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**SOLUTION OF MATHEMATICAL PROGRAMMING FORMULATIONS
OF SUBGAME PERFECT EQUILIBRIUM PROBLEMS**

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Charles M. Macal

Environmental Assessment and Information Sciences Division
Argonne National Laboratory
Argonne, IL 60439

Arthur P. Hurter

Department of Industrial Engineering and Management Sciences
Northwestern University
Evanston, IL 60208

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ABSTRACT

Mathematical programming models have been developed to represent imperfectly competitive (oligopolistic) market structures and the interdependencies of decision-making units in establishing prices and production levels. The solution of these models represents an economic equilibrium. A subgame perfect equilibrium formulation explicitly considers that each agent's strategies depend on the current state of the system; the state depends solely on previous decisions made by the economic agents. The structure of an industry-wide model that is formulated as a subgame perfect equilibrium problem is a matrix of simultaneous mathematical programming problems, where the rows represent time periods and the columns represent agents. This paper formally defines the subgame perfect equilibrium problem that includes mathematical programs for agent decision problems, and it characterizes the feasible space in a way that is conducive to the solution of the problem. The existence of equilibrium solutions on convex subspaces of the feasible region is proved, and this set is shown to contain the subgame perfect equilibrium solutions. A procedure for computing equilibrium solutions and systematically searching the subspaces is illustrated by a numerical example.

Keywords: Games/group decisions, Noncooperative: formulation of subgame perfect equilibria

Economics: competitive market models, economic equilibria

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Mathematical programming models have been developed to represent industries and markets at an ever increasing level of detail and sophistication. They are used by the U.S. Department of Energy and other governmental and private-sector organizations in making strategic policy decisions. Realistic representations of decision making in these models include resource constraints, such as available financial and physical resources, capacities of production facilities, and restrictions of physical units to non-negative values. These models also consider the imperfectly competitive nature of the markets and the interdependencies of decision-making units in establishing prices and production levels. The market representation is an oligopoly, and the solution of the models represents an economic equilibrium in the sense described below. Because there are a number of theories about how oligopoly prices and production come about and how agents interact, numerous equilibrium concepts have been adapted from the field of game theory. Selecting an equilibrium concept that is appropriate for modeling a particular market is a central problem in formulating an imperfect competition model.

The Nash equilibrium concept (Nash 1951) from noncooperative game theory is commonly applied to oligopolistic market models. Static Nash equilibrium models assume that each agent recognizes that its actions affect the market, but an agent takes the actions of other agents as constant, or fixed, when making decisions to maximize utility. Equilibrium is a condition in which no agent can unilaterally increase utility by adopting any strategy other than its equilibrium strategy, given that other agents adopt their equilibrium strategies. Dynamic Nash equilibrium formulations explicitly consider that each agent's strategies depend on the current state of the system which, in the deterministic case, depends solely on the decisions previously made by all agents. For rational agents, it is reasonable to model agents' strategies as being optimal with respect to the current state.

Selten (1965, 1973, 1975) introduced the notion of subgame perfect equilibrium (SPE) for games in extensive form, but this concept has direct applicability to imperfectly competitive markets as well. An SPE is a set of strategies that induces an equilibrium on every subgame of the game. In an oligopoly, a subgame is a decision point and is distinguished by an agent and a time period; an SPE is a set of strategies for all agents, where each strategy indicates the optimal decision of an agent for every time period as it depends on the state of the system at the time the decision is made.

Macal and Hurter (1991) formulated a subgame perfect equilibrium problem as a system of mathematical programming problems. An SPE problem can be thought of as a matrix of simultaneous mathematical programming problems, where the rows represent time periods and the columns represent agents. This paper describes a solution algorithm for SPE problems formulated as such a system of mathematical programs. A small problem is addressed consisting of two agents and two time periods, but application of the algorithm to arbitrarily large problems is straightforward.

This paper is organized as follows. Section 1 defines the SPE problem and characterizes the feasible space in a way that is conducive to its solution. Section 2 proves the existence of facet equilibrium solutions, the set of which contains the SPE solutions. Section 3 describes a procedure for computing facet equilibrium solutions and systematically searching facets for an SPE solution.

1 SUBGAME PERFECT EQUILIBRIUM FORMULATION

This section formulates the general subgame perfect equilibrium problem, presents an example for two agents and two time periods, and characterizes the feasible space in a way that is conducive to the solution of the SPE problem.

1.1 Notation

α	number of economic agents or firms, assumed to be finite.
τ	number of time periods, assumed to be finite.
$x_{it} \in \mathbf{R}^r$	decision variable (production) of agent (firm) i at time t .
$x_t = (x_{1t}, \dots, x_{it}, \dots, x_{\alpha t})$	production by all firms at t .
$x_{-it} = (x_{1t}, \dots, x_{i-1,t}, x_{i+1,t}, \dots, x_{\alpha t})$	production by firms other than i at t .
$x_{it}^* \in \mathbf{R}^r$	optimal production of firm i at t .
$X_{it}(x_1, \dots, x_{t-1})$	strategy mapping of all decisions preceding t (history at t) to a decision by agent i at t .
$X_t = (X_{1t}, \dots, X_{it}, \dots, X_{\alpha t})$	strategies of all firms at t .
$u_{it}(x_1, \dots, x_t)$	utility function of i at t assumed to depend on all decisions up to and including t .
β_{it}	discount factor for firm i at time t .
$U_i(x_1, \dots, x_\tau) = \sum_{t=1, \tau} \beta_{it} u_{it}$	utility function of i , assumed to be separable.
$G_{it}(x_1, \dots, x_t)$	constraint set for agent i at time t , assumed to be linear.
$z(x) = \text{Sol Max}_y g(x, y)$	z is the solution (assumed to be unique) of $\text{Max}_y g(x, y)$ as x varies.
$y_n^* = \text{Sol}_n \text{Max}_{y_1, \dots, y_n, \dots} u(y_1, \dots, y_n, \dots)$	y_n^* is the n^{th} solution variable of $\text{Max}_y u(y)$.
$(x_1^*, \dots, x_\alpha^*) = \text{Esol}_{i=1}^\alpha [\text{Max}_{x_i} u_i(x_i, x_{-i})]$	$(x_1^*, \dots, x_i^*, \dots, x_\alpha^*)$ is the solution to the equilibrium problem in brackets for agents 1 through α , that is $x_i^* = \text{Sol Max}_{x_i} u_i(x_i, x_{-i}^*)$ for $i = 1, \dots, \alpha$.

1.2 General Formulation

Definition 1: A subgame perfect equilibrium (SPE) outcome $(x_1^*, \dots, x_\tau^*)$ is a solution to the following system of optimization problems where x_1^* solves at $t = 1$:

$$(x_{11}^*, \dots, x_{i1}^*, \dots, x_{\alpha 1}^*) = \text{Esol}_{i=1}^\alpha \left[\begin{array}{ll} \text{Max}_{x_{i1} \geq 0} : & u_{i1}(x_{i1}, x_{-i1}) + \sum_{s=2}^\tau \beta_{is} u_{is}(x_{i1}, x_{-i1}, X_2, \dots, X_s) \\ \text{S.t.} : & G_{i1}(x_{i1}, x_{-i1}) \geq 0 \end{array} \right] \quad (1a)$$

where for $t > 1$ and $t < \tau$ ($\tau > 2$), the strategies X_t in (1a) are recursively defined as parameterized solutions to the following equilibrium system:

$$X_t(x_1, \dots, x_{t-1}) = \text{Esol}_{i=1}^\alpha \left[\begin{array}{ll} \text{Max}_{x_{it} \geq 0} : & u_{it}(x_1, \dots, x_{t-1}; x_{it}, x_{-it}) + \sum_{s=t+1}^\tau \beta_{is} u_{is}(x_1, \dots, x_{t-1}; x_{it}, x_{-it}, X_{t+1}, \dots, X_s) \\ \text{S.t.} : & G_{it}(x_1, \dots, x_{t-1}; x_{it}, x_{-it}) \geq 0 \end{array} \right] \quad (1b)$$

with $x_t^* = X_t(x_1^*, \dots, x_{t-1}^*)$

and for $t = \tau$, the strategies X_τ are defined as parameterized solutions to the following equilibrium system:

$$X_\tau(x_1, \dots, x_{\tau-1}) = \text{Esol}_{i=1}^\alpha \left[\begin{array}{ll} \text{Max}_{x_{i\tau} \geq 0} : & u_{i\tau}(x_1, \dots, x_{\tau-1}; x_{i\tau}, x_{-i\tau}) \\ \text{S.t.} : & G_{i\tau}(x_1, \dots, x_{\tau-1}; x_{i\tau}, x_{-i\tau}) \geq 0 \end{array} \right] \quad (1c)$$

with $x_\tau^* = X_\tau(x_1^*, \dots, x_{\tau-1}^*)$

□

A subgame perfect equilibrium problem is a system of mathematical programming problems, as illustrated in Figure 1 (Macal and Hurter, 1991). This system has structural characteristics of both (static) oligopoly equilibrium systems (Murphy et al., 1982) and multilevel mathematical programs (Bard and Falk, 1982).

The mapping $X_{it}(x_1, \dots, x_{t-1})$ is the equilibrium *strategy* for agent i at time t given the history $\{x_1, \dots, x_{t-1}\}$ of decisions preceding t . X_{it} is called a strategy because it specifies an optimal equilibrium decision in period t for any decisions $\{x_1, \dots, x_{t-1}\}$ made preceding period t . Conceptually, SPE strategies can be constructed by a process of backward recursion, as in dynamic programming. The equilibrium strategies for every possible state of the game (market) in the last period are constructed first; the equilibrium strategies for every possible state of the game in the next-to-last period are constructed next, based on the observation that only equilibrium strategies for the last period need to be considered in computing equilibrium strategies for the next-to-last period, and so on.

An equilibrium in which strategies depend only on time (that is, strategies are not functions of the state of the system) is known as an open-loop equilibrium. For example, in an open-loop model, each agent takes the decisions of the other agents for all time periods as fixed and chooses decisions that maximize discounted profit for all time periods. Typically, open-loop models are considerably easier to solve than dynamic equilibrium models, which have state-dependent strategies. The open-loop formulation is presented here because of its significance in computing SPE solutions.

Definition 2: An open-loop equilibrium (OLE) outcome $(x_1^*, \dots, x_T^*)$ is a solution to the following system of optimization problems:

$$(x_1^*, \dots, x_i^*, \dots, x_n^*) = \text{Esol}_{i=1}^n \left[\begin{array}{ll} \text{Max}_{(x_{i1}, \dots, x_{iT}) \geq 0} : & \sum_{t=1}^T \beta_{it} u_{it}(x_{i1}, x_{-i1}^*, x_{i2}, x_{-i2}^*, \dots, x_{iT}, x_{-iT}^*) \\ \text{S.t. :} & G_{i1}(x_{i1}, x_{-i1}^*) \geq 0 \\ & \dots \quad \dots \quad \dots \\ & G_{iT}(x_{iT}, x_{-iT}^*, \dots, x_{iT}, x_{-iT}^*) \geq 0 \end{array} \right] \quad \square$$

An open-loop equilibrium is a set of scalars, whereas an SPE is a set of mappings. This paper addresses computing SPE *outcomes*, which are scalars that result if all SPE strategies are acted upon.

1.3 Two-Period, Two-Agent Formulation

For the remainder of this paper we consider the SPE formulation for two agents ($\alpha = 2$, agents i and k) and two time periods ($\tau = 2$). We also assume that all constraints G are linear and utility functions U are quadratic. The period-two equilibrium problem from (1c) above is to find (x_{i2}^*, x_{k2}^*) for any (x_{i1}, x_{k1}) that satisfies:

$$x_{i2}^* = \text{Sol} \left[\begin{array}{l} \text{Max}_{x_{i2} \geq 0} \quad u_{i2}(x_{i1}, x_{k1}, x_{i2}, x_{k2}^*) \\ \text{S.t.:} \quad G_{i2}(x_{i1}, x_{k1}, x_{i2}, x_{k2}^*) \geq 0 \quad (\theta_i) \end{array} \right] \quad (2a)$$

$$x_{k2}^* = \text{Sol} \left[\begin{array}{l} \text{Max}_{x_{k2} \geq 0} \quad u_{k2}(x_{i1}, x_{k1}, x_{i2}^*, x_{k2}) \\ \text{S.t.:} \quad G_{k2}(x_{i1}, x_{k1}, x_{i2}^*, x_{k2}) \geq 0 \quad (\theta_k) \end{array} \right] \quad (2b)$$

where θ_i and θ_k are the Lagrange multipliers associated with the respective constraints. Under the following assumptions, the Karush-Kuhn-Tucker (KKT) conditions for (2) can be used as an equivalent substitute for the period-two equilibrium problem.

Assumption 3: *The KKT conditions for each of the respective agent's period-two optimization problems are necessary and sufficient. $\square$*

Definition 4: *The equivalent period-two equilibrium problem $E2(x_{i1}, x_{k1}, x_{i2}, x_{k2}, \theta_i, \theta_k)$, abbreviated $E2(x_1, x_2, \theta)$, consists of the KKT conditions for the agents' period-two optimization problems (2) for fixed x_{k1} (x_{i1}) in agent i 's (k 's) problem:*

$$\begin{aligned} -\frac{\partial u_{i2}}{\partial x_{i2}} - \theta_i \left(\frac{\partial G_{i2}}{\partial x_{i2}} \right) &\geq 0 & \left[\frac{\partial u_{i2}}{\partial x_{i2}} + \theta_i \left(\frac{\partial G_{i2}}{\partial x_{i2}} \right) \right] x_{i2} &= 0 & E2(x_1, x_2, \theta) \\ x_{i2} \geq 0, \quad G_{i2} &\geq 0, \quad \theta_i \geq 0, \quad G_{i2} \theta_i &= 0 \end{aligned}$$

$$-\frac{\partial u_{k2}}{\partial x_{k2}} - \theta_k \left(\frac{\partial G_{k2}}{\partial x_{k2}} \right) \geq 0 \quad \left[\frac{\partial u_{k2}}{\partial x_{k2}} + \theta_k \left(\frac{\partial G_{k2}}{\partial x_{k2}} \right) \right] x_{k2} = 0$$

$$x_{k2} \geq 0, \quad G_{k2} \geq 0, \quad \theta_k \geq 0, \quad G_{k2} \theta_k = 0$$

□

Assumption 5: For every (x_{i1}, x_{k1}) for which there exists a solution $(x_{i2}, x_{k2}, \theta_i, \theta_k)$ to $E2(x_1, x_2, \theta)$, that solution is unique. □

Assumption 5 is sufficient to ensure that strategies are well defined solutions to $E2(x_1, x_2, \theta)$ (that is, are functions) for fixed (x_{i1}, x_{k1}) , in that solutions in period-two variables are also consistent in both agents' period-two optimization problems.

Definition 6: Following (1a), substitution of the equivalent period-two equilibrium conditions $E2(x_1, x_2, \theta)$ into the period-one maximization problems of agents i and k yields the respective period-one maximization problems $P_i(x_{k1})$, in which x_{k1} is a fixed parameter, and $P_k(x_{i1})$, in which x_{i1} is a fixed parameter:

$$\begin{aligned} \text{Max}_{x_{i1}, x_{i2}, x_{k2}, \theta_i, \theta_k} \quad & u_{i1}(x_{i1}, x_{k1}) + \beta_{i2} u_{i2}(x_{i1}, x_{k1}, x_{i2}, x_{k2}) & P_i(x_{k1}) \\ \text{s.t.:} \quad & G_{i1}(x_{i1}, x_{k1}) \geq 0 \\ & E2(x_{i1}, x_{k1}, x_{i2}, x_{k2}, \theta_i, \theta_k) \end{aligned}$$

$$\begin{aligned} \text{Max}_{x_{k1}, x_{k2}, x_{i2}, \theta_k, \theta_i} \quad & u_{k1}(x_{i1}, x_{k1}) + \beta_{k2} u_{k2}(x_{i1}, x_{k1}, x_{i2}, x_{k2}) & P_k(x_{i1}) \\ \text{s.t.:} \quad & G_{k1}(x_{i1}, x_{k1}) \geq 0 \\ & E2(x_{i1}, x_{k1}, x_{i2}, x_{k2}, \theta_i, \theta_k) \end{aligned}$$

□

The two-agent, two-period SPE problem consists of simultaneously solving the two maximization problems $P_i(x_{kl})$ and $P_k(x_{il})$. That is, an SPE solution (x_{il}^*, x_{kl}^*) satisfies:

$$x_{il}^* = \text{Sol}_1 P_i(x_{kl}^*)$$

$$x_{kl}^* = \text{Sol}_1 P_k(x_{il}^*)$$

and under Assumption 5 it follows that x_{il}^* and x_{kl}^* satisfy (2) as required for an SPE solution.

An approach to simultaneously solving $P_i(x_{kl})$ and $P_k(x_{il})$ is to combine the KKT conditions for these two problems and to solve the resulting nonlinear complementarity problem. The drawbacks of this approach are the difficulty of formulating the complementarity problems and the fact that the KKT conditions for $P_i(x_{kl})$ and $P_k(x_{il})$ are necessary, but not sufficient, conditions for an optimal solution; that is, a solution to the complementarity problem is not necessarily an SPE solution. These approaches to solving *static* equilibrium formulations have been proposed by Harker (1984, 1986) and others. We are not aware of any research on solving *dynamic* equilibrium models formulated as mathematical programs other than that presented in this paper. The strategy described in the remainder of this paper simplifies the agent optimization problems by decomposing the feasible regions into convex subspaces, under appropriate assumptions.

1.4 Characterization of Feasible Sets

The strategy for solving the SPE problem is to first define a set Γ that contains the feasible regions of both agents' period-one optimization problems. Each agent's optimization problem is defined over a different subset of Γ . Next, the set Γ is divided into convex subsets Γ_i and for each subset, a convex optimization problem is defined for each agent.

Definition 7: *The set Γ consists of points that satisfy $E2(x_1, x_2, \theta)$ over the range of feasible (x_{il}, x_{kl}) :*

$$\Gamma = \{(x_{i1}, x_{k1}, x_{i2}, x_{k2}, \theta_i, \theta_k) \mid x_{i1} \geq 0, x_{k1} \geq 0, G_{i1}(x_1) \geq 0, G_{k1}(x_1) \geq 0, E2(x_1, x_2, \theta)\}$$

□

1.3 Facet Equilibrium

Definition 8: A facet index is an ordered m -tuple ($m > 0$) of binary variables $f = (f_1 \dots f_m)$ over the complementarity conditions in $E2(x_1, x_2, \theta)$ where

$f_j = 1$ if the j^{th} complementarity variable ≥ 0 , and the j^{th} complementarity constraint $= 0$,

$f_j = 2$ if the j^{th} complementarity variable $= 0$, and the j^{th} complementarity constraint ≥ 0 .

The set of all facet indices is denoted by the set F . □

A facet index can be used to denote a unique subset of the feasible regions in problems $P_i(x_{k1})$ and $P_k(x_{i1})$. The notion of a facet is defined for this purpose.

Definition 9: A facet $\Gamma_f, f \in F$, is the set of points which are feasible to the constraints of $E2(x_1, x_2, \theta)$ indicated by facet index f and the period-one constraints $x_{i1} \geq 0, x_{k1} \geq 0, G_{i1} \geq 0$, and $G_{k1} \geq 0$. □

For example, facet Γ_{1122} is the following set:

$$\Gamma_{1122} = \{(x_{i1}, x_{k1}, x_{i2}, x_{k2}, \theta_i, \theta_k) \mid x_{i1} \geq 0, x_{k1} \geq 0, G_{i1}(x_1) \geq 0, G_{k1}(x_1) \geq 0, E2_{1122}(x_1, x_2, \theta)\}$$

where $E2_{1122}(x_1, x_2, \theta) \equiv$

$$\{ x_{i2} \geq 0, \quad -\frac{\partial u_{i2}}{\partial x_{i2}} - \theta_i \left(\frac{\partial G_{i2}}{\partial x_{i2}} \right) = 0, \quad (3a)$$

$$x_{k2} \geq 0, \quad -\frac{\partial u_{k2}}{\partial x_{k2}} - \theta_k \left(\frac{\partial G_{k2}}{\partial x_{k2}} \right) = 0, \quad (3b)$$

$$\theta_i = 0, \quad G_{i2} \geq 0, \quad (3c)$$

$$\theta_k = 0, \quad G_{k2} \geq 0 \} \quad (3d)$$

In (3), the 1122 subscript on Γ_{1122} indicates that the first member of the complementary pairs in (3a) and (3b) have the inequality signs, and the second members in (3c) and (3d) have the inequality signs.

The next two propositions follow directly from the definitions of Γ , Γ_f , and an SPE solution (see Macal and Hurter (1991) for complete proofs):

Proposition 10: $\Gamma = \cup_{f \in F} \Gamma_f$. $\square$

Proposition 11: Any subgame perfect equilibrium solution is a member of some facet Γ_f , $f \in F$. $\square$

Facets are natural places to search for SPE solutions. Facet membership of an SPE outcome is not necessarily unique. Also, not every facet has a feasible point. For the sake of completeness, the notions of empty and nonempty facets are defined.

Definition 12: A point (x_1, x_2, θ) is feasible to facet Γ_f , $f \in F$, if $(x_1, x_2, \theta) \in \Gamma_f$. A facet Γ_f is a nonempty facet if a point feasible to it exists. Otherwise Γ_f is an empty facet. $\square$

The decomposition of the solution space into facets leads naturally to the formulation of equilibrium problems on facets, which are more easily solved than the original equilibrium problem.

A facet optimization problem can be specified for each agent in period one:

Definition 13: For $f \in F$, the problems $P_{\bar{f}}(x_{k1})$ and $P_{\bar{f}}(x_{i1})$ are the period-one optimization problems on the facet Γ_f :

$$\begin{aligned}
 & \text{Max}_{x_{i1}, x_{i2}, x_{k2}} \quad u_{i1}(x_{i1}, x_{k1}) + \beta_{i2} u_{i2}(x_{i1}, x_{k1}, x_{i2}, x_{k2}) & P_{\bar{f}}(x_{k1}) \\
 & \theta_i, \theta_k \\
 & \text{S.t.:} \quad x_{i1} \geq 0 \\
 & \quad \quad G_{i1}(x_{i1}, x_{k1}) \geq 0 \\
 & \quad \quad E2_f(x_{i1}, x_{k1}, x_{i2}, x_{k2}, \theta_i, \theta_k)
 \end{aligned}$$

$$\begin{aligned}
& \text{Max}_{x_{k1}, x_{k2}, x_{i2}, \theta_i, \theta_k} u_{k1}(x_{i1}, x_{k1}) + \beta_{k2} u_{k2}(x_{i1}, x_{k1}, x_{i2}, x_{k2}) & P_k(x_{i1}) \\
& \text{S.t.: } x_{k1} \geq 0 \\
& G_{k1}(x_{i1}, x_{k1}) \geq 0 \\
& E2_f(x_{i1}, x_{k1}, x_{i2}, x_{k2}, \theta_i, \theta_k)
\end{aligned}$$

An equilibrium problem can be defined on each facet for each set of agent optimization problems.

Definition 14: For $f \in F$, the problem of simultaneously solving $P_f(x_{k1})$ and $P_k(x_{i1})$ on the facet Γ_f is termed the facet equilibrium problem on facet Γ_f , or f . The solution of the facet equilibrium problem is a facet equilibrium solution (FES) on Γ_f . $\square$

For example, the simultaneous solution of $P_{1122,i}(x_{k1})$ and $P_{1122,k}(x_{i1})$ for (x_1^*, x_2^*) requires the following conditions to be satisfied:

$$\begin{aligned}
[x_{i1}^*, x_{i2}^*, x_{k2}^*] &= \text{Sol } P_{1122,i}(x_{k1}^*) \\
[x_{k1}^*, x_{k2}^*, x_{i2}^*] &= \text{Sol } P_{1122,k}(x_{i1}^*)
\end{aligned}$$

To this point, we have redefined the SPE problem to be one of simultaneously solving a set of optimization problems. We next simplify the problem further by reducing the space of the decision variables to those only indexed on the first period.

1.4 Common Solution Space

The set Ω is the projection of the set Γ onto the x_1 plane and consists of all the x_1 that are feasible to the period-one agent optimization problems $P_i(x_{k1})$ and $P_k(x_{i1})$:

Definition 15: The period-one feasible projection set is:

$$\Omega = \{x_1 \mid \text{there exists } (x_2, \theta) \text{ such that } (x_1, x_2, \theta) \in \Gamma\}$$

The period-one feasible projection set on facet $f \in F$ is defined analogously to Γ_f :

$$\Omega_f = \{x_1 \mid \text{there exists } (x_2, \theta) \text{ such that } (x_1, x_2, \theta) \in \Gamma_f\}$$

□

Definition 16: The set Λ consists of the original constraints of the SPE problem as follows:

$$\Lambda = \{x_1 \mid \text{there exists } x_2 \text{ such that } x_1 \geq 0, x_2 \geq 0, G_1(x_1) \geq 0, G_2(x_1, x_2) \geq 0\}$$

□

Because Λ is defined in terms of the original constraints of the SPE problem, properties of Λ are easier to verify than properties of the sets Ω_f and Ω . Assumption 5 can be restated in light of the definition of Λ :

Assumption 17: For every $x_1 \in \Lambda$ there exists a unique solution $[x_2(x_1), \theta(x_1)]$ to the equivalent period-two equilibrium problem $E2(x_1, x_2, \theta)$. □

1.5 Numerical Example

Figure 2 illustrates the set Ω_{1122} for a numerical example with the following parameters:

$$G_{i1}(x_1): \emptyset$$

$$G_{k1}(x_1): \emptyset$$

$$G_{i2}(x_1, x_2): B_i - x_{i1} - x_{i2} \geq 0, \quad B_i = 97$$

$$G_{k2}(x_1, x_2): B_k - x_{k1} - x_{k2} \geq 0, \quad B_k = 52$$

$$E2(x_1, x_2, \theta): \quad u = 14x_{i2} + 2x_{k2} + 10x_{i1} - 1000 \geq 0, \quad x_{i2} \geq 0, \quad ux_{i2} = 0$$

$$v = 5x_{k2} + 2x_{i2} + x_{k1} - 100 \geq 0, \quad x_{k2} \geq 0, \quad vx_{k2} = 0$$

$$w = 97 - x_{i1} - x_{i2} \geq 0, \quad \theta_i \geq 0, \quad w\theta_i = 0$$

$$z = 52 - x_{k1} - x_{k2} \geq 0, \quad \theta_k \geq 0, \quad z\theta_k = 0$$

This example consists of an oligopolistic market model with linear demand, quadratic costs, and resource constraints that limit the total amount of resource producible by each agent over the two periods. To construct Ω_{1122} , derive $E2_{1122}(x_1, x_2, \theta)$ from $E2(x_1, x_2, \theta)$:

$$\begin{aligned} 14x_{i2} + 2x_{k2} + 10x_{i1} - 1000 &= 0, & x_{i2} &\geq 0 \\ 5x_{k2} + 2x_{i2} + x_{k1} - 100 &= 0, & x_{k2} &\geq 0 \\ 97 - x_{i1} - x_{i2} &\geq 0, & \theta_i &= 0 \\ 52 - x_{k1} - x_{k2} &\geq 0, & \theta_k &= 0 \end{aligned} \quad E2_{1122}(x_1, x_2, \theta)$$

Solving $E2_{1122}(x_1, x_2, \theta)$ for x_2 in terms of x_1 gives the strategy mappings $X_{i2}(x_{i1}, x_{k1})$ and $X_{k2}(x_{i1}, x_{k1})$:

$$\begin{aligned} X_{i2}(x_{i1}, x_{k1}) &= -0.75758x_{i1} + 0.03030x_{k1} + 72.72727 \geq 0 \\ X_{k2}(x_{i1}, x_{k1}) &= 0.30303x_{i1} - 0.21212x_{k1} - 9.09091 \geq 0 \end{aligned}$$

where the inequalities follow from the non-negativity constraints on x_{i2} and x_{k2} , respectively.

Substituting $X_{i2}(x_{i1}, x_{k1})$ and $X_{k2}(x_{i1}, x_{k1})$ for x_{i2} and x_{k2} , respectively, into the G_2 constraints together with the non-negativity constraints on x_{i2} and x_{k2} yields the set Ω_{1122} :

$$\begin{aligned} \Omega_{1122} = \{ (x_{i1}, x_{k1}) \geq 0 \mid & 2400 - 25x_{i1} + x_{k1} \geq 0, \quad -300 + 10x_{i1} - 7x_{k1} \geq 0, \\ & 801 - 8x_{i1} - x_{k1} \geq 0, \quad 1008 - 5x_{i1} - 13x_{k1} \geq 0 \} \end{aligned}$$

For all $f \in F = \{(1 \ 1 \ 1 \ 1), (1 \ 1 \ 1 \ 2), \dots, (2 \ 2 \ 2 \ 2)\}$, the sets Ω_f are constructed similarly to Ω_{1122} . Figure 3 displays the sets Ω_f , Ω , and $\Lambda = \{(x_{i1}, x_{k1}) \mid 0 \leq x_{i1} \leq 97, 0 \leq x_{k1} \leq 52\}$. Note, that Ω_f , $f \in F$, tiles the set Ω . Some Ω_f are empty sets as no point satisfies all of the equations and inequalities defining the facet Γ_f . Also, some Ω_f are not of full dimension two, where full dimension is the dimension of the set Ω , the length of the x_1 vector, or the number of agents. These sets include Ω_{1221} of dimension one, a line in $\mathbb{R}^2$, and Ω_{2211} of dimension zero, a point in $\mathbb{R}^2$.

We have completed the decomposition of the space into convex subspaces parameterized on x_1 . Any equilibrium solution on a facet f can be defined as any (x_{i1}, x_{k1}) pair that simultaneously solves $P_{\pi}(x_{k1})$ and $P_{\kappa}(x_{i1})$. The transformation $P_f(x_1) \equiv [P_{\pi}(x_{k1}), P_{\kappa}(x_{i1})]$ takes any point $x_1 \in \Omega_f$ and maps it to another point in the same convex space Ω_f ; that is $P_f : \Omega_f \rightarrow \Omega_f, f \in F$. The properties of a mapping based on P_f and the existence of facet equilibrium solutions are addressed in Section 2.

2 EXISTENCE OF FACET EQUILIBRIUM SOLUTIONS

We will show that every nonempty facet contains a facet equilibrium solution, as stated in the following theorem.

Theorem 18: *Let Γ_f be a nonempty facet, $f \in F$, and the functions defining the facet be linear (as in the case of U quadratic) and the problem constraints G linear. Let Assumptions 3 and 17 hold. Then there exists a facet equilibrium solution on facet Γ_f . $\square$*

Theorem 18 will be proved in the following steps:

1. Show that Ω_f is a convex and compact set under appropriate assumptions on G_i and G_k .
2. Define a function T_f , based on P_f , over the set Ω_f for any nonempty facet.
3. Show that T_f is a continuous mapping on Ω_f .
4. Show that a fixed point of T_f exists and is a facet equilibrium solution.

It follows from Theorem 18 that any SPE solution is a fixed point of the mapping T_f for some $f \in F$. Therefore, it is natural to compute fixed points of T_f and test whether they are SPE solutions. This procedure will be described in Section 3.

2.1 Properties of the Feasible Space

Lemma 19: *If Assumption 17 holds, then $\Lambda = \Omega$.*

Proof that $\Omega \subseteq \Lambda$: Let $x_1 \in \Omega$. Then $x_1 \in \Omega_f$ for some $f \in F$. By the definition of Ω_f , there exists (x_2, θ) with $(x_1, x_2, \theta) \in \Gamma_f$. Hence, (x_1, x_2, θ) satisfies the constraints $\{x_1 \geq 0, x_2 \geq 0, G_1 \geq 0,$

$G_k \geq 0\}$ and $E2(x_1, x_2, \theta)$. Thus, given x_1 , there exists x_2 such that (x_1, x_2) satisfies the set of constraints $\{x_1 \geq 0, x_2 \geq 0, G_i \geq 0, G_k \geq 0\}$. This implies that $x_1 \in \Lambda$.

Proof that $\Lambda \subseteq \Omega$: Let $x_1 \in \Lambda$. By Assumption 17 there exists a solution to $E2(x_1, x_2, \theta)$. Denote this solution by $[x_2(x_1), \theta(x_1)]$. Then $[x_1, x_2(x_1), \theta(x_1)] \in \Gamma_f$ for some $f \in F$ because $\cup_{f \in F} \Gamma_f \subseteq \Gamma$. Hence, $x_1 \in \Omega_f \subseteq \Omega$. $\square$

We next make an assumption regarding the compactness of the constraint regions of the agent optimization problems of the SPE problem.

Assumption 20: Λ is a compact (closed and bounded) set in $\mathbb{R}^n$. $\square$

It follows from Assumptions 17 and 20 that Ω is a compact set. The next proposition follows from the assumption of the compactness of Λ and the results that $\Lambda = \Omega$, Γ_f is a (closed, compact) polyhedron in $\mathbb{R}^n$, and $\Omega = \cup_{f \in F} \Omega_f$. A complete proof is given in Macal (1989).

Proposition 21: If Λ is a compact (closed and bounded) set in $\mathbb{R}^n$, then Ω_f is compact and convex for $f \in F$. $\square$

The next section defines the mapping T_f over the search space Ω_f .

2.2 The Mapping T_f

Definition 22: For any nonempty facet Γ_f , define a vector mapping T_f on Ω_f as $T_f(x_{il}, x_{kl}) = [T_{fi}(x_{il}, x_{kl}), T_{fk}(x_{il}, x_{kl})]$ where for any fixed $\lambda \in (0, 1)$

$$T_{fi}(x_{il}, x_{kl}) \equiv \lambda x_{il} + (1 - \lambda) x_{il}(x_{kl}), \text{ and}$$

$$T_{fk}(x_{il}, x_{kl}) \equiv \lambda x_{kl} + (1 - \lambda) x_{kl}(x_{il})$$

with $x_{il}(x_{kl})$ and $x_{kl}(x_{il})$ defined as:

$$x_{il}(x_{kl}) \equiv \text{Sol}_i P_f(x_{kl}), \text{ and}$$

$$x_{kl}(x_{il}) \equiv \text{Sol}_k P_f(x_{il}). \quad \square$$

The function $[x_{i1}(x_{k1}), x_{k1}(x_{i1})]$ is the response function that given an input (x_{i1}, x_{k1}) produces a result for the period-one agent optimization problems.

Assumption 23: Let Γ_f be a nonempty facet. For any $(x_{i1}, x_{k1}) \in \Omega_f$ there exists a unique solution for x_{i1} to $P_{fi}(x_{k1})$ and there exists a unique solution for x_{k1} to $P_{fk}(x_{i1})$. That is $x_{i1}(x_{k1}) = \text{Sol}_i P_{fi}(x_{k1})$ and $x_{k1}(x_{i1}) = \text{Sol}_k P_{fk}(x_{i1})$ are unique given (x_{i1}, x_{k1}) . $\square$

The numerical example of the previous section satisfies Assumption 23. Next it will be shown that T_f is a continuous mapping of the set Ω_f into itself.

Theorem 24: Let Γ_f be a nonempty facet. Then T_f maps Ω_f into itself. That is, $T_f : \Omega_f \rightarrow \Omega_f$.

Proof: Let $x_1 \in \Omega_f$. The theorem requires that $T_f(x_1) \in \Omega_f$. Let

$$[x_{i1}(x_{k1}), x_{i2}(x_{k1}), x_{k2}(x_{k1}), \theta_i(x_{k1}), \theta_k(x_{k1})] \equiv \text{Sol } P_{fi}(x_{k1}), \text{ and}$$

$$[x_{k1}(x_{i1}), x_{k2}(x_{i1}), x_{i2}(x_{i1}), \theta_i(x_{i1}), \theta_k(x_{i1})] \equiv \text{Sol } P_{fk}(x_{i1}).$$

To prove that $T_f(x_{i1}, x_{k1}) \in \Omega_f$, it is sufficient to show that

$$[\lambda x_{i1} + (1 - \lambda) x_{i1}(x_{k1}), \lambda x_{k1} + (1 - \lambda) x_{k1}(x_{i1})] \in \Omega_f.$$

Because $\text{Sol } P_{fi}(x_{k1})$ is optimal to problem $P_{fi}(x_{k1})$, it is also feasible to the constraints of $P_{fi}(x_{k1})$. Hence

$$[x_{i1}(x_{k1}), x_{k1}, x_{i2}(x_{k1}), x_{k2}(x_{k1}), \theta_i(x_{k1}), \theta_k(x_{k1})] \in \Gamma_f.$$

Similarly for agent k's problem

$$[x_{i1}, x_{k1}(x_{i1}), x_{i2}(x_{i1}), x_{k2}(x_{i1}), \theta_i(x_{i1}), \theta_k(x_{i1})] \in \Gamma_f.$$

Note that $[x_{i1}(x_{k1}), x_{k1}] \in \Omega_f$ and $[x_{i1}, x_{k1}(x_{i1})] \in \Omega_f$. Since Ω_f is convex by Proposition 21, it follows that for any $\lambda \in (0, 1)$

$$[\lambda x_{i1} + (1 - \lambda) x_{i1}(x_{k1}), \lambda x_{k1} + (1 - \lambda) x_{k1}(x_{i1})] \in \Omega_f. \quad \square$$

2.3 Continuity of T_f

The next step in the proof of Theorem 18 is to show that T_f is a continuous mapping of the parameters (x_{i1}, x_{k1}) . A theorem by Fiacco is instrumental in this development.

Theorem (Fiacco, 1983): Consider the following mathematical program:

$$\text{Min}_x \quad q(x, \epsilon) \quad \text{P3}(\epsilon)$$

$$\text{s.t.:} \quad g(x, \epsilon) \geq 0,$$

$$h(x, \epsilon) = 0,$$

where the problem functions q , g , and h are continuously differentiable in (x, ϵ) , and

$$R(\epsilon) \equiv \{x \in \mathbb{R}^n \mid g(x, \epsilon) \geq 0, h(x, \epsilon) = 0\}$$

is defined to be the feasible region of P3(ϵ),

$$S(\epsilon) \equiv \{x \in R(\epsilon) \mid q(x) = q^*(\epsilon)\}$$

is defined to be the set of optimal solutions to problem P3(ϵ) above,

$$q^*(\epsilon) = \inf_x \{q(x) \mid x \in R(\epsilon)\}$$

is the optimal value function as ϵ varies, and the parameter ϵ is a vector in $\mathbb{R}^n$. Then, if $R(0) \neq \emptyset$,

with R uniformly compact near $\epsilon = 0$, and if the Mangasarian-Fromovitz constraint qualification

(Mangasarian and Fromovitz, 1967) holds for some $x^* \in S(0)$, q^* is continuous at $\epsilon = 0$. $\square$

In the facet equilibrium problem, U quadratic and G linear imply that the functions h_j and g_i above are affine. In turn g_i affine implies that g_i is concave and is therefore pseudo-concave (Fiacco, 1983). Based on these observations and following Fiacco, the facts that g_i is pseudo-concave and h_j is affine imply that the Mangasarian-Fromovitz constraint qualification is equivalent to the Slater condition, a general form of which is as follows:

Constraint Qualification (Slater, 1950): The Slater constraint qualification is satisfied at $x^* \in R$ $\equiv \{x \in \mathbb{R}^n \mid g(x) \geq 0, h(x) = 0\}$ if h_j is affine for each j , g_i is pseudo-concave for each i and there exists a point $y \in R$ with $g_i(y) > 0$ for each i in $B(x^*) \equiv \{i \mid g_i(x^*) = 0\}$. $\square$

Theorem 25: Assume that Assumptions 5 and 17 and the Slater constraint qualification hold for each agent's period-one optimization problem on the nonempty facet Γ_j . Denote $V_j(x_k)$ as the optimal

profit level for agent i obtainable on a nonempty facet f as a function of the parameter x_{k1} . That is

$$V_{\bar{f}}(x_{k1}) = U_{i1}(x_{i1}(x_{k1}), x_{k1}, x_{i2}(x_{k1}), x_{k2}(x_{k1}))$$

where $[x_{i1}(x_{k1}), x_{i2}(x_{k1}), x_{k2}(x_{k1})] = \text{Sol } P_{\bar{f}}(x_{k1})$. Then by Fiacco's theorem $V_{\bar{f}}(x_{k1})$ is a continuous mapping of x_{k1} .

Proof: Relate the variables of the SPE problem to the variables of Fiacco's Theorem as follows.

Take $P_{\bar{f}}(x_{k1})$ for $P3(\epsilon)$ with

$$x = (x_{i1}, x_{i2}, x_{k2}, \theta_i, \theta_k),$$

$$\epsilon = x_{k1},$$

$$q(x, \epsilon) = U_i(x_{i1}, x_{i2}, x_{k2}, x_{k1}),$$

$$h(x, \epsilon) = \text{equality constraints defining facet } \Gamma_f \text{ for } x_{k1} \text{ fixed,}$$

$$g(x, \epsilon) = \text{inequality constraints defining facet } \Gamma_f \text{ for } x_{k1} \text{ fixed,}$$

$$R(\epsilon) = \text{feasible region of agent } i\text{'s period-one optimization problem on facet } \Gamma_f \text{ for } x_{k1} \text{ fixed,}$$

$$S(\epsilon) = \text{set of optimal solutions to problem } P_{\bar{f}}(x_{k1}).$$

Since Γ_f is nonempty by assumption, the feasible region $R(\epsilon) \neq \emptyset$. U_i and the constraint functions are continuously differentiable for any $f \in F$. All inequality constraints of the original SPE problem are linear, thus g_i is pseudo-concave. Under the assumption that U_i is quadratic, h_i is affine. It can be shown that $R(x_{k1})$ is uniformly compact (Macal, 1989). Therefore by Fiacco's theorem, $V_{\bar{f}}$ is a continuous mapping of x_{k1} . The continuity of $V_{\bar{f}}$ on x_{i1} follows by a similar argument. $\square$

Theorem 25 used Fiacco's theorem to show that the optimal objective function value for agent i (agent k) on any nonempty facet is a continuous function of the parameter x_{k1} (parameter x_{i1}).

Theorem 26 will use this result to show that the optimal solution to agent i 's (agent k 's) optimization problem is a continuous function of x_{k1} (of x_{i1}) on any nonempty facet. This result will be used to show that T_f is a continuous mapping.

Theorem 26: Assume that Assumptions 5 and 17 hold. Also assume that the Slater constraint qualification holds for each agent's period-one optimization problem on the nonempty facet Γ_f . Then $x_{ij}(x_{kl})$ is a continuous mapping of x_{kl} where $x_{ij}(x_{kl}) = \text{Sol}_i P_{fi}(x_{kl})$.

Proof: The proof of continuity of the optimal solutions follows Murphy et al. (1982, Lemma 2).

For simplicity of exposition denote the following:

$$x = (x_{i1}, x_{i2}, x_{k2}),$$

$$\chi(x_{k1}) = [x_{i1}(x_{k1}), x_{i2}(x_{k1}), x_{k2}(x_{k1})], \text{ and}$$

$$\theta(x_{k1}) = [\theta_i(x_{k1}), \theta_k(x_{k1})].$$

Let $[\chi(x_{k1}), \theta(x_{k1})]$ denote the optimal solution to $P_{fi}(x_{k1})$ given x_{k1} , and let $V_i(x_{k1})$ denote the optimal objective function value as x_{k1} varies:

$$V_i(x_{k1}) = U_i(\chi(x_{k1}), x_{k1}).$$

Theorem 25 showed that $V_i(x_{k1})$ is a continuous function of x_{k1} . Thus, given any feasible convergent sequence on the facet Γ_f , $[x_{i1}(x_{k1}^n), x_{k1}^n] \in \Omega_f$ with $\{x_{k1}^n\}_n \rightarrow x_{k1}$, it follows that

$$\{V_i(x_{k1}^n)\}_n \equiv \{U_i(\chi(x_{k1}^n), x_{k1}^n)\}_n \rightarrow V_i(x_{k1}) \equiv U_i(\chi(x_{k1}), x_{k1})$$

The sequence $\{\chi(x_{k1}^n)\}_n$ is bounded by the compactness of Ω_f (Proposition 21) and Assumption 17 regarding the existence of a solution to $E2(x_i)$ for every $x_i \in \Lambda$. The sequence therefore has a convergent subsequence. If n indexes such a sequence, there exist limits as follows:

$$[x_{i1}(x_{k1}^n), x_{i2}(x_{k1}^n), x_{k2}(x_{k1}^n)] \rightarrow (y_{i1}, y_{i2}, y_{k2}). \quad (4)$$

That is, $\lim_n \chi(x_{k1}^n) \equiv (y_{i1}, y_{i2}, y_{k2})$. Since

$[x_{i1}(x_{k1}^n), x_{k1}^n, x_{i2}(x_{k1}^n), x_{k2}(x_{k1}^n), \theta_i(x_{k1}^n), \theta_k(x_{k1}^n)] \in \Gamma_f$ for each n it follows that $(y_{i1}, y_{k1}, y_{i2}, y_{k2}, \theta_i(x_{k1}), \theta_k(x_{k1})) \in \Gamma_f$ from the definition of Γ_f and the fact that for any continuous functions f and g , $f(\epsilon_n) = 0$, $g(\epsilon_n) \geq 0$, and $\epsilon_n \rightarrow \epsilon$ with $n \rightarrow \infty$ implies that $f(\epsilon) = 0$ and $g(\epsilon) \geq 0$. That is, the limit point $(y_{i1}, y_{k1}, y_{i2}, y_{k2}, \theta_i(x_{k1}), \theta_k(x_{k1}))$ is feasible to $P_{fi}(y_{k1})$. Suppose that the solutions to problem

$P_{\bar{n}}(x_{k1})$ are not continuous:

$$(y_{i1}, y_{i2}, y_{i2}) \neq [x_{i1}(x_{k1}), x_{i2}(x_{k1}), x_{i2}(x_{k1})].$$

From (4) above

$$\begin{aligned} U_i(\chi(x_{k1}), x_{k1}) &= \lim_n U_i(\chi(x_{k1}^n), x_{k1}^n) \\ &= U_i(\lim_n \chi(x_{k1}^n), \lim_n x_{k1}^n) \end{aligned} \quad (5)$$

$$= U_i(y_{i1}, y_{i2}, y_{i2}, y_{k1}) \quad (6)$$

where (5) follows from the continuity of U_i in all variables and (6) follows from (4). Equation 6

contradicts the uniqueness of $\chi(x_{k1}) = [x_{i1}(x_{k1}), x_{i2}(x_{k1}), x_{i2}(x_{k1})]$, Assumption 17, for the given x_{k1} .

Hence $x_{i1}(x_{k1})$ is continuous for all x_{k1} such that $[x_{i1}(x_{k1}), x_{k1}] \in \Omega_f$. The continuity of $x_{k1}(x_{i1})$ follows by a similar argument. The continuity of T_f follows since T_f is a continuous function of $x_{i1}(x_{k1})$ and $x_{k1}(x_{i1})$. $\square$

Next it will be shown that there exists a fixed point of T_f in Ω_f and that this fixed point is a facet equilibrium solution.

2.4 Existence of a Facet Equilibrium Solution

Theorem 27: *For any nonempty facet $\Gamma_f, f \in F$, there exists a fixed point of T_f in Ω_f .*

Proof: By Brouwer's Theorem (Brouwer, 1910), any continuous mapping H that maps a nonempty, compact, and convex set $D \subseteq \mathbb{R}^n$ into itself, $H: D \rightarrow D$, has a fixed point $x \in D$ such that $x = H(x)$. Applying Brouwer's Theorem with $D = \Omega_f$, Γ_f nonempty, and $H = T_f$, it follows by Proposition 21 that Ω_f is compact and convex. By Theorem 26, T_f is continuous, and by Theorem 24, T_f maps Ω_f into itself. Hence, there exists some $(x_{i1}, x_{k1}) \in \Omega_f$ such that $(x_{i1}, x_{k1}) = T_f(x_{i1}, x_{k1})$. $\square$

Theorem 28: *Any fixed point of T_f in Ω_f for $f \in F$ is a facet equilibrium solution.*

Proof: Let (x_{i1}, x_{k1}) be a fixed point of T_f in Ω_f such that $x_{i1} = T_{\bar{n}}(x_{i1}, x_{k1})$ and $x_{k1} = T_{\bar{n}}(x_{i1}, x_{k1})$.

By the definition of T_f it follows that

$$(1 - \lambda) x_{i1} = (1 - \lambda) x_{i1}(x_{k1}), \text{ and}$$

$$(1 - \lambda) x_{k1} = (1 - \lambda) x_{k1}(x_{i1})$$

which implies for $\lambda \in (0, 1)$ that

$$x_{i1} = x_{i1}(x_{k1}), \text{ and}$$

$$x_{k1} = x_{k1}(x_{i1}). \quad \square$$

This completes the proof of Theorem 18 concerning the existence of an FES on every nonempty facet.

2.5 Implications for Subgame Perfect Equilibrium Solutions

We will show that any SPE solution is in the set of FESs.

Theorem 29: *Define the following sets: $\text{FacE}(\Omega_f) = \text{set of FESs in the set } \Omega_f$, and $\text{SPE}(\Omega) = \text{set of SPE solutions in the set } \Omega$. Then, $\text{SPE}(\Omega) \subseteq \cup_{f \in F} \text{FacE}(\Omega_f)$.*

Proof: If $x_1 \in \text{FacE}(\Omega_f)$, then x_1 is a globally maximizing solution over Ω_f for each agent. That is, x_{i1} solves $P_{fi}(x_{k1})$ given x_{k1} , and x_{k1} solves $P_{fk}(x_{i1})$ given x_{i1} . For x_1 to be an SPE solution it is necessary that x_1 is a globally maximizing solution over the entire set Ω for each agent. Not all x_1 that are globally maximizing over Ω_f are also globally maximizing over the set Ω because $\Omega = \cup_{f \in F} \Omega_f$; a point $x_1 \in \text{FacE}(\Omega_f)$ could be globally maximizing, locally maximizing, a stationary point, or none of these with respect to the set $\text{SPE}(\Omega)$. On the other hand, any SPE solution is maximizing for each agent over the set Ω and therefore is maximizing for each agent over Ω_f for some f , again because $\Omega = \cup_{f \in F} \Omega_f$. $\square$

Theorem 29 has implications for a strategy to compute SPE solutions: compute FESs and test whether each solution is also an SPE solution. The next section describes a computational procedure for searching for SPE solutions.

3 SOLUTION ALGORITHM

The search for SPE solutions is conducted over the finite set of facets.

3.1 Steps of the Algorithm

The steps of the algorithm are illustrated in Figure 4:

Step 1: Find initial facet. Find a feasible starting facet projection set Ω_f for some $f \in F$.

Step 2: Solve facet equilibrium problem. Solve the facet equilibrium problem on Ω_f for an FES. This FES is a candidate for an SPE solution. There are three mutually exclusive and exhaustive cases for an FES. Typical examples of the cases are illustrated in Figure 5 and are described as follows.

Case 1: The FES is strictly interior to Ω_f . In this case, it follows that the FES is strictly interior to Ω . By the definition of a facet, a point that is strictly interior to a facet projection Ω_f is not contained in any facet projection other than Ω_f . In this case the FES is a local optimum for each agent with respect to the set Ω . However, an SPE solution is globally optimal for each agent on the set Ω . In this case, proceed to Step 3 and check whether the FES that is locally optimal on Ω is also globally optimal for each agent.

Case 2: The FES is on the boundary of Ω_f and is not contained in any facet projection other than Ω_f . In this case, the FES is also contained in the boundary of Ω and is a local optimum for each agent with respect to the set Ω . An SPE solution is globally optimal for each agent on the set Ω . Therefore, proceed to Step 3 and check whether the FES that is locally optimal on Ω is also globally optimal for each agent.

Case 3: The FES is on the boundary of Ω_f and is contained in some facet projection other than Ω_f , say the facet projection Ω_g , for $g \neq f$. In this case, the FES for Ω_f is also contained in the boundary of the adjacent facet projection set Ω_g . Although the FES is a local optimum for each agent

with respect to the set Ω_t , it is not necessarily a local optimum for each agent with respect to the set Ω_g . Thus the FES is not necessarily a local optimum for each agent with respect to the set Ω . To determine whether it is locally maximizing on Ω_g , the FES is computed for the facet projection set Ω_g . That is, Ω_g is assigned as the next set to be searched, and Step 2 is repeated.

Step 3: Check for global optimality. The FES is a local optimum for each agent on the set Ω . For each agent, check whether there is a point on any other feasible facet that results in a higher utility level than the utility level at the FES, holding every other agent's first-period decision variable fixed. If there is a point on another facet at which some agent can obtain higher utility, the candidate solution is not globally maximizing on Ω for every agent. In this case, select another facet projection set and repeat Step 2. If there is not a point on another facet for which some agent can obtain greater utility, the candidate solution is globally maximizing for all agents over the set Ω , and the FES is therefore an SPE solution.

3.2 Discussion of the Algorithm

The number of facets for a two-period SPE problem is $2^{(n+c)}$ where n is the number of period-two variables required to be non-negative, and c is the number of constraints included in the period-two optimization problems of all agents (excluding non-negativity constraints). This follows from the fact that for every non-negative period-two variable and constraint, there exists a complementarity condition in the KKT conditions for the period-two equilibrium problem. The number of facets is independent of the number of period-one constraints. For the case of no period-two constraints and unrestricted period-two variables ($n = 0$ and $c = 0$) there is only one facet. For the numerical example, $n = 2$ and $c = 2$ for a total of 16 facets.

Although the number of facets to be searched is finite, the number increases exponentially according to the number of constraints included in the period-two problem. As a practical matter, the

number of facets that needs to be searched may be far less than the maximum number of facets. The search for an SPE solution is conducted only on the feasible facets. In the numerical example, the number of facet equilibrium problems that need to be solved is significantly less than the number of facets. Additional aspects of applying the algorithm are discussed below.

3.2.1 Step 1: Find initial facet. The starting facet can be determined in a number of ways. It is relatively easy to solve an open-loop version of the equilibrium problem (Definition 2) assuming an open-loop solution exists. Any open-loop solution will be on a feasible facet. Therefore, one possibility is to start with the feasible facet that contains the open-loop solution. Alternatively, any facet could be selected initially and a Phase I feasibility problem, as in linear programming, could be solved to determine whether a feasible point existed on the facet. The Phase I problem constraints would consist of the equations and inequalities, which define the facet, augmented by infeasibility variables. If a facet was found to be infeasible, another facet would be tested for feasibility. Since the number of facets is finite, this procedure would either find a feasible facet or conclude that no feasible facet exists and therefore no SPE exists.

3.2.2 Step 2: Solve facet equilibrium problem

The solution approach to solving the facet equilibrium problem is based on a bargaining process among the agents:

Step 2.1: Start with an initial guess (x_{i1}, x_{k1}) on an initial feasible facet projection set Ω_f .

Step 2.2: Compute $T_{\bar{n}}(x_{k1})$ and $T_{\bar{k}}(x_{i1})$ on Ω_f .

Step 2.3: Compare the previous guess (x_{i1}, x_{k1}) with the computed result $[T_{\bar{n}}(x_{k1}), T_{\bar{k}}(x_{i1})]$:

Is $|(x_{i1}, x_{k1}) - [T_{\bar{n}}(x_{k1}), T_{\bar{k}}(x_{i1})]| < \text{tolerance?}$

If yes: The current guess is an FES.

Else: Assign a new guess as follows:

$$x_{i1} \leftarrow x_{i1} + \omega [T_{\bar{n}}(x_{k1}) - x_{i1}]$$

$$x_{k1} \leftarrow x_{k1} + \omega [T_{\bar{n}}(x_{i1}) - x_{k1}]$$

where ω is the result of a standard iterative approximation algorithm.

Repeat from Step 2.2.

In the following situation a choice must be made as part of Step 2. Suppose an FES for the set Ω_i occurs on the boundary of Ω_i and the FES is contained in more than one adjacent feasible facet projection set that has not previously been searched for an FES (Figure 5, Case 3). This situation was commonly encountered in solving the numerical example. For example, the FES on (1 1 2 1) is also contained in facets (1 1 2 2), (1 2 2 1), and (1 2 2 2). One selection strategy consists of choosing the facet that differs from the current facet by the least number of indices starting from the left. In the example given above, this facet selection rule leads to selecting facet (1 2 2 1) to search next. The use of this rule was found to eliminate cycles in the facet selection that were encountered using other facet selection rules. This selection rule is similar to Bland's pivoting rule for selecting a new basis at a degenerate feasible solution in linear programming (Bland, 1977). An interesting area of future research would be to show that a Bland-type rule or a rule similar to that used in Lemke's Algorithm for selecting almost-complementary solutions (Lemke, 1965) cannot result in a cycle of facets. A variant of Lemke's rule may also have potential application here because an agent's period-one optimization problem consists of an optimization problem on a linear complementarity problem. It would be desirable to show that a facet selection rule necessarily either results in an FES that is locally maximizing on Ω or concludes that no such solution exists. In this case, no SPE solution exists.

This algorithm could fail in Step 2 if either (1) the FESs had been computed for all facets adjacent to Ω_i and none was found to be an SPE solution or (2) a locally maximizing FES was found

to not be globally maximizing for each agent. The problem is in choosing which facet projection set to search next. If all feasible facets have not been searched, then one strategy is to arbitrarily select one of the unsearched facets as the next Ω_f and proceed to Step 2. If all feasible facets have been searched unsuccessfully, then two situations are possible: (1) if it is known that a unique FES exists on every feasible facet, then it can be concluded that an SPE solution does not exist, or (2) if it is not known whether FESs are unique, then nothing can be concluded about the existence of an SPE solution. In the second case, the strategy is to search the feasible facets for alternative FESs. Whether this case is likely or even possible to arise is unknown, in that neither of these problems was encountered with the small number of sample problems that were solved as part of this research.

3.2.3 Step 3: Checking for Global Optimality: Step 3 of the algorithm is based on the fact that any SPE solution is by definition globally maximizing for all agents on the set Ω . A test for the global optimality of a solution on facet projection Ω_f with respect to all agents' utility maximization problems could be conducted as follows. For agent i , hold x_{k1} fixed and solve $P_{gi}(x_{k1})$ for all $g \in F$, $g \neq f$. If the FES on Ω_f maximizes utility over all facets, the FES is globally maximizing for agent i . Repeat the process for agent k while holding x_{i1} fixed. If the FES is globally maximizing for both agents, declare the FES to be an SPE solution. If the local maximizing solution is found not to be globally maximizing for some agent on a particular facet, select that facet as the next facet to search.

3.3 Illustrative Results

The FESs that were computed for the numerical example are shown in Table I and Figure 6. Note from Figure 6 that all FESs are on facet boundaries, with the exception of the SPE solution. In the case where an FES occurs on the boundary of a facet and it is contained in one other facet, the following facet selection rule was successful at eventually leading to an SPE solution: the next facet along a path is selected from those that have a common boundary with the FES on the current facet.

However, the FESs for several facets were found to be contained in more than one other facet. In this case, the next facet to search needed to be selected from the alternatives. For example, the FES on facet (1 1 2 1) is also contained in facets (1 1 2 2), (1 2 2 1), and (1 2 2 2). As mentioned previously, the following rule was found to eliminate cycles in the facet selection process: the next facet selected along a path is the facet that differs from the current facet by the least number of indices starting from the left. For example, for facet (1 1 2 2) the facet selection rule leads to the selection of facet (1 2 2 1) as the next facet to search.

Figure 7 shows for each facet which unique facet would be selected as the next to be searched using the facet selection rule. Starting from any nonempty facet, a path exists to facet (1 1 2 2), where the SPE solution is located. Note that the FES on (1 1 2 2) is strictly interior to Ω_{1122} (Figure 6), and no other facet shares a common boundary with it.

Figure 8 shows the path generated by the algorithm for the starting facet (1 1 2 1). The starting point for the facet equilibrium algorithm on facet (1 1 2 1) was obtained by solving a Phase I feasibility problem. The starting point also happened to be the FES on facet (1 1 2 1), as shown in Figure 8. The path was generated by solving the facet equilibrium problems on the successive facets (1 1 2 1), (1 2 2 1), (1 2 2 2), and (1 1 2 2). The next facet on the path was selected as the one that shared a common boundary with the computed FES. A total of 55 iterations were required to go from the starting point on the initial facet to the SPE solution. An iteration consists of solving agent i's period-one maximization problem for x_{k1} fixed and agent k's period-one maximization problem for x_{i1} fixed.

Figure 9 shows the check for global optimality of the candidate SPE solution for both agents' utility maximization problems. Holding $x_{k1} = 5.33615$, the optimal x_{i1} values for agent i were computed on each feasible facet. The feasible facets and optimal solutions for agent i are as follows:

(1 2 2 2), $x_{i1} = 33.735305$, $U_i = 41090$; (1 1 2 2), $x_{i1} = 36.254677$, $U_i = 41110$; (2 1 2 2), $x_{i1} = 96.213446$, $U_i = 30387$; and (2 1 1 2), $x_{i1} = 97.0$, $U_i = 30101$. Thus, the solution $x_{i1} = 36.254677$ is globally optimal for agent i over all feasible facets given $x_{k1} = 5.33615$. The optimal x_{k1} values for agent k on each feasible facet are computed similarly for fixed $x_{i1} = 36.254677$. The feasible facets and optimal solutions for agent k are as follows: (1 1 2 2), $x_{k1} = 5.33615$, $U_k = 77$; (1 2 2 2), $x_{k1} = 8.935257$, $U_k = 46$; and (1 2 2 1), $x_{k1} = 52$, $U_k = -5330$. Based on these results, we conclude that the solution $x_{k1} = 5.33615$ is globally optimal for agent k over all feasible facets given $x_{i1} = 36.254677$. This establishes that $(x_{i1}, x_{k1}, x_{i2}, x_{k2}) = (36.254677, 5.33615, 45.42333, 0.763438)$ is an SPE solution.

The algorithm could fail if the FES on a facet was not unique. In this case the computed FES could depend on the starting point used for the facet equilibrium calculation. However, the primary candidate for an SPE solution is a strictly interior FES on any facet because any strictly interior FES must satisfy a set of linear equations that represent the first-order conditions for a maximization problem, the solution of which is known to exist.

4 SUMMARY AND CONCLUSIONS

In this paper we have described a systematic approach to solving a two-agent, two-period subgame perfect equilibrium formulation under assumptions on the agent objective functions and constraint sets. Specifically, these assumptions are as follows:

- 1) The KKT conditions for the period-two equilibrium problem are necessary and sufficient.
- 2) For each x_1 for which there exists a solution to the period-two equilibrium problem, that solution is unique.
- 3) Λ is a compact set.
- 4) The Mangasarian-Fromovitz (or Slater given the assumptions regarding functional form)

constraint qualification holds for each period-one agent optimization problem.

- 5) For any feasible (x_{i1}, x_{k1}) , there exists a unique solution in (x_{i1}, x_{k1}) to the period-one agent optimization problems.

Showing assumptions (1) and (2) to be true for a particular problem is facilitated if there exists a single equivalent optimization problem for the period-two equilibrium problem, as is often the case. Showing that (3) holds is a matter of having a closed and bounded constraint region. Showing that (4) and (5) are satisfied consists of proving properties of solutions to the period-one agent optimization problems. For example, U quadratic and G linear reduces the Mangasarian-Fromovitz constraint qualification to the Slater constraint qualification.

All results presented here for the two-agent, two-time-period case just discussed hold for any finite number of agents, as can be seen in the definitions and theorems by replacing the production levels attributable to agent k by the sum of production levels attributable to all other agents $k \neq i$ within a time period. For example, take $\sum_{k=1, k \neq i} x_{k1}$, rather than x_{k1} , as a fixed value in the period-one optimization problem. Regarding the number of time periods, the structure of the SPE problem is such that the number of facets for a multi-year formulation equals the number of facets in the two-period problem after appropriate reductions are made to the multi-year formulation (Macal, 1989). Although the problem size increases in terms of the number of variables as the numbers of agents and periods increases, the number of facets depends solely on the number of constraints. Therefore, the solution approach illustrated in this paper can be used for arbitrary finite numbers of time periods and agents to tile the feasible space of agent optimization problems.

Several extensions to this research are apparent, such as (1) relaxing the compactness assumption on the original constraint set Λ and applying an alternative fixed point theorem for unbounded regions, such as the Leray-Schauder Theorem, and (2) investigating alternative facet selection rules.

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Table I Facet Equilibrium Solutions

Facet Index ¹	x_{i1}	x_{k1}	x_{i2}	x_{k2}	θ_i	θ_k
1 1 1 1	95	41	2	11	0	0
1 1 1 2	97	25	0	15	0	0
1 1 2 1	66.4	52	24	0	0	0
1 1 2 2 ²	36.2547	5.3361	45.4233	0.7634	0	0
1 2 1 1	89.5	52	7.5	0	0	33
1 2 2 1	26.4999	52	52.5	0	0	0
1 2 2 2	35.5550	7.9357	46.0321	0	0	0
2 1 1 1	97	40	0	12	6	0
2 1 1 2	97	0	0	20	0	0
2 1 2 2	96	0	0	20	0	0
2 2 1 1	97	52	0	0	30	48

¹ Empty facets 1212, 2121, 2212, 2221, and 2222 are not included.

² Denotes subgame perfect equilibrium solution.

Figure 1 Matrix of Optimization Problems Comprising the Subgame Perfect Equilibrium Problem

Figure 2 Construction of the Set Ω_{1122}

Figure 3 Facet Sets

Figure 4 Algorithm Flowchart

Case 1:

$FES \in \text{int } \Omega_f$

Case 2:

$FES \in \partial \Omega_f$

$FES \notin \Omega_g$

Case 3:

$FES \in \partial \Omega_f$

$FES \in \Omega_g$

Figure 5 Cases for Facet Equilibrium Solutions
FES indicates facet equilibrium solution

Figure 6 Facet Equilibrium Solutions

Figure 7 Pointers of Facet Equilibrium Solutions to Other Facets
 $\emptyset$ indicates empty facet.
 Facet index- n indicates that Ω_f is of dimension n .

Figure 8 Solution Path Generated by Starting at Facet $f = (1 \ 1 \ 2 \ 1)$

Figure 9 Check for Global Optimality of the Subgame Perfect Equilibrium Solution Over the Set Ω

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