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**SAMPLING THE FERMI DISTRIBUTION
FOR β -DECAY ENERGY INPUT TO EGS4***

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Abstract

A general method is presented for sampling the kinetic energy of electrons emitted during the β -decay process. Two computer codes (BETACDF and BETASAM) have been created to demonstrate and check the sampling scheme. The main purpose of this exercise is to come up with a convenient way to incorporate β -spectra sampling as a front-end to the EGS4 code. This should aid in the solution of a number of problems of current interest, ranging from the design of detectors for experiments in search of 17-keV neutrinos, to establishing a better understanding of the role of β rays in the dosimetry around SLAC beam devices.

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Introduction

We have developed a general method for sampling the kinetic energy of electrons emitted during the β -decay process (allowed spectra only). The main purpose of this exercise is to come up with a convenient way to incorporate β -spectra sampling as a front-end to the EGS4 code. This will facilitate the solution of a number of problems of current interest, ranging from the design of detectors for experiments in search of 17-keV neutrinos, to establishing a better understanding of the role of β rays in the dosimetry around SLAC beam devices.

The Allowed Spectra

The Fermi theory^[1] of nuclear β decay has been extensively reviewed by Evans^[2]. The basic formula for the energy spectrum of the electrons can be expressed (in $c=1$ units) as

$$\frac{dN}{d\gamma} = C \frac{\gamma}{\eta} (\gamma_{max} - \gamma)^2 \eta^2 F(Z, \eta) , \quad (1)$$

where

$F(Z, \eta)$ = factor accounting for the nuclear coulomb potential,

$\gamma = E/m = 1 + T/m$ (and $\gamma_{max} = 1 + T_{max}/m$),

$\eta = p/m = \sqrt{\gamma^2 - 1}/m$,

p = electron momentum = $\sqrt{E^2 - m^2}$,

E = total energy = $T + m$,

T = kinetic energy (T_{max} = endpoint K.E.),

m = electron rest mass (0.511 MeV),

Z = atomic number of daughter isotope,

C = constant (for any given Z).

The factor $(\gamma_{max} - \gamma)^2 \eta^2$ is referred to as the *statistical factor* and is primarily responsible for the characteristic “bell shape” of the momentum spectrum.

The emitted β ray experiences a coulomb force while it is near the nucleus—classically, β^- particles are decelerated and β^+ particles are accelerated by the positively charged nucleus. If the effect of screening by the atomic electrons is not taken into account, this coulombic factor is given by

$$F(Z, \eta) = \phi(Z) \eta^{2S} e^{\pm \pi \delta} |\Gamma(1 + S + i\delta)|^2 = F(Z, \gamma) , \quad (2)$$

for a point charge, where

$$\phi(Z) = (4\pi m R/h)^{2S} [\Gamma(3)/\Gamma(3 + 2S)]^2 (1 + S/2),$$

$$S = \sqrt{1 - \alpha^2 Z^2} - 1,$$

$$\delta = \alpha Z \gamma / \eta,$$

$$\alpha = \text{fine structure constant} = 1/137,$$

$$R = \text{nuclear radius} = 1.37 A^{1/3} \times 10^{-13} \text{cm},$$

$$A = \text{atomic mass of daughter},$$

$$h = \text{Planck's constant}.$$

and where the $\pm$ sign applies to the spectra of β^- and β^+ particles, respectively.

The above expression for $F(Z, \eta)$ involves the use of the Γ function of a complex argument, a function which prior to 1952 was quite formidable to evaluate. To assist in the theoretical analysis of experimental data on β -ray spectra, Fano^[3] presented a set of tables giving the value of a related expression, the so-called "Fermi Function", defined as

$$f(Z, \eta) = \eta^2 F(Z, \eta) / \phi(Z) . \quad (3)$$

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The tabulation of $f(Z, \eta)$ covered all of the values of Z that were expected to be encountered in practice, with values of η sufficient to enable the reader to readily interpolate at intermediate values. To facilitate in this regard, use was made of the following mathematical identity

$$|\Gamma(a + ib)|^2 = |\Gamma(a)|^2 \prod_{n=0}^{\infty} [1 + b^2(n + a)^{-2}]^{-1} \quad (4)$$

for the Γ function of a complex argument^{*}.

In addition to the coulomb effect of the nucleus, the energy distribution is affected by the electrostatic field of the atomic electrons—i.e., *screening* reduces the emission probability for electrons and increases it for positrons. To account for this, Konopinski and Rose^[4] recommend replacing $F(Z, \gamma)$ with

$$F(Z, \gamma \mp V_o) \left[\frac{(\gamma \mp V_o)^2 - 1}{\gamma^2 - 1} \right]^{1/2} \left(\frac{\gamma \mp V_o}{\gamma} \right), \quad (5)$$

where, according to the Fermi-Thomas model, $V_o \approx 1.13\alpha^2 Z_p^{4/3}$ is the shift in the potential energy (in units of m) at the nucleus caused by the screening[†], and where the $\mp$ sign applies to the spectra of β^- and β^+ particles, respectively. Equation (1) then becomes

$$\begin{aligned} \frac{dN}{d\gamma} &= C \frac{\gamma}{\eta} (\gamma_{\max} - \gamma)^2 \eta^2 F(Z, \eta_o) \left[\frac{\gamma_o^2 - 1}{\gamma^2 - 1} \right]^{1/2} \left(\frac{\gamma_o}{\gamma} \right) \\ &= C \gamma_o \eta_o (\gamma_{\max} - \gamma)^2 F(Z, \eta_o), \end{aligned} \quad (6)$$

where $\gamma_o \equiv \gamma \mp V_o$ and $\eta_o = \sqrt{\gamma_o^2 - 1}$

* The Γ function of a complex argument is easily calculated on present-day computers using a simple iteration algorithm together with this identity, so the Fano tabulation is really no longer needed.

† Note that the appropriate atomic number in this expression is that of the *parent* isotope since the atomic structure does not react instantly to the change of atomic number.

The Sampling Scheme

Probability Density Function (PDF). Equation (6) is the PDF from which the kinetic energy of the β ray will be sampled. If we first define the variable $x = T/T_{max}$ over the range $x_{min} \leq x \leq x_{max} = 1.0$, the PDF (for $e^\pm$) can be written

$$\begin{aligned} p(x) &\equiv \frac{dN}{dx} = \left(\frac{T_{max}}{m} \right) \frac{dN}{d\gamma} \\ &= C \gamma_o \eta_o (1-x)^2 \eta_o^{2S} e^{\pm \pi \delta_o} \prod_{n=0}^{\infty} [1 + \delta_o^2 (n+1+S)^{-2}]^{-1}, \end{aligned} \quad (7)$$

where $\delta_o = \alpha Z \gamma_o / \eta_o$ and the constant, C , has been redefined. The smallest value of x_{min} is either set by the application—*e.g.*, the cutoff energy, PCUT, in the case of EGS4—or by

$$x_{min} = \begin{cases} 0, & \text{for } \beta^+; \\ mV_o/T_{max}, & \text{for } \beta^-. \end{cases} \quad (8)$$

Cummulative Distribution Function (CDF). It is virtually impossible to sample the PDF defined by Eqn.(7) using the direct method—*i.e.*, by analytic inversion. Instead, a computer program called BETACDF was written that constructs a fine-mesh table for the CDF with the normalization

$$\int_{x_{min}}^{1.0} p(x) dx = 1.0.$$

Sampling is then performed by selecting a random number, R_n , drawn uniformly on the interval $[0, 1]$, to obtain the value of x by table lookup from the CDF (with linear interpolation performed within the mesh interval selected).

A second program called BETASAM^{*} has been written to demonstrate the method of sampling and to provide an easy way to show that the sampling looks correct. As an example, Figs. 1 and 2 compare the sampled and theoretical PDFs for both the β^- and β^+ decay modes, respectively, of ^{64}Cu .

The theoretical PDFs are shown as a solid curves in the figures and the histograms are the sampled distributions for 10,000 events in both cases. These figures compare favorably with those of Reitz^[5] as shown on p. 538 of Evans^[2].

* BETASAM and BETACDF are Mortran3 programs that, together with their corresponding input files, are maintained by the SLAC Radiation Physics Department. On SLACVM they are located on the EGS4 disk and can be accessed with the command GIME EGS4. Documentation is provided within the code listings themselves.

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