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A CONTOUR COMPLETION ALGORITHM

by

R. Y. Dean

Operations Research and Synthesis Operation
Hanford Laboratories Operation
Richland, Washington

April 24, 1962

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A CONTOUR COMPLETION ALGORITHM

PURPOSE

It is the intent of this report to motivate, develop, and explain a mathematical algorithm which has been designed to complete contour descriptions of weapon components, which have been only partially specified by free-form data.

INTRODUCTION AND MOTIVATION

Customarily the dimensional and geometrical characteristics of those weapon components which are to be fabricated on rotary lathes as surfaces of revolution are specified by the design agencies with reference to a standard plane polar coordinate system, (r, θ) . With the polar axis, $\theta = 0$, chosen to coincide with the axis of rotational symmetry of the surface to be machined, and the pole (origin of coordinates) chosen at some convenient location on this axis, the surface is described by specifying a typical cross-section or "contour" in this coordinate system. Clearly, spherical, cylindrical, conical, elliptical and many other common surface areas can be described in terms of their corresponding circular, linear, etc. plane section contours by simply applying a modicum of analytic geometry. Thus, if a segment of contour has a known analytic form, it is a mere matter of computation to find the coordinates of any desired point on it.

Frequently, however, in the design of weapon components, a major segment of the desired contour is specified in the so-called free-form style. Such contour segments are incompletely described by giving only a finite number of equally or unequally spaced points along the contour together with a specified or implied direction at the two end points of the free-form segment. This situation may arise because the extreme analytic complexity in the design criteria virtually defies any simple or adequate mathematical description and essentially forbids the precise computation of more than a few points along the contour. Moreover, much contour design data are obtained empirically as a result of experimentation and are thus totally devoid of mathematical description. In any event, when a free-form segment of contour is presented for fabrication, it becomes incumbent upon the machinist to "fill in" or "complete" the contour in a continuous manner so that the finished surface will possess certain "smoothness" characteristics.

At present, there does not seem to exist any general agreement as to just what "smoothness" really means. Although the subject is under almost continuous discussion by representatives of the weapons design and fabrication agencies, quantitative criteria have as yet to be defined which have gained universal acceptance. A similar problem has existed and still exists in the aircraft industry in connection with the completion of air foil contours. Many techniques have been suggested and are in use, each apparently satisfactory to their originator and user. A fundamental difficulty appears to arise, however, whenever a particular technique is borrowed and applied under different circumstances. It is quite apt to be unsatisfactory. As a consequence, a somewhat heated academic argument has arisen as to just what method is best.

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It is the present writer's opinion that there is nor cannot be any one best method. The success of any one method is to a great extent a function of the type of contour it was designed to complete.

Repeated competitive tests have illustrated this point time and time again. A particular method may be admirably suited to completing a certain class of contours and yet may fail miserably when applied to one of a different species. The argument should cease as to which method is best, and attentions should be diverted to deciding which are more suitable to particular situations.

In reaching a decision on just what technique to use, the author believes the following three criteria to be the most important.

1. Was the technique designed to complete the particular type of contour under consideration?
2. Has the technique demonstrated its capabilities in competition with alternate techniques?
3. Can the necessary computations be efficiently and economically performed on the available computing equipment?

The following technique, developed at Hanford and tested in a variety of situations against alternate methods employed at other AEC installations, has met with considerable success. Moreover, it was designed to meet the requirements of the particular variety of free-form contours normally encountered at HAPO, and tailored for ease of handling on HAPO's computing equipment.

THE PROBLEM

Let

$$(1) \quad (r_i, \theta_i) \quad i = 1, 2, \dots, n+1$$

designate the polar coordinate of $n+1$ points in the plane (indexed according to increasing θ), and let

$$(2) \quad r'_1 \text{ and } r'_{n+1} \quad (' \equiv d/d\theta)$$

denote prescribed derivatives* at the first and last points respectively.

* Attention is called to the fact that $r' = dr/d\theta$ does not represent geometrically what is customarily called the "slope" at any point along the curve. Slope is a rectangular Cartesian coordinate concept and traditionally denoted by $y' = dy/dx$. The translation from one coordinate system to the other can be accomplished by means of the two formulas

$$r' = \frac{r(y' \sin \theta + \cos \theta)}{(y' \cos \theta - \sin \theta)}$$

and

$$y' = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$$

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Determine a mathematical expression, $r = r(\theta)$, from which

- (i) the value of r can be computed for any $\theta_1 \leq \theta \leq \theta_{n+1}$,
- (ii) the value r_i is obtained at each θ_i , $i = 1, 2, \dots, n+1$,
- (iii) r'_1 and r'_{n+1} are assumed at θ_1 and θ_{n+1} respectively, and
- (iv) the completed contour is as "smooth" as possible consistent with its constraints.

THE PROPOSED SOLUTION

The following development is based on three assumptions:

- (i) The segment of contour being completed is approximately a circular arc of radius a . This assumption is based purely on experience and observation of the type of contour found on blueprints and design data available at Hanford.
- (ii) The "smoothest" curve which goes through the prescribed points and obeys the assigned end conditions would be the one assumed by a thin elastic spline whose unstressed zero-energy state is a circular arc of radius a and which is then forced to obey the above constraints.
- (iii) The given data, i.e., the prescribed points, are themselves smoothly enough distributed in the plane so that the spline in its constrained state obeys the usual postulated restrictions of linear elasticity theory.

Based on these assumptions, it can be shown that each segment of the bent spline will assume that shape which will minimize its strain energy. If the bending is not too severe this energy is proportional to the line integral

$$(3) \quad \int_c \left(k - \frac{1}{a}\right)^2 ds$$

Here, c represents the sought curve between any two consecutive points, s denotes arc-length along this curve, and k represents the geometrical curvature of c . Expressed in polar coordinates, the integral in (3) can be represented in the more familiar form

$$(4) \quad \int_{\theta_1}^{\theta_{i+1}} \left[\frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} - \frac{1}{a} \right]^2 \sqrt{r^2 + r'^2} d\theta$$

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Choosing a curve, $r = r(\theta)$, so as to minimize (4) subject to the condition of a continuous bending moment is now a standard problem in the calculus of variations. Unfortunately, however, the Euler-Lagrange differential equation which determines $r(\theta)$ is nonlinear and presents extremely inconvenient mathematical difficulties. If the assumptions of (i) and (iii) above are valid however, (4) can be approximated without serious loss in accuracy by the following device. Along c , r is approximately a , and r'^2 is negligible when compared to r^2 , so that (4) collapses into the particularly simple form

$$(5) \quad \int_{\theta_i}^{\theta_{i+1}} (r'')^2 d\theta.$$

The Euler-Lagrange differential equation for the minimization of (5) is simply

$$(6) \quad \frac{d^4 r}{d\theta^4} = 0$$

with the general solution

$$(7) \quad r(\theta) = A + B\theta + C\theta^2 + D\theta^3.$$

Thus, in each segment, a cubic approximation to the desired curve is appropriate. Moreover, consistent with the constraints of the problem, the four arbitrary constants in (7) are determined by requiring each cubic to go through its end points, and to match its immediate neighbors in first and second derivatives. This not only assures a continuous slope, but continuous curvature as well.

It should be pointed out that if there are regions along the contour where the simplifying assumptions from (4) to (5) are not valid, the resulting curve approximation will naturally stray from the true spline fit. On the other hand, if the originally given data are given fairly densely along the desired curve, the above procedure at least represents a method of approximating by cubics which preserves continuity through the second derivative - a fair indicator of smoothness itself. In any event, the routine which follows for determining the successive cubics is relatively straight forward and possesses stability and accuracy properties which recommend it for computer use.

THE CUBIC APPROXIMATION ALGORITHM

Armed with the findings of the previous paragraph, we now assume that in each interval r is determined as a cubic polynomial in θ , and that it shares with its neighbor identical first and second derivatives at their common point. Denote

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by r'_i and r''_i the unique, but as yet unknown first and second derivatives at each of the points $i = 2, 3, \dots, n$. (r'_1 and r'_{n+1} are already known from (2).) If we let $c_i(\theta)$ represent the cubic for the i th interval, we may write for $\theta_i \leq \theta \leq \theta_{i+1}$

$$(8) \quad c_i(\theta) = r_{i+1} + r'_{i+1}(\theta - \theta_{i+1}) + r''_{i+1} \frac{(\theta - \theta_{i+1})^2}{2} + c'''_i(\theta_{i+1}) \frac{(\theta - \theta_{i+1})^3}{6}$$

and

$$(9) \quad c'_i(\theta) = r'_{i+1} + r''_{i+1}(\theta - \theta_{i+1}) + c'''_i(\theta_{i+1}) \frac{(\theta - \theta_{i+1})^2}{2}$$

Similarly, for $\theta_{i+1} \leq \theta \leq \theta_{i+2}$

$$(10) \quad c_{i+1}(\theta) = r_{i+1} + r'_{i+1}(\theta - \theta_{i+1}) + r''_{i+1} \frac{(\theta - \theta_{i+1})^2}{2} + c'''_{i+1}(\theta_{i+1}) \frac{(\theta - \theta_{i+1})^3}{6}$$

and

$$(11) \quad c'_{i+1}(\theta) = r'_{i+1} + r''_{i+1}(\theta - \theta_{i+1}) + c'''_{i+1}(\theta_{i+1}) \frac{(\theta - \theta_{i+1})^2}{2}$$

(Note that these two adjoining cubics have each been expanded about their common point, θ_{i+1} .)

For later convenience let

$$(12) \quad \left. \begin{aligned} p_i &= \theta_{i+1} - \theta_i \\ q_i &= r_{i+1} - r_i \end{aligned} \right\} \quad i = 1, 2, \dots, n.$$

Evaluating (8) and (9) at $\theta = \theta_i = \theta_{i+1} - p_i$ yields

$$(13) \quad r_i = r_{i+1} - r'_{i+1} p_i + r''_{i+1} \frac{p_i^2}{2} - c'''_i(\theta_{i+1}) \frac{p_i^3}{6}$$

and

$$(14) \quad r'_i = r'_{i+1} - r''_{i+1} p_i + c'''_i(\theta_{i+1}) \frac{p_i^2}{2}$$

Now evaluate (10) and (11) at $\theta = \theta_{i+2} = \theta_{i+1} + p_{i+1}$ to get

$$(15) \quad r_{i+2} = r_{i+1} + r'_{i+1} p_{i+1} + r''_{i+1} \frac{p_{i+1}^2}{2} + c_{i+1}''' (\theta_{i+1}) \frac{p_{i+1}^3}{6}$$

and

$$(16) \quad r'_{i+2} = r'_{i+1} + r''_{i+1} p_{i+1} + c_{i+1}''' (\theta_{i+1}) \frac{p_{i+1}^2}{2}$$

Multiplying (15) by $\frac{3p_i}{p_{i+1}}$, (13) by $\frac{-3p_{i+1}}{p_i}$, (14) by $-p_{i+1}$ and (16) by $-p_i$

and adding the resulting equations gives

$$(17) \quad p_{i+1} r'_i + 2(p_i + p_{i+1}) r'_{i+1} + p_i r'_{i+2} = \frac{3p_i q_{i+1}}{p_{i+1}} + \frac{3p_{i+1} q_i}{p_i}$$

The $n-1$ equations obtained by setting $i = 1, 2, \dots, n-1$ in (17) together with the given beginning and terminal derivatives form an $n+1$ square system of simultaneous linear equations for the determination of the r'_i , $i = 1, 2, \dots, n+1$. Arranged in matrix form these $n+1$ equations assume the tri-diagonal form

$$(18) \quad \begin{pmatrix} 1 & & & & \\ & p_2 & b_2 & p_1 & \\ & & p_3 & b_3 & p_2 \\ & & & & \\ & & & & p_n & b_n & p_{n-1} \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} r'_1 \\ r'_2 \\ r'_3 \\ \\ r'_n \\ r'_{n+1} \end{pmatrix} = \begin{pmatrix} r'_1 \\ d_2 \\ d_3 \\ \\ d_n \\ r'_{n+1} \end{pmatrix}$$

where

$$(19) \quad b_i = 2(p_{i-1} + p_i) \quad i = 2, 3, \dots, n$$

and

$$(20) \quad d_i = \frac{3p_{i-1} q_i}{p_i} + \frac{3p_i q_{i-1}}{p_{i-1}} \quad i = 2, 3, \dots, n$$

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It is seen from the form of its elements that the matrix in (18) is diagonally dominant, well conditioned, and hence subject to machine inversion with a minimum of error. In particular, it is easily arranged into the following equivalent system

$$(21) \quad \begin{pmatrix} 1 & & & \\ & 1 & C_2 & \\ & & 1 & C_3 \\ & & & \ddots \\ & & & & 1 & C_n \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} r'_1 \\ r'_2 \\ r'_3 \\ \vdots \\ r'_n \\ r'_{n+1} \end{pmatrix} = \begin{pmatrix} r'_1 \\ D_2 \\ D_3 \\ \vdots \\ D_n \\ r'_{n+1} \end{pmatrix} .$$

where

$$(22) \quad C_2 = p_1/b_2$$

and the remainder of the C_i are determined recursively by

$$(23) \quad C_i = \frac{p_{i-1}}{b_i - p_i C_{i-1}} \quad i = 3, 4, 5, \dots, n .$$

Similarly,

$$(24) \quad D_2 = \frac{d_2 - p_2 v'_1}{b_2}$$

and the remainder of the D_i are determined recursively by

$$(25) \quad D_i = \frac{d_i - p_i D_{i-1}}{b_i - p_i C_{i-1}} \quad i = 3, 4, 5, \dots, n .$$

It follows immediately then from (21) that the unknown derivatives at all the given intermediate points can be determined from the recursive relationship

$$(26) \quad r'_{n-1} = D_{n-1} - C_{n-1} r'_{n+1} \quad i = 0, 1, 2, \dots, n-2 .$$

If we now return to the cubic for the i^{th} interval expanded about its initial point, we may write for $\theta_i \leq \theta \leq \theta_{i+1}$

$$(27) \quad c_i(\theta) = r_i + r'_i (\theta - \theta_i) + A_i (\theta - \theta_i)^2 + B_i (\theta - \theta_i)^3$$

and

$$(28) \quad c'_i(\theta) = r'_i + 2A_i (\theta - \theta_i) + 3B_i (\theta - \theta_i)^2$$

where A_i and B_i are to be determined.

Setting $\theta = \theta_{i+1} = \theta_i + p_i$ in (27) and (28) yields

$$(29) \quad r_{i+1} = r_i + r'_i p_i + A_i p_i^2 + B_i p_i^3$$

and

$$(30) \quad r'_{i+1} = r'_i + 2A_i p_i + 3B_i p_i^2$$

The simultaneous solution of (29) and (30) yields

$$(31) \quad A_i = \frac{1}{p_i^4} \left\{ 3p_i^2 (q_i - r'_i p_i) - p_i^3 (r'_{i+1} - r'_i) \right\}$$

and

$$(32) \quad B_i = \frac{1}{p_i^4} \left\{ p_i^2 (r'_{i+1} - r'_i) - 2p_i (q_i - r'_i p_i) \right\}$$

We have thus attained the goal of having generated an analytic expression which can be used to continuously fill in each interval of the desired curve. Since in actual practice it is only desired to fill in each interval at a finite number of points, we may proceed as follows.

If it is desired to fill in the i^{th} interval at $m-1$ equally spaced points, let

$$(33) \quad \theta_{ij} = \theta_i + \frac{j}{m} p_i \quad j = 0, 1, 2, \dots, m.$$

Thus θ_{ij} designates the θ - coordinate of the j^{th} interpolated point in the i^{th} interval. ($\theta_{i0} = \theta_i$ and $\theta_{im} = \theta_i + p_i = \theta_{i+1}$ correspond to the beginning and ending of the interval.)

If we designate the radial coordinate corresponding to θ_{ij} by r_{ij} , it may be computed by substituting (33) in (27) and making use of (31) and (32). One easily derives

$$(34) \quad r_{ij} = r_i + \frac{1}{m} r'_i p_1 + \left(\frac{1}{m}\right)^2 \left\{ 3q_1 - p_1(r'_{i+1} + 2r'_i) \right\} + \left(\frac{1}{m}\right)^3 \left\{ p_1(r'_{i+1} + r'_i) - 2q_1 \right\}.$$

Similarly, substituting (33) in (28) we obtain

$$(35) \quad r'_{ij} = r'_i + \frac{2}{p_1} \left(\frac{1}{m}\right) \left\{ 3q_1 - p_1(r'_{i+1} + 2r'_i) \right\} + \frac{3}{p_1} \left(\frac{1}{m}\right)^2 \left\{ p_1(r'_{i+1} + r'_i) - 2q_1 \right\},$$

for the derivative at the j^{th} interpolated point in the i^{th} interval.

For the purposes of investigating curvature it is necessary to have available second derivatives. They can be computed in a like manner. In fact, if they are designated by r''_{ij} we have

$$(36) \quad r''_{ij} = \frac{2}{p_1} \left\{ 3q_1 - p_1(r'_{i+1} + 2r'_i) \right\} + \frac{6}{p_1^2} \left(\frac{1}{m}\right) \left\{ p_1(r'_{i+1} + r'_i) - 2q_1 \right\}.$$

In summary, formula (26) provides the means of assigning derivatives at each of the originally given points in terms of the coordinates of those points and the prescribed end conditions. Once these have been obtained they may be employed in (34) and (35) to "fill in" the radii and derivatives of the sought curve as densely as desired.

SOME COMMENTS

In the event that a desired contour is so extended that it is obviously no longer an approximate circular arc, there exists a routine almost identical to that outlined above for obtaining an approximate spline fit. The arguments are then most easily couched in Cartesian coordinates, (x, y) , and the assumption is made that an originally straight zero-energy state spline is bent to form the contour. Once again cubic approximations interval by interval follow from an energy principal and lead to formulas similar to (34) and (35).

Both routines have been programmed for the IBM 7090 and are available for contour completion use.

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