is modeled by the Richman & Marciniec (1990) in which, they consider a kinetic constitutive theory that includes the effects of particle transport and collisions. Richman & marciniec (1990) obtained closed form solution for the granular temperature profile, by replacing the volume fraction by its depth-averaged value in the balance equation's, and thereby from constitutive relations for the normal and shear stresses they obtained the volume fraction and velocity profiles. Here, we use the model proposed by Richman & marciniec (1990) to study the linearized stability for the flow of granular materials down an inclined plane. The governing equations are obtained from the conservation of mass, balance of linear momentum and balance of energy. The basic flow equations and the order of ε equations are derived, but we intend to use the approximate solution of Richman & Marciniec (1990) as the base solution for the linearized stability analysis.

INTRODUCTION

The key reason behind using a kinetic theory approach for granular material is the analogy between a *grain* and a *molecule*. So as a first approximation, both gas molecules and granular material can be considered single-sized hard spheres. If the flow of granular material is rapid, it can be characterized by an average or continuum velocity about which the velocities of the individual grains deviate because of random collisions. Since such motion closely resembles a flowing gas, the ideas incorporated into the kinetic theory of gases have been applied to the rapid flow of granular material. However, the analogy between a gas and granular material is not exact because the collisions between gas particles are energy conserving, whereas those between grains of material are energy dissipating. But the inelasticity of granular collisions prevents the theoretical methods developed for hard sphere gases from being directly applied to flowing granular materials. Boyle & Massoudi (1989) have written comprehensive survey article of the constitutive equations and the laws governing the flow of granular material. Also, the major difference between the continuum theories and the theories based on kinetic approach is the concept of "**granular temperature**". The granular temperature describes the fluctuating velocity of the flow of granular materials and in this sense is similar to the temperature in a gas due to fluctuating motion of the gas particles. This kinetic theory approach is applied to granular materials by various investigators [cf. Ogawa, Umemura & Oshima (1980), Shen (1981), Jenkins & Savage (1983), Ahmadi & Shahinpoor (1983a, 1984), Lun, Savage and co-workers (1984), Ahmadi and Ma (1986), Ma and Ahmadi (1985, 1988), Jenkins and Richman (1985, 1986), Jenkins & Mancini (1987), Johnson and Jackson (1987), Jenkins (1991), Pasquarell & Ackerman (1989) and Boyle and Massoudi (1989)].

GOVERNING EQUATIONS

The stress tensor governing the flow of granular materials down an inclined plane is given by [cf. Richman and Marciniec (1990)].

$$\mathbf{P} = \{ 4 \rho G F T \} \mathbf{1} - 2 \mu E \hat{\mathbf{D}} \quad (1)$$

$$\text{Where,} \quad F(v) = 1 + \frac{1}{4G} \quad (2)$$

$$E(v) = 1 + \pi \frac{(1+5/8G)^2}{12} \quad (3)$$

$$G(v) = \frac{v(2-v)}{2(1-v)^3} \quad (4)$$

$$\mu = \frac{8 \sigma \rho G T^{1/2}}{5 \pi^{1/2}} \quad (5)$$

In the above equation $\mathbf{P}$ denotes the Cauchy Stress, v the volume fraction of the solid, $\hat{\mathbf{D}}$ denotes the deviatoric part of stretching tensor $\mathbf{D}$ associated with the solid motion and T the granular temperature.

Consider the flow of granular material modeled by the above kinetic model down an inclined plane (cf. Figure 1) due to the action of gravity [cf. Savage (1979), Johnson and Jackson (1987), Johnson, et al. (1990), Hui, et al. (1984), Richman and Marciniec (1990), Hutter, et al. (1986a, b)]. In this problem we consider steady one dimensional flow of incompressible granular materials (*i.e.* $\rho_p = \text{constant}$) down an inclined plane, where the angle of inclination is α . The governing equations of motion are the conservation of mass, momentum, and energy. The conservation of mass is

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0, \quad (6)$$

where $\frac{\partial}{\partial t}$ is the partial derivative with respect to time. The balance of linear momentum is

$$-\text{div} \mathbf{P} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt}, \quad (7)$$

where $\frac{d}{dt}$ is the material time derivative and $\mathbf{b}$ is the body force. The energy equation in its general form is

$$-\text{tr}(\mathbf{P} \mathbf{D}) - \text{div} \mathbf{Q} - \gamma = \frac{3}{2} \rho \frac{dT}{dt}, \quad (8)$$

where $\mathbf{Q}$ is the flux of fluctuation energy, γ is the collisional rate per unit volume of energy dissipation. The derivation of these equations for the dissipative flows was first given by Jenkins & Savage (1983).

STABILITY ANALYSIS:

If the solution to the governing equations of motion is disturbed, then the solution is asymptotically stable if that disturbance eventually decays to zero and unstable if the disturbance grows in amplitude in such a way that the solution departs from initial state or reaches some constant value, yielding a new solution. Once the stable or unstable states are classified for the governing equations, then the locus which separates the two classes of states is defined as marginal stability or neutral stability curve. Here, in the present analysis we determine the marginal stability curves for the flow down an inclined plane. Linearized stability does not yield sufficient conditions for stability. So if the solution is unstable to small disturbances then it will be unstable to finite disturbances, while on the otherhand, if the solution is stable to small disturbances it is not necessarily stable to finite disturbances.

Consider solutions which consist of the basic flow plus an infinitesimal disturbance

$$v = v_0 + \varepsilon v_1 \quad (9)$$

$$u = u_0 + \varepsilon u_1 \quad (10)$$

$$V = \varepsilon u_2 \quad (11)$$

$$T = T_0 + \varepsilon T_1 \quad (12)$$

where v_0 , u_0 and T_0 correspond to the basic solution of the governing equations and v_1 , u_1 , u_2 , T_1 represent the disturbance. It is assumed that for infinitesimal disturbances, the equations may be linearized *i.e.* the terms of order ε^2 and higher order can be neglected. In general, in addition to perturbing the volume fraction, velocity field and granular temperature, it is necessary to perturb the free surface. Ofcourse, the base solution will not hold in all of the perturbed domain. Here, we do not allow for the perturbation of the free surface. Such a stability study is incomplete in that it addresses only the disturbances corresponding to a special case. Ofcourse, a small disturbance without such a constraint may prove to be unstable, nullifying the instability predictions. The basic flow is assumed to have the form

$$\begin{aligned} v_0 &= v_0(y) \\ u_0 &= u_0(y)i \\ T_0 &= T_0(y) \end{aligned} \quad (13)$$

Now substituting equations (9) through (12) into conservation of mass, balance of

linear momentum and balance of energy, the equations corresponding to the basic flow *i.e.* of order one are given by:

$$\frac{\bar{\sigma}}{20\sqrt{\pi}} \left\{ \frac{S_1}{2\sqrt{T_0}} \frac{du_0}{dy} \frac{dT_0}{dy} + \sqrt{T_0} S_1 \frac{d^2 u_0}{dy^2} + \sqrt{T_0} S_6 \frac{du_0}{dy} \frac{dv_0}{dy} \right\} + g v_0 \sin \alpha = 0, \quad (14)$$

$$S_2 \frac{dT_0}{dy} + T_0 S_3 \frac{dv_0}{dy} + g v_0 \cos \alpha = 0 \quad (15)$$

and,

$$\begin{aligned} & \frac{-\bar{\sigma}}{\sqrt{\pi}} \left\{ S_7 \sqrt{T_0} \frac{d^2 T_0}{dy^2} + \frac{S_7}{2\sqrt{T_0}} \left\{ \frac{dT_0}{dy} \right\}^2 + S_8 \sqrt{T_0} \frac{dT_0}{dy} \frac{dv_0}{dy} \right\} \\ & - \frac{\bar{\sigma}}{20\sqrt{\pi}} \sqrt{T_0} S_1 \left\{ \frac{du_0}{dy} \right\}^2 + \frac{12}{\bar{\sigma}\sqrt{\pi}} S_9 T \sqrt{T_0} = 0 \end{aligned} \quad (16)$$

where g denotes the acceleration due to gravity and α is the angle of inclination of the plane.

$$\text{With, } S_1 = \frac{16 v_0^2 (2-v_0)}{(1-v_0)^3} + \frac{\pi}{12} \frac{(-5v_0^3 + 11v_0^2 - 7v_0 + 5)^2}{(2-v_0)(1-v_0)^3}$$

$$S_2 = v_0 \left\{ \frac{2v(2-v)}{(1-v_0)^3} + 1 \right\}$$

$$S_3 = \frac{(7v_0^4 - 4v_0^3 + 4v_0^2 + 4v_0 + 1)}{(1-v_0)^4}$$

$$S_4 = \frac{16 v_0 (4-v_0)}{(1-v_0)^4}$$

$$S_5 = \frac{\pi}{12} \left\{ \frac{50 v_0^9 - 375 v_0^8 + 2000 v_0^7 - 4706 v_0^6 + 6500 v_0^5 - 5942 v_0^4 + 3584 v_0^3 - 1332 v_0^2 + 186 v_0 + 35}{(2-v_0)^2 (1-v_0)^6} \right\}$$

$$S_6 = S_4 + S_5$$

$$S_7 = \frac{2 v_0^2 (2-v_0)}{(1-v_0)^3} + \frac{\pi}{64} \frac{(-5v_0^3+9v_0^2-3v_0+5)^2}{(2-v_0)(1-v_0)^3}$$

$$S_8 = \frac{dS_7}{dv}$$

$$S_9 = \frac{v_0^2 (2-v_0)}{(1-v_0)^3} (1-e)$$

We need to solve the equations (14), (15) and (16) subject to the appropriate boundary conditions.

$$u_0 = \sqrt{\frac{\pi}{2}} \tan \alpha f \sqrt{T_0}$$

$$\frac{1}{\sqrt{T_0}} \frac{dT_0}{dy} = b \quad \text{at } y = 0 \text{ (on the inclined plane)} \quad (17)$$

$$N = \int_0^L v_0 dy. \quad (18)$$

and,

$$\frac{du_0}{dy} = 0$$

$$\sqrt{T_0} \frac{dT_0}{dy} = 0 \quad \text{at } y = L \text{ (at the free surface)} \quad (19)$$

$$T_0 = T_{01} \text{ (Specified)}$$

Equation (17)₁ indicates the slip condition on the inclined plane. Condition (18) gives the amount of material that is fed in and which remains the same at all locations of x .

In the above equations,

$$b = \frac{F}{M \sqrt{2}} \left\{ \pi (\tan \alpha)^2 \frac{f}{2} - 2(1-e_w)(1-\cos \theta) \csc^2 \theta \right\} \quad (20)$$

$$\text{where, } f = \frac{\left(1 - \frac{5 F \bar{a}}{2^{3/2} E} \right) \left\{ 1 + \frac{B}{a} \right\} \sin^2 \theta}{(2/3) [2 \csc^2 \theta (1 - \cos \theta) - \cos \theta]} + \frac{5 \bar{a} F}{\sqrt{2} E}$$

$$B(v) = \pi \frac{(1+5/8G)}{12 \sqrt{2}}$$

$$M(v) = 1 + 9 \pi \frac{(1+5/12G)^2}{32}$$

$$\bar{a} = \frac{1}{2} \left\{ 1 + \frac{1}{r} \right\}$$

$$r = \frac{\sigma}{d}$$

$$\bar{r} = \frac{s}{d}$$

$$\theta = \sin^{-1} \left\{ \frac{(1 + \bar{r})}{(1 + r)} \right\}$$

Here, $\bar{\sigma}$ is the diameter of the particle, d is the diameter of the hemispherical particles attached to the flat wall, and s is the average distance between the hemispherical particles. e_w is the coefficient of restitution that characterizes the energy dissipated in particle-wall collisions.

The equations corresponding to order of ε are:

$$\frac{\partial v_1}{\partial t} + v_0 \left\{ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right\} + u_0 \left\{ \frac{\partial v_1}{\partial x} \right\} + \frac{dv_0}{dy} u_2 = 0 \quad (21)$$

$$\begin{aligned} & \frac{\bar{\sigma}}{20 \sqrt{\pi}} \left\{ S_1 \sqrt{T_0} \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{4}{3} \frac{\partial^2 u_1}{\partial x^2} \right) + \left(S_6 \sqrt{T_0} \frac{dv_0}{dy} + \frac{S_1}{2 \sqrt{T_0}} \frac{dT_0}{dy} \right) \frac{\partial u_1}{\partial y} + \frac{4}{3} \frac{\partial^2 u_2}{\partial x \partial y} \right. \\ & + \frac{S_1}{2 \sqrt{T_0}} \frac{du_0}{dy} \frac{\partial T_1}{\partial y} - \frac{20 \sqrt{\pi} S_2}{\bar{\sigma} \sqrt{T_0}} \frac{\partial T_1}{\partial x} + \sqrt{T_0} A_1 \frac{du_0}{dy} \frac{\partial v_1}{\partial y} - \frac{20 \sqrt{\pi}}{\bar{\sigma}} A_2 \sqrt{T_0} \frac{\partial v_1}{\partial x} \\ & + \left(\frac{S_6}{2 \sqrt{T_0}} \frac{du_0}{dy} \frac{dv_0}{dy} + \frac{S_1}{2 \sqrt{T_0}} \frac{d^2 u_0}{dy^2} - \frac{S_1}{4 T_0 \sqrt{T_0}} \frac{du_0}{dy} \frac{dT_0}{dy} \right) T_1 \\ & + \left. \left(\frac{A_1}{2 \sqrt{T_0}} \frac{du_0}{dy} \frac{dT_0}{dy} + A_1 \sqrt{T_0} \frac{d^2 u_0}{dy^2} + S_{16} \sqrt{T_0} \frac{du_0}{dy} \frac{dv_0}{dy} + \frac{20 \sqrt{\pi} g}{\bar{\sigma}} \sin \alpha \right) v_1 \right\} \\ & = v_0 \left\{ \frac{\partial u_1}{\partial t} + u_0 \frac{\partial u_1}{\partial x} + \frac{du_0}{dy} u_2 \right\} \quad (22) \end{aligned}$$

$$\begin{aligned}
& \frac{\bar{\sigma}}{20\sqrt{\pi}} S_1 \sqrt{T_0} \left\{ \frac{\partial^2 u_2}{\partial x^2} - \frac{4}{3} \frac{\partial^2 u_2}{\partial y^2} + \frac{4}{3} \frac{\partial^2 u_1}{\partial x \partial y} \right\} - \frac{\bar{\sigma}}{15\sqrt{\pi}} \left\{ \frac{S_1}{2\sqrt{T_0}} \frac{dT_0}{dy} + S_6 \sqrt{T_0} \frac{dv_0}{dy} \right\} \frac{\partial u_2}{\partial y} \\
& + \frac{\bar{\sigma} S_1}{40\sqrt{\pi}\sqrt{T_0}} \frac{du_0}{dy} \frac{\partial T_1}{\partial x} + S_2 \frac{\partial T_1}{\partial y} + S_3 \frac{dv_0}{dy} T_1 + A_2 T_0 \frac{\partial v_1}{\partial y} + \left\{ A_2 \frac{dT_0}{dy} + T_0 S_{17} \frac{dv_0}{dy} + g \cos \alpha \right\} v_1 \\
& = v_0 \left(\frac{\partial u_2}{\partial t} + u_0 \frac{\partial u_2}{\partial x} \right)
\end{aligned} \tag{23}$$

$$\begin{aligned}
& S_7 \sqrt{T_0} \left\{ \frac{\partial^2 T_1}{\partial y^2} + 2 \frac{\partial^2 T_1}{\partial x \partial y} + \frac{\partial^2 T_1}{\partial x^2} \right\} + \left\{ \frac{S_7}{\sqrt{T_0}} \frac{dT_0}{dy} + S_8 \sqrt{T_0} \frac{dv_0}{dy} \right\} \left(\frac{\partial T_1}{\partial y} + \frac{\partial T_1}{\partial x} \right) \\
& + \left\{ \frac{S_8}{2\sqrt{T_0}} \frac{dT_0}{dy} \frac{dv_0}{dy} + \frac{S_7}{2\sqrt{T_0}} \frac{d^2 T_0}{dy^2} - \frac{S_7}{2T_0\sqrt{T_0}} \left\{ \frac{dT_0}{dy} \right\}^2 + \frac{S_1}{40\sqrt{T_0}} \left\{ \frac{du_0}{dy} \right\}^2 - \frac{18 S_9}{\bar{\sigma}^2 \sqrt{T_0}} \right\} T_1 \\
& + A_3 \sqrt{T_0} \frac{dT_0}{dy} \left\{ \frac{\partial v_1}{\partial x} + \frac{\partial v_1}{\partial y} \right\} + \left\{ A_3 \sqrt{T_0} \frac{d^2 T_0}{dy^2} + \frac{A_3}{2\sqrt{T_0}} \left\{ \frac{dT_0}{dy} \right\}^2 + \sqrt{T_0} S_{18} \frac{dv_0}{dy} \frac{dT_0}{dy} \right. \\
& + \left. \frac{A_1 \sqrt{T_0}}{20} \left\{ \frac{du_0}{dy} \right\}^2 - \frac{12 A_4 T_0 \sqrt{T_0}}{\bar{\sigma}^2} \right\} v_1 + \frac{S_1 \sqrt{T_0}}{10} \frac{du_0}{dy} \frac{\partial u_1}{\partial y} + \frac{S_1 \sqrt{T_0}}{20} \frac{du_0}{dy} \frac{\partial u_2}{\partial x} \\
& - \frac{S_2 T_0 \sqrt{\pi}}{\bar{\sigma}} \left\{ \frac{\partial u_2}{\partial y} + \frac{\partial u_1}{\partial x} \right\} + \frac{S_1 \sqrt{T_0}}{15} \left\{ \frac{\partial u_1}{\partial x} \right\}^2 = \frac{3}{2} v_0 \frac{\partial T_1}{\partial t}
\end{aligned} \tag{24}$$

subjected to the boundary conditions

$$\begin{aligned}
u_1 &= \sqrt{\frac{\pi}{2}} \tan \alpha \sqrt{T_0} \left\{ f_p v_1 + f \frac{T_1}{2 T_0} \right\} \\
u_2 &= 0 \quad \text{at } y = 0 \text{ (on the inclined plane)}
\end{aligned} \tag{25}$$

$$\int_0^L v_1 dy = 0 \tag{26}$$

$$\begin{aligned}
& S_1 \frac{\partial u_1}{\partial y} + S_1 \frac{\partial u_2}{\partial x} + A_1 \frac{du_0}{dy} v_1 + \frac{S_1}{2 T_0} \frac{du_0}{dy} T_1 = 0 \\
& \sqrt{T_0} \frac{\partial T_1}{\partial y} + \frac{1}{2\sqrt{T_0}} \frac{dT_0}{dy} T_1 = 0 \quad \text{at } y = L \text{ (at the free surface)} \\
& T_1 = 0
\end{aligned} \tag{27}$$

The system of equations subjected to the boundary conditions are non-dimensionalized by

$$\begin{aligned}
\bar{y} &= \frac{L-y}{\sigma}; & \bar{u}_0 &= \frac{u}{\sqrt{\sigma g}}; & \bar{W}_0 &= \left\{ \frac{T_0}{\sigma g} \right\}^{1/2}; & \bar{x} &= \frac{L-x}{\sigma}, \\
\bar{u}_1 &= \frac{u_1}{\sqrt{\sigma g}}; & \bar{u}_2 &= \frac{u_2}{\sqrt{\sigma g}}; & \bar{W}_1 &= \left\{ \frac{T_1}{\sigma g} \right\}^{1/2}; & \bar{t} &= t \sqrt{\frac{g}{\sigma}}
\end{aligned} \tag{28}$$

where L is the characteristic length. Now, the above system equations for the basic flow reduces to

$$S_1 \bar{W}_0 \frac{d^2 \bar{u}_0}{d\bar{y}^2} + \left\{ S_1 \frac{d\bar{W}_0}{d\bar{y}} + S_6 \bar{W}_0 \frac{d\nu_0}{d\bar{y}} \right\} \frac{d\bar{u}_0}{d\bar{y}} = -20 \sqrt{\pi} \nu_0 \sin \alpha, \tag{29}$$

$$S_3 \bar{W}_0^2 \frac{d\nu_0}{d\bar{y}} + 2 S_2 \bar{W}_0 \frac{d\bar{W}_0}{d\bar{y}} = -\nu_0 \cos \alpha \tag{30}$$

$$S_7 \bar{W}_0 \frac{d^2 \bar{W}_0}{d\bar{y}^2} + 2 S_7 \left\{ \frac{d\bar{W}_0}{d\bar{y}} \right\}^2 + S_8 \bar{W}_0 \frac{d\bar{W}_0}{d\bar{y}} \frac{d\nu_0}{d\bar{y}} + \frac{S_1}{40} \left\{ \frac{d\bar{u}_0}{d\bar{y}} \right\}^2 - 6 S_9 \bar{W}_0^2 = 0, \tag{31}$$

and the boundary conditions become

$$\bar{u} = \sqrt{\frac{\pi}{2}} \tan \alpha f \bar{W}_0$$

$$\frac{1}{\bar{W}} \frac{d\bar{W}_0}{d\bar{y}} = b \quad \text{at } \bar{y} = \beta \text{ (on the inclined plane)} \tag{32}$$

$$Q = \frac{1}{\beta} \int_0^\beta \nu \, d\bar{y} \tag{33}$$

and,

$$\frac{d\bar{u}_0}{d\bar{y}} = 0$$

$$\bar{W}_0^2 \frac{d\bar{W}_0}{d\bar{y}} = 0 \quad \text{at } \bar{y} = 0 \text{ (at the free surface)} \tag{34}$$

$$\bar{W}_0^2 = W_{01} \text{ (Specified)}$$

The non-dimensional equations for the order of ε are given by

$$\frac{\partial v_1}{\partial \bar{t}} - v_0 \left\{ \frac{\partial \bar{u}_1}{\partial \bar{x}} + \frac{\partial \bar{u}_2}{\partial \bar{y}} \right\} - \bar{u}_0 \left\{ \frac{\partial v_1}{\partial \bar{x}} \right\} - \frac{dv_0}{d\bar{y}} \bar{u}_2 = 0 \quad (35)$$

$$\begin{aligned} P_1 \left\{ \frac{\partial^2 \bar{u}_1}{\partial \bar{y}^2} + \frac{4}{3} \frac{\partial^2 \bar{u}_1}{\partial \bar{x}^2} \right\} + \frac{4}{3} P_1 \frac{\partial^2 \bar{u}_2}{\partial \bar{x} \partial \bar{y}} + P_2 \frac{\partial \bar{u}_1}{\partial \bar{y}} + 20 \sqrt{\pi} v_0 \bar{u}_0 \frac{\partial \bar{u}_1}{\partial \bar{x}} \\ + 2 \bar{W}_1 P_3 \frac{\partial \bar{W}_1}{\partial \bar{y}} + 40 \sqrt{\pi} S_2 \bar{W}_1 \frac{\partial \bar{W}_1}{\partial \bar{x}} + P_5 \frac{\partial \bar{v}_1}{\partial \bar{y}} + 20 \sqrt{\pi} S_{14} \bar{W}_0 \frac{\partial \bar{v}_1}{\partial \bar{x}} \\ + P_4 \bar{W}_1^2 + P_6 v_1 + 20 \sqrt{\pi} v_0 \frac{\partial \bar{u}_0}{\partial \bar{y}} \bar{u}_2 = 20 \sqrt{\pi} v_0 \frac{\partial \bar{u}_1}{\partial \bar{t}} \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{P_1}{20 \sqrt{\pi}} \frac{\partial^2 \bar{u}_2}{\partial \bar{x}^2} - \frac{P_1}{15 \sqrt{\pi}} \frac{\partial^2 \bar{u}_2}{\partial \bar{y}^2} + \frac{P_1}{15 \sqrt{\pi}} \frac{\partial^2 \bar{u}_1}{\partial \bar{x} \partial \bar{y}} - P_8 \frac{\partial \bar{u}_2}{\partial \bar{y}} + v \bar{u}_0 \frac{\partial \bar{u}_2}{\partial \bar{x}} \\ - 2 S_2 \bar{W}_1 \frac{\partial \bar{W}_1}{\partial \bar{y}} + 2 P_7 \bar{W}_1 \frac{\partial \bar{W}_1}{\partial \bar{x}} - P_{10} \frac{\partial \bar{v}_1}{\partial \bar{y}} - P_{11} \bar{W}_1^2 + P_9 v_1 = v_0 \frac{\partial \bar{u}_2}{\partial \bar{t}} \end{aligned} \quad (37)$$

$$\begin{aligned} P_{21} \bar{W}_1 \left\{ \frac{\partial^2 \bar{W}_1}{\partial \bar{x}^2} + 2 \frac{\partial^2 \bar{W}_1}{\partial \bar{x} \partial \bar{y}} + \frac{\partial^2 \bar{W}_1}{\partial \bar{y}^2} \right\} + P_{21} \left\{ \left\{ \frac{\partial \bar{W}_1}{\partial \bar{x}} \right\}^2 + 2 \frac{\partial \bar{W}_1}{\partial \bar{x}} \frac{\partial \bar{W}_1}{\partial \bar{y}} + \left\{ \frac{\partial \bar{W}_1}{\partial \bar{y}} \right\}^2 \right\} \\ + H_3 \bar{W}_1 \left\{ \frac{\partial \bar{W}_1}{\partial \bar{y}} + \frac{\partial \bar{W}_1}{\partial \bar{x}} \right\} + P_{19} \left\{ \frac{\partial \bar{v}_1}{\partial \bar{x}} + \frac{\partial \bar{v}_1}{\partial \bar{y}} \right\} + 2 P_{23} \frac{\partial \bar{u}_1}{\partial \bar{y}} + P_{28} \frac{\partial \bar{u}_1}{\partial \bar{x}} + P_{23} \frac{\partial \bar{u}_2}{\partial \bar{x}} + P_{28} \frac{\partial \bar{u}_2}{\partial \bar{y}} \\ + H_1 \bar{W}_1^2 + H_2 v_1 = 3 v_0 \bar{W}_1 \frac{\partial \bar{W}_1}{\partial \bar{t}} \end{aligned} \quad (38)$$

and the boundary conditions become

$$\begin{aligned} \bar{u}_1 = \sqrt{\frac{\pi}{2}} \tan \alpha \bar{W}_0 \left\{ f_p v_1 + f \frac{1}{2 \bar{W}_0^2} \bar{W}_1^2 \right\} \\ \bar{u}_2 = 0 \end{aligned} \quad (39)$$

$$\int_0^\beta v_1 d\bar{y} = 0 \quad (40)$$

$$S_1 \frac{\partial \bar{u}_1}{\partial \bar{y}} + S_1 \frac{\partial \bar{u}_2}{\partial \bar{x}} + A_1 \frac{d \bar{u}_0}{d \bar{y}} v_1 + \frac{S_1}{2 \bar{W}_0^2} \frac{d \bar{u}_0}{d \bar{y}} T_1 = 0 \quad (41)$$

$$2 \bar{W}_0 \bar{W}_1 \frac{\partial \bar{W}_1}{\partial \bar{y}} + \frac{d \bar{W}_0}{d \bar{y}} \bar{W}_0^2 = 0 \quad (42)$$

$$\bar{W}_1^2 = 0 \quad (43)$$

Equations (39)_{1,2} are the boundary conditions at $\bar{y} = \beta$ on the inclined plane and equations (41), (42) and (43) are the boundary conditions at $\bar{y} = 0$ at the free surface. We shall assume the disturbances to be spatially periodic. That is the perturbed quantities have the form

$$v_1 = v_p(\bar{y}) e^{s\bar{t}} e^{i\sigma\bar{x}} \quad (44)$$

$$\bar{u}_1 = U_{px}(\bar{y}) e^{s\bar{t}} e^{i\sigma\bar{x}} \quad (45)$$

$$\bar{u}_2 = U_{py}(\bar{y}) e^{s\bar{t}} e^{i\sigma\bar{x}} \quad (46)$$

$$\bar{W}_1 = W_p(\bar{y}) e^{s\bar{t}} e^{i\sigma\bar{x}} \quad (47)$$

Where, v_p is the amplitude of the volume distribution function,

U_{px} and U_{py} are the amplitudes of the perturbed velocity,

W_p is the amplitude of the granular temperature,

i is the imaginary number such that $i^2 = -1$,

σ is the wave number (real) and

$s = \zeta + i \omega$.

It would be appropriate at this juncture to observe that there is no equivalent of Squire's theorem for parallel flows of granular materials and thus we should in general study three dimensional disturbances. However, the equations, even for the class of disturbances (44) through (47), are so complicated that at this stage we wish to restrict ourselves to this case. Substituting (44) through (47) into the equations (35), (36), (37) and (38), and the corresponding boundary conditions (39) through (43) we end up with

$$v_0 \frac{dU_{py}}{d\bar{y}} + \frac{dv_0}{d\bar{y}} U_{py} + i \sigma v_0 U_{px} + (i \sigma \bar{u}_0 - s) v_p = 0 \quad (48)$$

$$\begin{aligned}
& P_1 \frac{d^2 U_{px}}{d\bar{y}^2} + i \frac{4}{3} \sigma P_1 \frac{dU_{py}}{d\bar{y}} + P_2 \frac{dU_{px}}{d\bar{y}} + P_3 \frac{dW_p}{d\bar{y}} + P_5 \frac{dv_p}{d\bar{y}} \\
& - \left\{ \frac{4}{3} \sigma^2 P_1 - i 20 \sqrt{\pi} \sigma v_0 \bar{u}_0 + 20 \sqrt{\pi} v_0 s \right\} U_{px} + 20 \sqrt{\pi} v_0 \frac{d\bar{u}_0}{d\bar{y}} U_{py} \\
& + \left\{ P_6 + i 20 \sqrt{\pi} S_{14} \sigma \bar{W}_0^2 \right\} v_p + \left\{ P_4 + i 20 \sqrt{\pi} S_2 \right\} W_p = 0
\end{aligned} \tag{49}$$

$$\begin{aligned}
& - \frac{P_1}{15 \sqrt{\pi}} \frac{d^2 U_{py}}{d\bar{y}^2} + i \frac{P_1 \sigma}{15 \sqrt{\pi}} \frac{dU_{px}}{d\bar{y}} - P_8 \frac{dU_{py}}{d\bar{y}} - P_{10} \frac{dv_p}{d\bar{y}} - S_2 \frac{dW_p}{d\bar{y}} \\
& + \left\{ i \sigma v_0 \bar{u}_0 - \frac{P_1 \sigma^2}{20 \sqrt{\pi}} - v_0 s \right\} U_{py} + P_9 v_p + \left\{ i \sigma P_7 - P_{11} \right\} W_p = 0
\end{aligned} \tag{50}$$

$$\begin{aligned}
& \frac{P_{21}}{2} \frac{d^2 W_p}{d\bar{y}^2} + \left\{ i \sigma P_{21} + H_3 \right\} \frac{dW_p}{d\bar{y}} + 2 P_{23} \frac{dU_{px}}{d\bar{y}} + P_{28} \frac{dU_{py}}{d\bar{y}} + P_{19} \frac{dv_p}{d\bar{y}} + \left\{ H_2 + i \sigma P_{19} \right\} v_p \\
& + i \sigma P_{28} U_{px} + i \sigma P_{23} U_{py} + \left\{ i \sigma H_3 + H_1 - \frac{\sigma^2 P_1}{2} - \frac{3}{2} v_0 s \right\} W_p = 0
\end{aligned} \tag{51}$$

and the boundary conditions become

$$\begin{aligned}
U_{px} &= \sqrt{\frac{\pi}{2}} \tan \alpha \bar{W}_0 \left\{ f_p v_p + f \frac{1}{2 \bar{W}_0^2} W_p \right\} \\
U_{py} &= 0
\end{aligned} \tag{52}$$

$$\int_0^\beta v_p d\bar{y} = 0 \tag{53}$$

$$S_1 \frac{dU_{px}}{d\bar{y}} + i \sigma S_1 U_{py} + A_1 \frac{d\bar{u}_0}{d\bar{y}} v_p + \frac{S_1}{2 \bar{W}_0^2} \frac{d\bar{u}_0}{d\bar{y}} W_p = 0 \tag{54}$$

$$2 \bar{W}_0 \frac{dW_p}{d\bar{y}} + \frac{d\bar{W}_0}{d\bar{y}} \bar{W}_p = 0 \tag{55}$$

$$W_p = 0 \tag{56}$$

Equations (52)_{1,2} are the boundary conditions at $\bar{y} = \beta$ on the inclined plane and equations (54), (55) and (56) are the boundary conditions at $\bar{y} = 0$ at the free surface. Equation (48) is used in equations (49) through (51) so that v_p is eliminated from the three equations and the boundary conditions. In the above equations we set $s = 0$ as we

are interested in the marginal stability curve and we end up with three second order ordinary differential equations. Also, in the present problem the free surface is fixed, *i.e.* same as the basic solution domain ($\bar{y} = 0$). But, in the real problem the perturbed domain will not be the same as the basic solution domain. The final equations are

$$P_1 \frac{d^2 U_{px}}{d\bar{y}^2} + i G_7 \frac{d^2 U_{py}}{d\bar{y}^2} + G_8 \frac{dU_{px}}{d\bar{y}} + (G_{10} + i G_9) \frac{dU_{py}}{d\bar{y}} + P_3 \frac{dW_p}{d\bar{y}} + (G_{11} + i G_{12}) U_{px} + (G_{13} + i G_{14}) U_{py} + (P_4 + i G_{15}) W_p = 0 \quad (57)$$

$$(G_{16} + i G_{17}) \frac{d^2 U_{py}}{d\bar{y}^2} + (G_{18} + i G_{19}) \frac{dU_{py}}{d\bar{y}} + (G_{20} + i G_{21}) \frac{dU_{px}}{d\bar{y}} - S_2 \frac{dW_p}{d\bar{y}} + (G_{22} + i G_{23}) U_{py} + G_{24} U_{px} + (G_{25} + i G_{26}) W_p = 0 \quad (58)$$

$$G_{27} \frac{d^2 W_p}{d\bar{y}^2} + i G_{30} \frac{d^2 U_{py}}{d\bar{y}^2} + (G_{31} + i G_{32}) \frac{dU_{py}}{d\bar{y}} + G_{33} \frac{dU_{px}}{d\bar{y}} + (G_{28} + i G_{29}) \frac{dW_p}{d\bar{y}} + (G_{34} + i G_{35}) W_p + (G_{36} + i G_{37}) U_{py} + (G_{38} + i G_{39}) U_{px} = 0 \quad (59)$$

and the boundary conditions become

$$U_{px} = \sqrt{\frac{\pi}{2}} \tan \alpha \bar{W}_0 \left\{ f_p v_p + f \frac{1}{2 \bar{W}_0^2} W_p \right\} \quad (60)$$

$$U_{py} = 0$$

$$\int_0^{\beta} \left\{ i v_0 \frac{dU_{py}}{d\bar{y}} + i \frac{dv_0}{d\bar{y}} U_{py} - \sigma v_0 U_{px} \right\} d\bar{y} = 0 \quad (61)$$

$$S_1 \frac{dU_{px}}{d\bar{y}} + i \frac{A_1 v_0}{\sigma u_0} \frac{d\bar{u}_0}{d\bar{y}} \frac{dU_{py}}{d\bar{y}} + i \left\{ \sigma S_1 + \frac{A_1}{\sigma u_0} \frac{d\bar{u}_0}{d\bar{y}} \frac{dv_0}{d\bar{y}} \right\} U_{py} - \frac{A_1 v_0}{u_0} \frac{d\bar{u}_0}{d\bar{y}} U_{px} + \frac{S_1}{2 \bar{W}_0^2} \frac{d\bar{u}_0}{d\bar{y}} W_p = 0 \quad (62)$$

$$2 \bar{W}_0 \frac{dW_p}{d\bar{y}} + \frac{d\bar{W}_0}{d\bar{y}} W_p = 0 \quad (63)$$

$$W_p = 0 \quad (64)$$

Equations (60)_{1,2} are the boundary conditions at $\bar{y} = \beta$ on the inclined plane and equations (62), (63) and (64) are the boundary conditions at $\bar{y} = 0$ at the free surface.

where,

$$\beta = \frac{L}{\sigma}$$

$$S_{10} = \frac{16 v_0}{(1-v_0)^3} \{ 4 + 3 v_0 (1-v_0) \}$$

$$S_{11} = \frac{\pi}{6} \frac{(5-7v_0+11v_0^2-5v_0^3)}{(2-v_0)(1-v_0)^2} (15v_0^2-22v_0+7)$$

$$S_{12} = \left\{ 1 - \frac{(7-4v_0)(5-7v_0+11v_0^2-5v_0^3)}{2(15v_0^2-22v_0+7)(2-v_0)(1-v_0)} \right\}$$

$$S_{13} = \frac{2 v_0}{(1-v_0)^3} \{ 4 + 3 v_0 (1-v_0) \}$$

$$S_{14} = \frac{\pi}{32} \frac{(5-3v_0+9v_0^2-5v_0^3)}{(2-v_0)(1-v_0)^3} (15v_0^2-18v_0+3)$$

$$S_{15} = \left\{ 1 - \frac{(7-4v_0)(5-3v_0+9v_0^2-5v_0^3)}{2(15v_0^2-18v_0+3)(2-v_0)(1-v_0)} \right\}$$

$$A_1 = S_{10} - S_{11} S_{12}$$

$$A_2 = 1 + \frac{2 v_0 (2+2v_0-v_0^2)}{(1-v_0)^4} + \frac{2 v_0 (2-v_0)}{(1-v_0)^3}$$

$$A_3 = S_{13} - S_{14} S_{15}$$

$$S_{16} = \frac{dA_1}{dv_0}$$

$$S_{17} = \frac{dA_2}{dv_0}$$

$$S_{18} = \frac{dA_3}{dv_0}$$

$$\phi_1 = \frac{\pi}{192} \frac{(5-7v_0+11v_0^2-5v_0^3)^2}{(2-v_0)^2 v_0^2}$$

$$\phi_2 = \frac{4}{v_0(2-v_0)} - \frac{2(7-22v_0+15v_0^2)}{(5-7v_0+11v_0^2-5v_0^3)}$$

$$\phi_3 = \frac{(1-v_0)^3}{2v_0(2-v_0)}$$

$$\phi_4 = \frac{3}{(1-v_0)} + \frac{2(1-v_0)}{v_0(2-v_0)}$$

$$\phi_5 = \frac{\phi_3 \phi_4}{1+\phi_3} + \frac{\phi_1 \phi_2}{1+\phi_1}$$

$$\phi_6 = \frac{5(1-v_0)^3}{2v_0(2-v_0)}$$

$$\zeta = \frac{2}{3} \{ 2 \csc^2 \theta (1 - \cos \theta) - \cos \theta \}$$

$$\phi_7 = \frac{5\bar{a}(1+\phi_3)}{2^{3/2}(1+\phi_1)} \left\{ \frac{\pi \phi_4 \phi_6}{12 \sqrt{2} \bar{a}} + \phi_5 + \frac{\pi(1+\phi_6)\phi_5}{12 \sqrt{2} \bar{a}} \right\}$$

$$f_p = \frac{\phi_7 \sin^2 \theta}{\zeta} - \frac{5\bar{a} \phi_5 (1+\phi_3)}{\sqrt{2} (1+\phi_1)}$$

$$P_1 = \bar{W}_0 S_1$$

$$P_2 = \bar{W}_0 S_6 \frac{dv_0}{d\bar{y}} + S_1 \frac{d\bar{W}_0}{d\bar{y}}$$

$$P_3 = \frac{S_1}{2\bar{W}_0} \frac{d\bar{u}_0}{d\bar{y}}$$

$$P_4 = \frac{S_6}{2\bar{W}_0} \frac{dv_0}{d\bar{y}} \frac{d\bar{u}_0}{d\bar{y}} + \frac{S_1}{2\bar{W}_0} \frac{d^2 \bar{u}_0}{d\bar{y}^2} - \bar{W}_0^2 \frac{d\bar{u}_0}{d\bar{y}} \frac{d\bar{W}_0}{d\bar{y}}$$

$$P_5 = A_1 \bar{W}_0 \frac{d\bar{u}_0}{d\bar{y}}$$

$$P_6 = A_1 \frac{d\bar{u}_0}{d\bar{y}} \frac{d\bar{W}_0}{d\bar{y}} + A_1 \bar{W}_0 \frac{d^2 \bar{u}_0}{d\bar{y}^2} + \bar{W}_0 S_{16} \frac{dv_0}{d\bar{y}} \frac{d\bar{u}_0}{d\bar{y}} + 20 \sqrt{\pi} \sin \alpha$$

$$P_7 = \frac{S_1}{40 \sqrt{\pi} \bar{W}_0} \frac{d\bar{u}_0}{d\bar{y}}$$

$$P_8 = \frac{S_1}{15 \sqrt{\pi}} \frac{d\bar{W}_0}{d\bar{y}} + \frac{\bar{W}_0 S_6}{15 \sqrt{\pi}} \frac{d\bar{v}_0}{d\bar{y}}$$

$$P_9 = -2 A_2 \bar{W}_0 \frac{d\bar{W}_0}{d\bar{y}} - \bar{W}_0^2 S_{17} \frac{d\bar{v}_0}{d\bar{y}} + \cos \alpha$$

$$P_{10} = A_2 \bar{W}_0^2$$

$$P_{11} = S_3 \frac{d\bar{v}_0}{d\bar{y}}$$

$$P_{12} = \frac{S_7}{\sqrt{\pi} \bar{W}_0} \left\{ \bar{W}_0 \frac{d^2 \bar{W}_0}{d\bar{y}^2} + \left\{ \frac{d\bar{W}_0}{d\bar{y}} \right\}^2 \right\}$$

$$P_{13} = \frac{S_8}{\sqrt{\pi}} \frac{d\bar{W}_0}{d\bar{y}} \frac{d\bar{v}_0}{d\bar{y}}$$

$$P_{14} = \frac{S_7}{\sqrt{\pi} \bar{W}_0} \left\{ \frac{d\bar{W}_0}{d\bar{y}} \right\}^2$$

$$P_{15} = \frac{2 S_7}{\sqrt{\pi}} \frac{d\bar{W}_0}{d\bar{y}}$$

$$P_{16} = \frac{2 A_3}{\sqrt{\pi}} \bar{W}_0 \left\{ \frac{d\bar{W}_0}{d\bar{y}} \right\}^2$$

$$P_{17} = \frac{2 A_3}{\sqrt{\pi}} \bar{W}_0 \left\{ \bar{W}_0 \frac{d^2 \bar{W}_0}{d\bar{y}^2} + \left\{ \frac{d\bar{W}_0}{d\bar{y}} \right\}^2 \right\}$$

$$P_{18} = \frac{2 \bar{W}_0^2}{\sqrt{\pi}} S_{18} \frac{d\bar{v}_0}{d\bar{y}} \frac{d\bar{W}_0}{d\bar{y}}$$

$$P_{19} = \frac{2 \bar{W}_0^2}{\sqrt{\pi}} A_3 \frac{d\bar{W}_0}{d\bar{y}}$$

$$P_{20} = \frac{2 S_8 \bar{W}_0}{\sqrt{\pi}} \frac{d\bar{v}_0}{d\bar{y}}$$

$$P_{21} = \frac{2 S_7}{\sqrt{\pi}} \overline{W}_0$$

$$P_{22} = \frac{S_1}{40 \sqrt{\pi} \overline{W}_0} \left\{ \frac{d\overline{u}_0}{d\overline{y}} \right\}^2$$

$$P_{23} = \frac{S_1 \overline{W}_0}{20 \sqrt{\pi}} \frac{d\overline{u}_0}{d\overline{y}}$$

$$P_{24} = \frac{A_1 \overline{W}_0}{20 \sqrt{\pi}} \left\{ \frac{d\overline{u}_0}{d\overline{y}} \right\}^2$$

$$P_{25} = \frac{S_1}{15 \sqrt{\pi}} \overline{W}_0$$

$$P_{26} = \frac{12 \overline{W}_0^3}{\sqrt{\pi}} A_4$$

$$P_{27} = \frac{18 \overline{W}_0}{\sqrt{\pi}} S_9$$

$$P_{28} = S_2 \overline{W}_0^2$$

$$H_1 = P_{12} + P_{13} - P_{14} + P_{22} - P_{27}$$

$$H_2 = P_{16} + P_{16} + P_{18} + P_{24} - P_{26}$$

$$H_3 = 2 P_{15} + P_{20}$$

$$G_1 = \frac{v_0}{\sigma \overline{u}_0}$$

$$G_2 = \frac{2}{\sigma \overline{u}_0} \frac{dv_0}{d\overline{y}} - \frac{v_0}{\sigma \overline{u}_0^2} \frac{d\overline{u}_0}{d\overline{y}}$$

$$G_3 = \frac{v_0}{\overline{u}_0}$$

$$G_4 = \frac{1}{\sigma \overline{u}_0} \frac{d^2 \overline{v}_0}{d\overline{y}^2} - \frac{1}{\sigma \overline{u}_0^2} \frac{dv_0}{d\overline{y}} \frac{d\overline{u}_0}{d\overline{y}}$$

$$G_5 = \frac{1}{\bar{u}_0} \frac{dv_0}{d\bar{y}} + \frac{v_0}{\bar{u}_0^2} \frac{d\bar{u}_0}{d\bar{y}}$$

$$G_6 = \frac{1}{\sigma \bar{u}_0} \frac{dv_0}{d\bar{y}}$$

$$G_7 = P_5 G_1$$

$$G_8 = P_2 - P_5 G_3$$

$$G_9 = \frac{4}{3} \sigma P_1 + P_5 G_2 + P_6 G_1$$

$$G_{10} = -20 \sqrt{\pi} \sigma S_{14} \bar{W}_0^2 G_1$$

$$G_{11} = -\frac{4}{3} \sigma^2 P_1 - P_5 G_5 - P_6 G_3$$

$$G_{12} = -20 \sqrt{\pi} \sigma S_{14} \bar{W}_0^2 G_3 + 20 \sqrt{\pi} \sigma v_0 \bar{u}_0$$

$$G_{13} = -20 \sqrt{\pi} \sigma A_2 \bar{W}_0^2 G_6 + 20 \sqrt{\pi} v_0 \frac{d\bar{u}_0}{d\bar{y}}$$

$$G_{14} = P_6 G_6 + P_5 G_4$$

$$G_{15} = 20 \sqrt{\pi} \sigma S_2$$

$$G_{16} = -\frac{P_1}{15 \sqrt{\pi}}$$

$$G_{17} = -P_{10} G_1$$

$$G_{18} = -P_8$$

$$G_{19} = -P_{10} G_2 + P_9 G_1$$

$$G_{20} = P_{10} G_3$$

$$G_{21} = \frac{\sigma P_1}{15 \sqrt{\pi}}$$

$$G_{22} = -\frac{\sigma^2 P_1}{20 \sqrt{\pi}}$$

$$G_{23} = P_9 G_6 + \sigma v_0 \bar{u}_0 - P_{10} G_4$$

$$G_{24} = P_{10} G_5 - P_9 G_3$$

$$G_{25} = -P_{11}$$

$$G_{26} = \sigma P_7$$

$$G_{27} = \frac{P_{21}}{2}$$

$$G_{28} = H_3$$

$$G_{29} = \sigma P_{21}$$

$$G_{30} = G_1 P_{19}$$

$$G_{31} = -\sigma P_{19} G_1 + P_{28}$$

$$G_{32} = H_2 G_1 + G_2 P_{19}$$

$$G_{33} = -G_3 P_{19} + 2 P_{23}$$

$$G_{34} = H_1 - \frac{\sigma^2 P_{21}}{2}$$

$$G_{35} = \sigma H_3$$

$$G_{36} = -\sigma P_{19} G_6$$

$$G_{37} = H_2 G_6 + \sigma P_{23} + G_4 P_{19}$$

$$G_{38} = -G_3 H_2 - G_5 P_{19}$$

$$G_{39} = -\sigma G_3 P_{19} + \sigma P_{28}$$

FUTURE WORK

The system of equations (57), (58) and (59) with the boundary conditions (60) through (64) are to be solved numerically to obtain marginal stability curves. For the base solution we use the approximate solution due to Richman and Marciniec (1990) in the stability analysis.

Figure 1. Flow Down An Inclined Plane

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