Quark-Antiquark Regge Trajectories in Large $N_c \, QCD^*$

MICHAEL MCGUIGAN AND CHARLES B. THORN

Department of Physics
University of Florida, Gainesville, FL 32611

ABSTRACT

We apply methods developed by Lovelace, Lipatov, and Kirschner to evaluate the leading Regge trajectories $\alpha(t)$ with the quantum numbers of nonexotic quark-antiquark mesons at $N_c = \infty$ in the limit $t \rightarrow -\infty$ where renormalization group improved perturbation theory should be valid. We discuss the compatibility of nonlinear trajectories with narrow resonance approximations.

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It is unlikely that Quantum Chromodynamics, the consensus theory of strong interactions, can be exactly solved with realistic values for all parameters. However, asymptotic freedom allows the application of weak coupling techniques such as perturbation theory to obtain the predictions of QCD for processes controlled by short distance dynamics. Besides high momentum transfer collision phenomena, one can hope to use such weak coupling techniques for computing the mass spectrum of hadrons containing only very heavy quarks. But for hadrons containing light quarks and also for glueballs, strong coupling dynamics is unavoidable.

It is reasonable to first confront these strong coupling issues making as many simplifying idealizations as possible. Thus we might idealize light quarks to massless quarks and delete the heavy quarks all together. Since \( m_u, m_d \ll \Lambda_{QCD} \) this idealized theory should give an excellent approximation to the dynamics of up and down quarks, which is to say all ordinary hadronic matter. These idealizations leave us with a theory with no further free parameters. Unfortunately, making the quarks massless does not simplify the dynamics enough for an analytical treatment: the S-matrix is non-trivial in all channels including those with particle production. Furthermore, the bound state spectrum includes all nuclei as well as the lowest mass hadron in each flavor sector.

That is why 't Hooft's idea of exploiting the \( N_c \rightarrow \infty \) limit is so attractive. In this limit the scattering amplitudes involving hadrons vanish and in lowest nonvanishing order are meromorphic in the channel invariants, just as the tree approximation to a quantum field theory. Nor do the nuclei bind in this limit. Thus an exact solution in this limit really would be significantly simpler than the exact solution at \( N_c = 3 \). Even if the definitive tests of QCD must come from large scale computation, a successful analytical understanding of all hadronic phenomena to 30% accuracy would be very desirable. Unfortunately, to date efforts to evaluate the large \( N_c \) limit have failed: at least with available methods the limiting theory seems almost as intractable as the finite \( N_c \) theory. Nonetheless, we think it is worthwhile to develop as much insight into the nature of the hoped for solution as possible.
String theory started as an effort to build exactly the sort of approximation to strong interaction dynamics that is provided by large \( N_c \) QCD. Since that approach led to the “wrong” answer, we should understand how the expected properties of QCD are different from those of string theory. In string theory the Regge trajectory functions \( \alpha_{\text{string}}(t) = \alpha' t + \alpha_0 \) where \( \alpha' = 1/2\pi T_0 \) with \( T_0 \) the rest tension in the string, play a central role in the string scattering amplitudes: they appear directly as the arguments of the gamma functions which characterize the Veneziano four string function \( A_4(s,t) = g_{\text{string}}^2 \Gamma(-\alpha(s))\Gamma(-\alpha(t))/\Gamma(-\alpha(s)-\alpha(t)) \). The meromorphy of \( A_4 \) in \( s \) and \( t \) follows directly from that of the gamma functions and the exact linearity of the Regge trajectories. Since the large \( N_c \) hadron amplitudes are also meromorphic in \( s \) and \( t \), the trajectory functions themselves should be good characteristics of the similarities and differences between string theory and QCD. Also they might carry some hints about the solution of large \( N_c \) QCD.

In this letter we study the Regge trajectories of large \( N_c \) QCD in the “meson” channels (i.e. those interpolating the rotational states of quark-antiquark mesons), in the limit of large negative \( t \) where perturbative QCD should be applicable. We follow ideas and methods developed by Lovelace, Lipatov and Kirschner and Lipatov. These methods are essentially renormalization group improved calculations based on summing leading logarithmic contributions of Feynman graphs. Such methods can only give the trajectory functions in the weak coupling approximation. Since the coupling “runs” with the scale \( \lambda(-t) = N_c g_s^2(-t)/4\pi^2 \sim 12/11 \ln(-t/\Lambda_{\text{QCD}}^2) \) this means that we can obtain only the large negative \( t \) behavior of the Regge trajectories using these methods.

Our first task is to identify the leading logarithmic contributions to a scattering process involving the exchange of a \( q\bar{q} \) pair. In any gauge theory the ladder diagrams, which iterate gauge boson exchange between two fermion lines,

* Note that the large positive \( t \) behavior of the trajectories is characterized by the confining force and should be asymptotically linear \( \alpha_{\text{QCD}}(t) \sim t/2\pi k \) where \( kR \) is the confining term in the \( q\bar{q} \) interaction energy.
contribute two powers of $\ln(s/\mu^2)$ for each additional rung. Thus the leading logarithms are actually double logarithmic and dominate the single logarithms of renormalization: the leading log sums will therefore \textit{not} include running coupling effects. Thus we proceed in two steps. First we evaluate the amplitudes to double logarithmic accuracy and then incorporate renormalization effects which make the coupling run in the second step. As shown in 1967[5] for \textit{QED}, the first step typically leads to a fixed square root branch point in the angular momentum plane. For \textit{QED} processes involving the exchange of total zero charge, the leading double logs come only from the ladder sum, which produces a branch point located at $J = \sqrt{2\alpha/\pi}$ where $\alpha \approx 1/137$ is the fine structure constant. When non zero charge is exchanged, there are additional double logarithmic contributions coming from soft “bremstrahlung” photons which either form crossed rungs in the ladder or Sudakov vertex corrections. Similarly, in \textit{QCD}, as discussed by Kirschner and Lipatov[6], the double logarithmic contributions of the basic ladder diagrams of Fig. 1 are supplemented by soft gluon bremstrahlung graphs. Fortunately for us, these additional diagrams are nonleading in the $1/N_\text{c}$ expansion when the ladder structure is “hooked on” to color singlet hadron vertices and so do not enter into our calculation of the Regge trajectories of large $N_\text{c}$ \textit{QCD}. Thus the double logarithmic sum is identical to the \textit{QED} zero charge exchange case with $\lambda \equiv N_\text{c} g^2/4\pi^2$ substituted for $2\alpha/\pi$.

Since renormalization effects are neglected in the leading double logarithmic approximation, the results depend on a fixed coupling constant $\lambda$. Asymptotic freedom must at least make the location of the singularities in the angular momentum plane vary with $t$ according to the replacement $\lambda \rightarrow \lambda(-t)$, but actually the cut is changed into a distribution of Regge poles accumulating at 0 as $t \rightarrow -\infty$. This phenomenon was first uncovered by Lovelace[2] for the case of $\phi^3$ theory in 6 space-time dimensions where the accumulation point is at $J = -1$. He analysed the Bethe-Salpeter (B-S) equation with a kernel improved to include the effects

\\[ \text{Note that these hadron “form factors,” represented by the left and right hand parts of Fig. 1, involve on-shell mesons and cannot be calculated perturbatively.} \]
of asymptotic freedom. This equation produced partial wave amplitudes with only pole singularities in the angular momentum plane. Since he only considered the case $t = 0$, his results for the pole locations (Regge intercepts) were untrustworthy. (The low momentum theory is a strong coupling one and the B-S equation is not valid there.) This shortcoming was removed by Kirschner and Lipatov, who incorporated $t$ dependence in leading order and obtained $\alpha(t)$ for large negative $t$ instead of $\alpha(0)$. With large enough $t$, the effective coupling is weak, justifying the B-S equation. Earlier, Lipatov had obtained similar results for the asymptotic behavior of the pomeron (glueball) trajectory in QCD. In this note we find the corresponding asymptotic behavior of the $q\bar{q}$ trajectories in large $N_c$ QCD.

In order to incorporate the running coupling, we consider the B-S equation which sums the ladder subgraphs in Fig. 1. We represent the Green’s function for the ladder subgraphs as a matrix $\Psi_{ab}$ in the Dirac indices of the $q\bar{q}$ lines coming in at the left. We apply the Dirac operators to these two lines in the coordinate representation to obtain (for simplicity we take all $m_q = 0$)

$$\gamma \cdot \partial_t \Psi(x_1, x_2; y_1, y_2) \gamma \cdot \mathcal{D} = -\delta(x_1 - y_1)\delta(x_2 - y_2) + \lambda(x_{12}^{-2})\gamma^\mu \Psi \gamma^\nu d_{\mu\nu}(x_{12})$$

where we have followed Lovelace’s treatment of $\phi^3$, replacing the coupling constant by the running coupling $\lambda(x^{-2}) \approx -12/11 \ln(x^2\Lambda^2_{QCD})$. In a general covariant gauge we define the coordinate space propagator by $d_{\mu\nu}(x)/4\pi^2$ with

$$d_{\mu\nu}(x) = -i \int \frac{d^4p}{(2\pi)^4} \epsilon_{\mu \nu} \frac{\eta_{\mu\nu} - \zeta p_\mu p_\nu/p^2}{p^2 - i\epsilon} = \frac{(1 + \zeta)\eta_{\mu\nu}}{2x^2} + \frac{(1 - \zeta)x_\mu x_\nu}{x^4}.$$

The B-S equation is not gauge invariant, but violations of gauge invariance will be small for weak coupling. Thus if we only keep leading order answers, our results should be gauge invariant. We keep $\zeta$ arbitrary so we can confirm this. We expect this equation to be accurate when $\lambda << 1$ i.e. for $x_{12}^2\Lambda^2_{QCD} << 1$. The
singularities in the $t$ channel are controlled by the solutions of the homogeneous equation

$$\gamma \cdot \partial_1 \Psi(x_1, x_2) \gamma \cdot \overrightarrow{\partial} _2 = \lambda(x_{1,2}^{-2})\gamma^\mu \Psi(x_1, x_2) \gamma^\nu d_{\mu\nu}(x_{1,2}).$$

It is convenient to work with $CM$ and relative coordinates $r = (x_1 + x_2)/2$ and $\rho = x_1 - x_2$ and to Fourier transform with respect to $r$, whose conjugate variable is $q$ so that $q^2 = -t$. Then the homogeneous equation reads

$$\gamma \cdot \left( \frac{\partial}{\partial \rho} - \frac{iq}{2} \right) \tilde{\Psi}(\rho, q) \gamma \cdot \left( -\frac{\overrightarrow{\partial}}{\partial \rho} - \frac{iq}{2} \right) = \lambda(\rho^{-2})\gamma^\mu \tilde{\Psi}(\rho, q) \gamma^\nu d_{\mu\nu}(\rho).$$

We can only make use of this equation for small $\rho$ and large $q$ when the effective coupling associated with both scales in the problem is small.

For $q\rho << 1$, but $q, \rho^{-1} >> \Lambda_{QCD}$, the B-S equation becomes quite manageable. This limit reduces it to the $q = 0$ case, with its $O(4)$ symmetry. As in Ref. 4 one can conveniently consider the $\ell^\text{th}$ partial waves in this limit by making the ansatz

$$\tilde{\Psi}_\ell(\rho) = \frac{(\rho \cdot \xi)_{\ell-1}}{|\rho|^\ell} \left[ \frac{\rho \cdot \gamma \rho \cdot \xi}{\rho^2} f(|\rho|) + \xi \cdot \gamma g(|\rho|) \right],$$

with $\xi^n$ a fixed light-like four-vector, so that one is forming traceless symmetric tensors of rank $\ell$. Plugging this ansatz into the B-S equation then yields the pair of equations

$$[(D - \ell)^2 - 4]f + 2[D - \ell][D - \ell - 2]g = \lambda(\rho^{-2})[\xi f - (1 - \xi)g]$$

$$2\ell[D + 1]f - [D - \ell]^2 g = \lambda(\rho^{-2})g.$$  \hspace{1cm} (2)

Here $D \equiv |\rho|\partial/\partial|\rho| = \partial/\partial \ln(|\rho|\Lambda)$ is simply the scaling derivative in the magnitude of $\rho$.

Eq. (2) is a spinor version of the small $\rho$ B-S equation for the $\phi^3$ theory analyzed in Ref. 2,4. With $\lambda(\rho^{-2})$ approximated by $-6/11 \ln(|\rho|\Lambda)$, the Laplace transforms of $\lambda f$, $\lambda g$ satisfy a pair of first order differential equations in $D \equiv -2i\tilde{\nu}$. 
Unlike the single equation in the $\phi^3_0$ case which can immediately be integrated, this pair of equations is equivalent to a single component second order equation which cannot be so readily solved. However, the leading angular momentum singularity (with largest Re $\ell$) is controlled (for small $\lambda$) by $\ell$ and $\tilde{\nu}$ small compared to unity. This is made clear by considering the determinant of the coefficient matrix on the l.h.s. of these equations:

$$\det \begin{pmatrix}
(-2i\nu - \ell)^2 - 4 & 2[-2i\nu - \ell][-2i\nu - \ell - 2] \\
2\ell[-2i\nu + 1] & [-2i\nu - \ell]^2
\end{pmatrix} = -[\ell^2 + 4\nu^2][(\ell + 2)^2 + 4\nu^2].$$

In the small $\rho, \nu$ limit, the second of (2) shows that $2\ell f \approx (\lambda(\rho^{-2}) + [D - \ell]^2) g$. Inserting this approximate form for $f$ into the first equation and making the same approximations there gives $(-D^2 + \ell^2) g \approx \lambda(\rho^{-2}) g$ the Laplace transform of which gives a first order equation in $\nu$. Thus we see that this leading Regge singularity is controlled by equations independent of the gauge $\zeta$, as we anticipated for weak coupling. Looking back to the full gauge dependent small $\rho$ equations, we notice that in Landau gauge ($\zeta = 0$) one can eliminate $f$ in favor of $g$ in a $\rho$ independent way. Thus in this gauge the Laplace transformed equations are first order in $\nu$ and can be directly integrated.

We exploit this simplification by setting $\zeta = 0$ in the following. Solving the second equation for $f$ and substituting in the first we obtain

$$[\ell^2 - D^2][(\ell + 2)^2 - D^2] g = [4 - D^2 - \ell^2 - 2\ell] \lambda(\rho^{-2}) g.$$

The equation for $g$ is now quite similar to the $\phi^3_0$ case and we can repeat the steps in Ref. 4 to derive the asymptotic behavior of the Regge trajectories. The equation for $g$ is first solved by writing it in the form

$$-\ln(\rho^2 \Lambda^2)(\lambda(\rho^{-2}) g) = \frac{12}{11} \frac{4 + 4\tilde{\nu}^2 - \ell^2 - 2\ell}{[\ell^2 + 4\tilde{\nu}^2][(\ell + 2)^2 + 4\tilde{\nu}^2]}(\lambda(\rho^{-2}) g),$$

where we have put $D = -2i\tilde{\nu}$. Since $[\tilde{\nu}, -\ln(\rho^2 \Lambda^2)] = -i$ we can interpret $R \equiv -\ln(\rho^2 \Lambda^2) \to i\partial/\partial\tilde{\nu}$, integrate the equation and then transform back to the
coordinate representation to obtain
\[
\lambda(\rho^{-2})g = \int_{-\infty}^{\infty} d\nu \exp \left\{ i\nu R - \frac{12i}{11} \int_{0}^{\nu} d\nu' \frac{4 + 4\nu' - \ell^2 - 2\ell}{[\ell^2 + 4\nu'^2][(\ell + 2)^2 + 4\nu'^2]} \right\}. \tag{4}
\]

This solution of the small \( \rho \) B-S equation is, in fact, a solution of the full B-S equation for \( q = 0 \). But of course the B-S equation is only a good approximation for large \( q \) and small \( \rho \). In Ref. 4 the analogous solution for the \( \phi_4^3 \) theory is used to gain information about the large \( q \) behavior of the trajectories by noting that for \( \rho q << 1 \) but \( \rho \) not too small one can have \( \lambda(\rho^{-2}) \approx \lambda(q^2) \) for a large range of \( \rho \) (essentially because the scale dependence of \( \lambda \) is only logarithmic). Thus instead of requiring regularity of the solution at \( R = 0 \) as in Ref. 2, the solution is matched to that of the B-S equation with a \( \rho \) independent coupling taken to be \( \lambda(q^2) << 1 \). For consistency of the weak coupling approximation this matching must be imposed at large \( R \) (small \( \rho \)).

The large \( R \) behavior of (4) is exponentially damped for extremely large \( R \) but there is oscillatory behavior for \( R \) not too large. This can be extracted by finding the saddle points
\[
\nu_0^2 = \frac{3}{22R} - \frac{1 + \ell}{2} - \frac{\ell^2}{4} \pm \sqrt{\left[ \frac{3}{22R} - \frac{1 + \ell}{2} - \frac{\ell^2}{4} \right]^2 + \frac{3(4 - \ell^2 - 2\ell)}{44R} - \frac{\ell^2(\ell + 2)^2}{16}}. \tag{5}
\]

We see that there are two real values of \( \nu_0 \) provided \( R < \frac{12}{11} \left[ \frac{4 - \ell(\ell + 2)}{\ell(\ell + 2)^2} \right] \) which is consistent with large \( R \) provided \( |\ell| << 1 \). In this regime, \( \lambda g \) is well approximated by the saddle point evaluation
\[
\lambda g \approx 2\sqrt{\frac{6\pi}{11\Phi''(\nu_0)}} \cos \left[ \frac{\pi}{4} + \nu_0 R - \frac{12}{11}\Phi(\nu_0) \right], \tag{6}
\]

where \( \nu_0 \) is the positive square root of (5) taken with the plus sign, and
\[
\Phi(\nu_0) = \frac{2 - \ell^2 - \ell}{8\ell(\ell + 1)} i\ln \frac{1 - 2i\nu_0/\ell}{1 + 2i\nu_0/\ell} + \frac{\ell(\ell + 3)}{8(\ell + 1)(\ell + 2)} i\ln \frac{1 - 2i\nu_0/\ell}{1 + 2i\nu_0/\ell}.
\]

For sufficiently large \( R \), the \( R \) dependence of \( \nu_0 \) can be neglected and (6) can
be matched to the small $\rho$ solution of the B-S equation with constant coupling $\lambda(q^2)$. Scale invariance of the finite $q$ B-S equation implies that the solution is a function of $\rho|q|$. In the small $\rho$ limit it therefore has the behavior

$$C[(\rho|q|)^{-2i\nu_1} + (\rho|q|)^{+2i\nu_1}e^{i8\ell\nu_1}] = 2C'e^{i8/2} \cos \left[ \frac{\delta}{2} + \nu_1 \ln |\rho|^2 |q|^2 \right]$$

(7)

where $\nu_1$ is $\nu_0$ with $12/11\pi$ replaced by $\lambda(q^2)$.

Replacing $\nu_0$ by $\nu_1$ in (6) and comparing to (7) gives the matching condition

$$\lambda(q^2)\Phi(\nu_1) - \nu_1 = \frac{1}{\ln(q^2/\Lambda^2)} \left\{ r\pi + \frac{\delta}{2} + \frac{\pi}{4} \right\}$$

(8)

where $r = \text{integer}$. We shall find that the consistent small coupling solution of these equations gives $\ell = O(\lambda^{1/2})$ and $\nu_1 = O(\lambda^{1/3})$.

Thus $\nu_1/\ell^2 = O(\lambda^{-1/3})$ and $\nu_3/\ell^4 = O(1)$. Neglecting all terms that vanish at zero $\lambda$ in $\Phi(\nu_1)$, the matching condition simplifies to

$$\lambda(q^2)\left( \frac{\nu_1}{\ell^2} - \frac{4\nu_1^3}{3\ell^4} \right) - \nu_1 = \frac{1}{\ln(q^2/\Lambda^2)} \left\{ r\pi + \frac{\delta}{2} + \frac{\pi}{4} \right\}$$

With these approximations $\lambda \approx \ell^2(1 + 4\nu_1^3/\ell^2)$ so the leading term on the l.h.s. cancels and we are left with

$$\frac{8\nu_1^3}{3\ell^2} \approx \frac{1}{\ln(q^2/\Lambda^2)} \left\{ r\pi + \frac{\delta}{2} + \frac{\pi}{4} \right\}$$

Replacing $\ell^2$ by $\lambda(q^2)$, we thus find the Regge trajectory asymptotics

$$\alpha_\nu(t) \rightarrow -\infty \sqrt{\lambda(-t) - 4\nu_1^2}$$

$$\approx \sqrt{\lambda(-t)} \left( 1 - 2 \left[ \frac{11}{32} \right]^{2/3} \lambda^{1/3}(-t) \left( r\pi + \frac{\delta}{2} + \frac{\pi}{4} \right)^{2/3} + \cdots \right)$$

(9)

Notice that we have an infinite number of trajectories accumulating at 0 in the limit $t \rightarrow -\infty$, with almost identical behavior to those of the $\phi^3_6$ theory obtained in Ref. 4.
The phase $\delta$ is not determined from the small $\rho$ dynamics considered so far. It must be determined by the dynamics at $\rho q = O(1)$. However, except in exceptional cases, $\delta = \pi$ in the limit we are considering. This is because this limit involves $\nu \approx 0$ in (7). For $\nu = 0$ the two behaviors $\rho^{\pm 2i\nu}$ are replaced by 1 and $\ln \rho$. Generically, both behaviors will be present, and unless the coefficient of $\ln \rho$ exactly vanishes, the behavior for $\nu$ slightly different from zero must be

$$N \left( \frac{c_1}{\nu} [ (q \rho)^{-2i\nu} - (q \rho)^{2i\nu} ] + c_2(q \rho)^{-2i\nu} + c_3(q \rho)^{+2i\nu} \right)$$

with $c_\alpha$ finite at $\nu = 0$. If the $\ln \rho$ term is present at $\nu = 0$, then $c_1 \neq 0$ there and $\delta = \pi$ at $\nu = 0$. This is analogous to the generic vanishing of phase shifts at zero energy. In that analogy the case $c_1 = 0$ corresponds to a "zero energy resonance." For the $\phi^3$ case, $\delta = \pi$ was shown by explicit solution of the constant coupling B-S equation using conformal invariance.\[^4\] In gauge theories, a conformal transformation changes the gauge condition, so the B-S equation, being gauge noninvariant, is scale invariant but not conformally covariant. Lacking an explicit solution, we can only state that it is likely, but not proven, that $\delta = \pi$ for large $N_c \, QCD$.

We close with some comments about the significance of nonlinear Regge trajectories for large $N_c \, QCD$. There is a common belief\[^7\] that narrow resonance approximations require exactly linear (or at worst polynomial) Regge trajectories. However this conclusion depends on a maximal analyticity assumption that the trajectory functions are free of singularities in the $t$ plane cut on the right at threshold branch points.\[^8\] Since $\text{Im} \, \alpha(t)$, the discontinuity across the threshold cut, is proportional to the resonance widths, the trajectories would then be entire functions in the limit of zero width resonances. We have seen that the $q\bar{q}$ trajectories of large $N_c \, QCD$ approach constants as $t \to -\infty$, and confinement together with infinite $N_c$ implies linear behavior as $t \to +\infty$ as well as no threshold branch points. The inescapable conclusion is that the maximal analyticity assumption fails for the Regge trajectories of infinite $N_c \, QCD$ and there
are additional singularities in the $t$ plane. This is probably also true at $N_c = 3$ since there is no good physical basis for the absence of additional singularities. In Ref. 9 (for earlier models see also Ref. 10,11) one of us discussed some examples of narrow resonance models with nonlinear trajectories with algebraic branch points in the complex $t$ plane. Some of these models have trajectories which are asymptotically linear for large positive $t$ and approach constants at large negative $t$. Unfortunately, they approach these constants as an inverse power of $t$ rather than an inverse power of $\ln(-t)$, so they are not candidates for large $N_c$ QCD. Nonetheless, they do show that nonlinear trajectories are compatible with narrow resonances, and indicate a direction toward solving large $N_c$ QCD.

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References


8. S. Mandelstam, in *Lectures on Elementary Particles and Quantum Field*


Figure Caption  A typical large $N_c$ diagram contributing to meson scattering with the exchange of a $q\bar{q}$ ladder structure. The leading log approximation as $s \to \infty$ is the sum of graphs with an arbitrary number of gluon rungs represented by the vertical double lines.