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An Electromagnetic Finite Difference Time Domain Analog Treatment of Small Signal Acoustic Interactions

Karl Kunz¹, David Steich¹, Kent Lewis², Charles Landrum², and Marvin Barth³

Introduction

Hyperbolic partial differential equations encompass an extremely important set of physical phenomena including electromagnetics and acoustics. Small amplitude acoustic interactions behave much the same as electromagnetic interactions for longitudinal acoustic waves because of the similar nature of the governing hyperbolic equations. Differences appear when transverse acoustic waves are considered, nonetheless the strong analogy between the acoustic and electromagnetic phenomena prompted the development of a Finite Difference Time Domain (FDTD) acoustic analog to the existing electromagnetic FDTD technique [1].

The advantage of an acoustic FDTD (AFDTD) code are as follows:

- 1) Boundary condition-free treatment of the acoustic scatterer
 - only the intrinsic properties of the scatterer's material are needed, no shell treatment or other set of special equations describing the macroscopic behavior of a sheet of material or a junction, etc. are required; this allows completely general geometries and materials in the model.
- 2) Advanced outer radiation boundary condition analogs
 - in the electromagnetics arena, highly absorbing outer radiation boundary conditions have been developed that can be applied with little modification to the acoustics arena with equal success.
- 3) A suite of preexisting capabilities related to electromagnetic modeling
 - this includes automated model generation and interaction visualization as its most important components and is best exemplified by the capabilities of the LLNL generated TSAR electromagnetic FDTD code [2].

The development of the acoustics FDTD code was accomplished in several steps. The first step was making the analogous connection between the electromagnetic and acoustic hyperbolic equations. Key to this development was noting how the two first order coupled acoustic partial differential equations for longitudinal waves, while differing substantially from the electromagnetic

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first order coupled curl equations, nonetheless led to a wave equation identical in form to the electromagnetic wave equation. The second step was noting that the wave equation for acoustics was also a force equation and that transverse wave behavior could be incorporated by the addition of a single term to the force equation for the compressional wave. The transverse waves are present only for an elastic media scatterer. In the treatment developed here it is assumed that the surrounding media, either gas or liquid, only supports compressional waves. This significantly eased the transfer of electromagnetic outer radiation boundary conditions to the media surrounding the elastic scatterer. The third step was validation.

Validation was performed in two stages. The first stage was to validate the compressional wave modeling against analytic results from an idealized configuration. The second stage was to validate the combined compressional and transverse model against experiment. The analytic results came from a two dimensional geometry and modeling resource requirements were quite modest. The experimental results were from a three dimensional system many wavelengths in size at the highest frequency of interest in all directions.

Compressional wave modeling results were excellent. Agreement in amplitude and phase throughout the space between computational modeling and analytic predictions were within a fraction of a dB and 15° in phase respectively. Agreement between the computational model and the experiment was observed in the spectrum of the scattered pressure. The agreement was good enough to conclude that the transverse waves were also modeled successfully.

The developments will now be examined in detail.

Acoustic FDTD Equations

The one dimensional force equation on an elastic media [1] is

$$\rho \frac{\partial^2 s_x}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 s_x \quad (1)$$

where $\Delta \equiv \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial s_x}{\partial x} + \frac{\partial s_y}{\partial y} + \frac{\partial s_z}{\partial z} \equiv \bar{\nabla} \cdot \bar{s}$ is the dilation

and λ and μ are the Lamé constants

The dilation can also be expressed as

$$\Delta \equiv \bar{\nabla} \cdot \bar{s} \equiv \frac{p}{\kappa} = \frac{p}{\left(\lambda + \frac{2\mu}{3}\right)} \quad (2)$$

where $\kappa = \lambda + \frac{2\mu}{3}$ is the bulk modulus or compressibility of the media and p is the pressure.

The force equation can therefore be written as

$$\rho \frac{\partial^2 s_x}{\partial t^2} = (\lambda + \mu) \frac{\partial}{\partial x} (\bar{\nabla} \cdot \bar{s}) + \mu \nabla^2 s_x \quad (3)$$

or in three dimensions

$$\rho \frac{\partial^2 \bar{s}}{\partial t^2} = (\lambda + \mu) \nabla (\bar{\nabla} \cdot \bar{s}) + \mu \nabla^2 \bar{s} \quad (4)$$

Since $\nabla^2 \bar{s} = \nabla (\bar{\nabla} \cdot \bar{s}) - \bar{\nabla} \times \bar{\nabla} \times \bar{s}$ the force equation can be expressed as

$$\rho \frac{\partial^2 \bar{s}}{\partial t^2} = (\lambda + 2\mu) \nabla (\bar{\nabla} \cdot \bar{s}) - \mu (\bar{\nabla} \times \bar{\nabla} \times \bar{s}) \quad (5)$$

This is the small amplitude equation of motion for elastic media under the influence of its own elastic restoring forces as given in [2, eqn. 2.2.1]

Equation (5) can be recast using the relations

$$\partial \bar{s} / \partial t = \bar{v} \text{ and } \Delta = \bar{\nabla} \cdot \bar{s} = \frac{p}{\left(\lambda + \frac{2\mu}{3}\right)}$$

into the form

$$\frac{\partial \bar{v}}{\partial t} = \frac{1}{\rho} \left[\left(\frac{\lambda + 2\mu}{\lambda + \frac{2\mu}{3}} \right) \nabla p - \mu (\bar{\nabla} \times \bar{\nabla} \times \bar{s}) \right] \quad (6)$$

This is one of two equations needed for a finite difference treatment of the media analogous to the two Maxwell curl equations of electromagnetics.

The second equation comes directly from equation (2) expressed as

$$p = \left(\lambda + \frac{2\mu}{3} \right) \bar{\nabla} \cdot \bar{s} \quad (7)$$

Taking the time derivative of this equation yields the second desired equation

$$\frac{\partial p}{\partial t} = \left(\lambda + \frac{2\mu}{3} \right) \bar{\nabla} \cdot \bar{v} \quad (8)$$

These two equations can be used in a time domain finite difference treatment of acoustics where the media can fall into any one of three categories:

- $\lambda \neq 0, \mu \neq 0$; metal for example
- $\lambda \neq 0, \mu = 0$; a fluid or gas for example
- $\lambda = 0, \mu \neq 0$; shear forces only as in an idealized string

When $\lambda \neq 0$ but $\mu = 0$ the equations simplify considerably to

$$\frac{\partial \bar{v}}{\partial t} = \frac{1}{\rho} \nabla p \quad (9)$$

$$\frac{\partial p}{\partial t} = \lambda \bar{\nabla} \cdot \bar{v} \quad (10)$$

These are the equations that describe sound waves in air for small displacements [3]. These waves are compressional only, as this is the only physical process allowed, and are longitudinal waves described by a scalar displacement potential ψ , where the longitudinal small signal displacement s_ℓ is defined by

$$s_\ell = \nabla \psi \quad (11)$$

In a fluid such as water when viscosity is ignored, these equations also describe the compressional waves or longitudinal waves in the water. Direct substitution of equation (10) into the time derivative of equation (9) yields the wave equation for compressional waves

$$\frac{1}{c_\ell^2} \frac{\partial^2 \bar{v}}{\partial t^2} = \nabla^2 \bar{v} \quad (12)$$

$$c_\ell = (\lambda / \rho)^{1/2}$$

where use has been made of the fact that $\nabla \times \bar{v} = 0$ when $s \equiv \nabla \psi$,

as $\nabla \times \bar{s} = \nabla \times \nabla \psi = 0$ and since $\nabla \times \bar{s} = 0$, we must have $\nabla \times \bar{v} = 0$.

When $\mu \neq 0$ but $\lambda = 0$ the media supports only shear waves, the transverse wave motion that corresponds to the behavior of an idealized string. Transverse wave motion displacement s_t arises from a vector displacement potential $\bar{A}$ where

$$\bar{s}_t = \nabla \times \bar{A} \quad (13)$$

When $\bar{s} = \nabla \times \bar{A}$, the term $\nabla \cdot \bar{s} = 0$, since $\nabla \cdot \nabla \times \bar{A} = 0$. Then the two equations (6) and (8) evaluated in their earlier forms in equations (5) and (7) become

$$\rho \frac{\partial^2 \bar{s}}{\partial t^2} = -\mu (\nabla \times \nabla \times \bar{s}), \quad \text{where } \bar{s} = \bar{s}_t = \nabla \times \bar{A} \quad (14)$$

and

$$p = 0 \quad \text{for transverse motion.} \quad (15)$$

Equation (14) is a form of the wave equation with velocity $c_t = (\mu / \rho)^{1/2}$.

In a material with both λ and μ non-zero, both compressional and transverse waves may exist simultaneously. Equations (6) and (8) are the governing equations describing this situation, and in the limiting case of $\mu \rightarrow 0$ or $\lambda \rightarrow 0$, describe the other media as well. It should be noted that in a media with $\lambda \neq 0$ and $\mu \neq 0$ the compressional wave velocity is $c_\ell = \left(\frac{\lambda + 2\mu}{\rho} \right)^{1/2}$.

Analogy to Electromagnetics

The analogy to electromagnetics only holds for the compressional waves. In electromagnetics the Maxwell curl equations for free space are

$$\frac{\partial \bar{E}}{\partial t} = \frac{1}{\epsilon} \nabla \times \bar{H} \quad (16)$$

$$\frac{\partial \bar{H}}{\partial t} = \frac{-1}{\mu} \nabla \times \bar{E} \quad (17)$$

The analogous compressional acoustic wave equations are

$$\frac{\partial \bar{v}}{\partial t} = \frac{1}{\rho} \nabla p \quad (18)$$

$$\frac{\partial p}{\partial t} = \lambda \nabla \cdot \bar{v} \quad (19)$$

The electromagnetic FDTD method [4] treats electromagnetic radiation scattering and coupling. Examples are antenna radiation patterns, aircraft RCS patterns and cavity coupling with an emphasis on the shielding properties of the cavity. An analogous situation in acoustics can be represented by a capped hollow gas-filled metal cylinder immersed in water. A compressional acoustic wave external to the cylinder is incident on the cylinder, producing scattered compressional waves in the water, a combination of compressional or longitudinal and transverse waves in the metal cylinder and finally compressional waves in the gas-filled interior of the cylinder.

In the electromagnetic treatment, the coupled curl equations can be treated directly. A prescribed E or H behavior along a surface of the problem space defines a wave that propagates into the space. It then encounters an object differing from the surrounding space in its constitutive properties (ϵ , μ , and σ) and scatters from and couples to the object. This is called a total field approach. Alternately, a separate field approach can be employed. Here there is an analytically prescribed wave incident on the object that is not numerically "marched" into the problem space. It is any physically realizable wave, meaning it satisfies the Maxwell curl equations, that may be analytically specified for the media in which the object resides. In electromagnetics this is typically free space but could just as well be a material with $\epsilon \neq \epsilon_0$ and $\mu \neq \mu_0$. The presence of the object produces a scattered wave that together with this incident wave satisfies the Maxwell equations in the object and throughout the surroundings.

In either case, the scattering and coupling arise from changes in the material properties of the object versus its surroundings. Only the Maxwell equations need be satisfied, there are no boundary conditions on the scattering and coupling to the object modeled. The scattering and coupling is determined strictly by the intrinsic material properties of each cell of the discretized

model of the object. Thus this formulation will be referred to as an intrinsic one, that is, boundary condition-free.

In electromagnetic treatments, a perfectly conducting material can be defined, and in this case there is an abrupt surface discontinuity on the object and the Maxwell equations in the limit of infinite conductivity do yield a surface boundary condition. This is a limiting case that should not obscure the basic boundary-condition-free nature of the interaction with the object.

Another potential source of confusion is that the scattered wave must be allowed to exit the problem space. Thus away from the object and having no part in the object interaction, which is boundary-condition-free, there must be an outer radiation boundary condition (ORBC) that provides field behavior at the outer edges of the problem consistent with outgoing scattered radiation that leaves the problem space.

The use of the separate field formalism allows for a particularly simple implementation of an ORBC. Popular ORBC's include Mur second order [5] and Liao [6] among others. For the acoustic system discussed here acoustic analogs of the Mur and Liao's boundary conditions can be readily effected since the fluid surrounding the object supports only compressional waves which are completely analogous to electromagnetic waves.

Implementation of the Acoustic FDTD Algorithm

A separate field formalism will be developed here. By setting the incident field to zero and prescribing a source region for the "scattered" field behavior, a total field formalism results in which the "scattered" field and the total field are identical. Thus providing the separate field formalism also provides, with slight modifications, the total field formalism.

In a separate field formulation, the scattered and incident components of the total field can be written

$$p = p^i + p^s; \quad \bar{v} = \bar{v}^i + \bar{v}^s; \quad \bar{u} = \bar{u}^i + \bar{u}^s$$

where p denotes pressure, $\bar{v}$ denotes velocity and $\bar{u}$ denotes displacement. Note that velocity and displacement are both vector quantities.

Water:

Presently water is used as the background "default" material in the code. The incident fields are analytically specified as the fields that would exist in the absence of a scatterer. This incident field travels in water with the pressure obeying the equation

$$p_t = -\gamma_o p_o \nabla \cdot \bar{v} \quad (20)$$

where the t subscript denotes a derivative with respect to time, p_o is the steady state pressure and γ_o the ratio of constant volume and pressure specific heats. Splitting into scattered and incident fields leads to

$$p_t^s = -\gamma_o p_o \nabla \cdot \bar{v}^s - \gamma_o p_o \nabla \cdot \bar{v}^i - p_t^i. \quad (21)$$

We also know

$$p_t^i = -\gamma_o p_o \nabla \cdot \bar{v}^i \quad (22)$$

and

$$\nabla \cdot \bar{v}^i = \frac{\bar{v}_t^i}{c_o} \quad (23)$$

where $\bar{v}$ is the direction magnitude of velocity and c_o is the speed of sound in water. Using 22, 21 simplifies to

$$p_t^s = -\gamma_o p_o \nabla \cdot \bar{v}^s \quad (24)$$

Also we have the equation

$$\bar{v}_t = -\frac{\nabla p}{\rho_o} \quad (25)$$

where ρ_o is background density. The same procedure as used above produces

$$\bar{v}_t^s = -\frac{\nabla p^s}{\rho_o} \quad (26)$$

Equations 24, 26 are the only equations needed to model water. The acoustic-elastic cell is described as shown in figure 1.

p and $\bar{u}$ are centered at $n + 1/2$, while $\bar{v}$ is centered at n . We define the operations

$$\begin{aligned}
\underline{\nabla \bullet \vec{v}} &\equiv \frac{v_x^s(I, J, K) - v_x^s(I-1, J, K)}{\Delta x} \\
&+ \frac{v_y^s(I, J, K) - v_y^s(I, J-1, K)}{\Delta y} \\
&+ \frac{v_z^s(I, J, K) - v_z^s(I, J, K-1)}{\Delta z}
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
\underline{\nabla p^s} &\equiv \frac{p^s(I+1, J, K) - (p^s(I, J, K))}{\Delta x} \hat{x} \\
&+ \frac{p^s(I, J+1, K) - p^s(I, J, K)}{\Delta y} \hat{y} \\
&+ \frac{p^s(I, J, K+1) - p^s(I, J, K)}{\Delta z} \hat{z}
\end{aligned} \tag{28}$$

Using 27 and 28, equations 24 and 26 become

$$p^{s, n+\frac{1}{2}} = p^{s, n-\frac{1}{2}} - \Delta t \gamma_o \rho_o (\underline{\nabla \bullet \vec{v}}^{s, n}) \tag{29}$$

$$v_x^{s, n+1} = v_x^{s, n} - \frac{\Delta t}{\rho_o} \left(\underline{\nabla p_x}^{s, n+\frac{1}{2}} \right) \tag{30}$$

where the other velocity components are defined similarly. The time derivatives in 29 and 30, were obtained by substituting

$$x_t^{n+\frac{1}{2}} \equiv \frac{x^{n+1} - x^n}{\Delta t} \tag{31}$$

which is the usual center differencing scheme.

Other compressional wave media

For media other than water that only support compressional waves, we proceed as follows.

Starting with the pressure we have

$$p_t = -\gamma'_{o'} \nabla \cdot \bar{v} \quad (32)$$

where p_o' is the steady state pressure of the new media. Splitting into incident and scattered components 32 becomes

$$p_t^s + p_t^i = \gamma'_{o'} \nabla \cdot \bar{v}^s - \gamma'_{o'} \nabla \cdot \bar{v}^i \quad (33)$$

Using 22 in 33 leads to

$$p_t^s = \gamma'_{o'} \nabla \cdot \bar{v}^s + \left(-1 + \frac{\gamma'_{o'}}{\gamma_o p_o} \right) p_t^i \quad (34)$$

In discretized form

$$p^{s,n+1/2} = p^{s,n-1/2} - \Delta t \gamma'_{o'} \left(\nabla \cdot \bar{v}^s \right) + \Delta t \left(\frac{\gamma'_{o'} - \gamma_o p_o}{\gamma_o p_o} \right) p_t^i \quad (35)$$

Also, equation 25 becomes

$$\bar{v}_t = \frac{-\nabla p}{\rho} \quad (36)$$

Splitting into incident and scattered components

$$\bar{v}_t^s = \frac{-\nabla p^s}{\rho} - \frac{\nabla p^i}{\rho} - \bar{v}_t^i \quad (37)$$

Using

$$\bar{v}_t^i = \frac{-\nabla p^i}{\rho_o} \quad (38)$$

leads to

$$\bar{v}_t^s = \frac{-\nabla p^s}{\rho} + \left(\frac{\rho_o - \rho}{\rho} \right) \bar{v}_t^i \quad (39)$$

Realizing that

$$\bar{v} = \bar{u}_t : \bar{v}_t = \bar{u}_{tt} \quad (40)$$

and discretizing 37 produces

$$v_x^{n+1} = v_x^n - \frac{1}{\rho} \Delta t \left(\nabla p_x^s \right) + \Delta t \left(\frac{\rho_o - \rho}{\rho} \right) \left(u_x^i \right)_{tt} \quad (41)$$

where other components are defined similarly. Equations 41 and 35 are the only difference equations needed to model acoustic materials.

Elastic media

For modeling elastic media we start with the equations

$$p_t = \kappa \nabla \bullet \bar{v} \quad (42)$$

and

$$\bar{v}_t = \frac{\mu}{\rho} \nabla^2 \bar{u} + \frac{\mu}{3\rho} \nabla \nabla \bullet \bar{u} - \frac{\nabla p}{\rho} \quad (43)$$

where μ denotes rigidity and κ denotes bulk modulus and is defined by

$$\kappa = \frac{2}{3}\mu + \lambda \quad (44)$$

Splitting 42 into scattered and incident parts we have

$$p_t^s + p_t^i = -\kappa \nabla \bullet \bar{v}^s - \kappa \nabla \bullet \bar{v}^i. \quad (45)$$

Substituting 22 into 45 leads to

$$p_t^s = -\kappa (\nabla \bullet \bar{v}^s) + \left(-1 + \frac{\kappa}{\gamma_o p_o}\right) p_t^i \quad (46)$$

In discretized form this becomes

$$p_s^{n+\frac{1}{2}} = p_s^n - \Delta t \kappa (\nabla \bullet \bar{v}^{s,n}) + \Delta t \left(\frac{\kappa}{\gamma_o p_o} - 1\right) p_t^{i,n+\frac{1}{2}}. \quad (47)$$

Splitting 43 into scattered and incident parts

$$\bar{v}_t^s + \bar{v}_t^i = \frac{\mu}{\rho} \nabla^2 (\bar{u}^s + \bar{u}^i) + \frac{\mu}{3\rho} \nabla \nabla \bullet (\bar{u}^s + \bar{u}^i) - \frac{\nabla p^s}{\rho} - \frac{\nabla p^i}{\rho} \quad (48)$$

Solving for $\nabla \bullet \bar{v}^s$ in 46 produces

$$\nabla \bullet \bar{v}^s = \frac{-p_t^s}{\kappa} + \left(\frac{1}{\gamma_o p_o} - \frac{1}{\kappa}\right) p_t^i. \quad (49)$$

Using 40 in 49 leads to

$$\nabla \bullet \bar{u}^s = \frac{-p^s}{\kappa} + \left(\frac{1}{\lambda_o p_o} - \frac{1}{\kappa}\right) p^i. \quad (50)$$

Also using 40 in 22 we have

$$\nabla \bullet \bar{u}^i = -\frac{p^i}{\gamma_o \rho_o}. \quad (51)$$

Substituting 50, 51 and 38 simplifies 48 to

$$v_t = \frac{\mu}{\rho} \nabla^2 (\bar{u}^s + \bar{u}^i) + \frac{\mu}{3\rho} \nabla \left(-\frac{p^s}{K} \right) + p^i - \left(\frac{1}{\rho_o} - \frac{1}{\rho} \right) \nabla p^i. \quad (52)$$

Simplifying

$$v_t = \frac{\mu}{\rho} \nabla^2 (\bar{u}^s + \bar{u}^i) - \left(\frac{\mu}{3\rho K} + \frac{1}{\rho} \right) \nabla p^s + \left[\frac{\mu}{3\rho} \left(\frac{-1}{K} \right) + \left(\frac{1}{\rho_o} - \frac{1}{\rho} \right) \right] \nabla p^i. \quad (53)$$

Noting that

$$\frac{1}{\rho} \left(\frac{\rho}{\rho_o} - 1 \right) = \frac{\rho - \rho_o}{\rho_o},$$

$$v_t = \frac{\mu}{\rho} \nabla^2 \bar{u}^s + \frac{\mu}{\rho c_o^2} \bar{u}_{tt}^i - \left(\frac{\mu + 3K}{3\rho K} \right) \nabla p^s - \left(\frac{\mu + 3K \left(\frac{\rho_o - \rho}{\rho_o} \right)}{3\rho K} \right) \nabla p^i \quad (54)$$

or

$$v_t = \frac{\mu}{\rho} \nabla^2 \bar{u}^s - \left(\frac{\mu + 3K}{3\rho K} \right) \nabla p^s + \left[\frac{\mu}{\rho c_o^2} + \left(\frac{\rho_o \mu + 3K(\rho_o - \rho)}{3\rho K} \right) \right] \bar{u}_{tt}^i \quad (55)$$

where c_o^2 is the speed of sound in water and

$$u_{tt}^i = c_o^2 \nabla^2 u^i. \quad (56)$$

and where (see equation 38)

$$\nabla p^i = -\rho_o \bar{u}_{tt}^i. \quad (57)$$

It is now clear by viewing 55 how both longitudinal and transverse waves are supported by the elastic equation. The longitudinal (or equivolume) velocity is defined

$$c_\ell \equiv \sqrt{\frac{\mu}{\rho}} \quad (58)$$

which is seen in 55. The transverse (or irrotational) velocity is defined

$$c_t \equiv \sqrt{\frac{\lambda+2\mu}{\rho}}. \quad (59)$$

This transverse speed of sound is not immediately obvious from 55 because

$$\frac{\mu+3K}{3\rho K} = \frac{\mu+3(2/3\mu+\lambda)}{3\rho K} = \frac{\lambda+\mu}{\rho K}. \quad (60)$$

So 55 can be rewritten

$$\bar{v}_t = c_1^2 \nabla^2 \bar{u}^s - \left(\frac{\lambda+\mu}{\rho K} \right) \nabla p^s + \left(\frac{c_1^2}{c_0^2} + \frac{\rho_o \mu + 3K(\rho - \rho_o)}{3\rho K} \right) \bar{u}_{tt}^i. \quad (61)$$

But writing the wave equation in terms of dilation defined as

$$\Delta \equiv \frac{p}{K}.$$

equation 55 can be written

$$\Delta_{tt} = \left(\frac{\lambda+2\mu}{\rho} \right) \nabla^2 \Delta$$

by taking dot product of both sides of 55.

In final discretized form, 55 becomes

$$v_x^{s,n+1} = v_x^{s,n} + \frac{\mu \Delta t}{\rho} \left(\frac{u_x^{n+\frac{1}{2}}(I+1, J, K) - 2u_x^{n+\frac{1}{2}}(I, J, K) + u_x^{n+\frac{1}{2}}(I-1, J, K)}{(\Delta x)^2} \right.$$

$$u_x^{n+\frac{1}{2}} \frac{u_x^{n+\frac{1}{2}}(I, J+1, K) - 2u_x^{n+\frac{1}{2}}(I, J, K) + u_x^{n+\frac{1}{2}}(I, J-1, K)}{(\Delta y)^2}$$

$$\left. + \frac{u_x^{n+\frac{1}{2}}(I, J, K+1) - 2u_x^{n+\frac{1}{2}}(I, J, K) + u_x^{n+\frac{1}{2}}(I, J, K-1)}{(\Delta z)^2} \right)$$

$$+\Delta t \left\{ -\left(\frac{\mu+3\kappa}{3\rho\kappa} \right) \left(\frac{\nabla p_x}{\rho c_o^2} \right)^{n+\frac{1}{2}} + \left[\frac{\mu}{\rho c_o^2} + \left(\frac{\rho\mu+3\kappa(\rho_o-\rho)}{3\rho\kappa} \right) \right] \left(u_x^i \right)_{tt}^{n+1} \right\} \quad (62)$$

Equations 47 and 62, along with

$$u_x^{s,n+\frac{1}{2}} = u_x^{s,n-\frac{1}{2}} + \Delta t v_x^{s,n} \quad (63)$$

make up the differenced equations for elastic materials.

Outer Radiating Boundary Conditions

Both 2nd Order Mur and a modified Nth order Liao ORBC were coded up under this effort. Liao was modified in two ways.

First, adiabatic loss terms were added (without degrading accuracy) to increase stability of the method. This allows a much larger number of timesteps than for the original Liao boundary condition.

Secondly, when modeling the elastic equation, the two first order Liao ORBC's were matched to the longitudinal and transverse wave speeds and then concatenated together to form a single boundary condition. When applying the code to the acoustic equation, the usual 2nd order Liao ORBC was used but each of the poles in the reflection coefficient were moved to obtain better "wide angle" absorption performance.

The code is set up so the Liao ORBC will work for either acoustic or elastic waves without modification of the boundary condition by the user. The performance of the ORBC's is similar for both electromagnetic and acoustic phenomenon. The only difference is that in the acoustic case we can apply the boundary condition to scalars (pressure). This allows slightly better accuracy because there is no polarization dependence on the reflection coefficient. Performance of the ORBC in elastic media is probably not as good as in the acoustic or electromagnetic cases. The reason is that the boundary condition must be matched to absorb two different waves traveling at different speeds.

Incident Field

The incident field has been set up to be a cylindrical shaped wave with a Gaussian spatial derivative. The incident displacement has a derivative of a Gaussian for temporal variation. The derivation of the incident field equations is involved and requires many details. Only an overview will be given here. The following is an equation for the displacement.

$$\vec{u}^i = A e^{-\alpha((\tau-\beta\Delta t)^2+r^2)} \quad (64)$$

where

$$\tau = t - \frac{(x-x_o)\cos\phi\sin\theta + (y-y_o)\sin\phi\sin\theta + (z-z_o)\cos\theta}{c_o} + t_o, \quad (65)$$

$$r^2 = \left\{ \left[(x - x_o) + \tan \theta (z - z_o) \right]^2 + (y - y_o)^2 \right\} \left(\frac{\beta \Delta s}{\beta \Delta t} \right)^2 \quad (66)$$

and c_o is speed of sound in water, α is defined

$$\alpha = \left(\frac{4}{\beta \Delta t} \right)^2 \quad (67)$$

$\beta \Delta s$ (length for 1 e-folding of amplitude),

$\beta \Delta t$ (time for 1 e-folding of amplitude),

$\frac{\beta \Delta s}{\beta \Delta t}$ is a normalization factor relating time and space.

In order to obtain equations for the incident pressure we use

$$p^i = -\gamma_o P_o \nabla \cdot \vec{u}^i. \quad (68)$$

But the Gaussian incident field has spatial variation (r^2 term in 47)

$$\nabla \cdot \vec{u}^i = \frac{(A_x + A_y + A_z) \left(-2\alpha(\tau - \beta \Delta t) \right)}{c_o^2} e^{-\alpha(\tau - \beta \Delta t)^2 + r^2}. \quad (69)$$

Note that $(\cos \phi \sin \theta)^2 + (\sin \phi \sin \theta)^2 + (\cos \theta)^2 = 1$ and (see 65) that the spatial variation is effectively a constant in 69. The reason for this is that the spatial variation is in the transverse direction so $\nabla \cdot A e^{-\alpha r^2} = 0$. This is an assumption.

For completeness,

$$\vec{u}_t^i = -2\alpha(\tau - \beta \Delta t)^2 A e^{-\alpha((\tau - \beta \Delta t)^2 + r^2)} \quad (70)$$

$$\vec{u}_{tt}^i = -2\alpha(-1 + 2\alpha(\tau - \beta \Delta t)^2) A e^{-\alpha((\tau - \beta \Delta t)^2 + r^2)} \quad (71)$$

and also note that

$$\vec{u}_{tt}^i = c_o^2 \nabla^2 \vec{u}^i. \quad (72)$$

Validation of the longitudinal wave modeling capabilities

A two dimensional rigid vibrating plate system was constructed (Figure 2) whose radiated pressure fields could be evaluated analytically [7]. The analytic predictions were made along the z

axis normal to the force of the rigid plate and along the x axis parallel to the plate. These pressures were normalized using the pressure just above the center of the plate. Comparison was then made to a model of the system (Figure 3) in the acoustic FDTD code with $\mu = 0$ so only longitudinal or compressional waves were possible. The acoustic FDTD model space was large enough that the Gaussian displacement specified for the plate produced a pressure wave short enough that sufficient clear time was obtained to see the wave pass a monitor point without any reflection from the problem space extremities contaminating the result. Thus the response predictions could be truncated just after the waveform passed to obtain clear time predictions or a longer record could be predicted and retained in which case the effects of the outer radiation boundary condition on reflections could be seen. The outer radiation boundary condition selected was Liao's second order condition and we will see it worked extremely well.

Comparison of analytic and acoustic FDTD predictions (Figures 4-11) was excellent for both clear time and second order Liao predictions. The acoustic FDTD code properly handles compression waves. The outer radiation boundary condition is working effectively at absorbing these compressional waves when they reach the problem space extremities.

Validation of the Transverse Wave Modeling Capabilities

A 1/4 inch thick aluminum plate was suspended in a water bath with an attached piezoelectric transducer coupled to the plate via a plexiglass wedge. Another piezoelectric transducer on a robot arm acted as a receiver. With the experimental setup (figure 12), waves propagating through the water and the plate could be monitored at various angles. Signals that traveled in the plate (Lamb waves) and then leaked backed into the water were of particular interest as they involved transverse wave motion in the plate.

The data was fed through a Hanning window (figure 13) to produce spectrograms that displayed the spectral content as a function of frequency, time and angle. By scanning the spectrograms for response peaks and using the largest response peak found among the different angles, a composite response of pressure amplitude vs. frequency was formed (figure 14). This composite response could be readily compared against the code predictions. The code predicts pressure at a point with no dependence on the angle of the incoming wave for a simple model of the experimental setup (figure 15). The same time history (derivative Gaussian) and beamwidth were used for the code's source as for the experiment.

The predicted data was processed to separate a fast moving component and a slower moving component (figure 16). The faster component involved propagation in the metal while the slower component was the reflected wave off the plate traveling in the water along with a surface wave

propagating along the water/plate interface and decaying exponentially above the interface and the signal immediately leaking back into the water.

Response predictions in the metal for the fast moving component revealed the highest frequency content (figure 17). Response predictions in the water for the fast moving component were similar but did not provide as high a frequency content (figure 18). For the slow moving component, the predictions show low frequency response peaks and valleys in the water that are roughly the opposite of those in the metal (figures 19, 20). It is posited that what is reflected by the plate at low frequencies does not get into the plate and hence the nature of the respective spectrums.

The comparison of predicted response to measured composite response was made using the low frequency, slow moving component in the water prediction and, because of the high frequency content, the high frequency, fast moving component in the metal prediction. The predictions contained a direct reflection as the source point in the model was well off the plate. In the experiment, the source transducer is effectively coupled directly to the plate and there is not a reflection. The result is the two spectrums are different and all that can be compared is the location of spectral resonances or peaks. There is good agreement in the two low frequency spectral peaks of the slow moving component in the water with the composite data. A number of the high frequency spectral peaks of the predicted pressure response coincide with the composite response.

The complexity of the predicted response can be seen (figure 21) in a black and white reproduction of a color graphic visualization of the response at successive times. The incident, reflected, transmitted (through the plate to the water below), surface wave, slow and fast moving components in the plate and the fast moving component "shed" back into the water are all visible.

Conclusions

For compressional waves, i.e. signals in water and air, AFDTD is a unqualified success. The comparison of experiment and predictions is less definitive. Nonetheless, there is enough of a match between the two to strongly suggest AFDTD also models transverse wave propagation. As such it makes a very powerful elastic media, small signal, modeling tool.

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figure 1

figure 2

figure 3

FDTD cell diagram with cubic cells and $y = 5\Delta l$

Figure 4: Magnitude pressure comparison between exact solution, FDTD using clear time, and FDTD using 2nd Order Liao boundary condition at frequency of $KL = 5.0$. The Liao boundary condition was applied one L away from piston head.

Figure 5: Phase of pressure comparison between exact solution, FDTD using clear time, and FDTD using 2nd Order Liao boundary condition at frequency of $KL = 5.0$. The Liao boundary condition was applied one L away from piston head.

Figure 6: Magnitude pressure comparison between exact solution, FDTD using clear time, and FDTD using 2nd Order Liao boundary condition at frequency of $KL = 5.0$. The Liao boundary condition was applied one L away from piston head.

Pressure Magnitude for 2-D Piston vs. Position

$KL = 5.0$ $L = 50$ cells/ λ .

Figure 7: Phase of pressure comparison between exact solution, FDTD using clear time, and FDTD using 2nd Order Liao boundary condition at frequency of $KL = 5.0$. The Liao boundary condition was applied one L away from piston head.

Figure 8: Magnitude pressure comparison between exact solution, FDTD using clear time, and FDTD using 2nd Order Liao boundary condition at frequency of $KL = 10$. The Liao boundary condition was applied one L away from piston head.

Pressure Magnitude for 2-D Piston vs. Position

$$KL = 10 \quad L = 50 \text{ cells}/\lambda.$$

Figure 9: Phase of pressure comparison between exact solution, FDTD using clear time, and FDTD using 2nd Order Liao boundary condition at frequency of $KL=10$. The Liao boundary condition was applied one L away from piston head.

Figure 10: Magnitude pressure comparison between exact solution, FDTD using clear time, and FDTD using 2nd Order Liao boundary condition at frequency of $KL = 10$. The Liao boundary condition was applied one L away from piston head.

Pressure Magnitude for 2-D Piston vs. Position

$KL = 10.0$ $L = 50$ cells/ λ .

Figure 11: Phase of pressure comparison between exact solution, FDTD using clear time, and FDTD using 2nd Order Liao boundary condition at frequency of $KL = 10$. The Liao boundary condition was applied one L away from piston head.

figure 12

figure 13

Time Gating Schematic

figure 14

Frequency spectrum, experimental, of maxima

figure 15

AFDTD Model

Expanded View of AFDTD Signals Above, In and Below the Plate

figure 16

Early time spectral signature of gated plate pressure.

Extracted and detrended gate

View 3: s[14]

Time signal at end of plate gated region shown between I-I

figure 17

Signal in water above plate
Early time signal

Extracted and detrended signal

Early time spectral signature
of gated pressure signal
in water above aluminum plate

figure 18

Signal in water above plate
Early time signal

1 10
0 5
0 0
e + 0
3 -5
-10
-15

Extracted and detrended signal

Early time spectral signature
of gated pressure signal
in water above aluminum plate

View 3: me

Extracted asnd detrended gate

View 3: s[14]

Time signal at end of plate
gated region shown between I-I

figure 20

Small angle, low amplitude leaky wave
incident and reflected wave

Large angle, high amplitude leaky wave
psuedo surface wave

figure 21

END

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