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Convection in Type II Supernovae

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(Ph.D. Thesis)

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Convection in Type II Supernovae

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Preface

Sometimes you get the bull. Sometimes the bull gets you.

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Abstract

Results are presented here from several two dimensional numerical calculations of events in Type II supernovae. A new 2-D hydrodynamics and neutrino transport code has been used to compute the effect on the supernova explosion mechanism of convection between the neutrinosphere and the shock. This convection is referred to as “exterior convection” to distinguish it from convection beneath the neutrinosphere. The model equations and initial and boundary conditions are presented along with the simulation results.

The 2-D code was used to compute an exterior convective velocity to compare with the convective model of the Mayle and Wilson 1-D code (which includes more of the relevant physics). Results are presented from several runs with varying sizes of initial perturbation, as well as a case with no initial perturbation but including the effects of rotation. The M&W code does not produce an explosion using the 2-D convective velocity. Exterior convection enhances the outward propagation of the shock, but not enough to ensure a successful explosion.

Analytic estimates of the growth rate of the neutron finger instability are presented. It is shown that this instability can occur beneath the neutrinosphere of the proto-neutron star in a supernova explosion with a growth time of ~ 3 microseconds.

The behavior of the high entropy bubble that forms between the shock and the neutrinosphere in one dimensional calculations of supernova is investigated. It has been speculated that this bubble is a site for r -process generation of heavy elements. Two dimensional calculations are presented of the time evolution of the hot bubble

and the surrounding stellar material. Unlike one dimensional calculations, the 2-D code fails to achieve high entropies (~ 400 k per baryon) in the bubble. When run in a spherically symmetric mode the 2-D code reaches entropies of ~ 200 . When convection is allowed, the bubble reaches ~ 60 then the bubble begins to move upward into the cooler, denser material above it. Once the bubble leaves the heating region its entropy is fixed. One or more jets of cold material pour down to fill in the space left behind by the hot bubble. More work is needed on late time convection to reconcile the differences between the 2-D code and the M&W code.

Part I

Preliminaries...

Chapter 1

Introduction

Supernovae are the cataclysmic death throes of an aged massive star. For a short time a star becomes brighter than the entire galaxy it inhabits; when one occurs in or near our own galaxy the explosion and its glowing remnant are visible during the day for weeks. Astronomers have studied these objects since at least 185 A.D., when the Chinese recorded the appearance of a “guest star” that appeared suddenly and then faded away during the following year. Over the last eighteen hundred years several supernovae have been reported; the brightest cast shadows on the ground at night in 1006. The most famous was the supernova of 1054. Its remnant can be seen today as the Crab Nebula.

The men who first measured the spectra of supernovae were Zwicky, Johnson and Minkowski at Caltech in the 1930's. They observed that some supernovae showed hydrogen lines while others did not and Zwicky, *et al*, divided supernovae on that basis; a Type II supernova shows hydrogen in its spectrum, a Type I supernova does not. Zwicky and Baade, in their 1934 paper [1], suggested that supernovae are powered by collapse of the inner portions of the star to a neutron star. In 1941, Gamow and Shoenberg [12] correctly identified neutrinos as important participants in the supernova process, pointing out that rapidly escaping neutrinos will carry the heat away from inside a stellar core causing it to gravitationally collapse. In the 1960's, Fowler and Hoyle suggested that supernovae are driven by thermonuclear reactions.

In modern theory, Type Ia's are indeed driven by a thermonuclear runaway, while Type II's and Type Ib's result from the gravitational collapse of a massive star's inner core.¹

It was only in the 1960's that Stirling Colgate and his collaborators, Johnson and White, offered a detailed model for a supernova explosion mechanism. (A "supernova mechanism" is any physical effect or combination of effects that produces a successful supernova explosion.) In 1960, Colgate and Johnson [9] produced the prompt shock model, which depended entirely upon the hydrodynamic shock generated by gravitational collapse to drive the supernova explosion. Neutrino transport was not included in the prompt shock work of the early 1960's. The first prompt shock models postulated a collapse to densities of about 3×10^{13} g/cc. This is about a factor of ten down from currently accepted values, which are higher than nuclear density ($\rho_{nuc} \approx 2.4 \times 10^{14}$ g/cc). The prompt shock model was followed in 1966 by Colgate and White's neutrino transport model [10]. The collapsed core releases most of its heat in the form of neutrinos over a few seconds. If a small fraction (say one percent) of this energy is re-absorbed into material energy as the neutrinos move outward, this is sufficient to power the explosion.

Key advances have been made since the 1960's and early 70's. The effects of weak neutral currents have been included, along with scattering of neutrinos by heavy nuclei. The biggest problem the early theorists faced was the lack of computers powerful enough to run realistic calculations. This hurdle is now behind us for one dimensional modeling, though calculations in two and three dimensions generally must limit the amount of correct microphysics included or run times become impractical. This is often reflected in simplifications of the neutrino transport calculation or the equation of state or both.

In 1971, Wilson[21], and in 1979, Bethe, et al, [3] pointed out that a supernova can be separated into two regimes; beneath the neutrinosphere, where neutrino mean

¹The awkward phrase "collapse initiated supernova" has arisen to distinguish Type II and Type Ib from Type Ia. Perhaps a new taxonomy is required.

free paths are small and diffusion accurately describes neutrino transport, and outside the neutrinosphere, where the neutrinos are free streaming. In the 1970's and early 80's many people did calculations of the prompt shock, hoping that the shock produced by the initial bounce would be sufficient to remove the stellar mantle. In numerical simulations this process was unsuccessful[4]. The shock stalled at about 200 Km and became an accretion shock (an accretion shock is stationary in space; material flows through the shock but the shock does not move in the rest frame of the observer). In 1982, Wilson found that at late times (a second or so after bounce) that neutrinos escaping from the neutrinosphere can heat the material beneath the shock and revive it[22], driving a successful explosion. Later work, which included the effects of neutrino redshifts and general relativistic effects, eliminated an energetic explosion[17] but some variant of the Wilson mechanism is still the most likely solution to the supernova problem.

Over the last twenty years several numerical models of supernovae have been constructed but so far none has yielded the answer to the basic question "How does a supernova explode?" Indeed, with one exception, current realistic 1-D simulations fail to explode rather dramatically. The prompt shock stalls and eventually falls back onto the surface of the proto-neutron star. If the calculation were continued long enough the star would collapse into a black hole.

The exception is the current model of Wilson and Mayle, who incorporate a rough 1-D convection model in their simulations. A convection model is an attempt to simulate convective transport in one dimension. Usually material is removed from the bottom of the convective region and added to the top (and vice versa) using some estimate of what the convective velocity ought to be. Convection models in realistic supernova simulations are a relatively new development[17], but it is already clear that convection is a significant process. This thesis is primarily an attempt to determine what effect convection outside the neutrinosphere has on the shock, and to discover how fluid instabilities affect nucleosynthesis.

Presented in the following chapters are physical models of supernovae, the numerical methods used to implement them, the initial and boundary conditions used in the calculations, and discussions of the simulation results. Also included is a chapter on analytic results obtained for the growth rates and length scales of neutron finger convection.

Part I is a supernova overview to provide a frame of reference for the work that follows. Part II is devoted to the supernova explosion mechanism. The model is described, along with the code built to implement it, the runs and results that concern the mechanism question. Part III takes up the question of nucleosynthesis. Most of the code is described in part II, with the changes necessary for part III described there.

Chapter 2

Supernova Overview

A star begins its life when a cloud of hydrogen gas coalesces into a large sphere. The star's self gravity drives increasingly rapid contraction until the central 10% of the star's mass becomes hot enough to ignite a nuclear reaction, burning hydrogen nuclei to helium. The hydrogen burning will last approximately 10 million years. This time can be estimated as follows; 7 MeV per nucleon is given up when 4 hydrogen nuclei become one helium nucleus. A 15 solar mass ($15 M_{\odot}$) star has a luminosity of about ten thousand times that of our own sun, $L \approx 10^4 L_{\odot}$ (our own sun has a luminosity of $L_{\odot} = 4.0 \times 10^{33}$ ergs/sec), so

$$\tau_{H \rightarrow He} = \left(\frac{0.1 \times 15 M_{\odot}}{m_B} \right) \left(\frac{7 \text{ MeV}}{10^4 L_{\odot}} \right) = 1.6 \times 10^7 \text{ years}$$

where m_B is the baryon mass. When the hydrogen is used up, the core contracts until the helium is hot enough to burn to carbon.

At this point the stellar evolution will take one of two paths depending upon the star's mass. In stars of mass greater than about $8 M_{\odot}$ the central temperature and density will be great enough to cause ignition while the core electrons are still non-degenerate, i.e., the pressure in the core is mostly thermal, $P \sim \rho T$. When the core ignites the temperature increases and so does the pressure. This causes the star to expand and reach a new pressure equilibrium, lowering the central density. The drop in density lowers the reaction rate, and this feedback between density and reaction

rate regulates the nuclear burning. For stars of lesser mass, the central temperature and density will not be sufficient to cause ignition until the core is supported mainly by degenerate electron pressure. When this happens, the heat from nuclear burning is not enough to change the pressure of the core appreciably, and thus neither is the density changed. The reaction rate is not regulated as before and this can result in a runaway nuclear burn, which can tear the star apart. While this phenomenon is extremely interesting, it is not a Type II supernova. This thesis is concerned with stars of high mass ($M > 8 M_{\odot}$) that do not undergo such an episode.

The evolution of a high mass star continues after the helium is exhausted in the core. The helium core contracts until it is hot and dense enough to ignite carbon. The cycle of contracting and fusing the ash from the previous burn repeats itself. While it takes about 10 million years to burn the hydrogen core to helium, successive burn phases are shorter because less energy is released per nucleon as heavier nuclei are fused together. For example, the helium to carbon burn releases only 0.6 MeV per nucleon[13] as compared with 7 MeV per nucleon for the hydrogen burning phase. Less energy released per nucleon causes the core to contract until it reaches hydrodynamic equilibrium; it turns out that this leaves the core energy output rate about the same as it was during the hydrogen burn. The energy per reaction is about ten times lower, hence the reaction rate must be ten times faster and as a result only a million years are needed to burn the helium to carbon. This trend continues with each successive burning phase. At the end, approximately one day of burning in the silicon shell is required to make an iron core of roughly 1.5 solar masses. By the end of a massive star's life, the stellar core has taken on an onion skin structure; working from the outside inward we find concentric shells of hydrogen, helium, carbon, neon, oxygen, silicon, and finally a core of iron. Iron does not burn (being the most tightly bound nucleus) hence there is no further source of heat to support the iron core against gravitational collapse. The iron core is now essentially an iron white dwarf with a temperature of 0.5 MeV (the silicon burning temperature) and a central density of about $2 \times 10^9 \text{ gcm}^{-3}$ [17].

The iron core, which has an initial radius of about 1000 Km, cools by a combination of neutrino emission and photodissociation of iron nuclei and begins to contract. Both effects make the core unstable to collapse. As the core contracts and heats up, iron nuclei begin to fall apart. It takes ~ 125 MeV to break an iron nucleus into 13 alpha particles and 4 neutrons[17]. This energy is taken from the thermal energy that supports the core against collapse. The drop in pressure causes the core to contract a little, which raises the temperature, which causes more iron nuclei to disintegrate, and so on. This vicious circle leads to a core collapsing at nearly free fall velocities. Once dynamic collapse begins the core shrinks from 1000 Km to 50 Km in about 0.1 seconds. The core collapse time can be estimated from dimensional analysis to be $1/\sqrt{G\rho_c} \approx 0.1$ sec, given an iron core central density of $\rho_c = 10^9$ g/cc.

Electron capture on protons also drives the collapse. The reaction ($e^- + p \rightarrow n + \nu_e$) not only deprives the core of the degenerate support of an electron, but more importantly thermal energy is carried away by the neutrino. The core contracts in response to the loss of energy, which raises the density and temperature. Higher temperature and density increase the reaction rate for electron capture, and again we have an unstable situation.

The collapsing core can be separated into a homologous core, which is defined to be that part of the core where the infall speed is less than the local speed of sound, and an exterior part where the infall is supersonic. The sonic point defines the boundary between the two regions. In numerical simulations the homologous core is generally the interior $0.5 M_\odot$ of the core[27]. Inside the sonic point, matter is in communication with itself via sound waves. Material outside the homologous core has no communication with matter inside it.

The pressure in the core rises dramatically when the central density reaches nuclear matter density ($\rho_c \sim 2.5 \times 10^{14}$ g/cc). The collapse stops and rebounds at this point. The bounce moves outward from the center of the core as a sound wave initially, but when this wave reaches the sonic point it steepens and soon becomes a shock. Simulations show that the shock wave forms around 40 Km from the center of the star

($\sim 0.60 M_{\odot}$) [15]. The shock moves out to ~ 300 Km in about 200 milliseconds. The exact stall radius is model dependent and varies from 100 Km to 800 Km. The proto-neutron star continues to cool by neutrino emission and to contract. At this point we have an unshocked inner core out to about $0.5 M_{\odot}$, (this mass point, which was at 40 Km at bounce has contracted to ~ 10 Km by 200 milliseconds) with entropy about 1 (in units of S/k per baryon), enclosed by an outer mantle in which the entropy has been raised by the shock to about 6 (there is also some entropy generated by neutrino heating which is discussed below). The shock encloses about $1.5 M_{\odot}$. Outside the shock material continues to fall inward, oblivious to the events below. See Figures 2.1 and 2.2 for profiles of various physical quantities at 175 milliseconds past bounce as computed by the Wilson and Mayle 1-D code [23]. The particular run shown has a vigorous shock which has only begun to stall at about 600 Km. Neutrino capture heating has raised the entropy behind the shock to about 15.

The shock loses energy as it moves outward through the star. The shock breaks infalling iron into free baryons and alpha particles at a cost of 8.8 MeV and 2.2 MeV per nucleon respectively. Neutrinos produced in electron capture on protons (the protons are freed by shock decomposition) carry away energy the shock needs for pressure support. The shock stalls for a few hundred milliseconds; however, it may be revived by neutrinos radiating out from the *neutrinosphere*. These neutrinos, which have energies of about 10 MeV, are absorbed by the cooler shocked material and heat it, via

The stalled shock is very sensitive to the radiation flow from the surface of the proto-neutron star which is defined by the *neutrinosphere*. The neutrino luminosity increases when matter drifting down from the shock accretes onto the hot surface and loses neutrinos through electron capture. This can cause oscillations in the shock position through the following mechanism. A strong shock moves out far from the surface of the proto-neutron star. A strong shock has a low accretion rate because

Figure 2.1: Plots from the Mayle and Wilson 1-D code of (a) mass density, (b) temperature, (c) pressure (d) entropy, (e) electron number fraction, and (f) velocity as functions of radius at $t = 0.175$ seconds postbounce. The progenitor was a $20 M_{\odot}$ model from Weaver and Woosley.

Figure 2.2: Plots from the Mayle and Wilson 1-D code of (a) mass density, (b) temperature, (c) pressure (d) entropy, (e) electron number fraction, and (f) velocity as functions of mass at $t = 0.175$ seconds postbounce.

the inward velocity is small after the shock (possibly nonexistent, *i.e.*, the shock can reverse the direction of the fluid velocity). Low accretion rates mean low neutrino luminosities, hence no reinforcement of the shock through neutrino heating; without this extra support the shock weakens and falls back nearer the neutrinosphere. A weak shock does less to slow down infalling matter and the accretion rate increases, raising the neutrino luminosity, which drives neutrino heating upward. The increase in neutrino energy deposited behind the shock reinvigorates the shockwave and it moves farther out again. This oscillation has been seen in 1-D calculations with a timescale of about 0.1 seconds per cycle. Figure 5.3c shows an example of this behavior. After neutrino heating of the material behind the shock (with or without shock oscillations) pressure support is restored and the shock continues outward. The spectacular optical display occurs when the shock reaches the stellar envelope, hours after it has left the inner core.

The problem with the scenario above is that it does not quite work. In numerical calculations the neutrinos do heat the shocked material significantly but not enough; the shock rises but eventually falls back and becomes an accretion shock on the proto-neutron star. Material rains down on the proto-neutron star and accumulates. Ultimately the proto-neutron star would form a black hole. The supernova explosion mechanism is delicate. The energy available from gravitational collapse is $\sim 3 \times 10^{53}$ ergs, while about 3×10^{52} ergs is required to break up the iron core and computer simulations show that at least a few times the break-up energy is emitted as neutrinos early in the explosion. A supernova's energy budget is finely balanced between sources and sinks. The optical display which so commands our attention in the night sky is a mere side effect costing less than 0.1% of the collapse energy, only $\sim 10^{49}$ ergs. More physics must be added to the model to reproduce the supernovae we observe.

Convection is necessary for a successful explosion [24] in the Mayle and Wilson supernova calculations. When I began this thesis work the M&W code produced an explosion if the interior convection mode was turned on, and failed to produce an explosion if interior convection was turned off, regardless of whether or not the

exterior convection was allowed or not. The interior convection mode, neutron fingers, was obviously the more important problem to examine. I began preliminary studies of this instability (which led to the work presented in chapter 6) but work by Herant, Benz and Colgate seemed to indicate that the exterior convection had a much greater effect on the supernova shock than we had previously believed[14]. This led us to drop the neutron fingers project temporarily and examine exterior convection with an eye towards improving the M&W convection model.

Convection is modeled in the one dimensional M&W code by a mixing length diffusion algorithm. There are uncertainties involved in using this method. The convective velocities are only rough estimates based on dimensional analysis. Convection does not grow naturally but is imposed at full strength as soon as the unstable condition appears. Recent work by Herant, Benz and Colgate has made it clear that it is not even certain what the proper mixing length ought to be. To resolve the uncertainties in convection modeling a two dimensional hydrodynamics code has been developed to study entropy driven convection which occurs above the neutrinosphere. This work is presented in part II. The 2-D results will be used to improve the M&W convection model.

This code has also been used to examine the formation and evolution of the hot bubble beneath the shock often deemed necessary for r-process nucleosynthesis during supernovae. This is discussed in Part III.

Part II

Convection and the Supernova Explosion Mechanism

Chapter 3

A Supernova Model

With current 1-D models generally failing to explode, it seems necessary to investigate what effect 2-D motion has on the success of the shockwave. In more than one dimension an unstable fluid system can relax into a more stable configuration, increasing the binding energy and possibly helping to drive the explosion. In the supernova an unstable region exists between the entropy maximum produced by neutrino heating and the shock. Convection driven by a negative entropy gradient should occur here and transfer energy from near the neutrinosphere out to the stalled shock, helping to revive it.

The initial conditions for the 2-D model are taken from the Mayle and Wilson supernova code, which is sometimes referred to as the “spherical code”, “M&W code” or “1-D code”. The 2-D model creates a shock that behaves like the shocks that marginally fail in the M&W code. The shock behavior differs somewhat from the M&W code because the interior core is not modelled, nor are all the neutrino species included, and the equation of state is simpler than the one used in the 1-D code. Small perturbations from spherical symmetry are introduced which grow into large scale fluid motion as the problem develops. The only difference between the 2-D spherically symmetric runs and the perturbed runs are the perturbations, so the effects of convection are isolated. An interesting question is how large must the perturbations be to have a significant effect on the success of the shock motion?

What follows is a discussion of the physical model used for the supernova shock mechanism study. These simulations are designed to investigate the behavior of convection and its effects on the supernova shock. The model begins with the shock just breaking out of the neutrinosphere at 50 Km, and covers the time from 0.005 to 0.400 seconds postbounce. The shock modeled is a marginal failure in 1-D runs. The modeled region goes from slightly beneath the neutrinosphere ($1.12 M_{\odot}$ or ~ 10 Km after 0.1 seconds) to 1500 Km.

There are some key differences between the model described here and the one presented later in chapter 7, which describes late time convection and its effects on nucleosynthesis. The explosion mechanism problem has a simpler inner boundary condition, the critical action takes place over a smaller region of space, requires neutrino diffusion as well as a free streaming neutrino model, and neutrino-electron scattering is relatively unimportant. The late time convection runs require an inner boundary that changes not only position but also density, energy and pressure as a function of time, uses an outer radius about thirty times larger, neutrino diffusion is unimportant, and neutrino-electron scattering is essential to the process. The late time problem requires that Y_e , the electron fraction, be evolved in time, and a better equation of state was needed. When the late time convection work is presented later in this thesis, changes made in the model for that project will be explained. The hydrodynamics remained essentially unchanged for both projects.

3.1 Hydrodynamics

The hydrodynamics model consists of the conservation of matter, momentum and energy equations, including the effects of Newtonian gravity, rotation, and neutrino radiation. For the supernova explosion mechanism simulations it is assumed that the electron number fraction, Y_e , is $Y_e = 0.5$ everywhere, and

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \quad (3.1)$$

$$\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot (\mathbf{vS}) = -\nabla P - \rho \nabla \Phi + \frac{(\mathbf{J} \times \hat{\mathbf{r}}) \times \mathbf{J}}{\rho r^3 \sin^4 \theta} \quad (3.2)$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\mathbf{v}\varepsilon) = -P \nabla \cdot \mathbf{v} + \dot{\varepsilon}_\nu \quad (3.3)$$

$$\frac{\partial J_z}{\partial t} + \nabla \cdot (\mathbf{v}J_z) = 0. \quad (3.4)$$

are solved, where ρ is the mass density, $\mathbf{v}$ is the fluid velocity, $\mathbf{S}$ is the momentum density ($\rho\mathbf{v}$), $\mathbf{J}$ is the angular momentum density (which is constrained to point along the $\theta = 0$ axis, hence the only non-zero component is J_z), ε is the specific energy (energy per mass), $\dot{\varepsilon}_\nu$ is the change in specific energy per time due to neutrino emission and absorption, and Φ is the gravitational potential.

In calculating the gravitational potential the star is assumed to be spherically symmetric. There is no loss of generality since most of the mass of the star is beneath the neutrinosphere and is essentially spherical. The gravitational potential Φ is obtained from the Poisson equation,

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \Phi}{\partial r}) = 4\pi \bar{\rho} \quad (3.5)$$

where $\bar{\rho}(r)$ is obtained by averaging the density over θ .

3.2 Neutrino Transport

A transport model was used to follow diffusion of the neutrinos from beneath the neutrinosphere out into the free streaming region. This model can follow the progress of a packet of neutrinos as it moves through the star. A discussion of the neutrino transport model now follows.

The radiation field is modeled with an implicit 2-D flux limited diffusion scheme which calculates the emission, absorption and scattering of neutrinos inside and outside the neutrinosphere. We begin with the equation of transport

$$\frac{1}{c} \frac{\partial f_\varepsilon}{\partial t} + \boldsymbol{\Omega} \cdot \nabla f_\varepsilon = K_a \rho (b - f_\varepsilon) \quad (3.6)$$

where f_ϵ is the neutrino distribution function and K_a is an absorption opacity. The left hand side of the transport equation is the convective derivative of f_ϵ , the right hand term $K_a \rho b$ is a source term due to blackbody cooling of the matter, and $K_a \rho f_\epsilon$ is a sink term representing the absorption of neutrinos by the fluid. Scattering terms have been ignored. In thermal equilibrium,

$$f_\epsilon = b \equiv \frac{4\pi\epsilon_\nu^3}{(hc)^3} \frac{1}{e^{(\epsilon_\nu - \mu_\nu)/kT} + 1}. \quad (3.7)$$

The distribution function is in general a function of five variables; neutrino energy, direction, and position. To reduce computation time to manageable values the 2-D code follows the time evolution of the neutrino energy density rather than the neutrino distribution function. This reduces the problem to one of two independent variables. To achieve this reduction we close a series of angular moments of f_ϵ with a diffusion approximation, following [6]. This method is outlined here. The neutrino energy density F is defined by

$$F(r, \theta) = \int f_\epsilon(\epsilon, \Omega, r, \theta) d\epsilon d\Omega. \quad (3.8)$$

Integrating (3.6) over angle and energy yields

$$\frac{1}{c} \frac{\partial F}{\partial t} + \nabla \cdot \mathcal{F} = K_a \rho (B - F) \quad (3.9)$$

where $\mathcal{F}$ is the neutrino energy flux. In thermal equilibrium this yields the familiar result

$$F = B \equiv \frac{7\pi^5 (kT)^4}{30(hc)^3} = aT_\nu^4 \quad (3.10)$$

where a is 7/16 of the usual blackbody coefficient calculated for photons. Note that F refers to a single neutrino species. Taking the second moment of (3.6) with respect to angle, *i.e.*,

$$\int \left[\frac{1}{c} \frac{\partial f_\epsilon}{\partial t} + \Omega \cdot \nabla f_\epsilon = K_a \rho (b - f_\epsilon) \right] \Omega d\Omega d\epsilon \quad (3.11)$$

gives

$$\frac{1}{c} \frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot P = -K_a \rho \mathcal{F}. \quad (3.12)$$

Making the assumption that f_ϵ is nearly isotropic one gets $P = F/3$ which can be used in (3.12). Then assuming that the time derivative is small enough to ignore entirely, which is true in the isotropic case, one can solve (3.12) for $\mathcal{F}$, use this result in (3.9) and assume the matter radiates as a blackbody to obtain a diffusion equation

$$\frac{\partial F}{\partial t} = \nabla \cdot (D \nabla F) + \rho c K_a (a T_m^4 - F). \quad (3.13)$$

This equation is then solved for the radiation energy density F . The diffusion coefficient is $D = c/3K_a\rho = \lambda c/3$. Equation (3.13) is an excellent approximation when the mean free path is small compared to the dimensions of our problem but it fails badly outside that region. A flux-limited diffusion coefficient is used to force F to the free streaming solution when the mean free path is large.

The flux-limited diffusion coefficient is taken as

$$D = \frac{\lambda c}{3[1 + \frac{\lambda}{3} |\frac{\nabla F}{F}|]}. \quad (3.14)$$

The flux-limited diffusion scheme is discussed in detail in chapter 4. The diffusion equation has two distinct temperatures for absorption and emission because the neutrinos will have a spectral temperature different from the local matter temperature.

3.2.1 Opacities

Now consider a model for the opacities, K_a , K_m , where K_a is the opacity for neutrinos being absorbed by matter and K_m is the opacity for neutrino emission. The neutrinos change the internal energy of a parcel of matter through emission and absorption interactions (and neutrino-electron scattering but scattering does not become important until the r -process is discussed in part III). The relevant reactions are

The cross sections for these reactions, ignoring degeneracies, are[20]

$$\begin{aligned} \sigma_{\nu_e n} &= \sigma_0 \left(\frac{\epsilon_\nu}{m_e c^2} \right)^2 \left(\frac{1 + 3\alpha^2}{4} \right) \\ \sigma_{\bar{\nu}_e p} &= \sigma_{\nu_e n} \end{aligned} \quad (3.16)$$

where $\alpha \approx 1.2$ and $\sigma_0 = 1.7 \times 10^{-44} \text{ cm}^2$. K_a is different from K_m because K_a is a function of the energy of the neutrinos but K_m involves the temperature of the matter. The neutrino absorption reactions (which go from left to right in the above equations) have twice the cross section of the emission reactions because the final state electron has two possible spin directions as compared to only one available to the final state neutrino for emission. Integrating the above cross sections over neutrino energy weighted by a neutrino blackbody distribution gives an average cross section and including contributions from the electron chemical potential μ , we get [23]

$$\sigma(T, \mu) = \sigma_0 \left[7 \left(\frac{T}{m_e c^2} \right)^2 + \frac{2}{3} \left(\frac{\mu}{m_e c^2} \right)^2 \right] \quad (3.17)$$

where

$$\mu = \left(\frac{3(hc)^3 Y_e \rho}{8\pi m_B} \right)^{\frac{1}{3}}. \quad (3.18)$$

An opacity K is defined to be the cross section per mass, so the absorption opacity K_a is

$$K_a(\varepsilon_\nu, \mu) = \frac{\sigma_0}{m_B} \left[\left(\frac{\varepsilon_\nu}{m_e c^2} \right)^2 + \frac{2}{3} \left(\frac{\mu}{m_e c^2} \right)^2 \right] \quad (3.19)$$

and the emission opacity K_m is

$$K_m(T, \mu) = \frac{\sigma_0}{m_B} \left[7 \left(\frac{T}{m_e c^2} \right)^2 + \frac{2}{3} \left(\frac{\mu}{m_e c^2} \right)^2 \right], \quad (3.20)$$

where T_m is the local matter temperature and μ is the local electron chemical potential.

When discussing absorption of neutrinos the chemical potential involved is the potential at the last point the neutrinos and matter are closely coupled, i.e., the chemical potential at the neutrinosphere. By contrast, when discussing the *emission* of neutrinos by matter the chemical potential referred to is the electron chemical potential of the matter.

3.2.2 Heating Rates

Next, bulk heating (or cooling) in the matter from the reactions given above is computed. Assume the neutrinos have a well defined average neutrino energy, $\bar{\varepsilon}_\nu$. Given a

neutrino number density n_ν , a number density of free baryons n_B , and a cross section $\sigma(\bar{\epsilon}_\nu)$, then the heating rate per gram due to capture processes is

$$\dot{\epsilon}_{\text{cap}}^{\text{heat}} = \frac{n_B n_\nu \bar{\epsilon}_\nu c}{\rho} \sigma(\bar{\epsilon}_\nu). \quad (3.21)$$

Re-writing this by combining the number density and average energy into a neutrino energy density and forming an absorption opacity $K_a = \sigma/m_B$, where m_B is the baryon mass, one gets

$$\dot{\epsilon}_{\text{cap}}^{\text{heat}} = F c K_a(\epsilon_\nu) X_B \quad (3.22)$$

where X_B is the free baryon fraction and F is the neutrino energy density. The cooling rate is calculated similarly

$$\dot{\epsilon}_{\text{cap}}^{\text{cool}} = -\frac{n_B n_\nu \epsilon_\nu^{\text{emitted}} c}{\rho} \sigma(T). \quad (3.23)$$

Assuming the matter cools as a blackbody, the neutrino number density and emitted neutrino energy can be re-written as an energy density determined by the blackbody relation,

$$\dot{\epsilon}_{\text{cap}}^{\text{cool}} = -X_B K_m(T) a T^4 c. \quad (3.24)$$

The final equation is then

$$\dot{\epsilon}_{\text{cap}} = X_B c [K_a F - K_m a T^4]. \quad (3.25)$$

T_ν , the neutrino temperature, is taken as a function of time from the M&W code. It varies from 3.2 to 4.5 MeV over the course of the simulations. $\frac{\partial F}{\partial r} = 0$ at the inner boundary. Neutrinos are taken as free streaming at the outermost radius of the grid, and reflected at the angular boundaries.

3.2.3 Boundary Luminosity

The interior of the proto-neutron star is not modelled, but neutrino luminosity as a function of time is imposed at the inner boundary. The boundary neutrino luminosity is emitted from the inner boundary at a temperature T_c and the boundary luminosity

decays exponentially as a function of time (all times refer to the amount of time since bounce). After a few tens of milliseconds the radiation field is dominated by neutrinos emitted from matter deposited on the initial inner core. By $t = 0.030$, the radiation emitted by the accreted matter is an order of magnitude more important than the initial inner core radiation, so the core luminosity only affects the evolution of the first 30 milliseconds. The total luminosity differs from the M&W 1-D code luminosity by no more than 10%. Figure 3.1 shows the total luminosity at the outer boundary.

Figure 3.1: Total luminosity, both core and accretion in units of 10^{51} ergs/s. The core luminosity is only important for 10 to 20 milliseconds.

The time rate of change of the specific energy due to the core luminosity ($\dot{\epsilon}_c$) is given by

$$\dot{\epsilon}_{\text{core}} = \frac{L_{\text{core}}}{4\pi r^2} K_a(T_{\text{core}}) X_B \quad (3.26)$$

where $T_{\text{core}} = 4.5$ MeV and K_a is the same as in (3.21) above.

3.3 Equation of State

In the mechanism simulations the equation of state includes matter composed of helium, free baryons, electrons, photons, and electron-positron pairs. Y_e is assumed to be $\frac{1}{2}$ everywhere and a Saha equation [18] provides the ratio of free baryons to helium nuclei. An approximate form is used for the electron-positron pairs and electron degeneracy energy and pressure. The pressure, specific energy, and free baryon fraction are given by:

$$P = \frac{\rho k T}{m_B} \left[X_B + \frac{X_{He}}{4} \right] + \frac{1}{3} a T^4 \left[1 + \left(\frac{7}{4} \right) e^{-\frac{m_e c^2}{3kT}} \right] + \frac{1}{3} \sqrt{(3Y_e k T / m_B)^2 + (A \rho^{\frac{1}{3}})^2} \quad (3.27)$$

$$\epsilon = \frac{1}{m_B} \left[X_B \left(\epsilon_0^B + \frac{3}{2} k T \right) + X_{He} \left(\frac{3}{8} k T \right) \right] + \frac{a T^4}{\rho} \left[1 + \left(\frac{7}{4} \right) e^{-\frac{m_e c^2}{3kT}} \right] + \sqrt{(3Y_e k T / m_B)^2 + (A \rho^{\frac{1}{3}})^2} \quad (3.28)$$

$$\frac{X_B^4}{1 - X_B} = \frac{1}{2} \left(\frac{m_B}{\rho} \right)^3 \left(\frac{m_B k T}{2\pi \hbar^2} \right)^{\frac{9}{2}} e^{-Q/kT} \quad (3.29)$$

The entropy due to nuclei, free baryons, electrons and photons is

$$s = \frac{k}{m_B} \left[\frac{3}{2} X_B + \frac{3}{8} X_{He} + X_B \log \left(\frac{T^{3/2}}{\alpha \rho X_B} \right) + \frac{1}{4} X_{He} \log \left(\frac{32 T^{3/2}}{\alpha \rho X_{He}} \right) + \frac{4 a T^3 m_B}{3 \rho} + \right. \quad (3.30)$$

$$\left. \frac{3}{2} \log \left[\frac{3 k T}{2 m_B} + \sqrt{\left(\frac{3 k T}{2 m_B} \right)^2 + (A \rho^{\frac{1}{3}})^2} \right] \right] \quad (3.31)$$

where;

$$X_{He} = 1 - X_B \quad (3.32)$$

$$Q = (4m_B - m_{He})c^2 = 27 \text{ MeV} \quad (3.33)$$

$$\epsilon_0^B = \frac{27}{4} \text{ MeV/baryon} \quad (3.34)$$

$$\alpha = \frac{(2\pi \hbar^2)^{3/2}}{m_B^{5/2} e k^{3/2}} = 9.3 \times 10^{-13} \text{ MeV}^{\frac{3}{2}} \text{ cm}^3/\text{gram} \quad (3.35)$$

$$a = 4\sigma_B/c = 1.37 \times 10^{26} \text{ ergs}/(\text{MeV}^4 \text{ cm}^3) \quad (3.36)$$

$$A = \left(\frac{81Y^4}{\pi m_B^4} \right)^{\frac{1}{3}} \frac{hc}{8} = 1.48 \times 10^{15} \text{ ergs cm/g}^{\frac{4}{3}} \quad (3.37)$$

ϵ_0^B is the binding energy per free baryon, A is defined such that the specific energy of the electrons is $\epsilon_e = A\rho^{1/3}$ for completely degenerate relativistic electrons with $Y_e = 0.5$, and σ_B is the Stefan-Boltzmann constant. Since some matter near the inner boundary rises to near nuclear density and $Y_e = 0.5$, the following piece is added to account for the zero temperature nuclear contribution to the pressure and energy. This term represents the nuclear symmetry energy that arises because $Y_e \neq 0.5$ below the neutrinosphere.

$$P_{\text{nuclear}} = \frac{5\eta^2}{1 + 5\eta} (2.89 \times 10^{19} \text{ erg/g}) \rho_{\text{nuc}} \quad (3.38)$$

$$\epsilon_{\text{nuclear}} = (2.89 \times 10^{19} \text{ erg/g}) \log(1 + 5\eta) \quad (3.39)$$

where $\eta = \rho/\rho_{\text{nuc}}$. This approximates the M&W code's nuclear pressure and energy for the density and temperature range encountered in the exterior convection problem.

3.4 Summary

A complete list of the model equations is provided below.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \quad (3.40)$$

$$\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{S}) = -\nabla P - \rho \nabla \Phi + \frac{(\mathbf{J} \times \hat{\mathbf{r}}) \times \mathbf{J}}{\rho r^3 \sin^4 \theta} \quad (3.41)$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\mathbf{v}\epsilon) = -P \nabla \cdot \mathbf{v} + \epsilon_\nu \quad (3.42)$$

$$\frac{\partial J_z}{\partial t} + \nabla \cdot (\mathbf{v}J_z) = 0. \quad (3.43)$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \Phi}{\partial r}) = 4\pi \bar{\rho} \quad (3.44)$$

$$\frac{\partial F}{\partial t} = \nabla \cdot (D \nabla F) + \rho c K_a (a T_m^4 - F). \quad (3.45)$$

$$D = \frac{\lambda c}{3[1 + \frac{\lambda}{3}|\frac{\nabla F}{F}|]}. \quad (3.46)$$

$$\sigma_{\nu_{en}} = \sigma_0 \left(\frac{\varepsilon_\nu}{m_e c^2} \right)^2 \left(\frac{1 + 3\alpha^2}{4} \right) \quad (3.47)$$

$$\sigma_{\bar{\nu}_{ep}} = \sigma_{\nu_{en}}$$

$$\sigma(T, \mu) = \sigma_0 \left[7 \left(\frac{T}{m_e c^2} \right)^2 + \frac{2}{3} \left(\frac{\mu}{m_e c^2} \right)^2 \right] \quad (3.48)$$

$$\mu = \left(\frac{3(hc)^3 Y_e \rho}{8\pi m_B} \right)^{\frac{1}{3}}. \quad (3.49)$$

$$K_a(\varepsilon_\nu, \mu) = \sigma_{\nu_{en}}(\varepsilon_\nu) / m_B \quad (3.50)$$

$$K_m(T, \mu) = \sigma_{\nu_{en}}(T) / m_B \quad (3.51)$$

$$\dot{\varepsilon}_{\text{cap}} = X_B c [K_a F - K_m a T^4]. \quad (3.52)$$

$$\dot{\varepsilon}_{\text{core}} = \frac{L_{\text{core}}}{4\pi r^2} K_a(T_{\text{core}}) X_B \quad (3.53)$$

$$\dot{\varepsilon}_\nu = \dot{\varepsilon}_{\text{core}} + \dot{\varepsilon}_{\text{cap}} \quad (3.54)$$

$$P = \frac{\rho k T}{m_B} \left[X_B + \frac{X_{He}}{4} \right] + \frac{1}{3} a T^4 \left[1 + \left(\frac{7}{4} \right) e^{-\frac{m_e c^2}{3kT}} \right] + \frac{1}{3} \sqrt{(3Y_e kT / m_B)^2 + (A\rho^{\frac{1}{3}})^2} \quad (3.55)$$

$$\varepsilon = \frac{1}{m_B} \left[X_B (\varepsilon_0^B + \frac{3}{2} kT) + X_{He} (\frac{3}{8} kT) \right] + \frac{a T^4}{\rho} \left[1 + \left(\frac{7}{4} \right) e^{-\frac{m_e c^2}{3kT}} \right] + \sqrt{(3Y_e kT / m_B)^2 + (A\rho^{\frac{1}{3}})^2} \quad (3.56)$$

$$\frac{X_B^4}{1 - X_B} = \frac{1}{2} \left(\frac{m_B}{\rho} \right)^3 \left(\frac{m_B k T}{2\pi \hbar^2} \right)^{\frac{3}{2}} e^{-Q/kT} \quad (3.57)$$

$$P_{\text{nuclear}} = \frac{5\eta^2}{1 + 5\eta} (2.89 \times 10^{19} \text{ erg/g}) \rho_{\text{nuc}} \quad (3.58)$$

$$\varepsilon_{\text{nuclear}} = (2.89 \times 10^{19} \text{ erg/g}) \log(1 + 5\eta) \quad (3.59)$$

Chapter 4

Numerical Methods

Hydrodynamic calculations in more than one dimension require significantly more computer time and memory than 1-D codes do. Someday fairly soon, more powerful computers will make it possible to run even the most complex current physical models in 2-D or 3-D, but at present this is not so. To obtain reasonable run times it is necessary to sacrifice some of the microphysics that is included in modern 1-D supernova evolution models (for example, full multi-group neutrino transport of all six species, general relativity, nuclear isotope burn networks, and accurate equations of state for matter above nuclear density).

To study the effects of convection on the supernova explosion, a hydrodynamics and neutrino transport code has been written in spherical coordinates with two space dimensions (r, θ) and three velocity dimensions (v_r, v_θ, v_ϕ) . All physical variables are independent of ϕ .

4.1 BUBBA: a 2-D Hydro/Radiation Code

BUBBA¹ is a 2-D code, written in spherical coordinates (r, θ) . It includes a hydrodynamics package, a gravitational field solver, two neutrino transport packages, and the usual graphics and dump-restart I/O routines. The hydrodynamics and neutrino

¹After Bubba Paris, right tackle for the 1988-89 San Francisco Forty-Niners

transport calculations are discussed here. The layout of the hydrodynamics section of the code is as follows.

HYDRO:

Contains the following sections which do the hydrodynamics.

Each routine operates once on the entire grid.

GETDT

Gets timestep. When running with multiple timestep regions this routine assigns the local timestep for each zone.

EOS

Returns new pressure and temperature from current density, energy, and electron fraction.

ACCEL

Calculates changes in momentum density due to pressure gradients, artificial viscosity, gravity (which is calculated here) and rotation.

PDVWORK

Computes PdV work changes to specific energy and shock heating caused by artificial viscosity.

TRANSPORT

Let ξ be ρ , $\mathbf{S}$, ε or Y_e .

This routine implements the $\nabla \cdot [(\mathbf{v} - \mathbf{v}_g)\xi]$ term of the ξ conservation equations.

GRIDMOVE

Corrects hydro variables for grid motion during a timestep.

This amounts to adding a $\xi \nabla \cdot \mathbf{v}_g$ term to the right hand side of the mass and momentum density equations. The specific energy and Y_e , not being volume densities, do not require this term.

4.2 Hydrodynamics Scheme

The scheme to solve the hydrodynamic equations is taken from *Numerical Modeling in Applied Physics and Astrophysics* (Bowers and Wilson, 1991). A semi-Eulerian finite difference method is used which allows the grid to move such that mesh orthogonality is always maintained. The radial zones are allowed to move during our runs to maintain good resolution as well as implement moving wall boundary conditions, which is necessary to follow the shrinking proto neutron star. The inner boundary moves to follow a specific mass point, the time history of which is taken from the M&W code. The angular zones are fixed. The scheme used to compute mass is described first, then energy and finally momentum density will be described in detail.

4.2.1 Mass Conservation and Transport

The mass conservation equation is the simplest of the conservation equations. In BUBBA the hydro equations are operator split into two parts; a local step, where changes within a zone are applied, and then a transport step, where the variable of interest is advected with the fluid velocity. The mass conservation equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \quad (4.1)$$

has no local part and requires only the transport step.

The transport step is often called an advection step, but the transport operator includes the effects of volume changes and is not precisely the same as advection. True advection of a variable ξ would mean applying

$$\frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla \xi = 0 \quad (4.2)$$

whereas the transport step is equivalent to applying the term

$$\frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla \xi + \xi \nabla \cdot \mathbf{v} = 0 \quad (4.3)$$

which is of course identical to (4.1).

The transport term $\nabla \cdot (\mathbf{v}\rho)$ corresponds physically to a flux of material across zone boundaries at velocity $\mathbf{v}$ with a volume density of some quantity, in this case mass, embedded in it. This physical picture provides a starting point from which to attack the problem of how to solve (4.1). Consider the one dimensional Cartesian coordinate case, shown in Figure 4.1.

Figure 4.1: 1-D computational grid for mass density. Dashed line shows position of $x_i + v_i \Delta t$. Slopes 1, 2, and 3 are used to calculate $\hat{\rho}$.

To calculate the new density at $x_{i+1/2}$, one needs to know the mass fluxed across the boundaries at x_i and x_{i+1} . At x_i , the fluid will have moved a distance $v_i \Delta t$. Multiplying this by the area of the wall at x_i , yields a volume $\Delta V_i = A_i v_i \Delta t$. Next, a density value is needed for this fluxed volume, and it is the selection of this density, $\hat{\rho}$, that will determine the accuracy of our scheme. To determine $\hat{\rho}$ we use a method similar to that of van Leer, which is described later. Once the amount of mass that has been fluxed into (or out of) the zone of interest is known, determining the new

density of the zone is trivial. Defining the mass fluxes as $F_i^M = \hat{\rho}_i \Delta V_i$, one has

$$\rho_{i+1/2}^{n+1} = \rho_{i+1/2}^n + \frac{F_i^M - F_{i+1}^M}{V_{i+1/2}} \quad (4.4)$$

where $V_{i+1/2}$ is the volume between x_{i+1} and x_i . By recasting this equation one can see its mathematical relationship to the differential equation (4.1).

$$\frac{\rho_{i+1/2}^{n+1} - \rho_{i+1/2}^n}{\Delta t} = \frac{\hat{\rho}_i A_i v_i - \hat{\rho}_{i+1} A_{i+1} v_{i+1}}{A_{i+1/2} \Delta x_{i+1/2}} \quad (4.5)$$

In Cartesian coordinates the areas cancel and the equivalence of the right hand side to a finite difference scheme for $\nabla \cdot (\mathbf{v}\rho)$ is obvious. In a spherical coordinate system the equivalence is not as immediate but if this expression for the divergence of $(\mathbf{v}\rho)$ is compared with a finite difference scheme generated by Taylor series expansion as shown below, it becomes apparent the two are the same within an error of order $(\Delta r)^2$.

Consider the divergence of a physical variable $\mathbf{f}$. The radial term of the divergence has the form $\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 f_r)$ in spherical coordinates. Taking the Taylor series expansion of $(r + \Delta r)^2 f_r(r + \Delta r)$ around r gives

$$(r + \Delta r)^2 f_r(r + \Delta r) = r^2 f_r + \Delta r \frac{\partial(r^2 f_r)}{\partial r} + \frac{\Delta r^2}{2} \frac{\partial^2(r^2 f_r)}{\partial r^2} + \dots \quad (4.6)$$

Solving for the first derivative term and dividing both sides by $\frac{1}{r^2}$ yields

$$\frac{1}{r^2} \frac{\partial(r^2 f_r)}{\partial r} = \frac{(r + \Delta r)^2 f_r(r + \Delta r) - r^2 f_r}{r^2 \Delta r} + \mathcal{O}(\Delta r^2) \quad (4.7)$$

Compare this to the result given by the physically motivated approach above. The radial part of the divergence is

$$(\nabla \cdot \mathbf{f})_r = \frac{A(r + \Delta r) f_r(r + \Delta r) - A(r) f_r(r)}{V(r + \frac{1}{2} \Delta r)} \quad (4.8)$$

In spherical geometry this becomes

$$(\nabla \cdot \mathbf{f})_r = \frac{4\pi(r + \Delta r)^2 f_r(r + \Delta r) - 4\pi r^2 f_r}{\frac{4\pi}{3}[(r + \Delta r)^3 - r^3]} \quad (4.9)$$

Expanding the products and eliminating terms higher than first order reveals that this is exactly (4.7) to within $\mathcal{O}(\Delta r^2)$.

A considerable advantage to this scheme for spherical or cylindrical coordinate systems is that the volume in the denominator is never zero, hence no singularity is encountered at the origin.

Now return to the problem of a suitable method for finding an interpolated flux quantity. One obvious method of obtaining $\hat{\rho}_i$ is to construct a straight line between $\rho_{i-1/2}$ and $\rho_{i+1/2}$, then move along that line to $x_i + v_i \Delta t/2$ to get $\hat{\rho}_i$. This method, which would use slope 3 in Figure 4.1, is second order and works well for smooth subsonic flows but fails badly when faced with problems involving shocks. What is needed is a scheme that will go to donor cell² advection when the situation has very sharp gradients but will tend to a second order method otherwise.

This can be accomplished through the following variation on the van Leer algorithm due to James LeBlanc and others at LLNL[16], in which three slopes are constructed as shown in Figure 4.1. Assuming that $v_i > 0$, i.e., the fluid is flowing to the right, then construct the slope between $\rho_{i-1/2}$ and $\rho_{i+1/2}$ (line 3), $\rho_{i-3/2}$ and $\rho_{i-1/2}$ (line 1), and also the slope of the line determined by the intersections of lines 1 and 3 with the zone walls at x_{i-1} and x_i , respectively. This last slope is line 2 in the figure. One then takes the slope with the smallest absolute value of slopes 1, 2 and 3, and uses it to extrapolate $\rho_{i-1/2}$ out to $x_i + v_i \Delta t/2$ to find $\hat{\rho}$. This method is implemented in an efficient way in the following pseudo-code

```
do i = 1,nzones
   $\rho_{\max} = \max(\rho_{i-3/2}, \rho_{i-1/2}, \rho_{i+1/2})$ 
   $\rho_{\min} = \min(\rho_{i-3/2}, \rho_{i-1/2}, \rho_{i+1/2})$ 
   $\Delta\rho_1 = 2.0 * \min(\rho_{\max} - \rho_{i-1/2}, \rho_{i-1/2} - \rho_{\min})$ 
   $\Delta\rho_2 = \bar{\rho}_i - \bar{\rho}_{i-1}$ 
   $\text{slope}(i) = \text{sign}(\min(\Delta\rho_1, |\Delta\rho_2|), \Delta\rho_2) / \Delta x_{i-1/2}$ 
enddo
```

²“Donor cell” means that no interpolation is used; one simply lets $\hat{\rho}_i$ equal the ρ value of the zone immediately upstream.

where $\bar{\rho}_i$ is the linearly interpolated density at the i^{th} node.

This is the basic algorithm used for transporting hydrodynamic quantities; mass density, energy, and momentum density are all done essentially the same way. An enhancement to the above scheme has been found which is remarkable in that the addition of a single line of code results in a visible improvement in square and Gaussian pulse advection tests. Before the last line inside the above loop, add the line

$$\Delta\rho_3 = \max(\rho_{\max} - \rho_{i-1/2}, \rho_{i-1/2} - \rho_{\min}, |\Delta\rho_2|)$$

and change the last line inside the loop to read

$$\text{slope}(i) = \text{sign}(\min(\Delta\rho_1, \Delta\rho_3), \Delta\rho_2) / \Delta x_{i-1/2}$$

This improvement is added only to the flux routine for scalar quantities, like energy and mass density. It is not added to vector quantities such as the momentum density.

4.2.2 Energy Conservation and Thermodynamics

The conservation of energy equation, including a neutrino heating term, can be written

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{v}E) = -P\nabla \cdot \mathbf{v} + \dot{E}_\nu \quad (4.10)$$

where E is the energy density. This equation is operator split into local and transport steps. The energy equation is differenced as

$$\frac{E' - E^n}{\Delta t} = -P\nabla \cdot \mathbf{v} \quad (4.11)$$

$$\frac{E'' - E'}{\Delta t} = \dot{E}_\nu \quad (4.12)$$

$$\frac{E^{n+1} - E''}{\Delta t} = -\nabla \cdot (\mathbf{v}E'') \quad (4.13)$$

where the first two equations are the local step and the final equation is the transport step. First solve equation (4.11) to get E' , then use E' when solving (4.12). Once the local changes to the hydro variables have been accounted for, the transport step (4.13) is applied; E'' is used to obtain E^{n+1} .

Solving the hydrodynamics equations requires that the energy density or specific energy (energy per mass) be known, but keeping the temperature is much more convenient when doing transport calculations. In BUBBA the hydrodynamics scheme is slightly unusual in that the temperature, not the energy density or specific energy, is kept as the thermodynamic variable. Define a pseudo heat capacity, C_v , such that

$$\varepsilon = C_v T \quad (4.14)$$

C_v is computed anew for each hydrodynamic cycle and is assumed constant during a time step. Given T one can always find ε if needed.

Starting with a new T at the beginning of a hydro cycle, the code applies the PdV work by moving the nodes of a zone with their current velocity for time Δt then computing the new Lagrangian volume of the zone. In computing the change in internal energy it is assumed that the fluid behaves like a gamma-law gas with a γ computed by

$$\gamma = \frac{P}{\rho \varepsilon} + 1 \quad (4.15)$$

This is assumed constant during a timestep. A new value of γ is computed for each cycle. Using the gamma law gas relationship (and $d\varepsilon = \frac{P}{\rho^2} d\rho$) results in a new temperature T' by

$$T' = T e^{-\Delta V(\gamma-1)/\bar{V}} \quad (4.16)$$

where V is the volume of the zone before the local step and V' is the volume after the nodes have been allowed to move a distance $v\Delta t$. An average is used to time center the mid-step zone volume $\bar{V}$, that is $\bar{V} = \frac{1}{2}(V' + V)$. Gamma varies slowly enough that this method gives good results without having to iterate between the PdV work and equation of state routines.

An artificial viscosity is used to account for shock heating.

$$\dot{\varepsilon}_Q = -Q_r \frac{\partial v_r}{\partial r} - Q_\theta \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \quad (4.17)$$

The method for calculating the amount of artificial viscosity present is described in the next subsection; the method for using the artificial viscosity, or Q , is described here.

Q is defined on the zone faces, as shown in Figure 4.3. To obtain the extra heating generated by the artificial viscosity in the angular direction, to take a specific example, the zone is split down the middle in the angular direction. Each half contributes to the new temperature, which is given by

$$\begin{aligned} \frac{T'_{i+1/2,j+1/2} - T_{i+1/2,j+1/2}}{\Delta t} = & - \left(Q_{i+1,j+1/2}^{\theta} V_{i+3/4,j+1/2} \frac{v_{i+1,j+1}^{\theta} - v_{i+1,j}^{\theta}}{r_{i+1} \Delta \theta_{j+1/2}} \right. \\ & + \left. Q_{i,j+1/2}^{\theta} V_{i+1/4,j+1/2} \frac{v_{i,j+1}^{\theta} - v_{i,j}^{\theta}}{r_i \Delta \theta_{j+1/2}} \right) \\ & / \left(\rho_{i+1/2,j+1/2} C_v v_{i+1/2,j+1/2} V_{i+1/2,j+1/2} \right) \end{aligned} \quad (4.18)$$

where the volumes centered on the quarter indices, such as $V_{i+3/4,j+1/2}$ indicate half volumes. Naturally, $V_{i+1/4,j+1/2} + V_{i+3/4,j+1/2} = V_{i+1/2,j+1/2}$, where $V_{i+1/2,j+1/2}$ is the total zone volume. The heating contribution of the radial artificial viscosity, Q_r , is done in the same manner.

In an Eulerian code like BUBBA all hydrodynamic quantities must be explicitly advected with the fluid. Failing to do so for the artificial viscosity introduces an error in shock problems which can be corrected by adding a pseudo-advection term to the Q . BUBBA uses

$$Q_{\text{corr}}^j = \frac{1}{2} \rho \Delta v^j (v^j - v_g^j) \quad (4.19)$$

where the j superscript refers to either the r or θ directions, v_g is the grid velocity and Δv is the interpolated velocity difference discussed below. The velocities v and v_g are not face centered with Q and are obtained by simple averaging of the velocities at the two nearest nodes. When calculating shock heating Q_{corr} is subtracted from the artificial viscosity used above before the artificial viscosity is applied to heat the zone. The correction term Q_{corr} is *not* applied to Q when calculating new velocities.

The radiation corrections are added directly after this step, using

$$T'' = T' + \frac{\dot{\epsilon}_v \Delta t}{C_v} \quad (4.20)$$

After the radiation effects are accounted for, nothing changes the energy of a zone until the transport routine. For transport one requires the specific energy, which is

obtained via 4.14. Transport of energy is done exactly as is transport of mass. A value for the fluxed specific energy $\hat{\epsilon}_i$ is found in the same way $\hat{\rho}_i$ is computed during mass transport. Then $\hat{\epsilon}_i$ is multiplied by the total mass fluxed F_i^M to get a total energy flux. This keeps the energy and mass fluxes in step with each other.

$$F_i^e = \hat{\epsilon}_i F_i^M \quad (4.21)$$

The energy fluxes are added to produce a new specific energy in the zone by

$$\epsilon_{i+1/2}^{n+1} = \epsilon_{i+1/2}'' \frac{\rho_{i+1/2}^n}{\rho_{i+1/2}^{n+1}} + \frac{F_i^e - F_{i+1}^e}{\rho_{i+1/2}^{n+1} V_{i+1/2}} \quad (4.22)$$

The zone energy is then put into the EOS routine which iterates with a Newton-Raphson algorithm to return a new temperature, T^{n+1} , and pressure P^{n+1} . A new C_v and γ are calculated using 4.14 and 4.15, and the cycle repeats.

4.2.3 Composition Solve

Only two species of nuclei are explicitly carried in the code, free baryons and alpha particles. These are assumed to be in nuclear statistical equilibrium, so a Saha equation can be used to compute the number fraction of each. From the equation of state (3.29), which is reproduced here, we have

$$\frac{X_B^4}{1 - X_B} = \frac{1}{2} \left(\frac{m_B}{\rho} \right)^3 \left(\frac{m_B k T}{2\pi \hbar^2} \right)^{\frac{3}{2}} e^{-Q/kT} \quad (4.23)$$

The right hand side is a function of ρ and T , call it $A(\rho, T)$. The function A is smooth and has interesting behavior only over a small range, so the solution to the fourth order algebraic equation

$$X_B^4 = (1 - X_B)A \quad (4.24)$$

can be approximated to within a few percent by a four piece fit, as follows

$$X_B = \begin{cases} \frac{A^{1/4}}{1.0+0.3721 A^{0.3}} & \text{if } A \leq 1 \\ .7288 A^{0.104} & \text{if } 1 < A \leq 10 \\ .8506 A^{0.0369} & \text{if } 10 < A \leq 20 \\ 1 - \frac{1}{A} & \text{if } A \geq 20 \end{cases} \quad (4.25)$$

This saves the time that would be spent either iterating or computing the algebraic solution directly.

4.2.4 Momentum Conservation and Transport

BUBBA transports momentum density, not velocity. Define the momentum density to be $\mathbf{S} = \rho \mathbf{v}$. The momentum density is defined on the nodes of the grid, unlike the scalar variables ρ and ϵ , which are defined at zone centers. Around each node we define a “momentum zone” which is the quadrilateral defined by the zone centers of the surrounding zones. See Figure 4.2.

Figure 4.2: Dashed line shows position of momentum zone about node i, j .

The momentum conservation equation is operator split into the local part and the transport step. The local part includes accelerations by pressure gradients, gravitational forces, artificial viscosity and rotation effects. The pressure and gravitational force steps solve

$$\frac{\partial \mathbf{S}}{\partial t} = -\nabla P - \rho \nabla \Phi \quad (4.26)$$

P , ρ , and Φ are zone centered and the accelerations due to pressure and gravity are both handled the same way. Take the pressure acceleration in the radial direction for example; this is calculated by

$$\frac{S'_{i,j} - S^r_{i,j}}{\Delta t} = \frac{(P_{i-1/2,j-1/2} - P_{i+1/2,j-1/2})a_{i,j-1/4}}{V_{i,j}} + \frac{(P_{i-1/2,j+1/2} - P_{i+1/2,j+1/2})a_{i,j+1/4}}{V_{i,j}} \quad (4.27)$$

where the $a_{i,j+1/4}$ is the area of half the zone edge, *i.e.*, that part of the zone edge that reaches from the node i, j to the midpoint of the zone edge, $i, j + 1/2$.

BUBBA uses an artificial viscosity devised by Randy Christensen of Lawrence Livermore National Laboratory. In one dimension it is calculated as follows.

$$Q_{i+1/2} = \rho_{i+1/2} \overline{\Delta v_{i+1/2}} (C_{\text{quad}} \overline{\Delta v_{i+1/2}} - C_{\text{lin}} c_s) \quad (4.28)$$

where $\overline{\Delta v_{i+1/2}}$ is the velocity difference calculated at the zone center, c_s is the local sound speed, and C_{quad} and C_{lin} are coefficients chosen by experience. In practice BUBBA uses $C_{\text{quad}} = (\gamma + 1)/2$ and $C_{\text{lin}} = 1$. A good artificial viscosity will “turn on” in the presence of a shock but will be insignificant otherwise. Computing the velocity difference correctly is the key to obtaining this behavior. First find a velocity interpolation slope for every momentum zone. Do this for a momentum zone at i by finding the slopes $m_- = (v_i - v_{i-1})/\Delta r_{i-1/2}$, $m_+ = (v_{i+1} - v_i)/\Delta r_{i+1/2}$, then taking as the interpolation slope m_i either m_- or m_+ , whichever has the smaller absolute value, or zero if m_- and m_+ are of opposite sign. This will yield a “monotonic Q ”, which has the important virtue that the interpolated velocity difference at the zone center will never be greater than the direct velocity difference $v_{i+1} - v_i$. The interpolated velocity difference is then calculated by

$$\Delta v_{i+1/2} = v_{i+1} - v_i - \frac{\Delta r_{i+1/2}}{2} (m_i + m_{i+1}) \quad (4.29)$$

If $\Delta v_{i+1/2} > 0$ then $\Delta v_{i+1/2}$ is set to zero. Using a difference of velocities without interpolation across a zone will turn Q on during adiabatic compression. This will generate entropy in a situation where by definition the entropy should remain constant. The scheme presented here ensures that Q is applied with full strength in

Figure 4.3: Q is face centered.

In practice BUBBA is a 2-D code which means an average must be taken to obtain $\rho_{i+1/2}$ in the scheme described above. In accordance with the face centering of Q , BUBBA takes the simple average of the mass density of the two zones adjacent to the face for which Q is currently being calculated.

Because the code uses spherical coordinates, the problem of parallel transport arises when trying to advect momentum. That is, given a momentum vector $\mathbf{S}$ with momentum components in spherical coordinates (S^r, S^θ) , in general if that vector is translated in space to a new location, $\mathbf{S}$ will be described by new components $(S^{r'}, S^{\theta'})$. One must take this into account in the momentum transport routine or serious errors occur. The simplest way around the problem is due to James LeBlond of LLNL. Convert the spherical momentum coordinates to Cartesian coordinates,

$$S^z = S^r \cos \theta - S^\theta \sin \theta \quad (4.30)$$

$$S^x = S^r \sin \theta + S^\theta \cos \theta$$

then do the momentum transport in Cartesian coordinates. In more than one spatial

dimension one has to keep track of both the direction in which one is transporting momentum and the direction of the momentum one is transporting. Define variables such as $F_r^{S^x}$ and $F_\theta^{S^x}$ to keep track of the x momentum flux in the r direction and the z momentum flux in the θ direction. Once the transport has been done one has the momentum at each node in terms of (S^x, S^z) , which can then be converted back to spherical coordinates. The spherical velocity components are necessary at many steps in the code so keeping both versions in memory saves time that would be spent on coordinate conversions. One might try using only Cartesian coordinates for the momenta if memory were a limiting factor (in a 3-D code for example).

Transport of momentum requires momentum fluxes computed at mid-zone. To calculate the momentum radially fluxed into (or out of) a momentum zone i, j to momentum zone $i + 1, j$, begin by finding the average mass flux $\bar{F}_{i+1/2,j}^M$ into the momentum zone by doing an area weighted average;

$$\begin{aligned} \bar{F}_{i+1/2,j}^M = & \frac{1}{2} \left(F_{i,j-1/2}^M \frac{a_{i,j-1/4}}{A_{i,j-1/2}} + F_{i,j+1/2}^M \frac{a_{i,j+1/4}}{A_{i,j+1/2}} \right. \\ & \left. + F_{i+1,j-1/2}^M \frac{a_{i+1,j-1/4}}{A_{i+1,j-1/2}} + F_{i+1,j+1/2}^M \frac{a_{i+1,j+1/4}}{A_{i+1,j+1/2}} \right) \end{aligned}$$

where the $a_{i,j+1/4}$ is the area of the part of the zone edge that reaches from the node i, j to the midpoint of the zone edge, $i, j + 1/2$, and $A_{i,j+1/2}$ is the area of the entire zone edge, from i, j to $i, j + 1$. The average mass flux is multiplied by the interpolated mid-zone velocity $\hat{v}_{i+1/2,j}$ to get the momentum fluxed between momentum zones i, j and $i + 1, j$.

$$\begin{aligned} F_{i+1/2,j}^{S^x} &= \bar{F}_{i+1/2,j}^M \hat{v}_{i+1/2,j}^z \\ F_{i+1/2,j}^{S^z} &= \bar{F}_{i+1/2,j}^M \hat{v}_{i+1/2,j}^x \end{aligned} \tag{4.31}$$

where $\hat{v}_{i+1/2}$ is found with the interpolation technique described above for the mass density. The interpolation is applied to the velocity using the grid defined by the momentum zones. Tying momentum transport to the mass fluxes used in the mass transport ensures that the two quantities will advect together. With the momentum fluxes in hand, the new momentum density is added up just like the mass density

above. Using the z momentum density for example, one has

$$S_i^{z(n+1)} = S_i^{z(n)} + \frac{F_{i-1/2}^{S^z} - F_{i+1/2}^{S^z}}{V_i} \quad (4.32)$$

where V_i is the volume of the momentum zone. After the new momentum densities are obtained in Cartesian coordinates they are transformed back to spherical coordinates using the inverse of (4.31).

Grid motion is often required in astrophysical problems to model moving boundary conditions or to maintain adequate zoning. It is required in the supernova problem for both of these reasons and is straightforward to include. Consider fluid flow on a grid with velocity $\mathbf{v}_g$. Then the equation for a volume density ξ (mass, energy or momentum) is modified from

$$\frac{\partial \xi}{\partial t} + \nabla \cdot (\mathbf{v} \xi) = \text{RHS} \quad (4.33)$$

to

$$\frac{\partial \xi}{\partial t} + \nabla \cdot [(\mathbf{v} - \mathbf{v}_g) \xi] = -\xi \nabla \cdot \mathbf{v}_g + \text{RHS} \quad (4.34)$$

This reduces to the Eulerian form in the case $\mathbf{v}_g = 0$ and the Lagrangian form when $\mathbf{v}_g = \mathbf{v}$. In the latter case the transport routines do nothing and can be skipped, which is convenient when running BUBBA with just one space dimension. An extra routine, **GRIDMOVE**, is used to update the effect of grid motion on the hydro variables after transport. This just amounts to applying the first term on the right hand side of 4.34 to the mass and momentum densities. This is done by allowing the grid nodes to move with their respective grid velocities for a time Δt , finding the new grid volumes, then setting

$$\begin{aligned} \rho^{n+1} &= \rho' \frac{V^{\text{old grid}}}{V^{\text{new grid}}} \\ S^{n+1} &= S' \frac{V^{\text{old grid}}}{V^{\text{new grid}}} \end{aligned} \quad (4.35)$$

where ρ' is the mass density after the mass transport step and S' is the momentum density after the momentum transport step. Keeping the temperature or specific energy of the fluid instead of the energy density makes a grid motion correction to the energy unnecessary; the extra term on the right hand side vanishes in this case.

4.2.5 Shock Tests

A new code must be tested or its results cannot be trusted. To test the hydrodynamics scheme I ran several problems for which the analytic answer is known.

The first test was a simple wall shock, which ran in both the radial and angular directions. A cold ideal gas, $\epsilon = 0.0$, $\gamma = 5/3$, with density $\rho = 1.0$, is give a velocity $v_x = -1.0$. The gas slams into a rigid wall on the left, generating a shock wave which travels to the right. This case is easily solved analytically using the shock jump conditions. If one separates the fluid into two parts, that which is the left of the shock wave subscripted with an L , and the fluid to the right of the shock subscripted with an R , then in the lab frame the jump conditions (mass, momentum and energy conservation across the shock) can be written as

$$\rho_R(v_s - v_R) = \rho_L(v_s - v_L) \quad (4.36)$$

$$\rho_R(v_s - v_R)(v_L - v_R) = P_L - P_R \quad (4.37)$$

$$\rho_R(v_s - v_R)(\epsilon_L + \frac{1}{2}v_L^2 - \epsilon_R + \frac{1}{2}v_R^2) = P_L v_L - P_R v_R \quad (4.38)$$

where v_s is the shock speed, ϵ is the energy per mass, ρ is mass density, and P is pressure. In the wall shock test case $v_L = P_R = \epsilon_R = 0$, $v_R = -1$, and $\rho_R = 1.0$. Inserting these values into the equations above and using the gamma law gas equation of state,

$$P = \rho\epsilon(\gamma - 1) \quad (4.39)$$

yields the state of the shocked fluid and the speed of the shock. See Figure 4.4.

$$\begin{aligned} \rho_L &= 4 & \epsilon_L &= \frac{1}{2} \\ P_L &= \frac{4}{3} & v_s &= \frac{1}{3} \end{aligned} \quad (4.40)$$

A related problem is the 1-D shock tube test. The computational region is split in half, the left half containing an ideal gas with $\rho_L = 10$, $\epsilon_L = 10$. The right half has $\rho_R = 1$, $\epsilon_R = 0$. All velocities are initially zero. A shock forms at the interface and moves right while a rarefaction wave moves into the gas on the left.

Figure 4.4: Density profile of wall shock test at $t = 0.5$. The result is within 1% of the analytic solution given above for the shocked fluid. The r range starts at $r = 100$ and extends to $r = 101$ to minimize spherical geometry effects. The fluid is initially a slab one unit long, with speed 1, moving to the left.

Consider Figure 4.5; the density just behind the shock (between (C) and (D)) should be 4.00. The density drops to 3.50 across the contact discontinuity, (C). (The contact discontinuity is the point which marks the interface between the hot fluid on the left and the cold fluid on the right.) (A) moves left at the speed of sound in the undisturbed medium, $c_s^a = 1.05$ in the units of this problem. (B) is a signal that moves left at the speed of sound in the expanded fluid $c_s^b = 0.75$, but this signal is embedded in fluid moving right at the speed of the contact discontinuity, $v_c = 0.93$. (B) ends up with an overall small speed (0.18) to the right. (D) is the shock wave, moving right at $v_s = 1.24$. Theoretical positions at $t = 0.32$ are $A = 100.18$, $B = 100.55$, $C = 100.80$, and $D = 100.90$. The shock is spread over three or four zones by the code's artificial viscosity. Like the wall shock test, the shock tube problem was done in both the radial and angular directions in 1-D.

Figure 4.5: Density profile of shock tube test at $t = 0.32$.

4.2.6 Rayleigh-Taylor Tests

It is important to know the magnitude of the error in the growth rates of hydrodynamic instabilities as computed by BUBBA. To test the code's ability to calculate growth rates a version of the Rayleigh-Taylor instability was used, in which a heavy fluid sits on top of a lighter one. This instability has for incompressible fluids an analytic solution for the growth rate of small perturbations[8].

$$\omega^2 = gk \left[\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} \right] \quad (4.41)$$

Here, k is the wavenumber, g is the acceleration due to gravity, ω is the growth rate and ρ_+, ρ_- are the heavy and light fluid densities.

The code is not designed for incompressible fluids, but if the fluid flow speeds in a problem are very subsonic then the fluid density of a gas is not changed significantly. In an ideal gas the sound speed can be set as high as desired by raising its temperature, so the incompressible fluid case can be modelled fairly well. Of somewhat more concern are the boundary conditions for this problem, because the growth rate above

pertains to a horizontal fluid interface with top and bottom walls infinitely far away. Reflecting walls were set at the top and bottom.

Several Rayleigh-Taylor runs were made with perturbations of different wavelengths and resolutions and the growth rate for each one was measured. $\rho_+ = 4$, $\rho_- = 1$ and $g = 10^6$ gives $\frac{\omega}{k} = 774.6$. If the situation were actually incompressible with infinitely distant walls this value would remain constant regardless of which wavelength is perturbed. The problems were run on a unit grid. The initial configuration was such that the fluid was in hydrostatic equilibrium, with the pressure at the bottom of the fluid $P = 10^7$. Figure 4.6 shows the initial conditions. A sinusoidal perturbation of 10^{-5} times the local sound speed was applied to the velocity at the midplane, $v_x = 10^{-5} c_s \sin kz$. After a few tens of cycles the system settled down to an exponential growth phase that was strongly dominated by the mode selected for in the initial velocity perturbation, though a Fourier transform of the velocity shows that some other modes were present. Figure 4.7 shows the velocity field for a $k = 2\pi$ perturbation on a 32×32 grid after $t = .001$.

The velocity can be broken down into its Fourier modes. Each mode grows with its own growth rate, given in 4.41. Thus

$$\hat{v}_x(k) \sim e^{\omega_k t} \quad (4.42)$$

The growth rates generated by the code were calculated by the following procedure;

1. For each time step take the spatial Fourier Transform of $v_x(z)$ across the midplane (at $x = 0.5$).
2. Store the magnitude squared of the perturbed velocity component; $|\hat{v}_x(k_p)|^2 \sim e^{2\omega_k t}$, where k_p is the wavenumber perturbed by the initial conditions.
3. Plot the time history of $\log(|\hat{v}_x(k_p)|^2)$. The exponential growth region will appear as a straight line. Estimate the slope.
4. The growth rate is the slope divided by two. Divided by two because the plotted quantity is $\sim 2\omega_k t$ rather than $\sim \omega_k t$.

Figure 4.6: Initial conditions for Rayleigh-Taylor runs. The density in the upper region (to the right on these graphs) is four times the density below. The region is one unit long, and begins at $r = 1000$ to minimize the effects of spherical geometry.

Figure 4.7: Velocity vectors for Rayleigh-Taylor run with $k = 2\pi$ at $t = .001$. $k\Delta x = \pi/16 = 0.20$. The grid is a unit square.

Figure 4.8: Growth rate plot for Rayleigh-Taylor runs. The actual growth rate is obtained by fitting a line to the linear region and taking half the slope.

The dimensionless quantity $k\Delta x$ is a measure of how well zoned a problem is (low $k\Delta x$ means better zoning). In Figure 4.9 the quantity $\frac{\omega}{\sqrt{k}}$ is plotted against $k\Delta x$. A perfect result would be a constant 775 (in this case where $\rho_+ = 4\rho_-$ and $g = 10^6$) regardless of $k\Delta x$ if the fluid were incompressible and the boundaries were infinitely far away. Neither of these assumptions is true in the calculation. The initial perturbations are not exactly right for selecting out one Fourier mode of the system because they do not satisfy $\nabla \cdot \mathbf{v} = 0$, nor are the initial perturbations exactly correct in the compressible case because corrections to the density should be made to match the velocity field. The results shown in Figure 4.9 should be compared against the runs done with no artificial viscosity ($Q = 0$; these runs are marked with squares in the figure) rather than against the 775 value. There are no $Q = 0$ results for the two smallest $k\Delta x$ values because smaller wavelength modes (which grow faster than long wavelengths in the Rayleigh-Taylor problem) swamp the configuration without artificial viscosity, making it impossible to pick out the growth of one mode clearly.

4.2.7 An Improvement on The Artificial Viscosity

The Rayleigh-Taylor problem is adiabatic; the entropy of a particular parcel of fluid never changes. A computer simulation of it should reproduce this behavior but in fact many methods will fail to do so because artificial viscosity tends to generate entropy when fluids form vortices. The failure of BUBBA's artificial viscosity to preserve entropy can be seen by considering a vortex centered about a single zone. See Figure 4.3.

The artificial viscosity, or Q , is face centered. Radial and angular artificial viscosities are calculated independently. Consider the radial Q ;

$$Q_{i+1/2,j}^r = \bar{\rho}_{i+1/2,j} \overline{\Delta v_{i+1/2,j}} (C_{\text{quad}} \overline{\Delta v_{i+1/2,j}} - C_{\text{lin}} \bar{c}_{i+1/2,j}^2) \quad (4.43)$$

The quantity $\overline{\Delta v_{i+1/2,j}}$ is an interpolated velocity difference. The velocity is extrapolated from both directions into the middle of the zone using the monotonic interpolation scheme discussed above.

Figure 4.9: Several Rayleigh-Taylor runs are shown at different resolutions and initial perturbation wavelengths. With better zoning the code approaches the theoretical result for the growth rate of the instability. The line is at 775, the theoretical value. The pluses mark the runs made with standard artificial viscosity, the diamonds mark runs made with the new artificial viscosity which takes the curl into account. The squares are runs in which no artificial viscosity was used.

In Figure 4.3 the difference between velocities $v_{i+1,j}$ and $v_{i,j}$ will generate some Q even though the flow is close to purely rotational. (A purely rotational flow satisfies $\nabla \cdot \mathbf{v} = 0$ so the density of a fluid parcel never changes; for an inviscid fluid such a flow is obviously adiabatic.) I have ameliorated this difficulty by changing the form of Q slightly.

Flow that is purely rotational should never generate any artificial viscosity. Divergent flows, under some circumstances, should. Therefore, let us create a new artificial viscosity, Q' , that uses a different definition for $\bar{\rho}_{i+1/2,j}$. Instead of taking

$$\bar{\rho}_{i+1/2,j} = \frac{1}{2}(\rho_{i+1/2,j+1/2} + \rho_{i+1/2,j-1/2}) \quad (4.44)$$

one could use

$$\bar{\rho}'_{i+1/2,j} = \frac{2\rho'_{i+1/2,j+1/2}\rho'_{i+1/2,j-1/2}}{\rho'_{i+1/2,j+1/2} + \rho'_{i+1/2,j-1/2}} \quad (4.45)$$

where

$$\rho'_{i+1/2,j+1/2} = \rho_{i+1/2,j+1/2} \left(\frac{|\nabla \cdot \mathbf{v}|}{|\nabla \cdot \mathbf{v}| + |\nabla \times \mathbf{v}|} \right)_{i+1/2,j+1/2} \quad (4.46)$$

and the curl and divergence are defined to be

$$\begin{aligned} (\nabla \times \mathbf{v})_{i+1/2,j+1/2} &= v_{i,j+1/2}^{\theta} r_i \Delta \theta_{j+1/2} + v_{i+1/2,j+1}^r \Delta r_{i+1/2} \\ &\quad - v_{i+1,j+1/2}^{\theta} r_{i+1} \Delta \theta_{j+1/2} - v_{i+1/2,j}^r \Delta r_{i+1/2} \end{aligned} \quad (4.47)$$

$$\begin{aligned} (\nabla \cdot \mathbf{v})_{i+1/2,j+1/2} &= v_{i+1,j+1/2}^r A_{i+1,j+1/2} - v_{i,j+1/2}^r A_{i,j+1/2} \\ &\quad + v_{i+1/2,j+1}^{\theta} A_{i+1/2,j+1} - v_{i+1/2,j}^{\theta} A_{i+1/2,j} \end{aligned} \quad (4.48)$$

The face centered velocities $v_{i+1/2,j}$ are obtained by setting $v_{i+1/2,j} = \frac{1}{2}(v_{i,j} + v_{i+1,j})$.

This cannot eliminate the problem entirely but tests with the Rayleigh-Taylor instability indicate that it does help. In Rayleigh-Taylor runs where the entropy generated by the artificial viscosity was measured by taking the integral

$$S_{\text{Tot}} = \int \int \frac{Q \nabla \cdot \mathbf{v}}{T} dV dt \quad (4.49)$$

over the entire computational grid, the new artificial viscosity created 21% less entropy than the old. Growth rates of Q' runs are closer to the expected theoretical value than standard Q growth rates. See Figure 4.9 for a comparison of growth rates.

4.2.8 Advection Test

An important difference between one and two dimensional calculations is that a 2-D code does not have the luxury of knowing that advection will always take place along a particular direction. A two (or more) dimensional code must advect physical quantities across its grid at any angle (often it requires two separate time steps to advect material in a direction diagonal to the grid). Advecting a block of material at an “unfriendly” angle across the computational grid, especially if it is done near the coordinate origin, is a severe test of a code’s advection scheme. To test BUBBA’s

advection scheme an empty spherical grid (50 radial zones, 75 angular zones) ranging from $0.1 \leq r \leq 5.0$ in radius and $0 \leq \theta \leq \pi$ in angle was set up with a velocity entirely in the x direction all over the grid. The mass density at a part of the boundary below the x -axis was set to $\rho = 1.0$ and the problem was allowed to evolve until a slab of matter with $\rho \geq 0.8$ existed from the $\theta = \pi$ boundary to the $r = 5.0$ wall. See

Figure 4.10: Density contour plot of the mass advection test. The block has been “grown” into position by setting $v_x = 1$ (positive x is to the right in the figure). The grid has 50 radial zones and 75 angular zones.

Figure 4.10. Any densities within the slab greater than 0.8 were then trimmed to 0.8, and the velocity was set to $v_z = 1$, which is towards the top of the page in Figures 4.10 and 4.11. The slab was advected through the x -axis and beyond to produce Figure 4.11. The curvature of the lower right corner of the slab is an artifact of its

initial shape; when the slab begins its upward motion it is bounded on the right by the $r = 5.0$ wall. The softening of the upper right corner is numerical diffusion after 3000 cycles.

Figure 4.11: Density contour plot of the mass advection test after being advected for 3000 cycles and passing the coordinate origin. The velocity has been set to $v_z = 1$ (positive z is towards the top of the page in the figure).

An even more stringent test is to advect both mass and momentum. In this case the grid velocity is not artificially set to a specific value but instead the slab of matter is given an initial velocity and the code must advect the velocity along with the mass. These results are shown in Figure 4.12.

Figure 4.12: Density contour plot of the mass and velocity advection test after being advected for 3000 cycles and passing the coordinate origin.

4.2.9 Sphericity Test

It is important to know whether the code is injecting perturbations of its own into a problem. To test this I ran the supernova mechanism model with no initial perturbations at all. This problem develops a negative entropy gradient after the shock has moved out to approximately 250 kilometers, which takes about 50 milliseconds. Given spherically symmetric initial conditions the code maintains spherical symmetry throughout the run (up to 400 milliseconds).

4.2.10 Multiple Time Step Scheme

The code uses a multiple time-step method, sometimes called “sub-cycling”, that lessens the severity of the Courant-Freidrich-Levy (CFL) condition limits on the Δt given by the high sound speed of the innermost zones. Designing an algorithm to automatically assign multiple time-step regions in a 2D code is difficult for the general case. Supernova modelers, however, can exploit the fact that in the supernova problem the local CFL condition on the time-step gives Δt as an almost monotonically increasing function of radius (and almost independent of angle). This has been used to create an algorithm that assigns multiple time-step regions in a straightforward way. Let the smallest Δt in the entire computation region be Δt_1 . In the supernova problem this will always be the first zone. The innermost region is assigned the minimum Δt in the entire problem. Then march outward in radius, inspecting the local time-step of each zone; four zones beyond the first cell with $\Delta t_{local} \geq 2\Delta t_1$ begin the next time-step region, which will have time-step $\Delta t_2 = 2\Delta t_1$. Continue in this manner until the end of the computational grid is reached, however one must be careful not to make the total (outermost) timestep so large that other physical processes are affected (neutrino transport for example). In a typical supernova problem the code ends up with three or four different time-step regions. Each region is run at its own time-step. When the innermost region has run two steps, the next region advances once, and so on. In shock test problems run with multiple time-step regions the code results show less than a one percent difference in the density, pressure, and temperature profiles from those run without. This technique cuts run times by a factor of more than ten.

4.3 Neutrino Transport Scheme

This section begins with an overview of the radiation packages. A finite difference scheme is developed and an operator splitting technique is shown that reduces the time it takes to obtain a solution.

RADIATION I:

Contains the following routines which handle neutrino transport. It includes a flux limited diffusion algorithm for solving the transport equation. **RADIATION I** was used for the supernova mechanism problem, for which the neutrino energy emission of accreted matter must be calculated carefully.

NEUTLUM

Assigns neutrino energies and luminosities to the “core” neutrino emissions.

OPACITY

Calculates neutrino opacity for each zone. As we currently do not have multi-group transport installed and the neutrino field is not differentiated by species, there is only one opacity to compute. This will change in the future.

GETRADFIELD

Calculates self-consistent neutrino energy density from the transport equation using an operator splitting scheme to implement flux limited diffusion.

FIELDHEAT

Calculates energy exchange between neutrino field and matter caused by $p + \bar{\nu}_e \rightarrow n + e^+$ and $n + \nu_e \rightarrow p + e^-$ reactions. Matter supplies neutrino field with black-body emissions.

COREHEAT

Calculates energy change in matter caused by above reactions with core neutrinos, as opposed to accretion generated neutrinos.

Equation 3.13 which we reproduce here, is

$$\frac{\partial F}{\partial t} = \nabla \cdot (D \nabla F) + \rho c (K_m a T_m^4 - K_a F) \quad (4.50)$$

This is solved for the radiation density with an implicit operator splitting scheme. By integrating over a cell volume and using the divergence theorem one obtains a difference equation

$$\begin{aligned} \frac{F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - F_{i+\frac{1}{2},j+\frac{1}{2}}^n}{\Delta t} = & \frac{1}{V_{i+\frac{1}{2},j+\frac{1}{2}}} \left(\frac{D_{i+1,j+\frac{1}{2}} A_{i+1,j+\frac{1}{2}}}{\Delta r_{i+1}} (F_{i+\frac{3}{2},j+\frac{1}{2}}^{n+1} - F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1}) - \right. \\ & \frac{D_{i,j+\frac{1}{2}} A_{i,j+\frac{1}{2}}}{\Delta r_i} (F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2},j+\frac{1}{2}}^{n+1}) + \frac{D_{i+\frac{1}{2},j+1} A_{i+\frac{1}{2},j+1}}{r_{i+\frac{1}{2}} \Delta \theta_{i+1}} (F_{i+\frac{1}{2},j+\frac{3}{2}}^{n+1} - F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1}) - \\ & \left. \frac{D_{i+\frac{1}{2},j} A_{i+\frac{1}{2},j}}{r_{i+\frac{1}{2}} \Delta \theta_i} (F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - F_{i+\frac{1}{2},j-\frac{1}{2}}^{n+1}) \right) + \rho c (K_m B_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - K_a F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1}) \end{aligned} \quad (4.51)$$

Here B is the energy integrated blackbody emission, so $B = aT^4$. This is an implicit equation, so B^{n+1} is unknown. B^{n+1} is approximated to first order with a Taylor series expansion in δT , which is the temperature change in the matter caused by radiation absorption (or emission). Define the change in F due only to absorption and emission to be ΔF_{ae} . Then

$$\Delta F_{ae} = \rho c \Delta t X_B \left[K_m (B(T) + \delta T \frac{\partial B}{\partial T}) - K_a F \right]. \quad (4.52)$$

The temperature change in the matter is

$$\rho C_v \delta T = \rho c \Delta t X_B \left[K_a F - K_m (B(T) + \delta T \frac{\partial B}{\partial T}) \right] \quad (4.53)$$

where C_v is the "heat capacity" defined such that $\varepsilon = C_v T$. Solving this equation for δT and substituting the solution into equation 4.52 yields

$$\Delta F_{ae} = \frac{\rho c \Delta t X_B (K_m B(T) - K_a F)}{1 + \frac{c \Delta t X_B}{C_v} K_m \frac{\partial B}{\partial T}} \quad (4.54)$$

The final difference equation is

$$\begin{aligned} F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - F_{i+\frac{1}{2},j+\frac{1}{2}}^n = & \frac{\Delta t}{V_{i+\frac{1}{2},j+\frac{1}{2}}} \\ & \times \left(\frac{D_{i+1,j+\frac{1}{2}} A_{i+1,j+\frac{1}{2}}}{\Delta r_{i+1}} (F_{i+\frac{3}{2},j+\frac{1}{2}}^{n+1} - F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1}) - \frac{D_{i,j+\frac{1}{2}} A_{i,j+\frac{1}{2}}}{\Delta r_i} (F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2},j+\frac{1}{2}}^{n+1}) \right) \end{aligned} \quad (4.55)$$

$$\begin{aligned}
& + \frac{D_{i+\frac{1}{2},j+1} A_{i+\frac{1}{2},j+1}}{r_{i+\frac{1}{2}} \Delta \theta_{i+1}} (F_{i+\frac{1}{2},j+\frac{3}{2}}^{n+1} - F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1}) - \frac{D_{i+\frac{1}{2},j} A_{i+\frac{1}{2},j}}{r_{i+\frac{1}{2}} \Delta \theta_i} (F_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - F_{i+\frac{1}{2},j-\frac{1}{2}}^{n+1}) \\
& + \frac{\rho c \Delta t X_B (K_m B^n - K_a F^{n+1})}{1 + \frac{c \Delta t X_B}{C_v} K_m \frac{\partial B^n}{\partial t}}.
\end{aligned}$$

Given n_r and n_θ radial and angular zones respectively, one could invert a pentadiagonal matrix of size $n_r \times n_\theta$ to solve this system, however this is computationally expensive. BUBBA uses a two-step iterative process instead. Re-write 4.55 as

$$\frac{\Delta F}{\Delta t} = (\nabla \cdot (D \nabla F))_r + (\nabla \cdot (D \nabla F))_\theta + \rho c X_B (\hat{K}_m B - \hat{K}_a F), \quad (4.56)$$

where the denominator in the last term of equation 4.55 is absorbed into K_m and K_a , now called $\hat{K}_m$ and $\hat{K}_a$. Now split the time evolution into two steps;

$$F^{n+\frac{1}{2}} - F^n = (\nabla \cdot (D \nabla F^{n+\frac{1}{2}}))_r + \omega \rho c X_B (\hat{K}_m B - \hat{K}_a F^{n+\frac{1}{2}}), \quad (4.57)$$

$$F^{n+1} - F^{n+\frac{1}{2}} = (\nabla \cdot (D \nabla F^{n+1}))_\theta + (1 - \omega) \rho c X_B (\hat{K}_m B - \hat{K}_a F^{n+1}), \quad (4.58)$$

where $\omega = \frac{1}{2}$. With this system one need only invert $n_r + n_\theta$ tri-diagonal matrices. This scheme was tested in both the diffusive and free streaming regimes. It is important for the reader to realize that this splitting is allowable only because the time step determined by the hydro equations is much smaller than the minimum timestep required for the splitting to be accurate.

The diffusion equation 4.50 is only valid when the mean free path is small compared to the characteristic lengths in the problem. This is emphatically not true for much of the computational region. The technique of flux-limited diffusion allows one to retain equation 4.50 by modifying the diffusion coefficient D such that the equation changes to the free streaming approximation in the correct circumstances. Define D as

$$D = \frac{\lambda c}{3[1 + \frac{\lambda}{3} |\frac{\nabla F}{F}|]}. \quad (4.59)$$

In diffusive regime 4.50 becomes

$$\frac{\partial F}{\partial t} = \frac{\lambda c}{3} \nabla^2 F + \rho c [K_m a T_m^4 - K_a (T_\nu) F] \quad (4.60)$$

while in the free streaming limit

$$\frac{\partial F}{\partial t} = c \nabla F + \rho c [K_m a T_m^4 - K_a(T_\nu) F]$$

The correct result is obtained for each extreme. In the intermediate region this method gives $\sim 10\%$ error. The neutrinosphere boundary is fairly sharp, so this region is small in the supernova problem.

The scheme for handling interaction of core neutrinos with nucleons is common to both transport packages, and we describe our method below. As described previously, the change in energy per gram due to core neutrino capture processes is

$$\dot{\epsilon}_{\text{core}} = X_B \left[\frac{L_{\text{core}}}{4\pi r^2} K_a(T_{\text{core}}) - K_m a c T^4 \right] \quad (4.61)$$

In this package the electron chemical potential is ignored as negligible compared to the matter temperature, so

$$K_m(T) = 7 \frac{\sigma_0}{m_B} \left(\frac{T}{m_e c^2} \right)^2 \quad (4.62)$$

To obtain an implicit scheme for the neutrino heating 4.61 can be re-written as

$$\rho C_v \frac{T' - T}{\Delta t} = A - (T')^6 B \quad (4.63)$$

which can be approximated to first order by defining $T' = T + \Delta T$ and expanding the second term on the right hand side then dropping terms higher than first order in ΔT to get

$$\rho C_v \frac{\Delta T}{\Delta t} = A - (T^6 + 6T^5 \Delta T) B \quad (4.64)$$

Solving for T' ,

$$T' = T + \Delta t \frac{A - BT^6}{\rho C_v + 6BT^5 \Delta t} \quad (4.65)$$

In practice this new temperature is converted to an energy change rate and is added into the hydrodynamics subcycling such that over a complete hydro cycle the correct amount of energy is added to a zone.

RADIATION II:

This package differs from RADIATION I in that it does not include a flux limited diffusion package. There is no neutrino field calculated in this package, only core neutrino emissions. RADIATION II was used for the late time convection problem, which has simpler transport requirements, but requires the addition of neutrino-electron scattering. Changes to the electron fraction Y_e are followed.

NEUTLUM

Assigns neutrino energies and luminosities to the core neutrino emissions.

YECHANGE

Finds changes in Y_e caused by neutrino interactions.

CAPHEAT

Calculates energy change in matter caused by electron capture type reactions (as listed under FIELDHEAT above).

ESCATT

Does neutrino-electron scattering. Calculates electron-positron pair density, then finds energy exchanged through scattering process averaged over neutrino energy and scattering angle.

The luminosities are specified as functions of time, which are given below. These are fits to the M&W code luminosities. Also given are the fits to the M&W code's averaged neutrino energies.

$$\begin{aligned}
 L_\nu &= 10^{52} \left(\frac{0.9}{t + 0.33t^2} + \frac{1}{5 + t} \right) \left(1 + \frac{(t/2)^2}{1 + (t/2)^4} \right) \text{ ergs/sec} & (4.66) \\
 L_{\nu_e} &= L_\nu / 1.075 \\
 L_{\bar{\nu}_e} &= 1.075 L_\nu
 \end{aligned}$$

$$\begin{aligned}
L_{\nu\mu\tau} &= 4.0L_\nu \\
\varepsilon_{\nu\mu\tau} &= 26.0 - 3.0/t \\
\varepsilon_{\nu e} &= 12.0 - 0.08/t \\
\varepsilon_{\bar{\nu} e} &= \begin{cases} 16.5 - 0.5/t & \text{if } t \leq 5.0 \\ 20.0 - 18.0/t & \text{if } t > 5.0 \end{cases}
\end{aligned} \tag{4.67}$$

Modeling neutrino-electron scattering is necessary for the nucleosynthesis runs. The following relation, which is explained in more detail in chapter 7, section 2, is used for an energy exchange rate[5];

$$\begin{aligned}
\dot{\varepsilon}_{\text{scatt}} &= \frac{1}{2} \left(1 - \frac{3T}{\bar{\varepsilon}_\nu} \right) \frac{3\sigma_0 \bar{n}_\nu n_{e^+e^-} \bar{\varepsilon}_\nu c}{8(m_e c^2)^2 \rho} \left(T + \frac{m_e c^2}{3} \right) \\
&\quad \times [\varepsilon_{\nu e} (C_{e-\nu e} + C_{e+\nu e}) + \varepsilon_{\bar{\nu} e} (C_{e-\bar{\nu} e} + C_{e+\bar{\nu} e}) + \varepsilon_{\nu\mu\tau} (C_{e-\nu\mu\tau} + C_{e+\nu\mu\tau})]
\end{aligned} \tag{4.68}$$

The energy per gram change can be implemented implicitly. Collect all of the factors on the right hand side into a term B to get

$$C_v \frac{\Delta T}{\Delta t} = B \left[1 - \frac{3(T + \Delta T)}{\bar{\varepsilon}_\nu} \right] \tag{4.69}$$

where $\bar{\varepsilon}_\nu = \frac{1}{3}(\varepsilon_{\nu e} + \varepsilon_{\bar{\nu} e} + \varepsilon_{\nu\mu\tau})$. This is then solved for $C_v \Delta T / \Delta t$ to get the heating rate.

$$\Delta \varepsilon_{\text{scatt}} = \frac{\Delta t B (1 - \frac{3T}{\bar{\varepsilon}_\nu})}{1 + \frac{3B \Delta t}{\bar{\varepsilon}_\nu C_v}} \tag{4.70}$$

This heating rate is applied during the hydrodynamics cycle after the PdV work and before the transport routines.

The electron number fraction Y_e (number of electrons per baryon) must be advected and also correctly modified by neutrino absorption reactions. Transport of Y_e , which is converted to electron number density for this purpose, is handled exactly as described above for mass transport. The change in Y_e caused by neutrino interactions is calculated explicitly zone by zone using the following equation.

$$\frac{\partial Y_e}{\partial t} + \nabla \cdot (\mathbf{v} Y_e) = \frac{\sigma_0}{3\pi R^2 (m_e c^2)^2} \left[X_n L_{\nu e} \varepsilon_{\nu e} - X_p L_{\bar{\nu} e} \varepsilon_{\bar{\nu} e} \right] \tag{4.71}$$

Chapter 5

Convection and the Supernova Mechanism

Convection, driven by a negative entropy gradient, occurs between the entropy maximum produced by neutrino heating and the shock, and transfers energy from near the neutrinosphere out to the stalled shock (helping to revive it). The initial conditions come from the M&W code, which is referred to as “the spherical code.”

What effect does exterior convection (convection above the neutrinosphere) have on the supernova mechanism? The M&W one dimensional code has been used in an attempt to answer this question but uncertainties persist in the spherical code’s convection model. The 2-D simulations described here have been performed to quantify the effects of convection on the stalled shock. The neutrino luminosity is fine tuned in the 2-D code to generate shock behavior that is as close as possible to a shock that marginally fails in the M&W code, in order to give convection the best possible chance to make a difference in the problem. Some differences remain in the shock time history of the two codes, primarily because the 2-D code does not model all of the neutrino species (μ and τ neutrinos are ignored in these runs), the 2-D code has a simpler equation of state, the 2-D code does not model the interior $1.12M_{\odot}$ of the star, and finally the 2-D code is not general relativistic. The only thing done to

introduce convection into the 2-D problem is the addition of a small velocity perturbation to the spherically symmetric initial conditions. A study was made of how large the initial perturbations need to be to have an effect on the supernova explosion.

All of the runs are made by perturbing the initial conditions of a particular 1D calculation. The Woosley and Weaver $20M_{\odot}$ progenitor model of 1987A is evolved up to five milliseconds past bounce with the M&W spherical code. Then the spherical code results are linked over to the 2D code by setting the 2D mass density, pressure and radial velocity equal to the 1D values. A spherical proto-neutron star of $1.12M_{\odot}$ is assumed to exist below the computational region; this mass is taken from the spherical code. The mass density, pressure and radial velocity are taken directly from the spherical code. The pressure is matched rather than the energy because the system is not in hydrostatic equilibrium and using the pressure reproduces the flow of momentum in the problem. This procedure gives reasonable agreement with the spherical code for the derived quantities T , the temperature, and X_B , the free baryon fraction. Both behind and in front of the shock the difference between the spherical model and the 2D code is less than 7% in T and less than 20% in X_B . The initial density, pressure, specific energy, entropy, and free baryon fraction are shown in Figures 5.1 and 5.2. Note that these figures are plotted against the logarithm of radial distance. The shock is spread over more than three or four zones initially due to the 1-D to 2-D code re-mapping but after a few hydro cycles the shock is only two to three zones wide as it should be. Some quantities, like the entropy, are not correctly defined in the zones initially being shocked and these show sharp spikes that die away rapidly after a few hydro cycles.

The inner boundary is a reflecting wall slightly inside the location of the neutrinosphere at the start of the run. The inner boundary follows the motion of its mass coordinate, which is taken from the spherical code. This simulates the shrinking of the proto-neutron star as it cools. The neutrinosphere moves outward in mass during the run as matter accretes. By the end of the runs (400 milliseconds post bounce) the neutrinosphere encompasses $1.5M_{\odot}$. The outer boundary is stationary in space

Figure 5.1: Mass, Pressure, Velocity, and Entropy initial conditions, $t = 0.005$ seconds, taken from the M&W code.

Figure 5.2: γ , Temperature, free baryon fraction, and specific energy initial conditions, $t = 0.005$ seconds, taken from the M&W code.

but has an inward flow of mass pouring through it with a density, temperature and velocity which is again taken from the spherical code. As a result, the calculation is believable only up to the point where the shock wave begins to interact with the last radial zones in the grid. The extent of the computational region is 50 Km to 1500 Km when the runs begin; the inner boundary falls to about 10 Km by 300 milliseconds. For most runs the angular boundaries are a reflecting wall at the equator ($\theta = \frac{\pi}{2}$) and the axis ($\theta = 0$).

Since the grid starts at the neutrinosphere and there is no initial neutrino flow in the problem a core luminosity is introduced to bridge the time gap between the starting time and the time it takes the neutrino energy densities to build up to appreciable levels. The core luminosity generated by the proto-neutron star as a function of time is taken as $L_{\text{core}} = 10^{52} e^{-t/.04}$ ergs/sec. No spectral information is carried; instead a core neutrino temperature of $T_c = 5$ MeV is assigned to the core neutrinos. The 2-D code produces a total neutrino luminosity that is within 10% of the luminosity generated by the M&W code. See Figure 3.1.

The effects of the core luminosity and core temperature combine with the accretion luminosity to give a shock position vs. time profile that is similar to the behavior of the spherical code in that it stalls and does not produce an explosion. The shock moves out to about 350 Km in 60 ms, then oscillates between 300 and 450 Km, driven by the neutrino radiation coming off the matter accreting onto the proto-neutron star. This continues up to 400 ms post bounce. The unperturbed run has a shock radius versus time profile that is slightly farther out than the M&W spherical calculation (see Figure 5.3). There are some important differences between the two codes which account for the gap in the shock profiles. The M&W code includes general relativity and the effects of μ and τ neutrinos, both of which degrade the shock; these are absent in the 2-D code. The equations of state are not exactly the same either, and the neutrino transport is more accurately done in the M&W code by using multiple energy groups.

Runs were performed with three kinds of perturbations to the velocity; sinusoidal,

Figure 5.3: Shock versus time profiles for a) the M&W code without convection, b) the M&W code with a 1-D convection model turned on, and c) BUBBA run in a 1-D mode.

random, and runs with non-zero rotation. Each initial perturbation is discussed with its results in detail below.

5.1 Sinusoidal Runs

A series of sinusoidal runs were done to find out how big the initial perturbation must be to develop significant convective effects and to find the growth rate of the instability. A sinusoidal perturbation was imposed on the radial velocity of the form

$$v_r(r, \theta) = v_r(r)(1 + a \sin 4\theta), \quad (5.1)$$

where a ranged from 10^{-3} to 10^{-8} . In Figure 5.4 one can see the effects of these perturbations on the shock. Very large perturbations from sphericity will develop significant convection after a few hundred milliseconds and this pushes the shock off our computational grid. Smaller perturbations require more time to develop strong

convection and the shock is less affected.

Figure 5.4: Shock positions versus time for six different initial sinusoidal velocity perturbation amplitudes: from top to bottom on the right hand edge of the plot, the perturbation amplitudes are 10^{-3} , 10^{-4} , 10^{-5} , 10^{-6} , 10^{-7} , and 10^{-8} .

A measure is necessary for how much convective activity exists at a particular point if one is to make a quantitative comparison with the spherical code convection model. Define the 2-D convective velocity to be

$$v_c = \frac{1}{n_\theta} \sum_{j=0, n_\theta} |\bar{v}_r(r_i) - v_r(r_i, \theta_j)|, \quad (5.2)$$

where $\bar{v}_r$ is the angle averaged radial velocity at r_i . This calculation is done halfway between the current zone of maximum entropy and the shock zone, hence r_i in Equation 5.2 changes during the calculation. The convective velocities plotted in Figure 5.5 show the growth of the sinusoidal perturbations. The exponential growth of the instability is clear in the 200 to 400 millisecond region of the $a = 10^{-6}$ to $a = 10^{-8}$ runs. Using just these exponential regions, the growth rate is 80 sec^{-1} , giving an e-folding time of 13 milliseconds. A dimensional analysis of the growth rate can be

made. Take

$$1/\tau \approx \sqrt{(g/l)}, \quad (5.3)$$

where g is the local acceleration due to gravity and l is the scale length of the vortices. From the 2-D code results $g \approx 2 \times 10^{11} \text{ cm/sec}^2$ and $l \approx 2 \times 10^7 \text{ cm}$, which gives $1/\tau = 100 \text{ sec}^{-1}$. The convection velocity does not grow indefinitely but saturates at the same order of magnitude in all cases. The final value varies directly with the size of the initial perturbation from $3 \times 10^8 \text{ cm/sec}$ to $6 \times 10^8 \text{ cm/sec}$.

Figure 5.5: Convective velocities vs. time for six different initial sinusoidal velocity perturbation amplitudes: from top to bottom on the left hand edge of the plot, the perturbation amplitudes are 10^{-3} , 10^{-4} , 10^{-5} , 10^{-6} , 10^{-7} , and 10^{-8} .

A negative entropy gradient generates convection cells and it has sometimes been assumed that the action of the vortices would quickly eliminate the condition that gave rise to them, leaving a relatively flat entropy profile in that region. Figure 5.6 shows that this is not true on the timescales involved here. The instability saturates at a fairly large convection velocity but the entropy gradient persists. Note in Figure 5.6 that the maximum value of the entropy does not change much over time (this is true

for all of the 2-D runs). This is because even though cooler material is brought down by convection, it is quickly heated through neutrino interactions. At the radius of peak entropy the net heating is set by the difference of large heating and cooling terms. This effect maintains the peak entropy.

Figure 5.6: Entropy, mass averaged over angle, at $t = 0.005$, $t = 0.050$, $t = 0.100$, $t = 0.150$, $t = 0.200$, $t = 0.250$, $t = 0.300$, $t = 0.350$, $t = 0.400$ seconds, for a 10^{-3} sinusoidal velocity perturbation run. Increasing times go with increasing values of the maximum entropy.

5.2 Random Runs

Two random velocity perturbation runs were performed. A random perturbation seems more realistic than a sinusoidal one and it has the added benefit of testing at once all the wavelengths at which the hydrodynamic configuration may be unstable. The results of these runs were very similar to the sinusoidal series; this gives some measure of confidence that the sinusoidal perturbation does not mask some undiscovered

behavior of the system. Shock position and convective velocity profiles for the random runs were similar to those for sinusoidal runs with perturbations of the same amplitude. In Figure 5.7 shows a combined entropy and velocity plot of the 10^{-4} random perturbation run at $t = .267$ sec. This run is similar to all of the 2-D runs in that the subsonic flow underneath the shock front barely deforms the shock from a perfect sphere.

A convection velocity analogous to the one used in the M&W spherical code is calculated during the 2-D runs (Wilson and Mayle 1992). The entropy gradient is used in the 2-D code to estimate what convection velocity the spherical code would calculate. This 1-D estimated convection velocity is several orders of magnitude larger than the actual 2-D convective velocity throughout most of the run because it does not allow for the growth time of the instability, but even at the end of the run it is about four times larger than the fastest convection velocity found in any of the 2-D calculations. See Figure 5.8.

One run was made with a random 10^{-4} perturbation on a full 180° grid to make sure the equatorial boundary condition was not affecting the results. The shock position versus time profile was not noticeably different from the 10^{-4} run shown in Figure 5.4. The time history of the convective velocity and the inferred M&W code convective velocity were very similar to those shown for the 10^{-4} random perturbation curve in Figure 5.8. During the 180° run ten vortices formed initially, all of about equal size, spanning from pole to pole in a half-circle about the center at $R \approx 175$ Km. These grew and coalesced, first into five, then into two large vortices, one in each hemisphere, and one small one at the equator dividing them. See Figure 5.9. This phenomenon of many small vortices combining into a few larger ones occurred in all of the runs; Herant, Benz and Colgate[14] noticed the same thing, though they were examining much later times. See Figure 5.10, which shows the 180° run at late time. Once again the shock remains almost perfectly spherical over the entire calculation. Overall the 180° run is not significantly different in any way from the 90° runs. Figures 5.12 through 5.19 show the time development of a typical random

Figure 5.7: Isoentropy contours and fluid velocity vectors for the 10^{-4} random perturbation run at $t = .267$ sec.

Figure 5.8: Convection velocity for the σ^{-4} random perturbation run is well below the estimated M&W code convection velocity.

run. The one-dimensional slices (Figures 5.12 and 5.13) show the configuration just before significant asymmetries begin to appear.

5.3 Rotational Run

Rotation was added to the unperturbed problem to see what effect, if any, it would have in driving the instability. A specific angular momentum of $6.3 \times 10^{12} \text{ cm}^2 \text{ sec}^{-1}$ was assumed above the cylindrical radius corresponding to the initial neutrinosphere radius of the star. Inside this radius the rotation velocity was set to a constant equal to the rotation velocity at the neutrinosphere. This roughly corresponds to the material on the surface of a 10 Km radius neutron star rotating once per second and gives an initial total rotational energy of 5×10^{43} ergs. There are no sources or sinks of angular momentum in the model. The rotation run generated convective action

Figure 5.9: Entropy and velocity plot at $t = 0.225$ seconds. The rising bubbles in the upwelling regions have entropies of about ten. The shock is clearly visible at 375 Km.

Figure 5.10: Entropy and velocity plot at $t = 0.400$ seconds. There are now only three bubbles, not five as in the previous figure. Entropies have risen slightly due to neutrino heating. The shock is at 1050 Km.

Figure 5.11: Zoomed shot of previous figure. Note the compression of downward flows and rapid expansion of rising bubbles.

Figure 5.12: Radial slice along 45° line of mass density, pressure, entropy, and velocity at $t = 0.170$ seconds. The star is still very symmetric at this time.

Figure 5.13: Radial slice along 45° line of γ , temperature, specific energy, and free baryon fraction at $t = 0.170$ seconds.

Figure 5.14: Entropy contours and velocity vectors at $t = 0.170$ seconds.

Figure 5.15: Entropy contours and velocity vectors at $t = 0.215$ seconds.

Figure 5.16: Entropy contours and velocity vectors at $t = 0.236$ seconds.

Figure 5.17: Entropy contours and velocity vectors at $t = 0.277$ seconds.

Figure 5.18: Entropy contours and velocity vectors at $t = 0.354$ seconds.

Figure 5.19: Entropy contours and velocity vectors at $t = 0.383$ seconds.

in much the same manner as the 10^{-5} sinusoidal perturbation run. The shock was similarly enhanced and both the shock versus time and convection velocity versus time plots are very much like the 10^{-5} sinusoidal run.

The possibility for convective growth close to the neutrinosphere has been pointed out by A. Burrows[7] and others[11]. In the M&W run a double bump in entropy forms early on, at about 20 ms post bounce, with the peaks at $1.1M_{\odot}$ and $1.3M_{\odot}$. The $1.3M_{\odot}$ peak is enhanced by neutrino transport and by 40 ms completely eclipses the inner peak, though a small valley may remain for some time afterwards. In the M&W 1-D code run this valley is actually stabilized by a positive gradient in Y_e . In the 2-D code the entropy behaves much the same as described above. The entropy gradient is negative behind the shock in the beginning but by 50 milliseconds the second peak has formed and is well on its way to dominating the calculation. Convection interior to this main peak occasionally appears in 2-D runs, presumably because variations are not allowed in Y_e . A thin layer, starting at about 30 Km and ending at 70 Km, develops a mild negative entropy gradient and becomes unstable. The growth time is very short due to the high sound speeds and small length scales at that radius but the instability saturates at a convective velocity that is two to three times lower than the outer convection (the entropy decline is not deep). Just outward of this inner convective layer, the entropy per baryon increases to its maximum. This entropy wall completely isolates the inner convection layer, which does not interact with the outer fluid motion. The thin layer of inner convection appears to be irrelevant to the success of the explosion.

It seems unlikely that exterior convection can save a stalled shock. The 2-D code is adequate to calculate the convection velocity beneath the shock in the first half second post bounce, because the convection is determined primarily by simple hydrodynamics and electron neutrino transport. However the 2-D code is not yet a complete supernova model. It lacks general relativity, mu and tau neutrinos, and forces the electron fraction to be constant, $\dot{Y}_e = 0$, hence one should not give much weight to the fact that it eventually produces explosions. The M&W 1-D code, which

is a thorough calculation, has a convection scheme that uses a convective velocity estimate that has turned out to be an order of magnitude higher than the actual convective velocity calculated by the 2-D code during the first 0.4 seconds after bounce (see Figure 5.8). Even with this overestimate of exterior convective effects the shock still does not continue outward in the M&W calculations unless neutron fingers are turned on. In short, the 2-D code has computed a convective velocity for use in the M&W code and the M&W code cannot produce an explosion using that convective velocity alone.

The growth rate for exterior convection is low, about 100 sec^{-1} , and the unstable conditions for convective growth are not well developed until about 0.10 seconds after bounce. The M&W code runs (see Figure 5.3) indicate that any mechanism that is going to act to save the shock must do so before 0.20 seconds past bounce. There are about 10 e-folding times in which to develop strong convection; this will increase the initial asymmetries by only 10^4 . It seems plausible that the ultimate success or failure of the shock will be decided before exterior convection can become a significant factor, unless the initial perturbations are very large. One can see by cross referencing Figure 5.4 and Figure 5.5 that the shock is not strongly affected by convection until the convective velocities reach $\sim 10^8 \text{ cm/sec}$. Ergo, the initial perturbation must have some characteristic velocity scale of about 10^4 cm/sec if it is to have a significant effect on the time history of the shock.

It is interesting that the shock is only barely deformed from a perfect sphere even when convective effects beneath it are quite pronounced, as in both the 180° and 10^{-3} sinusoidal perturbation run. It therefore seems safe to continue to believe that it is not necessary to go to two spatial dimensions computationally to get a good supernova explosion model as long as one includes an accurate 1-D representation of convective effects. What a self-consistent 1-D representation might be is still unclear. Further 2-D work is needed to better characterize the development and effects of convection so that an improved 1-D model of convection can be constructed.

The present M&W spherical code produces a successful explosion but it requires

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convection beneath the neutrinosphere to do so (as opposed to the exterior convection discussed in this thesis). Further study of convection beneath the neutrinosphere, in particular a numerical calculation of the neutron finger instability, is needed.

Chapter 6

Neutron Fingers

In the delayed bounce mechanism for Type II supernovae the shockwave stalls while trying to exit the iron core. It is then revived when neutrinos streaming from the proto-neutron star heat the material behind the shock through nuclear absorption and inelastic scattering off of electrons. Clearly, any process that increases the neutrino luminosity increases the likelihood that the stalled shock will start moving again and produce a supernova. Convection could bring significant amounts of proton rich material from within the inner core up to a region where electron capture is no longer blocked by neutrino degeneracy. It would also advect out the neutrinos trapped in the denser material below. Both effects would increase the neutrino flux appreciably over what would otherwise be expected.

The type of convection examined in this section is a doubly diffusive phenomenon that may occur beneath the surface of the proto-neutron star. Following Smarr[19], let us call this “neutron fingers”. Doubly diffusive situations are characterized by having two spatial gradients, often temperature and composition, in the same direction as the gravitational field. One gradient is stabilizing and the other is destabilizing. In this way it is possible for a fluid to be unstable to convection even though entropy increases with height (see Chandasekhar’s *Stellar Atmospheres* for a derivation of convection stability criteria).

It is useful to consider ordinary salt fingers before proceeding to neutron fingers,

because salt and temperature gradients in water are simpler to visualize. Imagine a layer of hot, salty water lying over a layer of cold fresh water. Salty water is more dense than fresh so it should sink, but cold water is more dense than hot. One gradient is stabilizing, the other is destabilizing. Heat diffuses much more quickly than salt in water; the ratio of the thermal diffusivity to the saline diffusivity is about 100. If a blob of the cold, fresh water should rise up a little into the hot, salty water, it will quickly become a hot fresh blob. The hot fresh blob will be less dense than the hot salty surroundings and it will continue to rise up. The opposite is true of a hot salty blob that enters the cold, fresh water. It becomes a cold salty blob that is more dense than the cold, fresh water around it and it continues to sink; the situation is unstable. Ultimately, long thin fingers of hot, salty water penetrate the cold, fresh, bottom layer and vice versa.

A similar initial arrangement exists in the proto-neutron star. The collapse stops when the center reaches nuclear density. The homologous inner core (roughly $0.5 M_{\odot}$) is falling in at subsonic speeds, so the information that the collapse has halted is transmitted as a relatively gentle sound wave. Outside the homologous core the infall is supersonic. This causes a shock to form just outside the sonic point. The shock moves outward, raising the entropy of the outer mantle to three to four times that of the inner core. Neutrinos trapped in the inner core block electron capture and prevent severe deleptonization there. The proton fraction, Y , remains about 0.3. The outer mantle is hotter and less dense, so neutrinos escape more easily. This leaves more room in phase space for the neutrinos released by electron capture, which drives the proton fraction down to about $Y \approx 0.1$. Hence, the proton fraction decreases with radius. The final picture is a sphere of about one and a half solar masses with opposing spatial gradients in entropy and proton fraction. Energy is transported via scattering interactions by all six neutrino species but composition is changed by only two, the electron and anti-electron neutrino through

hence the energy and composition transport rates are different. This situation is similar enough to the garden-variety salt fingers scenario that it seems prudent to find out whether or not significant convection can take place. See Figures 6.1 and 6.2. The first figure shows the shock at $1.2 M_{\odot}$ just 5 milliseconds after bounce. Deleptonization behind the shock has already begun to work its way inward causing the region from .8 to $1.2 M_{\odot}$ to possibly be unstable. By 175 milliseconds (Figure 6.2) all but the center of the core has been deleptonized and neutrino losses have brought down the entropy in the central region from about three units to about one. A much larger fraction of the core may now be unstable to the neutron finger instability. It is likely that the instability will develop at some time between the two figures shown. Once fully developed, the existence of convective action would most likely make significant changes to the one dimensional profiles shown here.

Figure 6.1: Electron number fraction (Y_e) and entropy (S) profiles from the M&W 1-D code at $t = 0.005$ seconds postbounce. The shock is clearly visible at $1.2 M_{\odot}$, where the entropy suddenly drops from 7 to 2.5 and Y_e suddenly rises to 0.5.

In this paper it is assumed that a neutron finger has already developed, as in Figure 6.3, then an estimate is made of the growth rate of the difference in density

Figure 6.2: Electron number fraction (Y_e) and entropy (S) profiles from the M&W 1-D code at $t = 0.175$ seconds postbounce. The lepton number of the core has decreased and the entropy of the central region has dropped a few units. The spike in entropy near $1.5 M_\odot$ marks the start of what will become the hot bubble behind the shock.

between the finger and its surroundings.

6.1 Physical Description, Definitions and Approximations

Approximate a fully developed neutron finger sinking into the low entropy inner core by considering two adjacent slabs of matter. Call the left slab a , the right b . Slab a consists of high entropy, neutron rich material (hence a lower proton fraction); this is the neutron finger. Slab b represents the surrounding material which has a higher proton fraction. Assume the material moves quickly enough that the finger retains most of its original composition, but slowly enough that the finger is in thermal (and pressure) equilibrium with its surroundings. One can then calculate gradients in temperature and neutrino chemical potential across the finger boundary in terms

Figure 6.3: Cold neutron poor fingers rise into higher entropy material that is more neutron rich and vice versa.

of the unconvected quantities. The outer walls are rigid and impermeable to heat, matter, and neutrinos; this is really just an assumption of symmetry with respect to the center of a finger. The flexible dividing wall allows neutrinos to pass, but not matter. The entire process is inside the neutrinosphere, so the diffusion approximation for neutrinos is valid. Take the chemical potential of the mu and tau neutrinos to be zero, and to simplify the algebra ahead take the mass of a horizontal slice in each slab to be the same, i.e. $M_a = M_b$. Assume the rough width of the neutron finger can be found by an expression Baines and Gill[2] derived for salt fingers. This is

$$L \approx \left(\frac{g \alpha}{\nu K_T} \frac{\partial T}{\partial z} \right)^{-1/4} \quad (6.2)$$

where K_T is the thermal diffusivity, ν is the specific viscosity, g is the local gravitational acceleration, α is the coefficient of thermal expansion, and $\frac{\partial T}{\partial z}$ is the temperature difference across the characteristic height. In the neutron fingers problem the specific

viscosity ν is replaced by neutrino viscosity, which is caused by the neutrinos having a different momentum vector than the fluid advecting them along with it. For the neutron finger case this gives a width L of approximately 6×10^5 cm. A typical radius is 2.4×10^6 cm, so about sixty fingers could be expected to form in the core.

6.2 Defining Equations

The equations that define this system break up in to three groups; the conservation equations, pressure balance and equation of state, and the equations that govern the rates of change of mass density, energy density, and proton number.

Consider a slab sliced out of the finger and its surroundings. The slab must be thin enough in z that physical quantities such as density and proton number are approximately constant in z . The conserved quantities are volume, total energy, and proton number. Total mass is implicitly conserved because the total volume doesn't change and no mass is considered to be flowing up into (or down out of) the slabs. Let L be the combined width of the two slabs, and L_a and L_b be the widths of slabs a and b , respectively. Let A be the area of contact between the two slabs. Then trivially

$$L_a + L_b = L \quad (6.3)$$

which with $M = M_a = M_b$ gives

$$\frac{1}{\rho_a A} + \frac{1}{\rho_b A} = L/M. \quad (6.4)$$

Taking the time derivative of this equation gives a condition on the densities

$$\frac{\dot{\rho}_a}{\rho_a^2} + \frac{\dot{\rho}_b}{\rho_b^2} = 0 \quad (6.5)$$

Because the walls are impermeable to heat, mass, and neutrinos, total energy must be conserved. Mass is constant, so total energy per mass in the slice must also be constant. E is the energy per volume. This gives

$$\frac{d}{dt} \left(\frac{E_a}{\rho_a} + \frac{E_b}{\rho_b} \right) = 0 \quad (6.6)$$

Proton number is also conserved. Y is the number of protons per baryon, which leads to

$$\dot{Y}_a + \dot{Y}_b = 0 \quad (6.7)$$

Impose pressure balance, and assume a simple ideal gas equation of state with relativistic electrons and non-relativistic baryons. This gives an energy density and pressure that is within 30% of the values seen in supernova code runs with much more realistic equations of state, so the simplified EOS should be accurate enough to get an idea of how a neutron finger will behave. A rough estimate of the neutrino contributions to pressure and energy density shows that the neutrino pressure and energy density are approximately 10^{27} dynes/cm² and 10^{34} MeV/cm³ respectively, which is three to four orders of magnitude smaller than the ideal gas pressure and energy density in the region of interest. The neutrino contributions are ignored.

$$P_a = P_b \quad (6.8)$$

$$P = \rho \frac{kT}{m_B} (1 + Y) \quad (6.9)$$

$$E = 3\rho \frac{kT}{m_B} \left(\frac{1}{2} + Y \right) \quad (6.10)$$

The rates of change are more complicated expressions. Starting with the first law of thermodynamics

$$d(EV) = dQ - PdV, \quad (6.11)$$

one obtains (for slab a , for example)

$$\frac{\dot{E}_a}{E_a} = \frac{\dot{Q}_a}{VE_a} + \frac{\dot{\rho}_a}{\rho_a} \left(\frac{P_a}{E_a} + 1 \right). \quad (6.12)$$

$\dot{Q}$ is the energy change due to the neutrino energy flux, Φ_E , so

$$\frac{\dot{Q}}{V} = -\frac{\Phi_E}{L_a} \quad (6.13)$$

which gives

$$\frac{\dot{E}_a}{E_a} = -\frac{\Phi_E}{L_a E_a} + \gamma \frac{\dot{\rho}_a}{\rho_a} \quad (6.14)$$

where $\gamma = \frac{P}{E} + 1$. Finally, the change in proton number due to a neutrino flux is needed. Calling the proton number flux due to neutrinos Φ_N , this is given by

$$\frac{\dot{Y}_a \rho_a}{m_B} = -\frac{\Phi_N}{L_a} \quad (6.15)$$

Equations 6.5 through 6.10, 6.14 and 6.15 define the system.

Now expressions for Φ_E and Φ_N must be found. Using the diffusion approximation, the neutrino energy flux is given by

$$\Phi_E = -\frac{\lambda_\nu c}{3} \nabla \varepsilon. \quad (6.16)$$

Consider only the x direction (perpendicular to the flow of the neutron finger) and use the Fermi-Dirac distribution function to get

$$\Phi_E = -\int_0^\infty \frac{\lambda_{\nu_e} c}{3} \frac{\partial}{\partial x} (F_{\nu_e} + F_{\bar{\nu}_e}) d\varepsilon - \int_0^\infty \frac{4\lambda_{\nu_{\mu\tau}} c}{3} \frac{\partial}{\partial x} F_{\nu_{\mu\tau}} d\varepsilon, \quad (6.17)$$

where

$$F_i = \frac{4\pi\varepsilon^3}{(hc)^3 (e^{(\varepsilon-\mu_i)/kT} + 1)}. \quad (6.18)$$

It has been assumed that $\lambda_{\nu_e} = \lambda_{\bar{\nu}_e}$, which is true to within 25 percent[20]. The mu and tau neutrinos (and their anti-particles) enter the scenario only through electron-positron pair creation and annihilation processes, hence their chemical potentials will be zero and $F_{\nu_\mu} = F_{\nu_\tau} = F_{\bar{\nu}_\mu} = F_{\bar{\nu}_\tau}$. This gives the factor of four in the second integral in 6.17.

$\mu_{\nu_{\mu\tau}}$ has been taken to be zero. Using $\lambda_{\nu_e} = \lambda_{\nu_0} (m_e c^2 / \varepsilon)^2$ and $\lambda_{\nu_{\mu\tau}} = \lambda_{\mu\tau_0} (m_e c^2 / \varepsilon)^2$, [20] yields an expression for the neutrino energy flux.

$$\Phi_E = -\frac{4\pi\lambda_{\nu_0} m_e^2 c^2}{3h^3} \frac{\partial}{\partial x} \left[\int_0^\infty \frac{\varepsilon d\varepsilon}{e^{(\varepsilon-\mu_{\nu_e})/kT} + 1} + \int_0^\infty \frac{\varepsilon d\varepsilon}{e^{(\varepsilon+\mu_{\nu_e})/kT} + 1} + \frac{\lambda_{\mu\tau_0}}{\lambda_{\nu_0}} \int_0^\infty \frac{\varepsilon d\varepsilon}{e^{\varepsilon/kT} + 1} \right] \quad (6.19)$$

The last integral is standard and equal to $(kT)^2 \pi^2 / 12$. Only the first integral is examined here since the second is done in a similar fashion. The first integral can be broken into two parts by substituting $w = (\varepsilon - \mu)/kT$ and $\xi = \mu/kT$. This leaves

$$\Phi_E = -\frac{4\pi\lambda_{\nu_0} m_e^2 c^2}{3h^3} (kT)^2 \frac{\partial}{\partial x} \left[\int_{-\xi}^0 \frac{(w + \xi) dw}{e^w + 1} + \int_0^\infty \frac{(w + \xi) dw}{e^w + 1} \right]. \quad (6.20)$$

The last integral on the right hand side is again a standard integral, equal to $\pi^2/12 + \xi \ln 2$. This gives the recalcitrant first integral on the right hand side. Approximate the integrand by expanding it about $w = 0$ and keeping only the first two terms. The total error is minor because the integral over the region from zero to infinity can be done exactly and it is this piece that contributes most of the value to the total integral. This procedure gives a 7% error in the value of the *total* integral from $-\xi$ to infinity when $\xi \approx 1.2$, which is the region of interest (this value for ξ comes from code runs done by Wilson and Mayle). The linear approximation is trivial to integrate and for Φ_E one gets

$$\Phi_E = -\frac{4\pi\lambda_{\nu_0}c}{3}\left(\frac{m_e c}{h}\right)^3 \left[\mu_{\nu_e} \frac{\partial}{\partial x} \left(\frac{\mu_{\nu_e}}{m_e c^2} \right) + \left(\frac{1}{3} + \frac{2\lambda_{\nu_0} \mu_{\nu_0}}{3\lambda_{\nu_0}} \right) \pi^2 kT \frac{\partial}{\partial x} \left(\frac{kT}{m_e c^2} \right) \right]. \quad (6.21)$$

Next an expression for the proton number flux is needed. Electron neutrinos lower the proton fraction and anti-electron neutrinos raise it. The other varieties of neutrino do nothing in this regard, so they need not be considered. Once again making use of the diffusion approximation,

$$\Phi_N = -\int_0^\infty \frac{\lambda_{\nu_e} c}{3} \frac{\partial}{\partial x} \left(\frac{F_{\nu_e} - F_{\bar{\nu}_e}}{\varepsilon} \right) d\varepsilon. \quad (6.22)$$

This integral can be done analytically by making use of the relation

$$\int_{-\xi}^{\xi} \frac{dw}{e^w + 1} = \xi. \quad (6.23)$$

Armed with this integral, it is easy to show

$$\Phi_N = -\frac{4\pi\lambda_{\nu_0}c}{3}\left(\frac{m_e c}{h}\right)^3 \frac{\partial}{\partial x} \left(\frac{\mu_{\nu_e}}{m_e c^2} \right). \quad (6.24)$$

Now a system of equations that fully describes the system has been obtained. The interesting questions concern the density behavior of the neutron finger; specifically, under what conditions will it become more dense with time and continue to sink.

6.3 Solving for the Density Difference Rate

This is nothing more than a lengthy algebra exercise. Some steps will be sketched and some of the results one obtains along the way will be presented. First, eliminate

$\dot{\rho}_b, \dot{T}_b, \dot{Y}_b$ in favor of $\dot{\rho}_a, \dot{T}_a, \dot{Y}_a$ via equations 6.5, 6.6, 6.7, and the time derivative of 6.10. This yields

$$\dot{\rho}_b = -\left(\frac{\rho_b}{\rho_a}\right)^2 \dot{\rho}_a, \quad (6.25)$$

$$\dot{Y}_b = -\dot{Y}_a, \quad (6.26)$$

$$\dot{T}_b = \frac{\dot{Y}_a(T_b - T_a) - \dot{T}_a(\frac{1}{2} + Y_a)}{(\frac{1}{2} + Y_b)}. \quad (6.27)$$

Take equation 6.14 and insert 6.10 to get

$$\frac{\dot{T}_a}{T_a} - (\gamma - 1)\frac{\dot{\rho}_a}{\rho_a} + \frac{\dot{Y}_a}{\frac{1}{2} + Y_a} = -\frac{\Phi_E}{L_a E_a}. \quad (6.28)$$

Starting with 6.8, substitute 6.9, then take the time derivative and use 6.25, 6.26, and 6.27 to eliminate b derivatives in favor of a derivatives.

$$\frac{\dot{\rho}_a}{\rho_a} \left(1 + \frac{\rho_b}{\rho_a}\right) + \frac{\dot{T}_a}{T_a} \left[1 + \frac{T_a(\frac{1}{2} + Y_a)}{T_b(\frac{1}{2} + Y_b)}\right] + \dot{Y}_a \left[\frac{1}{1 + Y_a} + \frac{1}{1 + Y_b} + \frac{\frac{T_a}{T_b} - 1}{\frac{1}{2} + Y_b}\right] = 0. \quad (6.29)$$

Equations 6.15, 6.28, and 6.29 give a system of three equations for the three unknowns $\dot{\rho}_a, \dot{T}_a, \dot{Y}_a$. Solve the system for $\dot{\rho}_a/\rho_a$.

$$\frac{\dot{\rho}_a}{\rho_a} = \frac{\frac{\Phi_E}{L_a E_a} \left(1 + \frac{T_a(\frac{1}{2} + Y_a)}{T_b(\frac{1}{2} + Y_b)}\right) - \frac{m_B \Phi_N}{2\rho_a L_a} \left(\frac{1}{(1+Y_a)(\frac{1}{2} + Y_a)} + \frac{1}{(1+Y_b)(\frac{1}{2} + Y_b)}\right)}{1 + \frac{\rho_b}{\rho_a} + (\gamma - 1) \left(1 + \frac{T_a(\frac{1}{2} + Y_a)}{T_b(\frac{1}{2} + Y_b)}\right)}. \quad (6.30)$$

The density difference is $\Delta\rho = \rho_a - \rho_b$, and $\Delta\dot{\rho} = \dot{\rho}_a - \dot{\rho}_b$. Rewrite this using 6.13,

$$\Delta\dot{\rho} = \dot{\rho}_a \left[1 + \left(\frac{\rho_b}{\rho_a}\right)^2\right]. \quad (6.31)$$

Consolidate the awkward equation 6.30 by defining two dimensionless quantities a_E and a_N such that

$$\frac{\Delta\dot{\rho}}{\rho_a} = a_E \frac{\Phi_E}{L_a E_a} - a_N \frac{m_B \Phi_N}{\rho_a L_a}. \quad (6.32)$$

Recall the expressions for Φ_E and Φ_N and define $\epsilon_m = m_B E_a / \rho_a$, to get a final equation for the density difference time rate of change,

$$\frac{\Delta\dot{\rho}}{\rho_a} = -\left(\frac{a_E}{L_a E_a}\right) \frac{4\pi\lambda_{\nu_0} c}{3} \left(\frac{m_e c}{h}\right)^3 \left[\left(\mu_{\nu_e} - \frac{a_N \epsilon_m}{a_E}\right) \frac{\partial}{\partial x} \left(\frac{\mu_{\nu_e}}{m_e c^2}\right) + \right. \quad (6.33)$$

$$\left(\frac{1}{3} + \frac{2\lambda_{\nu\mu_0}}{3\lambda_{\nu_0}} \right) \pi kT \frac{\partial}{\partial x} \left(\frac{\pi kT}{m_e c^2} \right) \Big]. \quad (6.34)$$

Now the question is whether or not $\Delta\dot{\rho} > 0$? If it is, then the neutron finger will continue to sink. If not, the finger will stop sinking and equilibrate with its surroundings. In one of the models at 100ms after bounce the M&W code gives the following conditions in the region where neutron finger convection is expected; $r = 2.4 \times 10^6$ cm, $\rho = 2.9 \times 10^{12}$ g/cc, $T = 11.3$ MeV, $\mu_{\nu_e} = 13.9$ MeV, $\mu_e = 40.7$ MeV, $Y = 0.173$, $E/\rho = 3.3 \times 10^{19}$ erg/g, $P = 4.00 \times 10^{31}$ dynes/cm², $m_B E/\rho = 34$ MeV, $L \approx \text{fingerwidth} \approx 6.4 \times 10^5$ cm.

Evaluate a_E and a_N by taking $\rho_a = \rho_b$, $T_a = T_b$, and $Y_a = Y_b = 0.2$, to get $a_E = 1.4$, $a_N = 0.86$, and $\frac{\lambda_{\nu\mu_0}}{\lambda_{\nu_0}} \approx 5$.

$$\frac{\Delta\dot{\rho}}{\rho_a} = (4.8 \times 10^5 \text{ sec}^{-1}) \left[\frac{\partial}{\partial x} \left(\frac{\mu_{\nu_e}}{m_e c^2} \right) - 3.1 \frac{\partial}{\partial x} \left(\frac{\pi kT}{m_e c^2} \right) \right] \quad (6.35)$$

This gives an approximate time scale for finger development of 2 μ s. By comparison, the core crossing time for a sound wave is roughly 500 μ s.

The $\partial_x T$ term is negative (temperature decreases to the right in the picture), so it will always contribute to finger development. The coefficient in front of $\partial_x \mu_{\nu_e}$ can be positive or negative. Under the core conditions taken from code runs this factor is negative. Taking into account the minus sign in front of the expression (6.34), the neutrino chemical potential assists finger development rather than works against it. Only when $\mu_{\nu_e} \gg T$ and $\mu_{\nu_e} > \epsilon_m$ will the $\partial_x \mu_{\nu_e}$ term dominate the $\partial_x T$ term, in which case the neutron finger will be stable. Using the numbers given above for T , a_E , a_N , ϵ_m , and μ_{ν_e} , it appears that once a neutron finger begins to sink into the inner core it will continue to do so.

6.4 A More Realistic Equation of State

One can replace the ideal gas equation of state with a more realistic version and investigate the stability using the method just described. The EOS has been improved by including the effects of electron degeneracy to the energy density and pressure.

While this approach is not based on *ab initio* physics, the result is good to within 10% for all values of the electron temperature and Fermi energy.

$$E = \sqrt{E_{deg}^2 + E_{th}^2} + \frac{3}{2}\rho RT \quad (6.36)$$

$$P = \frac{1}{3}\sqrt{E_{deg}^2 + E_{th}^2} + \rho RT \quad (6.37)$$

The degenerate electron energy is

$$E_{deg} = \left(\frac{81}{\pi m_B^4}\right)^{\frac{1}{3}} \frac{hc}{8} (Y\rho)^{\frac{4}{3}} \quad (6.38)$$

and the electron thermal energy is

$$E_{th} = 3Y\rho RT \quad (6.39)$$

To find the stability criterion one plows through the same procedure as before. The algebra is more tedious—it is omitted here.

The degenerate equation of state combined with the assumptions above gives

$$\frac{\Delta\dot{\rho}}{\rho_a} = -\left(\frac{b_E}{L_a E_a}\right) \frac{4\pi\lambda_{\nu_0}c}{3} \left(\frac{m_e c}{h}\right)^3 \left[\left(\mu_{\nu_e} - \frac{b_N \epsilon_m}{b_E}\right) \frac{\partial}{\partial x} \left(\frac{\mu_{\nu_e}}{m_e c^2}\right) + \right. \quad (6.40)$$

$$\left. \left(\frac{1}{3} + \frac{2\lambda_{\nu\mu\tau_0}}{3\lambda_{\nu_0}}\right) \pi kT \frac{\partial}{\partial x} \left(\frac{\pi kT}{m_e c^2}\right) \right] \quad (6.41)$$

where the constants b_E and b_N are

$$b_E = \frac{b_9 b_1 b_5}{\rho_a (b_2 b_5 - b_3 b_4)} \quad (6.42)$$

$$b_N = \frac{b_9 b_6 (b_7 b_4 - b_8 b_5)}{\rho_a (b_5 b_2 - b_3 b_4)}. \quad (6.43)$$

The b_i 's are defined as follows:

$$b_1 = -E_a \quad (6.44)$$

$$b_2 = \frac{3}{2}RT_a + \frac{4a^2}{3D_a\rho_a} Y_a^{\frac{8}{3}} \rho_a^{\frac{5}{3}} + (3Y_a RT_a)^2 / D_a - \frac{\gamma E_a}{\rho_a} \quad (6.45)$$

$$b_3 = RT_a + \frac{1}{3\rho_a D_a} \left[\frac{4a^2}{3} Y_a^{\frac{8}{3}} \rho_a^{\frac{5}{3}} + (3Y_a \rho_a RT_a)^2 / \rho_a \right] \quad (6.46)$$

$$+ (\rho_b/\rho_a)^2 RT_b - \rho_b R c_2 + \frac{1}{3\rho_b D_b} \left[\frac{4a^2}{3} Y_b^{\frac{5}{3}} \rho_b^{\frac{5}{3}} (\rho_b/\rho_a)^2 + (3Y_b \rho_b RT_b)^2 \left(\frac{(\rho_b/\rho_a)^2}{\rho_b} - c_2/T_b \right) \right] \quad (6.47)$$

$$b_4 = \frac{3}{2} R \rho_a + \frac{(3Y_a \rho_a RT_a)^2}{\rho_a D_a T_a} \quad (6.48)$$

$$b_5 = R \rho_a + \frac{(3Y_a \rho_a RT_a)^2}{3\rho_a D_a T_a} - c_3 R \rho_b - \frac{c_3 (3Y_b \rho_b RT_b)^2}{3\rho_b D_b T_b} \quad (6.49)$$

$$b_6 = \frac{-m_b}{\rho_a L_a} \quad (6.50)$$

$$b_7 = \frac{1}{3\rho_a D_a} \left[\frac{4a^2}{3} Y_a^{\frac{5}{3}} \rho_a^{\frac{5}{3}} + (3Y_a \rho_a RT_a)^2 / Y_a \right] - c_1 \rho_b R \quad (6.51)$$

$$+ \frac{1}{3\rho_b D_b} \left[\frac{4a^2}{3} Y_b^{\frac{5}{3}} \rho_b^{\frac{5}{3}} + (3Y_b \rho_b RT_b)^2 (1/Y_b - c_1/T_b) \right] \quad (6.52)$$

$$b_8 = \left(\frac{1}{D_a \rho_a} \right) \left[\frac{4a^2}{3} Y_a^{\frac{5}{3}} \rho_a^{\frac{5}{3}} + (3Y_a \rho_a RT_a)^2 / Y_a \right] \quad (6.53)$$

$$b_9 = 1 + (\rho_b/\rho_a)^2. \quad (6.54)$$

These constants have constants within them, c_1, c_2, c_3 . They are:

$$c_1 = \frac{T_b}{\frac{3}{2} RT_b D_b + (3Y_b RT_b)^2} \left[\frac{4a^2}{3} Y_b^{5/3} \rho_b^{2/3} + (3Y_b RT_b)^2 / Y_b \right] \quad (6.55)$$

$$- (D_b/D_a) \left(\frac{4a^2}{3} Y_a^{5/3} \rho_a^{2/3} + (3Y_a RT_a)^2 / Y_a \right) \quad (6.56)$$

$$c_2 = \frac{T_b}{\frac{3}{2} RT_b D_b + (3Y_b RT_b)^2} \left[\frac{a^2}{3} (Y_b \rho_b)^{5/3} / \rho_a^2 - (D_b/D_a) \left(\frac{a^2}{3} Y_a^{8/3} \rho_a^{-1/3} \right) \right] \quad (6.57)$$

$$c_3 = - \frac{D_b T_b (\frac{3}{2} R D_a T_a + (3Y_a RT_a)^2)}{D_a T_a (\frac{3}{2} R D_b T_b + (3Y_b RT_b)^2)}. \quad (6.58)$$

Finally,

$$D_i = \sqrt{a^2 Y_i^{8/3} \rho_i^{2/3} + (3Y_i RT_i)^2}. \quad (6.59)$$

Once again, the coefficients are evaluated assuming $\rho_a = \rho_b = 2.9 \times 10^{12}$ g/cm³, $T_a = T_b = 11.3$ MeV, and $Y_a = Y_b = 0.17$, using the values listed earlier in this paper. This gives $b_E = 1.49$, and $b_N = -1.02$. Inserting numerical values one gets

$$\frac{\Delta \dot{\rho}}{\rho_a} = -3.13 \times 10^5 \text{ sec}^{-1} \left[\frac{\partial}{\partial x} \left(\frac{\mu_{\nu_e}}{m_e c^2} \right) + 4.68 \frac{\partial}{\partial x} \left(\frac{\pi k T}{m_e c^2} \right) \right] \quad (6.60)$$

The timescale for growth is about $3 \mu\text{s}$. The temperature derivative is negative (decreasing to the right), hence $\Delta\dot{\rho} > 0$ is satisfied if the two derivatives are about equal in magnitude.

These two rough calculations concerning neutron finger convection show that it cannot be ruled out as a means of enhancing the neutrino luminosity during a supernova. While it is clear that extreme degeneracy in the core would kill the effect, the core is in fact only mildly degenerate, with $\mu_e \approx \pi kT$ and $\mu_{\nu_e} \approx kT$. The fact that the mean free path of the mu and tau neutrinos is five times longer than the mean free path for the electron neutrino more than makes up for the degeneracy effects in the calculation.

No numerical calculations were made using this model of a neutron finger, but the analytic estimates derived from the assumptions presented here indicate that a serious numerical study of the problem would be worthwhile.

Part III

Convection and Nucleosynthesis

Chapter 7

Late Time Convection and Nucleosynthesis

7.1 Introduction

The nuclear burning of hydrogen to helium to carbon, etc, on up to iron is fairly well understood. Fusion reactions beyond iron are not energetically favored (endothermic). This leaves the question of where elements with atomic number greater than iron come from. Researchers have for some time pointed to supernovae as the source and the *r*-process as the means (“*r*” for “rapid neutron accretion”, which means that accretion of neutrons happens faster than beta decay).

The *r*-process requires a site with a sufficiently high temperature and low density, plus a high ratio of neutrons to heavy nuclei ($\sim 100 - 200 : 1$). In the *r*-process, expanding material cools and drops out of nuclear statistical equilibrium (NSE). The temperature is high enough for helium to burn to carbon but the density is too low for this reaction to consume significant amounts of helium nuclei. The helium to carbon burn is a three-body interaction and is therefore very sensitive to the density. After the helium burn the resulting carbon quickly burns up the chain to heavier and heavier elements, topping out around cobalt ($Z=27$). The excess neutrons and alpha particles absorb onto the heavy nuclei, creating heavier and heavier nuclei. Beta

decay converts some neutrons to protons, raising the atomic number. In this way a nucleus moves up the periodic table; absorbed neutrons increase its atomic weight, decaying neutrons raise the atomic number.

In the current picture of a supernova, a site exists which appears to have all the prerequisites for the r -process. It is hot, not very dense, and the neutron excess is large. This situation comes about in the “hot bubble” which develops between the shock and the neutrinosphere about one second after bounce. At this time, about 1 second after bounce, the region just outside the neutrinosphere, extending roughly from 30 to 150 Km, has an entropy of about 100 in units of entropy/ k -baryon. Entropy is a convenient measure of how suitable a site is for the r -process because entropy combines the temperature and density such that

$$s \sim \log \frac{T}{\rho^{(\gamma-1)}} \quad (7.1)$$

hence the higher the entropy the better the chances for the r -process. At entropies of about 100, electron scattering of neutrinos becomes important compared to neutrino capture reactions because the matter has a much higher e^-e^+ pairs to nucleons ratio. Electron scattering raises the temperature and entropy of the material even higher, which causes the matter to generate electron-positron pairs even more copiously. The matter thus becomes ever more opaque to neutrinos of all species via scattering. By 3.5 seconds postbounce, this effect drives the entropy up to ~ 400 in 1-D calculations[23].

The late time convection is qualitatively different from the convection in the supernova mechanism problem. In the mechanism problem the heat source that drives the convection is very close to the unstable region. The flow pattern of a vortex is almost circular, with a radius of a few hundred kilometers at most. In the late time convection, the heat source sits at the bottom of a large region (the hot bubble) in which the entropy is nearly constant. Within the bubble itself, fluid is neutrally stable. Only the uppermost edge of the bubble, where the entropy falls off drastically with radius (recall $\frac{\partial s}{\partial r} < 0$ is the instability criterion), is actually unstable. Fluid descends as a jet through the bubble to the heating region at the bottom and then

must climb back up through it in an expanding, elongated plume that is thousands of kilometers long to complete one convective cycle. This is quite different from the tidy vortices that characterize early time convection behind the shock.

One dimensional calculations of the r -process in the hot bubble predict elemental abundances which compare well with observational data[26]. In Figure 7.1[26] the

Figure 7.1: Elemental abundances for nuclei from $A = 80$ to $A = 202$. Solid line is a calculated value from Woosley, *et al*, dots are observed values.

elemental abundances for nuclei above $A = 80$ are shown. The abundances were calculated with no free parameters using the M&W code time histories for density, temperature, and Y_e values for the hot bubble region as the background for a nuclear reaction network constructed by Woosley, Mathews, *et al*[26]. This network, which consisted of hundreds of isotopes and their appropriate reaction cross sections, followed the r -process in the hot bubble. The black dots show the observed abundances.

Figure 7.2: Mass Density, Pressure, Specific Energy, and Entropy at time $t = 2.00$ seconds, as calculated by BUBBA in 1-D mode. Note the high entropy/low density “hot bubble” forming.

Figure 7.3: Ratio of artificial viscosity to pressure (top) and velocity profile (bottom) for 1-D BUBBA calculation at $t = 2.00$ seconds. The peak velocity behind the shock coincides with the hot bubble. The shock is visible at $\sim 2.5 \times 10^9$ cm (25,000 Km).

The hot bubble lies beneath cooler, more dense material. The entropy gradient across the outer boundary of the hot region is strongly negative, hence the bubble is hydrodynamically unstable. See Figure 7.2. Multi-dimensional calculations which allow the cooler fluid to flow into the hot region will show events taking a very different course than 1-D calculations do. While this work is still somewhat preliminary, I have tried to reach some conclusions about what convection implies for the possibility of nucleosynthesis in the hot bubble.

Several 2-D supernova calculations were made, covering the region from just outside the neutrinosphere (around 10^6 cm) to either 3.5×10^9 cm or 6.3×10^9 cm in radius, $0 \leq \theta \leq \frac{\pi}{2}$, and starting at 0.2 seconds past bounce out to 6 seconds in time. These runs were designed to capture the development and subsequent behavior of the hot bubble (entropy ~ 400) seen beneath the shock in 1-D calculations.

7.2 Model Changes

Changes to the physical model were necessary to handle this problem. I added an improved equation of state, allowed Y_e to vary via neutrino interactions, included neutrino-electron scattering, replaced the neutrino diffusion physics with a free streaming “light bulb” model, included an improved advection method in the fluid dynamics, and took into account the Doppler shift in neutrino energies caused by high fluid velocities. Each of these is discussed below.

A 2-D code run begins with a link from the M&W code. In a perfect world one would therefore use the M&W EOS in the 2-D code, but this would put a 2-D problem out of reach because the run times would become very long. Instead, I have implemented a tabulated version of the M&W equation of state. This makes EOS calls very fast, at the cost of a few percent loss of accuracy in the thermodynamics. The new EOS is a table of $T(\rho, \varepsilon_{th}, Y_e)$, $\gamma(\rho, \varepsilon_{th}, Y_e)$, and $S(\rho, \varepsilon_{th}, Y_e)$ constructed from the M&W code’s EOS routines[5]. The zero temperature contribution to the energy is subtracted off from ε before the call to the interpolation routine is made,

so only the thermal energy is used in making the table. The pressure is obtained from γ , which is defined by $\gamma \equiv P/(\rho\epsilon_{th}) + 1$. The tables cover the range in density $10^3 \text{ g/cc} \leq \rho \leq 10^{12} \text{ g/cc}$, in thermal energy $0.0 \leq \epsilon_{th} \leq 1.5 \times 10^{20} \text{ ergs/g}$, and electron fraction $0.15 \leq Y_e \leq 0.5$, using 120 densities spaced logarithmically, 50 equally spaced specific energy values, and 20 equally spaced entries for Y_e . A 3-D linear interpolation is used to find T , γ , and S inside the box defined by the density, thermal energy and Y_e grids.

The density occasionally drops below 10^3 during the late time convection runs. When this happens the code extrapolates off the table using

$$T = T_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{1}{4}} \quad (7.2)$$

where T_0, ρ_0 are the values at the edge of the 3-D table nearest the actual triplet (ρ, ϵ, Y_e) for which T is desired. For γ extrapolation, BUBBA takes the value of γ associated with the nearest (ρ, ϵ, Y_e) triplet. The entropy is extrapolated in a manner identical to the temperature.

The number of electrons per baryon is carried as a free parameter in the late time runs. Y_e is advected with the fluid and modified by neutrino interactions according to

$$\frac{\partial Y_e}{\partial t} + \nabla \cdot (\mathbf{v} Y_e) = \frac{\sigma_0}{3\pi R^2 (m_e c^2)^2} \left[X_n L_{\nu_e} \epsilon_{\nu_e} - X_p L_{\bar{\nu}_e} \epsilon_{\bar{\nu}_e} \right] \quad (7.3)$$

where X_n and X_p are the neutron and proton number fractions respectively.

The most important improvement to the code in terms of following the development of the hot bubble is the inclusion of neutrino-electron scattering. This reaction is relatively unimportant below entropies of about 100, but above that it is the dominant neutrino heating effect. Without it there would be no hot bubble. To include electron scattering heating by a single neutrino species, begin with the number of scattering events per second per gram and multiply by the average energy per event to get

$$\dot{\epsilon}_{\text{scatt}} = \frac{n_\nu n_{e+e-\text{CS}}(T, \epsilon_\nu) \Delta \epsilon}{\rho} \quad (7.4)$$

where n_ν and $n_{e^+e^-}$ are number densities of neutrinos and electron pairs respectively, σ is the cross section, and $\Delta\epsilon$ is the energy transferred by the interaction. Adding the effects of all six neutrino species, integrating over neutrino energies, and assuming that an average over scattering angle results in half the neutrino energy being transferred one can obtain (Tubbs and Schramm[20]),

$$\begin{aligned} \dot{\epsilon}_{\text{scatt}} = & \frac{1}{2}(\bar{\epsilon}_\nu - 3T) \frac{3\sigma_0 \bar{n}_\nu n_{e^+e^-} c}{8(m_e c^2)^2 \rho} \left(T + \frac{m_e c^2}{3}\right) \\ & \times [\bar{\epsilon}_{\nu_e}(C_{e^-\nu_e} + C_{e^+\nu_e}) + \bar{\epsilon}_{\bar{\nu}_e}(C_{e^-\bar{\nu}_e} + C_{e^+\bar{\nu}_e}) + 4\bar{\epsilon}_{\nu_{\mu\tau}}(C_{e^-\nu_{\mu\tau}} + C_{e^+\nu_{\mu\tau}})] \end{aligned} \quad (7.5)$$

where symbols with an overbar are arithmetic averages over species, except for $\bar{\nu}_e$ which has its usual meaning of electron anti-neutrino. The $C_{e\nu}$ factors are interaction coefficients calculated from electroweak theory[20] with values of

$$\begin{aligned} C_{e^-\nu_e} &= \frac{7}{3} & C_{e^+\nu_e} &= 1 \\ C_{e^-\bar{\nu}_e} &= 1 & C_{e^+\bar{\nu}_e} &= \frac{7}{3} \\ C_{e^-\nu_{\mu\tau}} &= \frac{1}{3} & C_{e^+\nu_{\mu\tau}} &= \frac{1}{3} \end{aligned} \quad (7.6)$$

The neutrino transport in the nucleosynthesis runs is much simpler than in the previous problem of the supernova mechanism. The optical depth of the modeled region for a 15 MeV neutrino is

$$\tau = \int \frac{dr}{\lambda} = \int \frac{\rho \sigma}{m_B} dr \quad (7.7)$$

where m_B is the baryon mass and σ is given by the approximation $\sigma = 1.7 \times 10^{-44}(\epsilon_\nu/m_e c^2)^2 \text{ cm}^2$. Integrating over the computational grid at $t = 0.200$ seconds we get $\tau = 0.012$. In other words the entire modeled region can be thought of as only about one one hundredth of a mean free path to a typical neutrino so the chance of the matter significantly altering the radial neutrino flow is zero for all practical purposes. The free streaming approximation is valid so the neutrinos are treated as being entirely radially directed outward from the center of the star. For the nucleosynthesis runs the neutrino advection and diffusion routines were eliminated and the self-consistent neutrino field was replaced with neutrino luminosities

and temperatures taken from a fit to the M&W code values as functions of time.

$$\begin{aligned}
 L_\nu &= 10^{52} \left(\frac{0.9}{t + 0.33t^2} + \frac{1}{5 + t} \right) \left(1 + \frac{(t/2)^2}{1 + (t/2)^4} \right) \text{ ergs/sec} & (7.8) \\
 L_{\nu_e} &= L_\nu / 1.075 \\
 L_{\bar{\nu}_e} &= 1.075 L_\nu \\
 L_{\nu_{\mu\tau}} &= 4.0 L_\nu \\
 \varepsilon_{\nu_{\mu\tau}} &= 26.0 - 3.0/t \text{ MeV} \\
 \varepsilon_{\nu_e} &= 12.0 - 0.08/t \text{ MeV} \\
 \varepsilon_{\bar{\nu}_e} &= \begin{cases} 16.5 - 0.5/t \text{ MeV} & \text{if } t \leq 5.0 \\ 20.0 - 18.0/t \text{ MeV} & \text{if } t > 5.0 \end{cases} & (7.9)
 \end{aligned}$$

where the units of t are seconds. L_ν was then multiplied by a constant factor to fine tune the luminosity so that the 2-D code, when run in 1-D mode, would reproduce the M&W entropy profiles. The multiplier was set to 1.2.

A term is included to take into account the accretion of matter onto the proto neutron star (PNS), but this effect is not very important in the late time runs because the inner boundary is set at a radius 1.25 times the neutrinosphere radius, unlike the supernova mechanism calculations. To estimate the energy emitted in neutrinos by accreting matter it was assumed that the infalling matter gives up all its potential energy into neutrinos. This is valid because the inner boundary is very close to the neutrinosphere. Multiplying the rate of infalling mass by the gravitational potential at the bottom of the computational grid results in an overestimate of the effect but the total effect is always small compared to L_ν given above. A simple estimate using values at the beginning of a run ($t = 0.2$ seconds post-bounce) shows that the accretion luminosity, L_{acc} is

$$L_{\text{acc}} = \dot{M}_{\text{infall}} \Phi \approx (4 \times 10^{32} \text{ g/sec}) \times (4 \times 10^{19} \text{ ergs/g}) \approx 2 \times 10^{51} \text{ ergs/sec} \quad (7.10)$$

In fact the numerical results show that the accretion luminosity never rises above 1.5×10^{51} ergs/sec. Compared to the core luminosity at $t = 0.2$ seconds, which is $\sim 25 \times 10^{51}$ ergs/sec, the accretion luminosity does not seem to play a major role. L_{acc} is added to L_{ν_e} and $L_{\bar{\nu}_e}$.

A significant difference between the nucleosynthesis problem and those previously discussed in this thesis is that high fluid velocities are generated during the late time runs ($\sim 10^{10} \text{ cm s}^{-1}$) and the Doppler shift of the neutrinos must be taken into account. The neutrino energy was multiplied by $1 - v_r/c$ wherever the neutrino energy was used (v_r is the radial fluid velocity). This increases heating by electron capture by a factor of $(1 - v_r/c)^3$ (recall $v_r < 0$ for infalling material), because the reaction cross section contains two powers of the neutrino energy; this times the energy transferred in the interaction gives the three in the exponent. The neutrino energy shift makes a pronounced difference in the calculation. Without the Doppler effect properly accounted for, the maximum entropies achieved were much lower (about a third of the highest values) and the fluid velocities were much higher.

The final difference between the mechanism and nucleosynthesis models is the incorporation of a slightly better advection routine in the hydrodynamics package of BUBBA. This improvement is discussed in chapter 4, section 2.

7.3 Initial and Boundary Conditions

The initial conditions come from the Mayle and Wilson 1-D code. The M&W code is used to evolve a Woosley and Weaver 20 solar mass SN1987A[25] progenitor model up to 0.2 seconds past bounce. The density, velocity, Y_e , and specific energy are linked over to the 2-D code. In the unstable region ($\frac{\partial s}{\partial r} < 0$) the r velocity is given a sinusoidal perturbation, $v_r = 0.25c_s \sin 4\theta$, where c_s is the local sound speed. The initial condition of the fluid is show in Figures 7.4 and 7.5. Neutrino luminosities and neutrino temperatures are as shown above. A model that successfully explodes was chosen as the initial condition for this problem. In these calculations the shock is already out to 1000 Km.

A guard cell was used to set the interior boundary condition. The density, Y_e , and pressure of this cell were set as functions of time taken from the M&W code. The mass zone corresponding to the inner boundary at $t = 0.200$ was followed with

Figure 7.4: Mass, Pressure, Specific Energy, and Entropy initial conditions, $t = 0.200$ seconds, taken from the M&W code.

Figure 7.5: γ , Temperature, free baryon fraction, and electron number fraction (Y_e) initial conditions, $t = 0.200$ seconds, taken from the M&W code.

the M&W code to produce Table 7.1, then linear interpolation was used to find the guard cell quantities at any particular time t . The position of the inner boundary is 1.25 times the neutrinosphere radius as computed by the M&W code. This is about 40 Km and decreases to about 10 Km as the run progresses. The velocity at the boundary in the 2-D code was allowed to fluctuate freely in response to pressure gradients between the guard cell and the first computed zone. While this occasionally led to oscillations and some material flowing inward off the grid, the vast majority of the time this scheme resulted in a small outward velocity at the boundary, with guard cell material flowing onto the grid at low speeds. This reproduced the 1-D code behavior in this region, which occasionally showed oscillations in the velocity (sound waves) but by and large displayed a steady outward flow.

Time	$\rho_{\text{BC}} \times 10^9 \text{ g/cc}$	$\epsilon_{\text{BC}} \times 10^{19} \text{ ergs/g}$	Y_e	$P_{\text{BC}} \times 10^{28} \text{ cgs}$
0.20	9.4	1.9	0.22	5.0
0.24	35	1.9	0.24	21
0.50	14	2.1	0.26	9.0
1.00	6.6	2.5	0.34	5.1
1.50	3.6	2.9	0.40	3.3
2.00	1.3	4.3	0.42	1.7
2.50	0.66	5.7	0.44	1.2
3.00	0.60	5.5	0.45	1.0
4.00	0.22	8.3	0.46	0.58
5.00	0.12	9.0	0.47	0.34

Table 7.1: The inner boundary condition as a function of time is governed by a guard cell. The exact density, specific energy, and pressure at a given time are linearly interpolated between the values shown here, which are taken from the M&W code.

The outer boundary was fixed at $r = 6.4 \times 10^9 \text{ cm}$. Data beyond the spatial extent of the M&W code results were filled in with values from the Weaver and Woosley progenitor model.

7.4 Simulation Results

One dimensional and two dimensional results are presented here. The plan was to first experiment with boundary conditions and neutrino luminosities to find a 1-D BUBBA run that emulated the M&W calculations. BUBBA was then run in 2-D mode with the model that best reproduced the M&W results.

7.4.1 One-Dimensional Runs

The point of the 1-D BUBBA runs was to emulate the M&W code results. To this end initial and boundary conditions were taken from the M&W code as described above. In both codes an outward flowing region develops behind the shock, in the region between 100 Km and 600 Km at $t > 0.6.0$. This is driven by neutrino heating, and the high velocity region coincides with the hot bubble. See Figure 7.3. The material is pushed at high (but subsonic) velocities across the hot bubble by the “neutrino wind”[26]. In a typical 1-D BUBBA run however, the outward wind-driven velocity was four to ten times higher than the velocity in the same region of a M&W code run. In Figure 7.3, for example, the peak velocity of the hot bubble region is nine times higher than the corresponding M&W value. At the same time entropy generation in BUBBA runs was lower; the M&W code achieves a peak entropy per baryon of ~ 180 units by 2 seconds postbounce whereas in Figure 7.2 BUBBA’s peak entropy is only ~ 95 . The first hypothesis raised was that the neutrino luminosity was too large, driving the velocities so high that material spent too little time in the heating region and this was the reason BUBBA entropies failed to reach the M&W values, but experiments with lower (and higher) neutrino luminosities failed to correct the problem. The best result BUBBA achieved was a velocity across the bubble that was four times too high ($\sim 5 \times 10^8$ cm/sec) and a peak entropy per baryon that was about three-fourths of the M&W peak entropy, (60 as opposed to 80) at one second after bounce. More work needs to be done to reconcile the M&W code results and BUBBA’s calculations.

7.4.2 Two-Dimensional Runs

The two-dimensional runs were done on a grid with 75 radial zones and 15 or 30 angular zones, using the same initial and boundary conditions described above. These runs developed just as their 1-D counterparts did until the entropy of the bubble reached 50 – 70, then the unstable nature of the bubble configuration began to make itself felt.

See Figures 7.6 through 7.10 for entropy and velocity plots of the hot bubble and jet formation. Material close to the neutrinosphere began gaining entropy due to neutrino heating, as expected, until the entropy reached about 60. Then this “warm bubble” began to rise up into the colder and denser material above it. At the same time cold material began to penetrate downward around the bubble. This unstable situation accelerated until cold matter poured down in a strong jet, and the warm bubble had risen far above the heating region, trailing a plume of less warm material behind it. The jet was pushed to the side by the rapidly expanding bubble, and usually flowed down the axis or the equator. It became very narrow as it moved down into the star and adiabatically climbed the density gradient. The jet did not stop until it reached the bottom of the computational grid. It is suspected that if more of the star at the bottom were included that the rapid increase in density at the proto-neutron star boundary would turn the jet in much the same way the bottom of the grid did. By $t = 2.0$ sec the warm bubble was over 1000 Km long and its hottest region, which was near its center, had reached about 3000 Km. The best thing about this picture is that the jet neatly removed most of the material at entropies between 20 and 40. The M&W calculations typically end up with too much of this material. In spite of that, this “jet and warm bubble” result is not exciting because the entropy of the bubble was not high enough for the r -process to work well.

7.4.3 Power Law Models

To understand how the instability grows in the presence of different initial conditions the density profile of the star outside the shock was changed. In the original model the

Figure 7.6: Entropy contours and velocity vectors at $t = 0.81$ seconds postbounce, out to 2,000 Km. The shock is at ~ 6000 Km, beyond the frame of the figure. The calculation has been seeded with a perturbation but at this point the problem is still quite symmetric. The hot bubble is in an early stage, having reached peak entropies of ~ 60 units per baryon.

Figure 7.7: Entropy contours and velocity vectors at $t = 1.08$ seconds postbounce. The perturbation is making itself apparent. The bubble has broken spherical symmetry and has begun to move up along the equator of the star.

Figure 7.8: Entropy contours and velocity vectors at $t = 1.28$ seconds postbounce. The bubble has broken entirely free of the heating region; the entropy of the bubble is now fixed. The bubble moves up the equator and a jet of cold (low entropy) material can be seen accelerating down the axis. The jet velocity near the center of the star is large enough that the velocity arrows poke out of the frame at the bottom.

Figure 7.9: Entropy contours and velocity vectors, again at $t = 1.28$ seconds. This close up view shows the jet, which has punched through all the way to the bottom of the computational grid. The rapidly expanding material rising out of the heating region pushes the jet to one side. The jet becomes more compressed as it moves to the center of the star. (The strange markings at the top and right edge of the figure are contour line labels mashed up against the frame edge; please ignore these.)

Figure 7.10: Entropy contours and velocity vectors at $t = 1.95$ seconds. Note the change in length scale; the figure now encloses 10,000 Km. The bubble has moved out to ~ 6900 Km. The jet velocity near the center of the star is so high that the velocity vectors elsewhere are too small to be read accurately. The bubble displays the characteristic mushroom shaped head of a flow in which one fluid penetrates another.

Figure 7.11: Entropy contours and velocity vectors, again at $t = 1.95$ seconds. This close up shows the jet at the bottom of the computational grid. This figure is much the same as the $t = 1.28$ close-up. Once the jet reaches the bottom of the grid the situation shown here remains essentially unchanged up to at least 4.5 seconds postbounce, which was the longest run time achieved.

density falls off as an almost perfect power law, $\rho = Ar^n$, with $n = -1.75$. Different values for n were tried, from $n = -1.5$ to $n = -2.5$, changing the density outside the shock but keeping the original specific energy and velocity. The more rapid the density fall off, the higher the entropy developed in the bubble before the bubble begins to rise. For $-1.5 \geq n \geq -2.0$ essentially the same scenario as described above occurs, though it takes longer for the bubble to break away and the jet to form as n becomes more negative (the density gradient becomes more steep). For $n = -2.25$ entropies of ~ 300 were achieved in the bubble before it broke away. The downward flowing jet is still present, which helps to remove the troublesome middle entropy material. The success of the $n = -2.25$ model in generating high entropies leads one to suspect that the r -process may take place inside a supernova's hot bubble (or perhaps a stream of bubbles) in spite of that configuration's blatantly unstable nature. It is unclear whether the $n = -2.25$ initial condition should be taken seriously. It is not known how close this case is to an initial condition that could be arrived at "honestly", i.e., starting with a carefully calculated progenitor such as one of those published by Nomoto or Weaver and Woosley and evolving it with a realistic model.

7.5 Summary

The one dimensional BUBBA calculations were not successful in replicating the M&W results. This raises some doubts about the 2-D calculations, but the 2-D results are so violent, and their character remains unchanged under so many different changes to the initial and boundary conditions that I feel they should be presented.

In every run, at least one jet formed and forced its way to the bottom of the computational grid. The hot bubble always escaped the region where it first formed and moved up into the cooler, denser material above. This happened every single time on roughly the same time scale of about a third of a second past the time at which the entropy entered the 50 – 70 range, in spite of changes to the density profile, the neutrino luminosity and the boundary conditions. It is my belief that while the

details of these calculations are flawed, the general trend towards a jet and bubble scenario is the same result that a better calculation will obtain.

The implications of these calculations for supernovae as a site for the r -process are not encouraging. It is likely that the hot bubble rises out of the heating region before high entropies can be achieved. This is difficult to understand in light of the excellent agreement of theory with observation in recent 1-D nucleosynthesis calculations, see Figure 7.1. The agreement is obviously good for nuclei with $A > 120$, yet the 2-D results argue that no one dimensional code can produce a good representation of the state of the hot bubble in a supernova. A resolution to this contradiction has yet to be found. Clearly, further work in this area remains to be done.

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