*A Simple Model for Determining
Photoelectron-Generated Radiation
Scaling Laws*

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Contents

ABSTRACT	1
Introduction	2
Development of Theory Equations	5
Radiation Equations	5
Single-Pulse Trajectory Equations	13
Steady-State Equations	16
PIC Parameter Set Equations	23
UV Parameter Set Equations	27
Natural Parameter Set Equations	31
Conversion to Natural Parameters	32
Natural Parameter Graphs	33
Optimization Trends	39
Theory Model Constraints	43
List of Model Constraints	43
Analysis of Model Constraints	48
Optimization with Model Constraints	50
Generalized Scaling Laws	57
Conclusions	62
Appendix A	67
Appendix B	71

LIST OF FIGURES

List of Figures

1	1-D cylindrical current source and spherical observer reference frame coordinate systems shown with respect to Cartesian coordinates, as used in the derivations.	6
2	Emitted electron model of infinite, 2-D planar, virtually thin, rigid charge sheet for a single pulse.	14
3	Current density versus time for emitted electron beam emission comparing a rectangular pulse train with a sinusoidal pulse train having the same average current density $\langle j \rangle$	18
4	Theory natural parameter nomogram for small spots and steady state showing the relationship among the natural parameters and model constraints.	35
5	Graph of the relationship among x , ν_{nat} , and r . Also displayed on the graph are scales showing the effect of model constraints x_{10} and x_{12} in selecting valid choices of x , ν_{nat} , and r	38

LIST OF TABLES

List of Tables

1	Two Cases used in Natural Parameter Graphs	33
2	Theoretical Values of Coefficients and Exponents	60
3	Summary of Numerical Small-Spot Theory Predictions	64

A SIMPLE MODEL FOR DETERMINING PHOTOELECTRON-GENERATED RADIATION SCALING LAWS

by

Thomas M. Dipp*

ABSTRACT

The generation of radiation via photoelectrons induced off of a conducting surface was explored using a simple model to determine fundamental scaling laws. The model is one-dimensional (small-spot) and uses monoenergetic, nonrelativistic photoelectrons emitted normal to the illuminated conducting surface. Simple steady-state radiation, frequency, and maximum orbital distance equations were derived using small-spot radiation equations, a $\sin^2$ type modulation function, and simple photoelectron dynamics. The result is a system of equations for various scaling laws, which, along with model and user constraints, are simultaneously solved using techniques similar to linear programming. Typical conductors illuminated by low-power sources producing photons with energies less than 5.0 eV are readily modeled by this small-spot, steady-state analysis, which shows they generally produce low efficiency ($\eta_{rsL} < 10^{-10.5}$) pure photoelectron-induced radiation. However, the small-spot theory predicts that the total conversion efficiency for incident photon power to photoelectron-induced radiated power can go higher than $10^{-5.5}$ for typical real conductors if photons having energies of 15 eV and higher are used, and should go even higher still if the small-spot limit of this theory is exceeded as well. Overall, the simple theory equations, model constraint equations, and solution techniques presented provide a foundation for understanding, predicting, and optimizing the generated radiation, and the simple theory equations provide scaling laws to compare with computational and laboratory experimental data.

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Introduction

This paper presents a very simple derivation of the theory equations and scaling laws that describe the generation of radiation via photoelectrons emitted off of a conducting surface, as well as an analysis of the resulting theory equations. Originally, this analysis began as a need to understand some of the basic electron dynamics so that Particle-In-Cell (PIC) code simulations of the physical problem could be modeled and programmed. In order to set up a PIC code simulation that will run and produce useful results, a computationalist running the code needs some idea of the basic physics involved as well as the spatial and temporal scales that exist in the problem. This need to understand basic physics of the problem to be simulated in order to perform the computer simulations led to some back-of-the-envelope equations for the electron beam dynamics of the problem. A postprocessing code for converting PIC code simulation quantities into far-field radiation parameters was developed using a rigorous numerical implementation of the original radiation equations presented at the start of the next section. Next, simple analytic equations were derived for various radiation parameters to test and confirm the computational results obtained from the more rigorous numerical implementation of the original radiation equations. Later these theory equations became more useful as simple steady-state, analytic equations were developed. The theory was soon useful, if simple, for predicting, validating, and comparing scaling laws and PIC results, and for understanding and optimizing various parameters subject to constraints.

The strength of the theory is its simplicity, which focuses on key parameters in a way that makes their effects easy to understand. Of course, its simplicity is also the theory's main weakness. The theory was only designed to predict radiation quantities for small spots in steady state, though some of the photoelectron orbital quantities are also useful to know even for large spots. This simplicity is inherited from and consistent with the original goal of the theory's development, since *realistic, self-consistent* PIC code runs were limited to one-dimensional computer simulations. However, the theory shows that the maximum steady-state radiation and radiation conversion efficiencies for typical conductors and low-power sources of low-energy, 5.0 eV and less, photons are generally small-spot limited (i.e. dominated by constructive interference and having a dipole far-field radiation pattern) and are therefore modeled by this simple theory. Further, trying to achieve predictions of the

maximum radiated power leads one to want to develop a simple, ideal theory that may not be physically rigorous but that provides an approximation of the upper limit on radiated power that can be physically generated from a pure photoelectric effect. For these reasons a simple, idealized theory works well.

Therefore, these theory results came from a real world need to set up, predict, and cross check PIC code simulations and postprocessor radiation results, and the theory has performed well in that role. Further, the final equations were also meant to provide approximate theory scaling laws involving the important parameters, thereby allowing theoretical and experimental generated radiation comparisons and optimization analysis. All in all, where the assumptions and models used in the theory are maintained, these theory equations adequately describe the idealized optimum power that can be predicted by analytic theory for steady state and small spots. And the theory is useful because typical conductors and many photon sources are modeled by the steady-state, small-spot theory.

Confidence in the theory results presented in this paper will be established in follow-on papers. For instance, in one follow-on paper [Ref. 1], adjustments to the theory scaling laws presented in this paper were performed using PIC code simulation results. That paper found that the adjustments to the theory equations to reproduce the better modeled, fully nonlinear, self-consistent results from the PIC code computer simulations were minor. And in a second follow-on paper [Ref. 2], the theory equations derived in this paper are shown to closely match the theory equations that were developed by another researcher to describe single-pulse photoelectron-generated radiation. These three papers form a basic foundation or set of *inter-supporting research results*, based on two different theories and PIC code computer simulation results, that together establish the basic photoelectron-generated radiation scaling laws for small-spots. Further, I have performed other research that shows a large-spot theory [Ref. 3] also matches the small-spot theory scaling laws when the large-spot equations are reduced to small-spot equations, and that a slightly modified form of the large-spot theory equations shows agreement with large-spot data as well. Time permitting, I will also present my analysis of the large-spot theory in a follow-on paper.

The sequence of presentation for this paper is given below. First, I will develop the theory equations describing photoelectron-generated radiation using PIC parameters, and summarize the resulting system of equations.

INTRODUCTION

Writing the system of equations in the compact PIC parameter set facilitates working with and comparing PIC code simulation results with the theory results. Then I will convert the system of equations to my photon parameter set, also referred to in the paper as my ultraviolet (UV) or laboratory parameter set, since it is a more practical set of parameters for interfacing with experimentalists and for working with experimental results. Next the system of equations will be converted to my natural parameter set, specific example problem cases will be graphed, and optimization trends will be discussed. This natural parameter set allows the reader to more easily understand, use, make predictions with, and optimize the derived theory system of equations. Then model constraints applied in developing the simple theory equations will be listed, analyzed, and used to discuss general optimization, which involves model and user constraints. Generalized scaling laws will then be presented that can be used in testing the theory versus computational and laboratory experimental data. Last, I will present my conclusions, and the appendices will present Fortran programs that simplify using and optimizing the theory system of equations derived in this paper, subject to constraints. Unless noted otherwise, all equations, parameters, and constants used in this paper are expressed in the International Metric System (SI) system of measurement.

Development of Theory Equations

Radiation Equations

This section begins the derivation of the simple small-spot, steady-state theory by starting with general far-field radiation equations. From Chapter 14, Eq. (14-43), *Classical Electricity and Magnetism* [Ref. 4, p. 248], in SI:

$$\mathbf{B}_{\text{rad}} = \frac{\alpha}{c} \int \frac{[\dot{\mathbf{j}}] \times \mathbf{r}}{r^2} dv' \quad (1)$$

$$\mathbf{E}_{\text{rad}} = \alpha \int \frac{([\dot{\mathbf{j}}] \times \mathbf{r}) \times \mathbf{r}}{r^3} dv' \quad \text{where} \quad (2)$$

$$\alpha \equiv \frac{1}{4\pi\epsilon_0 c^2} \equiv \frac{\mu_0}{4\pi} \quad (3)$$

$$[\dot{\mathbf{j}}] = \dot{j}_{\rho'}(\rho', \phi', z', t')\hat{\rho}' + \dot{j}_{\phi'}(\rho', \phi', z', t')\hat{\phi}' + \dot{j}_{z'}(\rho', \phi', z', t')\hat{z}' \quad (4)$$

$$\mathbf{j} = (j_{\rho'} \cos \phi' - j_{\phi'} \sin \phi')\hat{i} + (j_{\rho'} \sin \phi' + j_{\phi'} \cos \phi')\hat{j} + j_{z'}\hat{k} \quad (5)$$

$$\mathbf{r} = \mathbf{R} - \boldsymbol{\rho} \quad (6)$$

$$\mathbf{R} = R \cos \phi \sin \theta \hat{i} + R \sin \phi \sin \theta \hat{j} + R \cos \theta \hat{k} \quad (7)$$

$$\boldsymbol{\rho} = \rho' \cos \phi' \hat{i} + \rho' \sin \phi' \hat{j} + z' \hat{k} \quad (8)$$

$$r = |\mathbf{r}| \quad (9)$$

$$t' = t - r/c \quad (10)$$

where I have used cylindrical and spherical coordinate systems for the source and observer respectively. $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$ are the radiated far-field electric and magnetic field strength vectors generated from an accelerating charge source, $[\dot{\mathbf{j}}]$ is the retarded time $\dot{\mathbf{j}}$, $\dot{\mathbf{j}}$ is the time derivative of the source current density vector, dv' is an infinitesimal volume element of the source, and the primed and unprimed parameters represent parameters in the source and observer frames, respectively. Also, $\mathbf{R}$ is the origin to field point vector, and $\boldsymbol{\rho}$ is the origin to source point vector. The radiation fields become transverse to the direction of propagation at distances far from the source such that $|\mathbf{E}_{\text{rad}}|/|\mathbf{H}_{\text{rad}}| = \sqrt{\mu_0/\epsilon_0} = \mu_0 c$. Finally, $\hat{i} = \hat{x}' = \hat{x}$, $\hat{j} = \hat{y}' = \hat{y}$, and $\hat{k} = \hat{z}' = \hat{z}$. Figure 1 shows the coordinate systems used.

Next, approximations will be made to simplify applying the equations. In particular, photoemission will be modeled as a perpendicular, essentially

DEVELOPMENT OF THEORY EQUATIONS

Figure 1: 1-D cylindrical current source and spherical observer reference frame coordinate systems shown with respect to Cartesian coordinates, as used in the derivations.

one-dimensional (uniform in ϱ' and ϕ') j_z , source of current density along the z -axis. Also, θ is the polar angle measured from the z -axis, making it the angle measured from the normal of the conducting photoemission surface that occupies the $x - y$ plane. For a one-dimensional charge source, the source is symmetric in ϱ' and ϕ' , and the equations can be simplified by setting $\phi' = 0$ and using the observer $x - z$ plane. Then the following equations can be used,

$$j_{\varrho'} = j_{\phi'} = 0.0 \quad (11)$$

$$[\dot{j}] = \dot{j}_z(z', t') \hat{k} \quad (12)$$

$$\mathbf{r} = (R \sin \theta - \varrho') \hat{i} + (R \cos \theta - z') \hat{k} \quad (13)$$

$$\hat{R} = \sin \theta \hat{i} + \cos \theta \hat{k} \quad (14)$$

$$\hat{\theta} = \cos \theta \hat{i} - \sin \theta \hat{k} \quad (15)$$

Next, to easily perform the source integrations, retarded time must be ignored. Ignoring retarded time can be accomplished by modeling the source as a 'point' charge, where $\rho = |\boldsymbol{\rho}| \rightarrow 0.0$, so that $\varrho' \rightarrow 0.0$ and $z' \rightarrow 0.0$. Then $\mathbf{r} \rightarrow \mathbf{R}$, $r \rightarrow R$, and r or R can be used interchangeably. Equivalently, one could impose the more quantitative $\rho \ll \lambda_{\text{mod}} \ll R$ small-spot charge

Radiation Equations

source and far-field radiation condition, where λ_{mod} is the wavelength of the generated radiation associated with the source's temporal modulation frequency. The $\rho \ll \lambda_{\text{mod}}$ condition indicates a criteria for limiting the maximum size of the small-spot source: to prevent retarded time effects that would destroy constructive interference of the integrated source fields at a far-field observer's location and time. Either way, since retarded time effects can be essentially ignored when integrating over the volume of a small-spot or 'point' charge source, then $t = t'$ of the 'point' source can be used. The far-field radiation condition is also necessary since Eqs. (1) and (2) show radiated far-field quantities, but this condition is easy to satisfy since R can be selected as large as needed when deriving the theory results. So a small-spot or 'point' charge source of $\dot{j}_z$, the time derivative of the current density component in the z direction, will be used in the derivations. Therefore, letting $j_z \hat{k} = j_z(z', t') \hat{k} = \mathbf{j}$ and $\rho \ll \lambda_{\text{mod}} \ll R$, then for Eqs. (1) and (2),

$$\mathbf{j} \times \mathbf{r} = \dot{j}_z R \sin \theta \hat{j} \quad (16)$$

$$(\mathbf{j} \times \mathbf{r}) \times \mathbf{r} = (\dot{j}_z R \sin \theta) (R \sin \frac{\pi}{2}) \hat{\theta} \quad (17)$$

$$= \dot{j}_z R^2 \sin \theta \hat{\theta} \quad (18)$$

$$= \dot{j}_z R^2 \sin \theta (\cos \theta \hat{i} - \sin \theta \hat{k}) \quad (19)$$

and since $j_z(z', t') \hat{k}$ is a 1-D cylindrical source of current density,

$$\int F(z') dv' = \int_{\phi'=0}^{2\pi} \int_{\varrho'_1}^{\varrho'_2} \int_{z'_1}^{z'_2} F(z') dz' \varrho' d\varrho' d\phi' \quad (20)$$

$$= \pi (\varrho')^2 \int_{z'_1}^{z'_2} F(z') dz' \quad (21)$$

$$= A'_{\text{spot}} \int_{z'_1}^{z'_2} F(z') dz' \quad (22)$$

Now Eq. (1) for $\mathbf{B}_{\text{rad}}$ can be rewritten as

$$\mathbf{B}_{\text{rad}} = \frac{\alpha}{c} A'_{\text{spot}} \int_{z'_1}^{z'_2} \frac{\dot{j}_z R \sin \theta \hat{j}}{R^2} dz' \quad (23)$$

$$= \frac{\alpha}{c} \int_{z'_1}^{z'_2} \frac{\dot{I}_z \sin \theta}{R} dz' \hat{j} \quad (24)$$

$$= \frac{\alpha \sin \theta}{cR} \int_{z'_1}^{z'_2} \dot{I}_z dz' \hat{j} \quad (25)$$

DEVELOPMENT OF THEORY EQUATIONS

Similarly for Eq. (2) for $\mathbf{E}_{\text{rad}}$,

$$\mathbf{E}_{\text{rad}} = \alpha A'_{\text{spot}} \int_{z'_1}^{z'_2} \frac{j_z R^2 \sin \theta \hat{\theta}}{R^3} dz' \quad (26)$$

$$= \frac{\alpha \sin \theta}{R} \int_{z'_1}^{z'_2} \dot{I}_z dz' \hat{\theta} \quad (27)$$

Having equations for both the far-field magnetic and electric field strength vectors allows the Poynting vector $\mathbf{P}$, the instantaneous power per unit solid angle $dP/d\Omega$, and the total instantaneous power radiated through a sphere in the far-field P_{rs} , to be written as:

$$\mathbf{P} = \frac{1}{\mu_0} (\mathbf{E}_{\text{rad}} \times \mathbf{B}_{\text{rad}}) \quad (28)$$

$$= \frac{\alpha^2 \sin^2 \theta}{\mu_0 c R^2} \left(\int_{z'_1}^{z'_2} \dot{I}_z dz' \right)^2 \hat{R} \quad (29)$$

$$\frac{dP}{d\Omega} = \mathbf{P} \cdot R^2 \hat{R} \quad (30)$$

$$= \frac{\alpha^2 \sin^2 \theta}{\mu_0 c} \left(\int_{z'_1}^{z'_2} \dot{I}_z dz' \right)^2 \quad (31)$$

$$P_{\text{rs}} = \oint_s \mathbf{P} \cdot d\mathbf{S} \quad (32)$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{\alpha^2 \sin^2 \theta}{\mu_0 c R^2} \left(\int_{z'_1}^{z'_2} \dot{I}_z dz' \right)^2 R^2 \sin \theta d\theta d\phi \quad (33)$$

$$= \frac{2\pi \alpha^2}{\mu_0 c} \int_{\theta=0}^{\pi} \sin^3 \theta d\theta \left(\int_{z'_1}^{z'_2} \dot{I}_z dz' \right)^2 \quad \text{for } \phi \text{ symmetric} \quad (34)$$

$$= \frac{2^3 \pi \alpha^2}{3 \mu_0 c} \left(\int_{z'_1}^{z'_2} \dot{I}_z dz' \right)^2 \quad (35)$$

Next, the small-spot or 'point' source approximation also simplifies the previous equations. The small-spot approximation implies that no integral over the source z' coordinates is needed because the charge at every small-spot source point is accelerating uniformly (as well as having no size related field propagation retarded time phase differences), and the source current can be thought of as a single, uniform 'point' charge located near the origin

Radiation Equations

of the coordinate system. Consequently, the results for all of the 1-D 'point' charge equations are summarized below:

$$\mathbf{B}_{\text{rad}} = \frac{\alpha \dot{I}_z \Delta z'}{cR} \sin \theta \hat{j} \quad (36)$$

$$\mathbf{E}_{\text{rad}} = \frac{\alpha \dot{I}_z \Delta z'}{R} \sin \theta \hat{\theta} \quad (37)$$

$$\mathbf{P} = \frac{\alpha^2 \dot{I}_z^2 \Delta z'^2}{\mu_0 c R^2} \sin^2 \theta \hat{R} = \frac{\dot{I}_z^2 \Delta z'^2}{2^4 \pi^2 \epsilon_0 c^3 R^2} \sin^2 \theta \hat{R} \quad (38)$$

$$\frac{dP}{d\Omega} = \frac{\alpha^2 \dot{I}_z^2 \Delta z'^2}{\mu_0 c} \sin^2 \theta = \frac{\dot{I}_z^2 \Delta z'^2}{2^4 \pi^2 \epsilon_0 c^3} \sin^2 \theta \quad (39)$$

$$P_{\text{rs}} = \frac{2^3 \pi \alpha^2 \dot{I}_z^2 \Delta z'^2}{3 \mu_0 c} = \frac{\dot{I}_z^2 \Delta z'^2}{6 \pi \epsilon_0 c^3} \quad \text{and since} \quad (40)$$

$$c^2 = \frac{1}{\mu_0 \epsilon_0} \quad \text{and} \quad \alpha^2 = \frac{\mu_0^2}{2^4 \pi^2} \quad \text{then} \quad (41)$$

$$\frac{\alpha^2}{\mu_0 c} = \frac{\mu_0}{2^4 \pi^2 c} = \frac{1}{2^4 \pi^2 \epsilon_0 c^3} \quad (42)$$

where all quantities that are a function of $\dot{I}_z$ evolve in time since $\dot{I}_z = f(t')$.

Next, it will be assumed that the temporal function of an actual current source can be modeled by $I_z(t')$, the temporal function of the 1-D 'point' charge source, as a current that varies periodically at ν , a primary sinusoidal modulation frequency. This will then allow the temporal dependence to be removed from $\mathbf{B}_{\text{rad}}$, $\mathbf{E}_{\text{rad}}$, $\mathbf{P}$, $dP/d\Omega$, and P_{rs} by calculating the expectation value of those equations. A sinusoidally varying monode current source $I(t')$ will be used for $I_z(t')$ to model the photoelectron-generated radiation's photoemission current source. Also, a modulation factor can be defined and modeled as well. But the modulation factor can be defined several ways. Which to use is simply a matter of convenience to the user. Letting I_0 be the amplitude of the alternating or sinusoidal component of the current source and letting $\langle I \rangle$ be the average component of the current source, such that $I_0 \equiv F_D \langle I \rangle$ where $0 \leq F_D \leq 1$, then the current as a function of time can be expressed as a $\sin^2$ type of function,

$$I(t') \equiv \langle I \rangle [1 - F_D \cos(2\pi \nu t')] \quad (43)$$

$$= \langle I \rangle [2F_D \sin^2(\pi \nu t') + (1 - F_D)] \quad \text{or} \quad (44)$$

$$I(t') \equiv F_D \langle I \rangle \sin(2\pi \nu t') + \langle I \rangle \quad (45)$$

DEVELOPMENT OF THEORY EQUATIONS

where the modulation function F_D is a function of the modulation depth D and depends on how D is defined.

If the modulation depth is defined in terms of the above periodic functions' minimum and maximum $I(t')$, then

$$D \equiv \frac{I_{\max} - I_{\min}}{I_{\max}} = \frac{2I_0}{I_0 + \langle I \rangle} \quad \text{so} \quad (46)$$

$$F_D \equiv \frac{I_0}{\langle I \rangle} = \frac{D}{2 - D} \quad (47)$$

But if D is simply written in terms of both the amplitude of the oscillating component of the current and the average current, then by definition

$$D \equiv \frac{I_0}{\langle I \rangle} \quad \text{and} \quad (48)$$

$$F_D = D \quad (49)$$

where D is constrained as $0 \leq D \leq 1$ in both cases. For these choices, $F_D = D$ is simpler to use in doing and understanding the analysis but $F_D = D/(2 - D)$ could be easier to measure. As such, F_D will be used in the rest of the paper, and the reader can select which to use.

And so, for the monode current source having $I_0 \equiv F_D \langle I \rangle$

$$I(t') \equiv F_D \langle I \rangle \sin(\omega t') + \langle I \rangle \quad (50)$$

$$\dot{I} = F_D \langle I \rangle \omega \cos(\omega t') \quad (51)$$

$$\dot{I}^2 = F_D^2 \langle I \rangle^2 \omega^2 \cos^2(\omega t') \quad (52)$$

$$\langle \dot{I}^2 \rangle = \frac{1}{2} F_D^2 \omega^2 \langle I \rangle^2 \quad (53)$$

$$= 2\pi^2 F_D^2 \nu^2 \langle I \rangle^2 \quad (54)$$

where $\omega = 2\pi\nu$ was used. Note that any definition of $I(t')$ in Eqs. (43-45) will give the same result for $\langle \dot{I}^2 \rangle$. Also, the equation is derived for the case when an average current $\langle I \rangle$ already exists and becomes modulated up to $F_D = 100\%$, and therefore if $\langle I \rangle \rightarrow 0$, then the emitted electron beam would disappear and no radiation would be generated.

In the previous equations, $F_D^2 = D^2$ or $F_D^2 = D^2/(2 - D)^2$, and it could be useful if a simple scaling term in D could be found to replace the more complex F_D function above. Some function of the form $F_D^2 = aD^b$ or just

Radiation Equations

$F_D^2 = D^a$ would be a simple replacement function. To find a suitable coefficient that reproduces the more complex function over the range of D , the following condition is used:

$$\frac{\partial}{\partial a} \int_{D=0}^{D=1} \left[\frac{D^2}{(2-D)^2} - D^a \right]^2 dD = 0 \quad (55)$$

And when solved numerically, $a \approx 3.45$, so

$$F_D^2 = \frac{D^2}{(2-D)^2} \approx D^{3.45} \quad (56)$$

for $0 \leq D \leq 1$, with an average RMS error of around 0.0152 from the actual function being approximated. So $D^{3.45}$ can be used for simplicity when measuring the emitted electron current's approximate periodic minimum and maximum values.

By using a sinusoidally varying current function in time and the previous summary of the 1-D 'point' charge equations to calculate time averaged quantities of interest, the explicit dependence on time can be removed from the equations. These new far-field, time-averaged radiation equations for a 'point' charge, sinusoidally varying monode current source in time with modulation function F_D are summarized below, where all of the primed variables will now be written as unprimed variables for the rest of the paper.

$$\langle \mathbf{B}_{\text{rad}} \rangle = \frac{2\pi\alpha}{cR} F_D \nu \cos(2\pi\nu t) \langle I \rangle \Delta z \sin \theta \hat{j} \quad (57)$$

$$\langle \mathbf{E}_{\text{rad}} \rangle = \frac{2\pi\alpha}{R} F_D \nu \cos(2\pi\nu t) \langle I \rangle \Delta z \sin \theta \hat{\theta} \quad (58)$$

$$\langle \mathbf{P} \rangle = \frac{2\pi^2\alpha^2}{\mu_0 c R^2} F_D^2 \nu^2 \langle I \rangle^2 \Delta z^2 \sin^2 \theta \hat{R} \quad (59)$$

$$\frac{d\langle P \rangle}{d\Omega} = a_\Omega F_D^2 \nu^2 \langle I \rangle^2 \Delta z^2 \sin^2 \theta \quad (60)$$

$$a_\Omega \equiv \frac{2\pi^2\alpha^2}{\mu_0 c} = \frac{\mu_0}{2^3 c} = \frac{1}{2^3 \epsilon_0 c^3} \approx 5.2396 \times 10^{-16} \quad (61)$$

$$\langle P_\tau \rangle_s = a_p F_D^2 \nu^2 \langle I \rangle^2 \Delta z^2 \quad (62)$$

$$a_p \equiv \frac{\pi}{3\epsilon_0 c^3} \approx 4.3895 \times 10^{-15} \quad (63)$$

$$\alpha \equiv \frac{1}{2^2 \pi \epsilon_0 c^2} = \frac{\mu_0}{2^2 \pi} \equiv 1 \times 10^{-7} \quad (64)$$

DEVELOPMENT OF THEORY EQUATIONS

$$\frac{a_p}{a_\Omega} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin^3 \theta d\theta d\phi = \frac{8\pi}{3} \quad (65)$$

$$I(t) \equiv F_D \langle I \rangle \sin(2\pi\nu t) + \langle I \rangle \quad (66)$$

How good is the model for radiated power so far? The steady-state theory equations developed later in this paper indicate that experiments involving low-energy photoelectrons should typically have $\Delta z_{\max} \leq 100$ microns, so the generated radiation's wavelength can be a few millimeters and larger and satisfy the $\Delta z_{\max} \ll \lambda_{\text{rf}}$ 'point' charge constraint. Further, 1-D PIC code runs having $r_{\text{spot}} \ll \lambda_{\text{rf}}$, as well as spot radii that are 'small' by several criteria (constructive interference, coherence), also satisfy and are well approximated by a 'point' charge constraint. Large-spot diffraction effects are not modeled by the above small-spot radiation equations and predictions. An upper limit therefore exists on the maximum small-spot size to prevent significant destructive interference effects not modeled by the theory. Later, when the natural radiation frequency and the maximum orbital distance of the emitted photoelectrons are calculated, the models used to derive these will also impose their own lower and upper limits on the spot radius. So other approximations will be used that characterize the 'point' source as having both a primary frequency and a uniform, finite extent based on the photoelectron orbital dynamics. In general, all of the model approximations that will be used will act as additional constraint equations in a mathematical programming optimization sense. Mathematical programming is the general case of linear programming, and as such, can be used with nonlinear equations and constraints. These approximations and constraints will be explored in detail near the end of this paper, after the steady-state theory equations for both the photoelectron dynamics and generated radiation have been fully developed.

But the primary purpose here is not to derive the exact solutions for all parameters and parameter regions, as such solutions would be impossible to derive analytically in a practical, easy-to-understand form. Instead, it is quite useful to have a set of simple theoretical scaling laws that approximate the solutions, allow testing the PIC and laboratory results, and provide insight into the use and optimization of the parameters and radiation. For instance, these radiation equations for a pure sinusoidally modulated 'point' charge already display useful scaling laws for helping to theoretically predict the maximum possible $\langle P_r \rangle_s$ since the best that can be done, according to the

Single-Pulse Trajectory Equations

simple theory so far, is to maximize F_D , ν , $\langle I \rangle$, and Δz of the photoemitted electrons. Finally, steady-state constraints for resonance will be determined and imposed, since *both single frequency operation and maximum power can be improved by using resonance*. Therefore, the primary importance of the theory equations being developed will be to test, predict, and optimize various small-spot, steady-state radiation quantities by allowing one to scale, constrain, and understand the basic physics involved. Lastly, once developed, these simple theory equations should provide the reader with a solid foundation for both the design of more efficient and effective experiments and for the construction of more advanced analytic equations.

Single-Pulse Trajectory Equations

This subsection will model and develop the the single-pulse equations for the time $t_{\max}$ and distance $d_{\max}$ to maximum orbit of an emitted, thin, rigid charge sheet off of a conducting surface. Later, these single-pulse equations will be used to calculate an approximate steady-state, sinusoidal natural radiation frequency ν_{nat} and current source size Δz to use with the previously derived radiation equations. First, the simple model and assumptions used will be presented. Next, the trajectory and energy equations will be solved for emitted electron orbital parameters. Finally, these simple single-pulse equations will be summarized.

Listed below are the key assumptions used in developing the single rectangular pulse model.

1. $0 < \beta \ll 1$, where $\beta \equiv v_e/c$, so model assumes classical treatment of nonrelativistic photoemission.
2. The $+\rho_s$ charge sheet is fixed, not moving, and only the emitted photoelectron $-\rho_s$ charge sheet moves.
3. $r_{\text{spot}} \geq d_{\max}$, or the radius of the illuminated spot is greater than or equal to the maximum orbit distance of the emitted 2-D charge sheet so that the perpendicular electric field strength component E_z is constant (electrostatic) like an infinite 2-D sheet, and real world fringing effects are minimized.
4. Small-spot, essentially 1-D model where only electric forces are involved.

DEVELOPMENT OF THEORY EQUATIONS

5. $v_e \Delta t_p \ll d_{\max}$ or $\Delta t_p \ll t_{\max}$ for a thin charge sheet, where Δt_p is the pulse emission time duration.
6. A single emitted charge sheet pulse per orbit, having a uniform area charge density ρ_s for the charge sheet.
7. Both ρ_s charge sheets are rigid.

Figure 2 shows the single rectangular pulse model used to approximate the current source and forces.

Figure 2: Emitted electron model of infinite, 2-D planar, virtually thin, rigid charge sheet for a single pulse.

As such, the following equations for the model are:

$$\mathbf{E} = E_z = \left| \frac{\rho_s}{\epsilon_0} \right| \hat{k} \quad (67)$$

$$\rho_s = \langle j \rangle \Delta t_p \quad (68)$$

$$E_z = \left| \frac{\langle j \rangle \Delta t_p}{\epsilon_0} \right| \hat{k} \quad (69)$$

$$\langle j \rangle = \frac{I}{A} \quad (70)$$

$$\epsilon_0 \approx 8.8542 \times 10^{-12} \quad \text{in SI} \quad (71)$$

Single-Pulse Trajectory Equations

where the photoelectron-created ρ_s has dimensions of charge per area A of the illuminated spot on the conducting surface, and the E_Z electric field strength component normal to the conducting surface, or $+\rho_s$ charge sheet, is for the model a constant and essentially maximum value between the $+\rho_s$ and $-\rho_s$ charge sheets.

Consequently, the trajectory equations are simply

$$\frac{dv_z}{dt} = \frac{q_e E_Z}{m_e} \quad (72)$$

$$\frac{dz}{dt} = \frac{q_e E_Z}{m_e} t + v_e \quad (73)$$

$$z = \frac{q_e E_Z}{2m_e} t^2 + v_e t + z_0 \quad (74)$$

and setting $dz/dt = 0$, and noting that $d^2z/dt^2 = q_e E_Z/m_e$ is negative since E_Z is positive and q_e is negative, then the time to maximum orbit is

$$t_{\max} = -\frac{m_e v_e}{q_e E_Z} \quad (75)$$

and to find $d_{\max} = z_{\max}$, the kinetic and potential energy balance equation can be used to get

$$\frac{1}{2} m_e v_e^2 = -q_e E_Z d_{\max} \quad \text{so} \quad (76)$$

$$d_{\max} = -\frac{m_e v_e^2}{2q_e E_Z} \quad (77)$$

Next, using absolute values for all numbers, replacing q_e by the elementary charge e , and substituting for v_e and E_Z , the thin, single-pulse equations can be summarized as

$$d_{\max} = \frac{m_e v_e^2}{2e E_Z} \quad (78)$$

$$= \frac{\epsilon_0 m_e c^2}{2e} \frac{\beta^2}{\langle j \rangle \Delta t_p} \quad (79)$$

$$t_{\max} = \frac{m_e v_e}{e E_Z} \quad (80)$$

$$= \frac{\epsilon_0 m_e c}{e} \frac{\beta}{\langle j \rangle \Delta t_p} \quad \text{where} \quad (81)$$

DEVELOPMENT OF THEORY EQUATIONS

$$v_e = \beta c \quad (82)$$

$$E_Z = \frac{\langle j \rangle \Delta t_p}{\epsilon_0} \quad (83)$$

These equations describe the behavior of the model with its various assumptions. Some of those assumptions are very good with respect to the real problem, such as assumptions 1 and 2. Other assumptions that are good are assumptions 3 and 4. Assumptions 5 through 7 are probably the worst with respect to real problems, as they model a thin, single, rigid charge sheet. But again, this model will help to derive the basic scaling laws, and later the equations will be modified by relaxing assumptions 5 through 7 to obtain a simple steady-state theory.

Steady-State Equations

This subsection will look at photoemission and model it so that basic scaling laws can be developed for steady-state parameters of interest. First, simple steady-state resonance conditions will be presented and discussed. Next, these simple resonance conditions will be used to determine the approximate maximum steady-state orbital period of the emitted electrons. Then the maximum steady-state orbital period can be used to determine the other approximate steady-state quantities of interest such as the natural orbital frequency, maximum orbital distance or size of the photoemission layer, and approximate maximum electric field strengths at the conducting surface. Finally, these approximate steady-state equations that will be derived can then be used to derive the steady-state 'resonance' radiation scaling laws.

Since this analysis is primarily interested in general scaling laws and approximate predictions, and not in precise predictions, a very simple approximation for a *resonance condition* will be made that equates

$$\Delta t_p = \frac{T_{\text{mod}}}{2} = \frac{1}{2\nu_{\text{mod}}} \equiv \frac{1}{2\nu_{\text{nat}}} = \frac{T_{\text{nat}}}{2} = t_{\text{max}} \quad (84)$$

where $\nu_{\text{mod}} \equiv \nu_{\text{nat}}$ for resonance. This resonance condition will allow the single-pulse trajectory equations developed in the last subsection to be converted into approximate steady-state resonance equations. Since ν_{nat} represents the frequency that a steady-state $-\rho_s$ charge sheet orbit has, driving

Steady-State Equations

the current source at a modulation frequency ν_{mod} equal to the natural orbit frequency ν_{nat} should result in maximum, or resonance, generated power. This resonance condition works well for a constant, monoenergetic, perpendicularly emitted photoelectron beam with some average current density $\langle j \rangle$, which will then self-oscillate at ν_{nat} due to nonlinear beam bunching of the photoelectron sheath. A spread of energies and velocity vectors for the emitted photoelectrons could destroy the self-oscillation ν_{nat} but the electron sheath would still have characteristic photoelectron time and distance scales that would constrain the ability to arbitrarily choose any modulated driving frequency ν_{mod} for the emitted current density. So it is assumed that the resonance condition is an optimum radiation condition but that the ν_{mod} frequency should be able to be varied with less degradation to optimal radiation values because the ideal photoemission used in these models is only a simple approximation of real photoemission. In summary, with real photoemission one could expect the self-oscillation frequency ν_{nat} to wash out or disappear, so that a wider range of ν_{mod} frequencies could be used (the resonance curve broadens) that would generate optimal radiation, but that *there are still limits* based on the photoelectron dynamics and other electron sheath characteristics that will again *eventually* limit the choice of values for ν_{mod} that allow optimal radiated power and conversion efficiency for a given set of independent parameters.

Of course, instead of the single, $\Delta t_p \ll t_{\text{max}}$, rectangular pulse modeled in the previous subsection, now a steady-state, sinusoidal pulse train with $\Delta t_p = T_{\text{mod}}/2 \approx t_{\text{max}}$ needs to be modeled. This change to a sinusoidal pulse train should not significantly affect the fundamental scaling laws but will change the calibration or proportionality constant coefficients in the theory equations. For the single-pulse equations, the pulse emission time is a small fraction of the orbit time, and the full forces begin acting on the rectangular pulse immediately. Since the sinusoidal, steady-state pulse builds slower and longer, approximately up until the time required to reach maximum orbit, j and E_z of the pulse grow slower, so the actual values of the constant coefficients for steady-state ν_{nat} and d_{max} should slightly decrease and increase, respectively. Figure 3 shows how a sinusoidal pulse is approximated as a pulse train of rectangular pulses having the same amount of current.

From Fig. 3 it is seen that the sinusoidal electron emission current density versus time $j(t)$, can be viewed as rectangular square pulse train emission,

DEVELOPMENT OF THEORY EQUATIONS

Figure 3: Current density versus time for emitted electron beam emission comparing a rectangular pulse train with a sinusoidal pulse train having the same average current density $\langle j \rangle$.

especially when $F_D = 100\%$, such that

$$\langle j \rangle T_{\text{mod}} = 2\langle j \rangle \frac{T_{\text{mod}}}{2} = 2\langle j \rangle \frac{1}{2\nu_{\text{mod}}} = \frac{\langle j \rangle}{\nu_{\text{mod}}} \quad (85)$$

where the relationship between the external modulation period and external modulation frequency is $T_{\text{mod}} = 1/\nu_{\text{mod}}$. This view allows an approximate calculation of the steady-state t_{max} and d_{max} . By requiring that the natural frequency that the collection of emitted electrons, as a single pulse, wants to oscillate at in steady state matches an external modulating frequency, resonance can be achieved. In other words, $\nu_{\text{mod}} \equiv \nu_{\text{nat}}$ for resonance. Essentially then,

$$2\langle j \rangle \Delta t_p = 2\langle j \rangle \frac{T_{\text{mod}}}{2} = \frac{\langle j \rangle}{\nu_{\text{mod}}} \equiv \frac{\langle j \rangle}{\nu_{\text{nat}}} = 2\langle j \rangle \frac{T_{\text{nat}}}{2} = 2\langle j \rangle t_{\text{max}} \quad (86)$$

and dividing out $2\langle j \rangle$ gives the resonance condition listed at the beginning of this subsection that equates

$$\Delta t_p = \frac{T_{\text{mod}}}{2} = \frac{1}{2\nu_{\text{mod}}} \equiv \frac{1}{2\nu_{\text{nat}}} = \frac{T_{\text{nat}}}{2} = t_{\text{max}} \quad (87)$$

Steady-State Equations

so that

$$\Delta t_p \equiv t_{\max} \quad \text{and} \quad \nu_{\text{mod}} \equiv \frac{1}{2t_{\max}} \quad (88)$$

Simultaneously solving both the single-pulse $t_{\max}$ equation and the steady-state resonance condition $\Delta t_p \equiv t_{\max}$ gives the approximate steady-state $t_{\max}$ equation, which is equal to the approximate steady-state Δt_p equation. This approximate steady-state $t_{\max}$ equation is

$$t_{\max} \equiv \Delta t_p \quad (89)$$

$$= a_t \beta^{1/2} \langle j \rangle^{-1/2} \quad (90)$$

$$a_t \equiv \left(\frac{\epsilon_0 m_e c}{e} \right)^{1/2} \approx 1.2285 \times 10^{-7} \quad (91)$$

Further, since $1/(2\nu_{\text{mod}}) \equiv t_{\max}$, and for resonance $\nu_{\text{mod}} \equiv \nu_{\text{nat}}$, then the steady-state natural frequency for self-oscillation is approximately

$$\nu_{\text{nat}} = a_\nu \beta^{-1/2} \langle j \rangle^{1/2} \quad (92)$$

$$a_\nu \equiv \left(\frac{e}{2^2 \epsilon_0 m_e c} \right)^{1/2} \approx 4.0700 \times 10^6 \quad (93)$$

where for the reader's convenience, the SI numerical values of the a_t and a_ν coefficients are shown, without listing the associated dimensions.

Note that ν_{nat} scales as the plasma frequency ν_{plasma} but is not the same as the plasma frequency. Specifically, the plasma frequency, using the same parameters as for ν_{nat} , is

$$\nu_{\text{plasma}} = \left(\frac{e}{2^2 \pi^2 \epsilon_0 m_e c} \right)^{1/2} \beta^{-1/2} \langle j \rangle^{1/2} \quad (94)$$

So, $\nu_{\text{nat}} = \pi \nu_{\text{plasma}}$, and is not twice the plasma frequency as one might quickly guess for a 'full oscillation' ν_{plasma} versus a 'half oscillation' ν_{nat} .

Also, the steady-state value of Δt_p can be substituted into the single-pulse equations for maximum E_Z and $d_{\max}$ to get the following approximate steady-state equivalents

$$E_Z = a_E \beta^{1/2} \langle j \rangle^{1/2} \quad (95)$$

$$a_E \equiv \left(\frac{m_e c}{\epsilon_0 e} \right)^{1/2} \approx 13875 \quad (96)$$

DEVELOPMENT OF THEORY EQUATIONS

$$d_{\max} = a_d \beta^{3/2} \langle j \rangle^{-1/2} \quad (97)$$

$$a_d \equiv \left(\frac{\epsilon_0 m_e c^3}{2^2 e} \right)^{1/2} \approx 18.415 \quad (98)$$

There are now steady-state theory equations approximating the sinusoidal resonance frequency $\nu_{\text{nat}} \equiv \nu_{\text{mod}}$, and the maximum current distance above the emitting surface $d_{\max}$, for this simple, sinusoidal, steady-state model. Consequently, $\langle P_r \rangle_s$ can finally be solved for an approximate average steady-state, 'resonance' power radiated through a sphere.

$$\langle P_r \rangle_s = a_p F_D^2 \nu^2 \langle I \rangle^2 \Delta z^2 \quad \text{substituting} \quad (99)$$

$$\nu \equiv \nu_j = \nu_{\text{mod}} = \nu_{\text{nat}} \quad (100)$$

$$\langle I \rangle = A \langle j \rangle \quad (101)$$

$$\Delta z \equiv d_{\max} \quad \text{then} \quad (102)$$

$$\langle P_r \rangle_s = a_p F_D^2 \left(a_\nu^2 \frac{\langle j \rangle}{\beta} \right) A^2 \langle j \rangle^2 \left(a_d^2 \frac{\beta^3}{\langle j \rangle} \right) \quad (103)$$

$$= a_4 F_D^2 A^2 \beta^2 \langle j \rangle^2 = a_4 F_D^2 \beta^2 \langle I \rangle^2 \quad \text{where} \quad (104)$$

$$a_4 \equiv a_p a_\nu^2 a_d^2 = \left(\frac{\pi}{3 \epsilon_0 c^3} \right) \left(\frac{e}{2^2 \epsilon_0 m_e c} \right) \left(\frac{\epsilon_0 m_e c^3}{2^2 e} \right) \quad (105)$$

$$= \left(\frac{\pi}{3 \epsilon_0 c^3} \right) \left(\frac{c^2}{2^4} \right) = \frac{\pi}{2^4 3 \epsilon_0 c} \approx 24.657 \quad (106)$$

Similarly, for $d\langle P \rangle/d\Omega$, the average power radiated per unit solid angle

$$\frac{d\langle P \rangle}{d\Omega} = a_3 F_D^2 A^2 \beta^2 \langle j \rangle^2 \sin^2 \theta \quad \text{where} \quad (107)$$

$$a_3 \equiv a_\Omega a_\nu^2 a_d^2 = \left(\frac{1}{2^3 \epsilon_0 c^3} \right) \left(\frac{e}{2^2 \epsilon_0 m_e c} \right) \left(\frac{\epsilon_0 m_e c^3}{2^2 e} \right) \quad (108)$$

$$= \left(\frac{1}{2^3 \epsilon_0 c^3} \right) \left(\frac{c^2}{2^4} \right) = \frac{1}{2^7 \epsilon_0 c} \approx 2.9432 \quad (109)$$

The equation for conversion efficiency η_{rsB} , calculated using the average emitted photoelectron beam power $\langle P_B \rangle$ and the average generated power radiated through a sphere $\langle P_r \rangle_s$, is listed below. Equations for other relationships, such as emitted velocity v_e and $\beta_{\text{classical}}$ of the emitted photoelectrons,

Steady-State Equations

are listed below as well.

$$\eta_{rsB} = \frac{\langle P_r \rangle_s}{\langle P_B \rangle} \quad \text{where} \quad (110)$$

$$\langle P_B \rangle = |e|eV_b \frac{A\langle j \rangle}{e} \quad (111)$$

$$\frac{1}{2}m_e v_e^2 = |e|eV_b \quad (112)$$

$$v_e = \left(\frac{2|e|}{m_e} eV_b \right)^{1/2} \quad (113)$$

$$\beta \equiv v_e/c \quad (114)$$

$$\beta_{\text{classical}} = b_2 \left(\frac{|e|eV_b}{e} \right)^{1/2} \quad (115)$$

$$b_2 \equiv \left(\frac{2e}{m_e c^2} \right)^{1/2} \quad (116)$$

$$eV_b = \frac{e}{|e|} \frac{1}{b_2^2} \beta^2 \quad (117)$$

$$\langle P_B \rangle = \frac{1}{b_2^2} A \beta^2 \langle j \rangle \quad \text{and finally} \quad (118)$$

$$\eta_{rsB} = \frac{\langle P_r \rangle_s}{\langle P_B \rangle} = a_5 F_D^2 A \langle j \rangle \quad (119)$$

$$a_5 \equiv a_4 b_2^2 = \left(\frac{\pi}{2^3 3 \epsilon_0 c} \right) \left(\frac{2e}{m_e c^2} \right) \quad (120)$$

$$= \frac{\pi e}{2^3 3 \epsilon_0 m_e c^3} \approx 9.6505 \times 10^{-5} \quad (121)$$

where the final form of the steady-state PIC parameter equations are written in terms of β since β is a nice number to work with, being *limited for these classical expressions* and this paper's theory equations to values of $0 < \beta \leq 0.3$. Note that the elementary charge e in SI units, when written as $|e|$ in this paper, becomes a conversion number having units of joules per eV. So $|e|$ is a number having the magnitude of e in SI, that is multiplied with quantities having units of eV to convert those quantities to values having dimensions of joules. Consequently, the parameter eV_b which represents the energy per emitted photoelectron in units of eV, is always associated with $|e|$ in the equations. The units of eV are *always* used with eV_b , and later

DEVELOPMENT OF THEORY EQUATIONS

on with the work function of the photoemission surface ϕ , because they are more convenient numbers to work with and for looking up the published ϕ values, which are usually listed in units of eV. Except as noted above, the equations and parameters in this paper should be applied using SI units.

As a last caution, it is important to note that while this paper calls these steady-state 'resonance' equations and used a simple resonance condition, these equations may only represent steady-state self-oscillation equations. The equations may only represent sinusoidally imposed steady-state self-oscillation equations without resonance because the equations may not model the effects of an actual, externally imposed sinusoidal frequency that matches the natural frequency. Since this is a problem with the simple nature of the theory, the simple theory would need further analysis to settle this issue. Regardless, the approximate resonance frequency ν_{nat} and the resonance condition $\nu_{\text{mod}} = \nu_{\text{nat}}$ for the theory models are not in doubt. Further, from PIC simulations [Ref. 1] it was found that the resonance generated radiation using $\nu_{\text{mod}} = \nu_{\text{nat}}$ was only three to four times larger than the steady-state self-oscillation generated radiation without an externally imposed ν_{mod} . So the issue of whether the equations represent 'resonance' or only steady-state generated radiation is less important than having derived useful predictive scaling laws that are representative, with minor proportionality constant adjustments, of both resonance and self-oscillation driven steady-state photoelectron-generated radiation.

PIC Parameter Set Equations

This section will summarize and briefly discuss the several nonrelativistic, small-spot, steady-state equations, or *system* of equations, developed in the last section. It is this system of equations, in fundamental units, that describes the behavior of parameters associated with photoelectron-generated radiation. This system of equations for the simple theory is summarized below using the PIC code parameter set, where the dependent variables have been expressed in terms of the four independent variables D , A , β , and $\langle j \rangle$.

$$\Delta t_p \equiv t_{\max} \quad (122)$$

$$= a_t \beta^{1/2} \langle j \rangle^{-1/2} \quad (123)$$

$$\nu_{\text{mod}} \equiv \nu_{\text{nat}} \quad (124)$$

$$= a_\nu \beta^{-1/2} \langle j \rangle^{1/2} \quad (125)$$

$$E_Z = a_E \beta^{1/2} \langle j \rangle^{1/2} \quad (126)$$

$$d_{\max} = a_d \beta^{3/2} \langle j \rangle^{-1/2} \quad (127)$$

$$\frac{d\langle P \rangle}{d\Omega} = a_3 F_D^2 A^2 \beta^2 \langle j \rangle^2 \sin^2 \theta \quad (128)$$

$$\langle P_r \rangle_s = a_4 F_D^2 A^2 \beta^2 \langle j \rangle^2 \quad (129)$$

$$eV_b = \frac{e}{|e|} \frac{1}{b_2^2} \beta^2 \quad (130)$$

$$\langle P_B \rangle = \frac{1}{b_2^2} A \beta^2 \langle j \rangle \quad (131)$$

$$\eta_{\text{rsB}} = a_5 F_D^2 A \langle j \rangle = \frac{\langle P_r \rangle_s}{\langle P_B \rangle} \quad (132)$$

$$\beta \equiv \frac{v_e}{c} = \beta_{\text{classical}} \quad (133)$$

$$0 < \beta_{\text{classical}} \leq 0.3 \quad (134)$$

$$F_D \equiv \frac{j_0}{\langle j \rangle} \quad (135)$$

$$j(t) \equiv j_0 \sin(2\pi \nu_{\text{mod}} t) + \langle j \rangle \quad (136)$$

$$0 \leq j_0 \leq \langle j \rangle \quad \text{so} \quad (137)$$

$$0 \leq F_D \leq 1 \quad (138)$$

where F_D is a function of D . Note that D depends on how the modulation depth or percentage is measured and was discussed in detail in the *Radiation*

PIC PARAMETER SET EQUATIONS

Equations subsection of the *Development of Theory Equations* section. Since the above equations are a *system of equations*, if $\langle j \rangle$ is *increased*, for example, then ν_{nat} , E_Z , $d\langle P \rangle/d\Omega$, $\langle P_r \rangle_s$, P_B , and η_{rsB} *increase*, while t_{max} and d_{max} *decrease*, and eV_b remains *unchanged*.

The constant coefficients for the above equations, along with their SI magnitudes, are

$$a_t \equiv \left(\frac{\epsilon_0 m_e c}{e} \right)^{1/2} \approx 1.2285 \times 10^{-7} \quad (139)$$

$$a_\nu \equiv \left(\frac{e}{2^2 \epsilon_0 m_e c} \right)^{1/2} \approx 4.0700 \times 10^6 \quad (140)$$

$$a_E \equiv \left(\frac{m_e c}{\epsilon_0 e} \right)^{1/2} \approx 13875 \quad (141)$$

$$a_d \equiv \left(\frac{\epsilon_0 m_e c^3}{2^2 e} \right)^{1/2} \approx 18.415 \quad (142)$$

$$a_3 \equiv \frac{1}{2^7 \epsilon_0 c} \approx 2.9432 \quad (143)$$

$$a_4 \equiv \frac{\pi}{2^4 3 \epsilon_0 c} \approx 24.657 \quad (144)$$

$$b_2 \equiv \left(\frac{2e}{m_e c^2} \right)^{1/2} \approx 1.9784 \times 10^{-3} \quad (145)$$

$$a_5 \equiv \frac{\pi e}{2^3 3 \epsilon_0 m_e c^3} \approx 9.6505 \times 10^{-5} \quad (146)$$

Since the summary equations are a system of equations, some general comments about systems of equations will be made. When a system of equations is manipulated to change various dependent and independent parameters to better suit some purpose, various parameters become fixed while others are allowed to vary. Some ways of writing a system of equations may be more convenient, such as for prediction or comparison purposes, or may provide additional information, such as during optimization analysis. However, it is important to understand that a system of equations is unique no matter how the dependent and independent parameters are arranged. Consequently, any way the parameters and equations are written always gives the same results, for that system of equations has a unique solution space. This information is useful to know because if you could graph all of the parameters at once, that graph is unique and represents any possible rearrangement

or interpretation of the 'full' system of equations. So in a system of equations, one needs to be aware of all the significant equations and important parameters being modeled, since they act as constraints on each other. In addition, this awareness of the full system of equations should include model assumptions as well, which are often not shown explicitly, and so require care not to be violated. This paper will handle the total system of equations in two parts: the theory results essentially given by the summary equations, and later the 'hidden' model (and user) constraint equations. Then when the system of equations is eventually graphed, that will be a full description of the parameters *allowed to vary on the graph* as long as the graphed constraint equations (and experimental constraints on parameters) are used to select *feasible* parameters where the model assumptions (and experimental limitations) are all satisfied.

Eventually this system of equations, after being converted to other parameter sets, will be used to predict and optimize the best radiation that can be produced in a given experimental set-up, something that is not necessarily a simple matter to determine. Succinctly put, one cannot simply write down a radiated power equation in terms of some arbitrary independent parameter set and optimize that equation by itself and hope to get a feasible 'best' solution. The full optimization process involves considering all significant parameters and theory (and model constraint) equations, identifying the parameter to maximize and the possible ranges of the other parameters, and using the full information available to get a feasible 'best' solution for a well posed problem. This general type of problem can be solved by mathematical programming, which is the general case of linear programming as applied to the nonlinear system of equations typical of this paper. However, in order to point out the importance of the summary equations and model constraint equations forming a *system* of equations that need to be analyzed together, topics that will be covered and worked out in later sections have been touched upon, and this paper will now continue with the current discussion.

Having briefly overviewed some concepts concerning generic systems of equations, the variables in the system of equations will now be rearranged so that $\langle j \rangle$ is a dependent variable, and ν_{nat} is an independent variable. Why? For fitting PIC code data or graphing results, the summary equations are fine. But if it were easier to adjust a power knob in an experiment than to change the measurements of a varying resonance frequency, then one would want to write all of the equations having ν_{nat} as an independent parameter,

PIC PARAMETER SET EQUATIONS

letting $\langle j \rangle$ vary as a dependent variable. Also, it is instructive to see how the apparent scaling laws for various parameters change depending upon how the system is expressed in terms of different sets of independent parameters, even though the solution of the system of equations is unique and does not change with respect to dependent or independent variable choices. Therefore, the equations will be rewritten as a function of the defined independent parameters D , A , β , and ν_{nat} . Rearranging the system of equations is easily accomplished by rearranging the equation for ν_{nat} as

$$\langle j \rangle = \frac{1}{a_\nu^2} \beta \nu_{\text{nat}}^2 \quad (147)$$

and substituting this equation for $\langle j \rangle$ into the equations making up the system of equations summarized at the beginning of this section.

The new system of steady-state equations, for some of the more important summarized equations, using the PIC parameter set, but with ν_{nat} replacing $\langle j \rangle$ as an independent parameter, is

$$\frac{d\langle P \rangle}{d\Omega} = \frac{a_3}{a_\nu^4} F_D^2 A^2 \beta^4 \nu_{\text{nat}}^4 \sin^2 \theta \quad (148)$$

$$\langle P_r \rangle_s = \frac{a_4}{a_\nu^4} F_D^2 A^2 \beta^4 \nu_{\text{nat}}^4 \quad (149)$$

$$\eta_{\text{rsB}} = \frac{a_5}{a_\nu^2} F_D^2 A \beta \nu_{\text{nat}}^2 \quad (150)$$

$$\langle j \rangle = \frac{1}{a_\nu^2} \beta \nu_{\text{nat}}^2 \quad (151)$$

$$d_{\text{max}} = a_\nu a_d \frac{\beta}{\nu_{\text{nat}}} \quad (152)$$

where all of the coefficients are in terms of the previously summarized coefficients.

It now appears from the above rearranged equations that the generated radiation's scaling with respect to β is larger than indicated by the previously summarized equations. For instance, the radiated power $\langle P_r \rangle_s$ now seems to scale as β^4 instead of β^2 . So, when parameter scalings are quoted, it is important to know with respect to what independent parameters the equations are written. *And if individual parameter scalings are to be usefully compared, it is necessary to have the various equations being compared all expressed in terms of the same set of independent parameters.*

UV Parameter Set Equations

The parameter space used so far has been derived in terms of fundamental parameters that are *based on beam emission only*. Advantages to using this PIC parameter set are that the parameter space is small in dimensions and that it matches PIC code beam emission simulations well. But it does have its drawbacks. For one, it is not a laboratory or photon parameter set, so discussing results with experimentalists can be difficult. Also, since it is not the more general photon or UV parameter space, a quantity like photon-source-to-generated-radiation power conversion efficiency does not exist without mixing various parameter sets, which is *not recommended*. So quantities such as

$$\frac{\langle P_r \rangle_s}{\langle P_L \rangle} = \frac{\langle P_r \rangle_s \langle P_B \rangle}{\langle P_B \rangle \langle P_L \rangle} \quad (153)$$

can best be written only in terms of a fundamental parameter set that includes photoelectron parameters like $\langle P_L \rangle$, the average power of the photon source. The wording 'only' is used because to see clearly the scaling involved requires all terms in the system of equations to be written without 'mega-terms' or 'mega-parameters' which hide other fundamental parameters that can cancel or change the apparent 'mega' and mixed term scaling laws.

So by picking an appropriate fundamental parameter set, the systems of equations from the last section can be rewritten in terms of a laboratory or UV parameter set. To convert to the UV parameter space, all that is needed are equations that relate beam emission to photon or UV induced photoemission. The PIC parameter set to UV parameter set conversion equations used are:

$$\langle j \rangle = \frac{1}{b_1} \frac{\lambda_L Y \langle P_L \rangle}{A} \quad (154)$$

$$\beta_{\text{classical}} = b_2 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{1/2} \quad (155)$$

$$b_1 \equiv \frac{hc}{e} \approx 1.2398 \times 10^{-6} \quad (156)$$

$$b_2 \equiv \left(\frac{2e}{m_e c^2} \right)^{1/2} \approx 1.9784 \times 10^{-3} \quad (157)$$

$$\phi_s \equiv \frac{|e|\phi}{e} \quad (158)$$

UV PARAMETER SET EQUATIONS

$$\langle P_L \rangle = \frac{E_L}{\Delta t_E} \quad (159)$$

where E_L , the total pulse train energy output of the photon source reaching the conducting surface, is summed over the characteristic or defined time duration of the total emitted pulse train having length Δt_E , and results in an average pulse train photon source power $\langle P_L \rangle$ reaching the conducting surface. Also, λ_L is the wavelength of the photons produced by the source, Y is the average photoelectron yield (or quantum efficiency) of photoelectrons to incident photons, A is the illuminated circular spot area on the conducting surface, and ϕ is the work function of the illuminated conducting surface expressed in units of eV. Finally, note that the equation for $\beta_{\text{classical}}$ is only good for *nonrelativistic emitted* photoelectron velocities where approximately $0 < \beta \leq 0.1$, or $0 < \beta \leq 0.3$ at most.

Using the conversion equations, and converting the summarized steady-state PIC parameter set equations having $\langle j \rangle$ as an independent parameter, gives the following selected and summarized nonrelativistic, steady-state, small-spot, UV parameter set system of equations

$$\frac{d\langle P \rangle}{d\Omega} = a_6 F_D^2 \left(\frac{b_1}{\lambda_L} - \phi_s \right) \lambda_L^2 Y^2 \langle P_L \rangle^2 \sin^2 \theta \quad (160)$$

$$\langle P_r \rangle_s = a_7 F_D^2 \left(\frac{b_1}{\lambda_L} - \phi_s \right) \lambda_L^2 Y^2 \langle P_L \rangle^2 \quad (161)$$

$$\eta_{\text{rsL}} = a_7 F_D^2 \left(\frac{b_1}{\lambda_L} - \phi_s \right) \lambda_L^2 Y^2 \langle P_L \rangle = \frac{\langle P_r \rangle_s}{\langle P_L \rangle} \quad (162)$$

$$\nu_{\text{nat}} = a_8 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{-1/4} \left(\frac{\lambda_L Y \langle P_L \rangle}{A} \right)^{1/2} \quad (163)$$

$$d_{\text{max}} = a_9 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{3/4} \left(\frac{A}{\lambda_L Y \langle P_L \rangle} \right)^{1/2} \quad (164)$$

$$\langle P_B \rangle = \frac{1}{b_1} \left(\frac{b_1}{\lambda_L} - \phi_s \right) \lambda_L Y \langle P_L \rangle \quad (165)$$

$$\eta_{\text{BL}} = \frac{1}{b_1} \left(\frac{b_1}{\lambda_L} - \phi_s \right) \lambda_L Y = \left(1 - \frac{\lambda_L \phi_s}{b_1} \right) Y = \frac{\langle P_B \rangle}{\langle P_L \rangle} \quad (166)$$

$$\eta_{\text{rsB}} = a_{10} F_D^2 \lambda_L Y \langle P_L \rangle = \frac{\langle P_r \rangle_s}{\langle P_B \rangle} \quad (167)$$

where the independent parameters are D , λ_L , ϕ_s , Y , $\langle P_L \rangle$, A , and θ .

The corresponding constant coefficients, coefficient definitions, coefficient SI magnitudes, and useful conversion equations are

$$a_6 \equiv \frac{b_2^2}{b_1^2} a_\Omega a_\nu^2 a_d^2 = \frac{e^3}{2^6 \epsilon_0 m_e h^2 c^5} \approx 7.4937 \times 10^6 \quad (168)$$

$$a_7 \equiv \frac{b_2^2}{b_1^2} a_p a_\nu^2 a_d^2 = \frac{\pi e^3}{2^3 3 \epsilon_0 m_e h^2 c^5} \approx 6.2779 \times 10^7 \quad (169)$$

$$a_8 \equiv \frac{a_\nu}{\sqrt{b_1 b_2}} = \left(\frac{e^3}{2^5 m_e \epsilon_0^2 h^2 c^2} \right)^{1/4} \approx 8.2179 \times 10^{10} \quad (170)$$

$$a_9 \equiv \sqrt{b_1 b_2} a_d = \left(\frac{\epsilon_0^2 h^2 c^2}{2 m_e e} \right)^{1/4} \approx 1.8043 \times 10^{-6} \quad (171)$$

$$a_{10} \equiv \frac{b_2^2}{b_1} a_p a_\nu^2 a_d^2 = \frac{\pi e^2}{2^3 3 \epsilon_0 m_e h c^4} \approx 77.836 \quad (172)$$

$$b_1 \equiv \frac{h c}{e} \approx 1.2398 \times 10^{-6} \quad (173)$$

$$b_2 \equiv \left(\frac{2e}{m_e c^2} \right)^{1/2} \approx 1.9784 \times 10^{-3} \quad (174)$$

$$a_\Omega \equiv \frac{1}{2^3 \epsilon_0 c^3} \approx 5.2396 \times 10^{-16} \quad (175)$$

$$a_p \equiv \frac{\pi}{3 \epsilon_0 c^3} \approx 4.3895 \times 10^{-15} \quad (176)$$

$$a_\nu \equiv \left(\frac{e}{2^2 \epsilon_0 m_e c} \right)^{1/2} \approx 4.0700 \times 10^6 \quad (177)$$

$$a_d \equiv \left(\frac{\epsilon_0 m_e c^3}{2^2 e} \right)^{1/2} \approx 18.415 \quad (178)$$

$$a_\nu a_d = \frac{c}{2^2} \approx 7.4948 \times 10^7 \quad (179)$$

Again, the system of equations can be rewritten using ν_{nat} as an independent parameter, instead of $\langle P_L \rangle$, where the lower level coefficients are as before. This rearranged, nonrelativistic, steady-state, small-spot, UV parameter set system of equations is summarized below:

$$\frac{d\langle P \rangle}{d\Omega} = a_{11} F_D^2 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^2 A^2 \nu_{\text{nat}}^4 \sin^2 \theta \quad (180)$$

UV PARAMETER SET EQUATIONS

$$\langle P_\tau \rangle_s = a_{12} F_D^2 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^2 A^2 \nu_{\text{nat}}^4 \quad (181)$$

$$\eta_{\text{rsl}} = a_{13} F_D^2 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{3/2} \lambda_L Y A \nu_{\text{nat}}^2 = \frac{\langle P_\tau \rangle_s}{\langle P_L \rangle} \quad (182)$$

$$\langle P_L \rangle = a_{14} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{1/2} \frac{A \nu_{\text{nat}}^2}{\lambda_L Y} \quad (183)$$

$$d_{\text{max}} = a_{15} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{1/2} \frac{1}{\nu_{\text{nat}}} \quad (184)$$

$$\langle P_B \rangle = \frac{1}{b_1} \left(\frac{b_1}{\lambda_L} - \phi_s \right) \lambda_L Y \langle P_L \rangle \quad (185)$$

$$\eta_{\text{BL}} = \frac{1}{b_1} \left(\frac{b_1}{\lambda_L} - \phi_s \right) \lambda_L Y = \left(1 - \frac{\lambda_L \phi_s}{b_1} \right) Y = \frac{\langle P_B \rangle}{\langle P_L \rangle} \quad (186)$$

$$\eta_{\text{rSB}} = a_{16} F_D^2 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{1/2} A \nu_{\text{nat}}^2 = \frac{\langle P_\tau \rangle_s}{\langle P_B \rangle} \quad (187)$$

where the independent parameters are now D , λ_L , ϕ_s , Y , ν_{nat} , A , and θ .

Lastly, the corresponding constant coefficients, coefficient definitions, and coefficient SI magnitudes are

$$a_{11} \equiv \frac{a_6}{a_8^4} = \frac{b_2^4 a_\Omega a_d^2}{a_\nu^2} = \frac{\epsilon_0}{2c^3} \approx 1.6431 \times 10^{-37} \quad (188)$$

$$a_{12} \equiv \frac{a_7}{a_8^4} = \frac{b_2^4 a_p a_d^2}{a_\nu^2} = \frac{2^2 \pi \epsilon_0}{3c^3} \approx 1.3765 \times 10^{-36} \quad (189)$$

$$a_{13} \equiv \frac{a_7}{a_8^2} = \frac{b_2^3 a_p a_d^2}{b_1} = \left(\frac{\pi^2 \epsilon^3}{32^2 m_e h^2 c^8} \right)^{1/2} \approx 9.2960 \times 10^{-15} \quad (190)$$

$$a_{14} \equiv \frac{1}{a_8^2} = \frac{b_1 b_2}{a_\nu^2} = \left(\frac{2^5 m_e \epsilon_0^2 h^2 c^2}{c^3} \right)^{1/2} \approx 1.4807 \times 10^{-22} \quad (191)$$

$$a_{15} \equiv a_8 a_9 = b_2 a_\nu a_d = \left(\frac{e}{23 m_e} \right)^{1/2} \approx 1.4827 \times 10^5 \quad (192)$$

$$a_{16} \equiv \frac{a_{10}}{a_8^2} = b_2^3 a_p a_d^2 = \left(\frac{\pi^2 e}{32^2 m_e c^6} \right)^{1/2} \approx 1.1526 \times 10^{-20} \quad (193)$$

Natural Parameter Set Equations

This section will convert the theory equations from laboratory or UV parameters to natural parameters so that the equations can be easily used and optimized. Previously the PIC parameter theory equations were converted to the laboratory parameter set, and in order to more easily *understand, make predictions with, and optimize* the theory equations, the laboratory parameter theory equations will now be converted to and rewritten in terms of a 'natural' parameter set. Primarily, these natural parameters are so useful because they reduce the number of parameters to analyze, graph, and simultaneously solve and optimize. This results in requiring a minimum number of free parameters to be treated as constants when graphed, so these graphed 'natural' parameter equations can be used to simultaneously 'solve' and optimize the theory system of equations, uniquely in natural parameter space, for most UV parameter relationships. Besides greatly increasing the amount of UV parameter space that can be simultaneously graphed and analyzed, the small natural parameter set also simplifies the analytical use of the theory equations for optimization analysis.

Often, natural parameters will play a key role in later parts of the paper as more topics that help to understand, predict, optimize, and in general use the theory equations are discussed. For instance the next section, which will refine the ability to make valid predictions and to do valid optimization by listing and quantifying the theory's model constraints, will use natural parameters. And eventually **Appendix A** and **Appendix B** will automate the entire process of using the natural parameter equations in two small computer programs that can be used to understand, predict, and optimize the theory equations without violating model or user input constraints.

This section is organized as follows. First this section will present the natural parameter definitions and convert the steady-state, UV parameter set theory equations to the natural parameter set. Then this section will graph the natural parameter theory equations and specifically discuss an example of how the graph can be used to manually predict the light-source-to-generated-radiation power conversion efficiency, η_{rsL} . Finally, the natural parameter equations and graphs will be used to discuss optimization trends. In summary, this section will first convert the UV parameter theory equations to natural parameters, then graph the converted equations and show how to make specific predictions, and finally discuss optimization trends.

NATURAL PARAMETER SET EQUATIONS

Conversion to Natural Parameters

This subsection begins with the definition of the natural parameters for the theory equations. Perhaps the easiest way of thinking about the **natural parameters** is to imagine that they are a carefully selected set of 'mega-parameters' chosen to represent logical and practical UV parameter groupings that create a largely decoupled set of 2-D, y versus x equations that must be solved simultaneously subject to user and model constraints. This set of 2-D graphs can then be plotted as a multiscale nomogram, which when read vertically, solves those parameters simultaneously. As such, the mega-parameters will be defined as x and y_i , where each parameter is itself an equation made up of several UV parameters. For instance, the natural parameter x is a logical grouping of the UV parameters ν_{nat} and A , which combine to determine the diffraction of a radiating spot. This conversion to and use of 'natural' parameters will simplify using the theory equations. The complete natural parameter definitions for the theory equations are:

$$\lambda_L \equiv \lambda_L \quad (194)$$

$$\phi_s \equiv \phi_s \quad (195)$$

$$x \equiv \nu_{\text{nat}}^2 A \quad (196)$$

$$y_1 \equiv Y \langle P_L \rangle \quad (197)$$

$$y_2 \equiv \frac{d\langle P \rangle}{d\Omega} / (F_D^2 \sin^2 \theta) \quad (198)$$

$$y_3 \equiv \langle P_r \rangle_s / F_D^2 \quad (199)$$

$$y_4 \equiv \eta_{\text{rsL}} / (F_D^2 Y) \quad (200)$$

where $\theta = \theta_{\text{optimum}} = 90^\circ$ for maximum $d\langle P \rangle / d\Omega$ versus θ , and where $\nu_{\text{mod}} = \nu_{\text{nat}}$ for steady-state, 'resonance' y_2 , y_3 , and y_4 .

Substituting the above definitions into some of the more important summarized theory UV parameter set equations, and solving for y_i as a function of the independent parameters λ_L , ϕ_s , and x results in the following natural parameter equations, where all of the constant coefficients and coefficient definitions were previously defined.

$$y_1 = a_{14} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{1/2} \frac{1}{\lambda_L} x \quad (201)$$

Natural Parameter Graphs

$$y_2 = a_{11} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^2 x^2 \quad (202)$$

$$y_3 = a_{12} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^2 x^2 = \frac{2^3 \pi}{3} y_2 \quad (203)$$

$$y_4 = a_{13} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{3/2} \lambda_L x = \frac{y_3}{y_1} \quad (204)$$

By replacing the coefficients with their SI numerical values, the theory equations converted to natural parameters finally become,

$$y_1 \approx 1.4807 \times 10^{-22} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{1/2} \frac{1}{\lambda_L} x \quad (205)$$

$$y_2 \approx 1.6431 \times 10^{-37} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^2 x^2 \quad (206)$$

$$y_3 \approx 1.3765 \times 10^{-36} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^2 x^2 \quad (207)$$

$$y_4 \approx 9.2960 \times 10^{-15} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{3/2} \lambda_L x \quad (208)$$

Natural Parameter Graphs

This subsection will present and discuss the theory equations as displayed in the form of natural parameter graphs. These graphs will be constructed for two cases, where two of the natural parameters shown in Table 1 are set constant for the graphs, while the x and y_i parameters can vary. These cases

Table 1: Two Cases used in Natural Parameter Graphs

Case Number	Case eV_b	λ_L	ϕ_s	x	y_i
Case 1	1.3305	532.0×10^{-9}	1.0	variable	variable
Case 2	0.46106	266.0×10^{-9}	4.2	variable	variable

explore the theory predictions for the case of a very optimistic metal (case 1) and for the case of a typical real metal (case 2). So while the graphs will only explore two cases, those two cases will allow the reader to span an interesting parameter range when using the graphs.

NATURAL PARAMETER SET EQUATIONS

The natural parameter equations for case 1 reduce to:

$$y_1 = 3.2106 \times 10^{-16} x \quad (209)$$

$$y_2 = 2.9086 \times 10^{-37} x^2 \quad (210)$$

$$y_3 = 2.4367 \times 10^{-36} x^2 \quad (211)$$

$$y_4 = 7.5901 \times 10^{-21} x \quad (212)$$

and the natural parameter equations for case 2 are:

$$y_1 = 3.7799 \times 10^{-16} x \quad (213)$$

$$y_2 = 3.4928 \times 10^{-38} x^2 \quad (214)$$

$$y_3 = 2.9261 \times 10^{-37} x^2 \quad (215)$$

$$y_4 = 7.7413 \times 10^{-22} x \quad (216)$$

Both of these cases are graphed in Fig. 4. For dynamic range and simplicity, the graph uses logarithmic scales. Note that the graph in Fig. 4 has the form of a set of scales representing the dependent parameters y_i all plotted versus a single x scale. However, a graphical representation of a complete set of simultaneous equations provides a *unique mapping among all of the natural parameters* shown, and the independent and dependent parameters can be chosen arbitrarily by the graph's user. Therefore the natural parameter equations, when graphed simultaneously, allow researchers using the graph to select from the natural parameters varied on the graph, a set of independent variables that suits their needs. Of course, λ_L and ϕ_s , which are set constant, are not shown and cannot be varied when using the Fig. 4 nomogram.

For the models used, the graph in Fig. 4 displays the optimum, or maximum, purely photoelectron-generated radiation and radiation conversion efficiencies that can be achieved for the two cases because:

1. The simple theory was developed for monoenergetic, perpendicular photoelectron emission, simple yield and work functions, perfect conductors in perfect vacuum, and a single sinusoidal operating frequency which together should provide an estimate of the upper limit on the maximum radiation that it is possible to generate from a pure photoelectric effect.
2. The graph in Fig. 4 solves the simultaneous equations for the resonance condition, $\nu_{\text{mod}} = \nu_{\text{nat}}$, thereby using the approximately optimal, single

Natural Parameter Graphs

Figure 4: Theory natural parameter nomogram for small spots and steady state showing the relationship among the natural parameters and model constraints.

NATURAL PARAMETER SET EQUATIONS

best frequency for modulation that the photoemission electron sheath and dynamics modeled naturally support, to consequently predict the approximate *maximum steady-state* radiation and conversion efficiency possible with respect to modulation frequency.

3. Setting $\theta = \theta_{\text{opt}} = 90$ degrees optimizes the small-spot $d\langle P \rangle / d\Omega$ with respect to θ .
4. The graph not only allows optimization of a typical case, but also allows optimization of an *optimistic* case as well.

As such, in addition to providing predictions and optimizing parameter selections, the graph can also be used to provide approximate upper bounds on both the maximum radiation and total light-to-generated radiation conversion efficiencies that can reasonably be achieved for a nonrelativistic, small-spot problem having a steady-state, 'resonance' pulse train and low-energy (less than 5.0 eV) photons.

So, Fig. 4 presents a lot of information as a concise, multidimensional, multiscale nomogram that is read vertically. As mentioned previously, the nomogram provides a unique mapping among the natural parameters. Constraints for constant E_z fields, and for constructive interference based on the small-spot approximation, are both developed in the next section, and have been included on the theory nomogram. These constraints limit the dynamic range, in terms of x on the graph, to approximately $10^{10.97} \leq x \leq 10^{15.64}$ for both cases. All natural parameter selections within this approximate range are valid choices for the theory models and resulting theory equations.

To use the graph to make specific predictions, one selects any one of the natural parameters for a specific case, and reads off the corresponding values of the other natural parameters that lie in a vertical line intersecting the initial selected natural parameter. For example, for case 1, selecting $y_1 = 10^0$ requires setting $x \approx 10^{15.5}$ to get $y_4 \approx 10^{-4.6}$ for optimum radiation at the selected y_1 value. In the case where you have both an x and a y_1 value, and you want the most radiation or best efficiency, you must select the largest value of either x or y_1 that allows both natural parameters to take on permissible values. In other words, if both parameters are not on a vertical line, then both parameters cannot be simultaneously optimized under mathematical programming, and one parameter must be selected at a lesser value. *Remember, the scales should only be read vertically.* However,

Natural Parameter Graphs

if the two values initially line up vertically, then the radiation is optimized with respect to both quantities without wasting excess parameter capability in either natural parameter.

Once you know the values for the natural parameters, it is easy to convert them to UV parameters. For the previous example, since $y_1 \equiv Y\langle P_L \rangle$, letting $Y = 0.1$ requires $\langle P_L \rangle = 10.0$ watts. Then any desired $\nu_{\text{mod}} = \nu_{\text{nat}}$ and A can be selected as long as $x \equiv \nu_{\text{nat}}^2 A \approx 10^{15.5}$. For example, if one wanted $\nu_{\text{nat}} = 2.4 \times 10^9$ hertz, then that would require a spot radius of $r \approx 0.0132$ meters. Figure 5 makes selecting appropriate ν_{nat} and r values simple and has been included in this paper to simplify working with the x parameter in Fig. 4. Figure 5 is a graph showing the relationship among x , ν_{nat} , and r , where r is the illuminated spot radius on the conducting surface. On the graph, r is expressed in meters, and ν_{nat} is expressed as hertz. By simply picking values for two of the three parameters, the value of the other parameter is determined. Or as in the above example, if a certain value of x is required, then combinations of the ν_{nat} and r variables can be easily scanned to help the user pick the most suitable combination of ν_{nat} and r . Note that the constraints shown on Fig. 4 are also shown on Fig. 5 so that valid parameters can be easily selected when working with both graphs.

For the example being used, once x and y_1 are correctly selected and converted, then other corresponding radiation quantities can also be 'decoded' to UV parameters. For the example being used, $y_4 \approx 10^{-4.6}$, and is converted by using the definition of y_4 listed on Fig. 4 and shown here as,

$$y_4 \equiv \frac{\eta_{\text{rsL}}}{F_D^2 Y} \quad (217)$$

rearranging to find the value of η_{rsL} gives,

$$\eta_{\text{rsL}} = F_D^2 Y y_4 \quad (218)$$

and by using $Y = 0.1$ from before, and $D = 1.0$ and $F_D = D$, then $\eta_{\text{rsL}} \approx 10^{-5.6}$ at 100% pulse train modulation. The example that has been used was picked for a purpose: it is close to an optimized example of the maximum light-to-generated radiation conversion efficiency predicted by the theory equations for the optimistic case 1, as calculated at $y_1 = 10^0$, or $x \approx 10^{15.5}$, near the upper small-spot constraint limit of the theory model. In the next section, after the model constraints are presented and analyzed,

NATURAL PARAMETER SET EQUATIONS

Graph of the relationship among x , ν_{nat} , and r . Also displayed on the graph are scales showing the effect of model constraints x_{10} and x_{12} in selecting valid choices of x , ν_{nat} , and r .

Optimization Trends

a more precise and detailed example of the maximum small-spot radiation possible for the optimistic case 1, subject to the model constraints and using these ideal theory equations, will be presented as an upper limit on the maximum small-spot radiation that is possible for the low-energy (less than 5.0 eV) photons used in the two graphed cases. But for now, the last subsection of this section will next explore the optimization trends inherent in the theory system of equations.

Optimization Trends

To optimize the theory equations for maximum radiation and efficiency requires the user to understand and exploit the trends inherent in the *combined scaling laws for the entire system of equations* as seen in the natural parameter equations and, partially, in Fig. 4. Consequently, besides making specific predictions, Fig. 4 also allows the user to understand and optimize the theory system of equations for specific cases of constant λ_L and ϕ_s by showing many of the trends in the combined scaling laws. This understanding of the trends in the combined scaling laws helps the user to optimize parameters of interest. For instance, the Fig. 4 nomogram shows that for a fixed case having constant λ_L and ϕ_s , as x and y_1 increase, then y_2 , y_3 , and y_4 also increase. In other words, the maximum possible radiation and radiation conversion efficiency occur by maximizing the values possible for ν_{nat} , A , Y , and $\langle P_L \rangle$ that make up x and y_1 . In addition, from the natural parameter equations it is seen that the values possible for λ_L and ϕ_s need to be minimized to maximize the possible small-spot photoelectron-generated radiated power and the total light-to-generated radiation conversion efficiency. Decreasing λ_L and ϕ_s also cause y_1 to increase, but this should not pose a problem when using typical metals and photons having energies less than 15 to 20 eV. Also, one should set $\theta = \theta_{\text{opt}} = 90$ degrees for maximum $d\langle P \rangle/d\Omega$. And both the modulation function F_D and its parameter D should be set to 100% to maximize radiated power and efficiency. So for the example nomogram, with λ_L and ϕ_s held constant, $y_4 \propto x$ and $y_1 \propto x$, both increasing linearly with respect to the natural parameter x , and optimization is easy to understand. Of course model and experimental constraints, as well as other physics and environmental considerations, would need to be considered and traded-off in selecting an optimal set of *useful experimental* parameters. In summary, for Fig. 4 and its equations, increasing y_1 , y_2 , y_3 , and y_4 can be accomplished by

NATURAL PARAMETER SET EQUATIONS

increasing x , decreasing λ_L , and decreasing ϕ_s .

Similarly, the same natural parameter equations used for the nomogram can be rearranged in terms of new independent parameters, such as λ_L , ϕ_s , and y_1 to see new relationships. Of course a graph in hyperspace that simultaneously varies *all* of the parameters is unique, but rearranging the equations can show new information when such a complex graph of all relationships is not possible. If, for the system of newly rearranged equations, λ_L and ϕ_s are then held constant, then $y_4 \propto y_1$ and $x \propto y_1$.

But when the optimization of y_4 is performed with respect to either λ_L or ϕ_s , the optimization becomes more complicated. With respect to ϕ_s , y_4 increases as ϕ_s decreases, and when λ_L and x are held constant then y_1 also increases as ϕ_s is decreased, but when λ_L and y_1 are held constant then x decreases as ϕ_s is decreased. Last and most interesting is how y_4 is optimized with respect to λ_L and has two behaviors depending on which other natural parameters are held constant. First, when ϕ_s and x are held constant, then both y_4 and y_1 increase as λ_L is decreased. As an example, one could hold ϕ_s , $x \equiv \nu_{\text{nat}}^2 A$, A , and Y constant, while decreasing λ_L to increase the emitted photoelectron energy, thereby requiring $\langle P_L \rangle$ to be increased so that ν_{nat} and t_{max} remain constant as the conversion efficiency increases. Secondly, and unlike most of the previous optimizations that called for continuous increases and decreases in various natural parameters to increase y_4 , when ϕ_s and y_1 are held constant and y_4 is optimized with respect to λ_L , it is seen that y_4 has a global maximum, resulting in y_4 , λ_L , and x each having an optimum value. This global maximum exists because the equation used for maximizing y_4 , holding ϕ_s and y_1 constant, is a parabola with respect to λ_L . Lastly, this global y_4 maximum is potentially useful because it occurs at an optimum λ_L that always produces photoemission.

The primary equations describing the *optimum theoretical values* of λ_L , x , and y_4 when ϕ_s and y_1 are held constant and y_4 is maximized with respect to λ_L are

$$\lambda_L = \frac{b_1}{2} \frac{1}{\phi_s} \quad (219)$$

$$x = \frac{b_1}{2a_{14}} \frac{y_1}{\phi_s^{3/2}} \quad (220)$$

$$y_4 = \frac{b_1^2 a_{13}}{4a_{14}} \frac{y_1}{\phi_s} \quad (221)$$

Optimization Trends

which can be derived directly by using the rearranged system of equations having independent parameters λ_L , ϕ_s , and y_1 . Another way to derive the above results involves using the rearranged system of equations but letting $\lambda_L \equiv f_\lambda b_1 / \phi_s$, where for finite photoelectron emission to occur, $0 < f_\lambda < 1$. In the latter case, the equation for y_4 can then be written as $y_4 = a_7 b_1^2 (f_\lambda - f_\lambda^2) y_1 / \phi_s$, and the y_4 maximum with respect to f_λ occurs at $f_\lambda = 1/2$, resulting in an optimum $\lambda_L = b_1 / (2\phi_s)$, as previously listed.

The maximum y_4 values, using the optimum λ_L condition that was just derived, are obtained by using λ_L photon energies that are twice as large as the conductor's work function and that cause a model photoelectron to be emitted with an energy level that is equal to the value of the conductor's work function. For instance, a value of $\phi = 1.0$ eV would require $h\nu_L = 2.0$ eV (or 620 nm) and would emit a 1.0 eV idealized photoelectron. And a value of $\phi = 4.2$ eV would require $h\nu_L = 8.4$ eV (or 148 nm) and would emit a 4.2 eV idealized photoelectron. Since many typical metals have work functions ranging from approximately 4.0 to 6.0 eV, the theory indicates that it would be best to use optimum λ_L values ranging from approximately 155 to 100 nm respectively. Of course, other factors may diminish the value of generating and using these higher energy, UV wavelengths. So when doing *general* optimization, using the optimum $\lambda_L = b_1 / (2\phi_s)$ for a given ϕ_s can be investigated *but may not generally result in any practical advantage*, especially if the optimum λ_L is itself impractical or when enforced constraints and the general optimization process act to find feasible solutions where $f_\lambda \neq 1/2$.

Some of the major *optimization trends* based upon the theory system of equations were presented, but *until model constraints and user constraints are defined and imposed, which of the trends dominates for a given set of constraints cannot be known for the general optimization process, nor can the feasible optimum solution set of natural parameters be determined*. Therefore, the various model constraints will be explored, quantified, and discussed in more detail in the next section. So how does this discussion of optimization trends affect using the Fig. 4 nomogram and what do the Fig. 4 nomogram results mean? Figure 4 is a *special case of the general optimization process*, since it sets λ_L and ϕ_s to *constant values* in the equations. Although the model constraints were not fully discussed in this section, Fig. 4 was drawn for cases producing nonrelativistic photoemission and also includes scales for two of the most limiting model constraints, so that when using the nomogram, the user can avoid parameters for which the theory models and

NATURAL PARAMETER SET EQUATIONS

equations are invalid. In general, if a Fig. 4 type nomogram is constructed where the constant values for λ_L and ϕ_s are chosen to produce nonrelativistic photoemission, and where the primary model constraints are drawn on the graph and are not subsequently violated, the nomogram will allow the user to find optimal feasible solutions for the special case where λ_L and ϕ_s are set constant for the experiment. So whether making specific predictions or optimizing various parameters under different constraints, Fig. 4 and the natural parameter equations it is based on both provide an easy way to understand, use, and explore the optimization trends for the nonrelativistic, small-spot, sinusoidal, steady-state, 'resonance' pulse train equations. Further, the specific optimization process discussed for Fig. 4 provides a foundation for building a general optimization program that varies all of the λ_L , ϕ_s , x , and y_1 inputs when searching for a maximum cost function result, such as y_4 , while enforcing the model and user constraints. It is only in this general optimization process that all of the optimization trends or scaling law behaviors inherent in the full theory system of equations, as seen through various rearrangements of the theory system of equations, and which would create a unique graph in hyperspace, can potentially express themselves subject to the constraints determining feasible regions of the hyperspace.

Consequently, general optimization or general mathematical programming would find that to maximize y_4 subject to varying constraints, where λ_L is always allowed to vary, that sometimes the derived 'optimum' λ_L would maximize y_4 , while for other constraints, values of λ_L less than or greater than the derived optimum λ_L , between and including the minimum or maximum constrained range values imposed on λ_L , would produce the maximum y_4 . In short, λ_L or any of the other natural parameters *could take on practically any value* in a generalized search for a maximized y_4 when subjected to various sets of constraint conditions. Therefore, the reader is *strongly encouraged* to run the program in **Appendix B** to determine actual general maximized y_4 for the parameters when model and experimental constraints are to be considered, as the model constraints are enforced by the program, and natural parameter constraints on λ_L , ϕ_s , x , and y_1 can be easily input into the program as well. This program also dumps, as a diagnostic, the optimized value of f_λ , which due to the various constraints, may not optimize to a value of $f_\lambda = 1/2$ even when ϕ_s and y_1 are held constant, let alone when those parameters are allowed to vary as well when searching for an optimum constrained y_4 .

Theory Model Constraints

This section will quantify and analyze the various model assumptions or constraints affecting the simple theory system of equations, as these act as additional constraint equations in the system of equations that have been developed. Without these constraints, it is difficult to say where the equations and predictions are valid and where they break down. Consequently, *before the theory can be used* to make predictions or to maximize radiation, the model limitations must be known, quantified, and not violated. On a computer, these constraints could all be easily enforced during numerical optimization. But for understanding and manual optimization, analytic or graphed constraints will be needed. Therefore this section will list the model approximations, from excellent to poor, and quantify the worst of these constraints. The most limiting of these constraints have already been included on both Figs. 4 and 5 that were presented in the previous section.

The approximations selected support the goal of understanding the physics of the problem as simply as possible by trying to select and model the minimal, important physics that can reproduce computational or laboratory experimental results. So if the model and experimental constraints are known, valid theory predictions can be made and tested versus PIC data and experiments. In summary, constraints are needed to make valid theory predictions and for optimization purposes, and PIC code simulation and experimental data are needed to test the valid theory predictions.

List of Model Constraints

Model constraints are important to consider whenever approximations or assumptions are introduced into models to make a theory analytically tractable. These model limitations impose additional constraint equations into the total solution matrix. Only by understanding and quantifying these constraints, on a level *appropriate* to the quality of the rest of the theory equations, can the limits of the theory be quantified. However, even rough estimates are better than no estimates.

Accordingly, this subsection will prioritize and list most of the model approximations and assumptions used in this paper. These approximations are listed starting with those that are almost always satisfied and ending with those that cannot be ignored without risk. Those last constraints were

THEORY MODEL CONSTRAINTS

graphed as appropriate so that someone using the theory can adhere to the constraints, thereby making correct use of and valid predictions with, the simple theory equations and theory nomogram. The model assumptions are listed below.

1. Nonrelativistic photoelectron motion. First, nonrelativistic motion is an excellent assumption allowing classical analysis. The approximate upper limit on the electron velocity during its lifetime is its emitted photoelectron velocity, v_e . Basically then,

$$\beta \equiv \frac{v_e}{c} \leq 0.1 \quad \text{or} \quad eV_b \leq 2600 \text{ electron volts} \quad (222)$$

which would require using photon lengths of $\lambda_L < 1.0$ nm and shorter, in the X-ray and higher energy regions of the electromagnetic spectrum, to violate this condition. Further, even $\beta \leq 0.3$, for $eV_b \leq 25000$ electron volts, could be used with less than a 4% error where compared with calculating β relativistically. So using photons from the visible and UV portion of the electromagnetic spectrum will not violate this model assumption, regardless of the conductor's work function. Secondly, since photoemission is assumed to occur such that a photoelectron with nonzero velocity is assumed to be emitted, the lower limits on β and eV_b are $\beta > 0$ and $eV_b > 0$, and these lower limits must be enforced to generate radiation using the theory equations.

2. $\eta_{rsL} \ll 1$, so the theory can ignore radiation damping affecting the electron motion. Approximately,

$$x_2 \ll \frac{1.0758 \times 10^{14}}{F_D^2(b_1/\lambda_L - \phi_s)^{3/2} \lambda_L Y} \quad (223)$$

3. $\langle P_r \rangle_s \ll \langle P_B \rangle$, so the generated radiation does not significantly affect the electron trajectories and energies. Similar to constraint 2, and an excellent assumption for small spots. Approximately,

$$x_3 \ll \frac{8.6767 \times 10^{19}}{F_D^2(b_1/\lambda_L - \phi_s)^{1/2}} \quad (224)$$

4. A perfectly conducting metal photoemission surface and image charge are assumed.

List of Model Constraints

5. Only vacuum physics is modeled, with no atmospheric or gas effects.
6. Only non-plasma forming parameter values that create a purely photoelectron space charge limited electron sheath above the conducting surface are assumed. This is usually a good assumption for low-power UV light sources and relatively large illuminated spot areas.
7. Monoenergetic, perpendicularly emitted electrons having a natural resonance frequency are assumed. Even angular emission would still have perpendicular components. Also, a *maximum radiation analysis* such as this theory does would still look at 100% perpendicular photoemission off of the conducting surface so as to calculate an *approximate upper bound on the maximum generated radiation*. As for monoenergetic and resonance, these assumptions work together. However, even if there is no characteristic resonance function, then a larger range of useful ν_{mod} frequencies can be used without severe resonance constraints. In effect, with relaxed natural frequency limits that hardly reduce the optimum generated radiation, any ν_{mod} used may then be considered as ν_{nat} in these theory equations. Note that $\nu_{\text{rf}} = \nu_{\text{mod}}$, as seen from the PIC code simulations [Ref. 1], even when $\nu_{\text{mod}} \neq \nu_{\text{nat}}$ for nonresonance PIC simulations. Also, the PIC simulations self-oscillate at ν_{nat} if they are not driven by an external ν_{mod} .
8. The $j(t)$ current density source can be modeled well as a 'point' charge current source (see constraints 9, 11, and 12) with a single, dominant sinusoidal frequency driven at ν_{mod} .
9. The distance d_{max} of the 'point' charge source is basically limited as $d_{\text{max}} \ll \lambda_{\text{rf}} \ll R$ because $\rho \ll \lambda_{\text{rf}} \ll R$ was used to develop the theory system of equations. Since $\lambda_{\text{rf}} = \lambda_{\text{mod}}$, and for this steady-state theory $\lambda_{\text{mod}} = \lambda_{\text{nat}}$, then

$$d_{\text{max}} \ll \frac{c}{\nu_{\text{nat}}} \quad (225)$$
10. Constant E_z fields and forces are assumed, due to the 'infinite', thin, rigid, charge sheets used in the model to calculate t_{max} and d_{max} . This model assumption is sometimes referred to as the 'pancake' approximation, and it places a limit on how small the spot size can get (as do

THEORY MODEL CONSTRAINTS

vaporization and plasma formation considerations). Therefore,

$$r_{10} \geq d_{\max} \quad (226)$$

11. Coherence of the moving electrons with respect to each other is assumed. So all electron charge layers, assumed to be uniform in the $x - y$ plane, should see the same electric fields and forces at a given distance z above the conducting surface. The requirement that coherent electrons should experience the same electric fields can be used as an approximate measure of the expected coherence length of the in-phase spot size of oscillating charge layers. For example, if the constraint is violated for a self-oscillating injected constant electron beam, then out of phase spots, or areas, of oscillating charge could form.

$$r_{11} \leq \frac{d_{\max}}{\beta} \quad (227)$$

12. Lastly, similar to constraint 9 for $d_{\max}$, a radial constraint on the spot size was also assumed so as to ignore retarded time effects. This constraint comes from the constraint $\rho \ll \lambda_{\text{mod}} \ll R$ that was used in deriving the radiation equations. Substituting $2r$ for ρ gives the approximate constraint

$$r_{12} \ll \frac{c}{2\nu_{\text{nat}}} \quad (228)$$

However, constraints 9 and 12 are very roughly expressed above and can be better approximated by realizing that the significance of having no retarded time effects from a small-spot source is that these constraints imply no substantial far-field phase differences to sum from a finite sized current source that generates radiation. Therefore, the maximum phase difference between two points of finite extent in z and r that an observer in the far-field would see is constrained to avoid destructive interference.

In keeping with the simplicity of the theory itself, approximate constraints based on the maximum phase difference in radians, between two planar, parallel, sinusoidal electric field point sources having the same frequency and phase offset, will be developed where

$$\delta_{\max} = k\Delta x_{\max} \quad (229)$$

$$k \equiv \frac{2\pi\nu_{\text{nat}}}{c} \quad (230)$$

List of Model Constraints

Here, $\Delta x_{\max}$ represents the maximum path length difference the radiated fields travel from any two source point emitters. As an approximation for constraint 9, $\Delta x_{\max} = d_{\max}$ for a one-dimensional model in z at $\theta = 0$ degrees. And as an approximation for constraint 12, $\Delta x_{\max} = 2r_{\max}$ for a one-dimensional model in r at $\theta = 90$ degrees.

Further, the maximum phase difference allowed for no significant retarded time source effects can be defined as

$$\delta_{\max} = \frac{\pi}{3} \quad \text{to} \quad \frac{\pi}{2} \quad \text{radians} \quad (231)$$

where $\delta_{\max}$ is an arbitrary number that is to be picked as large as possible without causing *substantial* destructive interference. Of course, at $\delta_{\max} = \pi$, two point sources completely destructively interfere, forming the first completed destructive interference lobe's null. From observation, $\delta_{\max}$ at $\pi/3$ to $\pi/2$ for two waves is not too destructive, and since *most* of the source points lie *inside* $\Delta x_{\max}$, these limits on $\delta_{\max}$ are quite acceptable.

Solving for $\Delta x_{\max}$ gives

$$\Delta x_{\max} = \frac{\delta_{\max} c}{2\pi \nu_{\text{nat}}} \quad (232)$$

using this definition, and the definitions of $d_{\max}$ and ν_{nat} , it is seen that for constraint 9 where $\Delta x_{\max} = d_{\max}$

$$\delta_{\max} = \frac{\pi}{2} \beta \quad \text{classically} \quad (233)$$

and that constraint 9 is always satisfied for nonrelativistic electron beam emission velocities, since $\delta_{\max} \leq 0.15\pi$ for $\beta \leq \beta_{\max} = 0.3$, and so $\delta_{\max}$ is *always* less than the maximum $\pi/3$ to $\pi/2$ values that were previously found to be acceptable.

Similarly, constraint 12 can finally be written as

$$r_{12} \leq \frac{\delta_{\max} c}{2^2 \pi \nu_{\text{nat}}} \quad (234)$$

and this completes the list of model constraints.

THEORY MODEL CONSTRAINTS

Analysis of Model Constraints

The last several pages listed various model constraints. Of these, the last three are probably the most easily violated when using the theory equations, and so will be examined further. The last three constraints are:

$$r_{10} \geq d_{\max} \quad (235)$$

$$r_{11} \leq \frac{d_{\max}}{\beta} \quad (236)$$

$$r_{12} \leq \frac{\delta_{\max} c}{4\pi} \frac{1}{\nu_{\text{nat}}} \quad (237)$$

where r_{10} enforces the 'pancake' approximation for the fields and forces involved, r_{11} is related to particle communication and coherence with other particles, and r_{12} is the constraint on the source's spot radius for no substantial destructive interference of the generated far-field radiation.

Substituting the $d_{\max}$ and ν_{nat} equations, and rewriting all spot radius constraints as a function of ν_{nat} gives

$$r_{10} \geq \frac{c}{4} \frac{\beta}{\nu_{\text{nat}}} = \frac{1}{4} \frac{v_e}{\nu_{\text{nat}}} \quad (238)$$

$$r_{11} \leq \frac{c}{4} \frac{1}{\nu_{\text{nat}}} \quad (239)$$

$$r_{12} \leq \frac{c}{4\pi} \frac{\delta_{\max}}{\nu_{\text{nat}}} \quad (240)$$

Note that r_{11} could also have been defined as

$$r_{\max} = ct_{\max} \quad \text{with} \quad (241)$$

$$t_{\max} = \frac{1}{2\nu_{\text{nat}}} \quad \text{so} \quad (242)$$

$$r_{11} < \frac{c}{2} \frac{1}{\nu_{\text{nat}}} \quad (243)$$

From these constraints, it is seen that a small spot is necessary for no significant destructive interference from a coherent in-phase current source, but that the spot cannot be too small or the fields and forces no longer approximate those of an infinite charge sheet in two dimensions.

Analysis of Model Constraints

Rewriting the constraints in terms of the natural variable x generates the primary natural parameter constraint equations

$$x_{10} \geq \frac{\pi c^2}{16} \beta^2 \quad \text{where} \quad \beta \equiv \frac{v_e}{c} \quad (244)$$

$$x_{11} \leq \frac{\pi c^2}{16} \quad \text{or} \quad x_{11} < \frac{\pi c^2}{4} \quad (245)$$

$$x_{12} \leq \frac{c^2}{16\pi} \delta_{\max}^2 \quad (246)$$

Since constraint x_{11} is less strict than constraint x_{12} , and since constraint x_{11} is a crude constraint that is not really violated for the modeled beam emission, it will not be considered further. This leaves x_{10} and x_{12} as the primary constraints, where eV_b can be used instead of β in writing constraint x_{10} . Finally, *the two primary constraints that affect using the theory equations are listed below:*

$$x_{10} \geq \frac{\pi}{8m_e} |e| eV_b \equiv x_{\min} \quad (247)$$

$$x_{12} \leq \frac{c^2}{16\pi} \delta_{\max}^2 \equiv x_{\max} \quad \text{where} \quad (248)$$

$$eV_b \equiv \frac{e}{|e|} \left(\frac{b_1}{\lambda_L} - \phi_s \right) \quad (249)$$

And as these equations are for nonrelativistic eV_b as set by constraint 1, eV_b will generally be limited as $0.0 < eV_b \leq 2600$ electron volts so that β is limited to nonrelativistic values of $0.0 < \beta \leq 0.1$ (or potentially $0.0 < \beta \leq 0.3$ at most). This upper limit on eV_b sets an upper limit on $x_{\min}$. Where these constraints were graphed, $x_{\min}$ was shown in terms of eV_b , and $x_{\max}$ in terms of $\delta_{\max}$. As stated previously, $\delta_{\max}$ is the maximum path difference two ν_{nat} wavelengths travel from a circular spot, so $\delta_{\max} = \pi$ would cause destructive interference and those two waves would cancel, while $\delta_{\max}$ at $\pi/3$ to $\pi/2$ for two waves is not too destructive, especially since most of the source points lie inside $\Delta x_{\max}$. Consequently, $\delta_{\max} \leq \pi/2$ will generally be considered an acceptable upper limit for no substantial destructive interference from a radiating spot. Finally, valid choices of x must now satisfy the constraint equation

$$x_{\min} \leq x \leq x_{\max} \quad (250)$$

THEORY MODEL CONSTRAINTS

and it is assumed that the choices of λ_L and ϕ_s are also valid with respect to satisfying the photoemission condition:

$$\frac{b_1}{\lambda_L} - \phi_s > 0 \quad (251)$$

Optimization with Model Constraints

The Fig. 4 and Fig. 5 graphs both incorporate the model constraints to aid in selecting feasible natural parameters. Some of the model constraints are inherent in the developed system of equations, one is enforced by selecting appropriate λ_L and ϕ_s constants, some are 'primary' constraints and are graphed on the figures, and one is enforced after the feasible natural parameters are determined. Here is the specific breakdown of how the model constraints have been incorporated into Figs. 4 and 5. The 'maximal analysis', ideal model constraints 4, 5, 7, and 8 are inherent in the system of theory equations that were developed. Appropriate λ_L and ϕ_s values were selected for both cases that satisfy model constraint 1 which requires non-relativistic photoemission. Both Figs. 4 and 5 graph the 'primary' model constraints 10 and 12, so that the user can avoid violating those constraints. Effectively, enforcing the primary or more severe model constraints 1, 10, and 12 also enforces model constraints 2, 3, 9, and 11 as well. Therefore, only model constraint 6, requiring no plasma formation, is left and needs to be enforced by the user *when applying* the feasible x and y_1 solutions by selecting appropriate values for A and $\langle P_L \rangle$.

Let us examine the primary x_{10} and x_{12} model constraints, which have been graphed and which the user must not violate, in more detail. First, they were included on the Fig. 4 theory nomogram as two scales near the x scale. The x_{10} values of $x_{\min}$ corresponding to $eV_b = 1.3305$ and 0.46 electron volts have been annotated for cases 1 and 2 respectively. For case 1 then, using of $x \geq 10^{10.97}$ and $y_1 \geq 10^{-4.54}$ will not violate the constant E_Z force assumption used in developing the theory equations. Constraint x_{12} has an absolute $x_{\max}$ of no more than $x \approx 10^{15.64}$ to keep from violating the small-spot criteria used in developing the theory equations. So for *both* cases 1 and 2, all values of x satisfying $10^{10.97} \leq x \leq 10^{15.64}$ will produce valid predictions that do not violate the theory models and equations. Secondly, to help use the theory nomogram, the Fig. 5 logarithmic graph of the relationship between x , ν_{nat} , and r was created. This graph helps to convert x values to ν_{nat} and r

Optimization with Model Constraints

values, and vice versa. On the Fig. 5 graph the x_{10} and x_{12} constraints were included to facilitate choosing valid conversions.

It is now a simple matter to use Figs. 4 and 5 for the system of theory equations and constraints to create valid theory predictions of the maximum *small-spot* radiation powers and conversion efficiencies for the two specific cases. Simply determine where the maximum x and y_1 occur that do not violate the model constraints, and use the maximum overlap value. A typical example of this process is worked out below for optimistic case 1. Assume $Y = 0.1$, $0.0 \leq \langle P_L \rangle \leq 1000$ watts, $0.0 \leq r \leq 0.0425$ meters, $\nu_{\text{mod}} = \nu_{\text{nat}}$ for steady-state $\sin^2$ type modulation with $1.0 \leq \nu_{\text{nat}} \leq 6.0$ gigahertz, $F_D = D$ with $D = 100\%$ current modulation, and $\theta = \theta_{\text{opt}} = 90^\circ$ for y_2 . Of course for case 1, $\lambda_L = 532$ nanometers and $\phi_s = 1.0$ volts. From Fig. 5, x varies from 0.0 to $10^{17.3}$ m^2/s^2 . And from the definition of y_1 , y_1 varies from 0.0 to 10^2 watts.

With these numbers, Fig. 4 can then be used to determine the approximate maximum steady-state radiation possible for the given parameters. From the graph it is seen that both the x and y_1 maximums exceed the x_{12} constraint. Consequently, all values of x and y_1 *within* the x_{10} and x_{12} constraints are possible. So for predicting maximum small-spot, steady-state radiation, x is chosen as large as possible, at the x_{12} upper limit of $\delta_{\text{max}} \approx \pi/2$. This produces the maximum small-spot, steady-state radiation possible for the case 1 constants, and occurs at $x \approx 10^{15.64}$.

With this maximum x value now known, all the rest of the corresponding scale values can be read vertically to get $y_1 \approx 10^{0.15}$, $y_2 \approx 10^{-5.25}$, $y_3 \approx 10^{-4.32}$, and $y_4 \approx 10^{-4.48}$. Applying the natural variable definitions as labeled on the Fig. 4 graph results in the following *set of simultaneous equations*:

$$x \equiv \nu_{\text{nat}}^2 A \approx 10^{15.64} \quad (252)$$

$$y_1 \equiv Y \langle P_L \rangle \approx 10^{0.15} \quad (253)$$

$$y_2 \equiv \frac{d\langle P \rangle}{d\Omega} / (F_D^2 \sin^2 \theta) \approx 10^{-5.25} \quad (254)$$

$$y_3 \equiv \frac{\langle P_r \rangle_s}{F_D^2} \approx 10^{-4.32} \quad (255)$$

$$y_4 \equiv \frac{\eta_{\text{rsl}}}{F_D^2 Y} \approx 10^{-4.48} \quad (256)$$

THEORY MODEL CONSTRAINTS

Since $F_D = D = 1.0$, $Y = 0.1$, and $\theta = 90^\circ$, then

$$\nu_{\text{nat}}^2 A \approx 10^{15.64} \quad (257)$$

$$\langle P_L \rangle \approx 10^{1.15} \quad (258)$$

$$\frac{d\langle P \rangle}{d\Omega} \approx 10^{-5.25} \quad (259)$$

$$\langle P_r \rangle_s \approx 10^{-4.32} \quad (260)$$

$$\eta_{\text{rsL}} \approx 10^{-5.48} \quad (261)$$

Finally, any $\nu_{\text{nat}}^2 A \approx 10^{15.64}$ combination can be used. So if you wanted to use 6.0 GHz, a small spot with a 0.62 cm radius would be required having an incident light source average power of $\langle P_L \rangle \approx 14.1$ watts. If a more accurate 1.0 GHz measurement could be made than a 6.0 GHz measurement, then using an $r \approx 3.73$ cm radius spot would generate the 1.0 GHz ν_{nat} frequency using the same required average input power of $\langle P_L \rangle \approx 14.1$ watts. If an arbitrary frequency is set too high, the area required would shrink to a point where a plasma would be generated, violating the theory model. At the other extreme, a very large spot would require a frequency lower than this example problem's 1.0 GHz lower limit. Potentially, if enough radiated power were generated, even a 0.37 m radius spot with $\nu_{\text{mod}} = \nu_{\text{nat}} = 100$ MHz, the same frequency that a hand-held FM radio receives, could be used to pick up the radiated signal in scaling experiments. According to the simple theory, *as long as $x \approx 10^{15.64}$ remains unchanged, the generated radiation and conversion efficiency will not change as $\nu_{\text{mod}} = \nu_{\text{nat}}$ and A are varied.*

Since x_{12} limited the x and y_1 values, the above example represents the maximum *small-spot* steady-state power and efficiency predictions for radiation using case 1 with $D = 100\%$. It should be noted that if both x and y_1 maximums fall within the x_{10} and x_{12} model constraints, then for overlap to occur when drawing a vertical line on the Fig. 4 theory nomogram, the x or y_1 value that gives the smaller predicted radiation must be used. This process is actually a simplified representation of the mathematical programming optimization process that would produce the same results for the theory system of equations and model constraint equations. So from an efficient use of resources viewpoint the x and y_1 maximums should be close since having one be much larger than the other may represent expensive capability of experimental equipment that cannot be utilized to optimally increase the generated radiation. Note that **Appendix A** presents a program that

Optimization with Model Constraints

automates the mathematical programming process when λ_L and ϕ_s are set constant, like using the Fig. 4 nomogram, where the maximum experimental x and y_1 possible are also input into the program. The program then enforces the model and maximum experimental constraints and returns the optimized feasible values of x , y_1 , y_2 , y_3 , and y_4 . **Appendix B** takes this process one step further by allowing the user to set ranges for λ_L , ϕ_s , x , and y_1 to search, while enforcing model constraints, in maximizing y_4 . These programs were designed to be easy to type into a workstation having Fortran and are recommended when doing constrained parameter optimization using the theory equations developed in this paper.

Next, note in this example that using the natural parameter equations for case 1, that both x and y_1 maximum values are close to each other at $x = 10^{17.3}$, where they overlap (the maximum y_1 extends to $x \approx 10^{17.5}$). This overlap point has been circled on the Fig. 5 graph. While efficiently creating or overlapping the x and y_1 example assets, as discussed above, this x value falls out of the small-spot region, violating model constraint x_{12} . In essence, diffraction lobes begin to form and the maximum small-spot steady steady theory predictions break down. However, the trend toward increasing $d\langle P \rangle / d\Omega$ with increasing x and y_1 should continue past $\delta_{\max} \approx \pi/2$, but the rate of increase in the maximum lobe will likely change. And $\langle P_r \rangle_s$ and η_{rsL} , integrated over θ having diffracted lobes, may continue to increase for a while but at what rate and for how long is unknown and will require new theory and models that accurately represent the physics involved in large spots as well as small spots. So if $x = 10^{17.3}$ were used, significant diffraction effects would occur (approximately 1 or 2 radiation lobes would form with the primary lobe lifting away from the conducting surface located at $\theta = 90$ degrees), but would probably result in even more radiated power and better conversion efficiency as x is still close to the small-spot region. However, this optimistic case is *not* very representative of the case of typical metals illuminated by low-power sources of low-energy (less than 5.0 eV) photons, which are small-spot limited. Therefore, reaching the large-spot region for typical metals and low-power sources of low-energy photons is very difficult, and the idealized optimum generation of radiation via photoelectrons for most typical metals and low-power sources of low-energy photons is described and limited by these nonrelativistic, small-spot, steady-state equations driven at (or near) the idealized monoenergetic, perpendicular "resonance" condition.

Finally, what are the current limitations on, and potential ways of in-

THEORY MODEL CONSTRAINTS

creasing, the generated radiation and radiation conversion efficiency? *One of the worst factors is the low photoelectron yield of typical real metals when exposed to low-energy photoelectrons.* The low Y values directly degrade the amount of efficiency possible, as seen in the expression for the natural variable y_4 , where $y_4 \equiv \eta_{rsL}/(F_D^2 Y)$. Another problem is that the low Y also makes it difficult to maximize the generated radiation and radiation conversion efficiency by making it difficult to reach, let alone surpass, the small-spot constraint x_{12} upper limit for $y_1 \equiv Y \langle P_L \rangle$. Ideally then, to dramatically increase the radiated power and conversion efficiencies for typical real metals, the photon energies of the source must be increased so that Y , and hence the y_4 and y_1 values, can be increased while maintaining reasonable powers. The problem is that generating higher energy photons or using a photon source with higher energy photons may not be practical, and that even if it were practical, the theory itself would not be able to accurately model and optimize the resulting large-spot radiated powers and efficiencies that could be generated. However, the theory can be used up to the small-spot upper limit, and as previously discussed, radiated powers and efficiencies should continue to observe the small-spot theory trends and therefore continue to increase when starting to enter into the large-spot region.

In choosing a value for λ_L , it is useful to ask if there exists an *optimum experimental value* of λ_L that should be exploited to maximize η_{rsL} , the total light-to-generated radiation conversion efficiency? It would appear that there is an optimum range of experimental photon energies for various typical real metals. From published results, it appears that for typical real metals the photoelectron yield values rapidly increase up to a maximum of approximately 0.1 through 0.3 as λ_L reaches 83 to 50 nm, or about 15 to 25 eV photon energies, and then begin to slowly decline. And depending upon the metal (such as stainless steel and aluminum), even 12 to 13 eV photons can generate yield values above 0.1. Interestingly, *the above optimum experimental values for the photon energies that maximize photoelectron yield almost overlap the corresponding optimum theoretical values for the λ_L photon energies that maximize y_4* , where maximizing y_4 essentially maximizes η_{rsL} . The derived optimum theoretical values for λ_L would require photon energies of 8.0 to 12.0 eV for typical real metals having ϕ values ranging from 4.0 to 6.0 eV so as to produce emitted photoelectrons having 4.0 to 6.0 eV energies. Further, the optimum theoretical values for λ_L were derived from the theory equations for *perpendicularly* emitted photoelectrons, so *real*,

Optimization with Model Constraints

non-perpendicularly emitted photoelectrons generated from 12 to 25 eV photons would have perpendicular energy components that are even closer to the optimum theoretical, perpendicularly emitted, photoelectron energy values. Even so, the primary advantage to be gained comes from using the optimum experimental values of λ_L to increase Y , which will then help to increase both y_4 and y_1 . Consequently, the optimum experimental values for the photon energies should be used, along with photon source powers $\langle P_L \rangle$ that cause y_1 to reach (and surpass) the small-spot limit, and by using corresponding values of $x \equiv \nu_{\text{nat}}^2 A$ as well, so that η_{rsL} conversion efficiencies greater than $10^{-5.5}$, as presented in the optimized case 1 example just worked in this subsection, should then be possible for many useful real metals.

Optimizing specific small-spot examples using experimental constraints for photon sources and typical metals to determine the resulting small-spot x , source power requirements, and generated radiation and conversion efficiencies will demonstrate the above optimization trends. Running the program in **Appendix B** for 15 to 25 eV photon sources and typical metals, and using $Y = 0.1$, will also help to demonstrate the above optimization trends. For example, using the **Appendix B** program, and maximizing y_4 for $\lambda_L = 82.66$ nm, $4.2 \leq \phi_s \leq 6.0$ volts, and letting x and y_1 reach the small-spot, x_{max} limit ($10^{10} \leq x \leq 10^{20}$ and $10^1 \leq y_1 \leq 10^{20}$) gives the following rounded, rough results, where $Y = 0.1$ is assumed for the 15 eV photon energies used: $\phi_s = 4.2$ volts, $\langle P_L \rangle \approx 260$ watts, $x = x_{\text{max}} = 10^{15.64}$ m²/s² so $\nu_{\text{mod}} = \nu_{\text{nat}} = 1.0 \times 10^9$ Hz requires an illuminated spot radius of $r \approx 0.037$ m to produce, at $F_D = D = 1.0$, $\langle P_r \rangle_s \approx 3.1 \times 10^{-3}$ watt, and $\eta_{\text{rsL}} \approx 10^{-4.92}$ for a small-spot limited, typical real metal.

Again, if it is experimentally possible, entering the large-spot region should provide even better improvements to the radiated power and conversion efficiency. Entering the large-spot region should not be a completely blind undertaking, as the scaling laws that do not deal with radiated power and conversion efficiency should be less affected by entering the large-spot region, and at least near the small-spot region, the small-spot equations should continue to provide rough estimates for the large-spot radiation trends. The program in **Appendix B**, modified so that $x_{\text{max}} = 10^{17}$, is next used to maximize y_4 and obtain very rough trend estimates for a nearly small-spot limited example. Setting $\lambda_L = 82.66$ nm, $4.2 \leq \phi_s \leq 6.0$ volts, letting x and y_1 reach the new program x_{max} limit ($10^{10} \leq x \leq 10^{20}$ and $10^1 \leq y_1 \leq 10^{20}$), and assuming $F_D = 1.0$ and $Y = 0.1$, a solution is obtained that shows the fol-

THEORY MODEL CONSTRAINTS

lowing trend results: $\phi_s = 4.2$ volts, $\langle P_L \rangle \approx 5900$ watts, $x = 10^{17.00} \text{ m}^2/\text{s}^2$ so $\nu_{\text{mod}} = \nu_{\text{nat}} = 1.0 \times 10^9 \text{ Hz}$ requires an illuminated spot radius of $r \approx 0.18 \text{ m}$ to produce $\langle P_r \rangle_s \approx 1.6$ watts, with an efficiency $\eta_{\text{rsL}} \approx 10^{-3.56}$ *for a nearly small-spot limited, typical real metal*. And since $r \propto 1/\nu_{\text{nat}}$ for a fixed x , if $\nu_{\text{mod}} = \nu_{\text{nat}} = 2.0 \times 10^9 \text{ Hz}$ were used, then an illuminated spot radius of $r \approx 0.089 \text{ m}$ would be required. Depending on the potential power $\langle P_L \rangle$ available from a 15 to 25 eV photon source, the theory seems to indicate that even better photoelectron-induced radiated power and radiation conversion efficiencies should be possible. In summary, using photon energies of 15 to 25 eV and x and y_1 values surpassing the small-spot limit should dramatically improve the photoelectron-generated radiation and conversion efficiency possible for typical real metals.

Generalized Scaling Laws

Eventually, the theory predictions from the preceding sections should be tested and compared with both computer simulation data and laboratory experimental data. In a follow-on paper [Ref. 1], the theory predictions will be compared with numerical Particle-In-Cell (PIC) code computer simulation results. For a variety of good reasons that will be discussed in the follow-on paper, the method used to test the theory equations will involve curve fitting the generalized theory scaling laws to the PIC data to see if the theory values for the constants in the generalized theory scaling laws agree with the values of those constants obtained by curve fitting the generalized theory scaling laws to the PIC data. This method of testing is a general, powerful technique others can use to test the theory scaling laws with both computational and laboratory experimental data. As such, this section will now present the generalized theory scaling laws that can be curve fitted to experimental data in order to compare the theory and data.

The approximate steady-state theory equations for maximum electric field strength component $E_{Z \max}$, maximum electron orbit distance $d_{\max}$, natural or resonance frequency ν_{nat} , and average small-spot, far-field power radiated through a sphere $\langle P_r \rangle_s$, can be written as the following generalized scaling laws

$$E_{Z \max} = a' \beta^{b'} \langle j \rangle^{c'} \quad (262)$$

$$d_{\max} = a' \beta^{b'} \langle j \rangle^{c'} \quad (263)$$

$$\nu_{\text{nat}} = a' \beta^{b'} \langle j \rangle^{c'} \quad (264)$$

$$\langle P_r \rangle_s = a' F_D^{d'} A^{e'} \beta^{b'} \langle j \rangle^{c'} F_R \quad (265)$$

The parameters used in the above equations have the following definitions. F_D is the modulation depth function involving the independent parameter D , where D is the modulation depth of the sinusoidally modulated photoemission current density versus time having modulation frequency ν_{mod} . A is the electron photoemission area on the conducting surface. β represents the emitted electron velocity divided by the speed of light in a vacuum. The parameter $\langle j \rangle$ is the average emitted current density. ν_{nat} is the natural or self-oscillation primary frequency component that spontaneously arises when an unmodulated, constant electron beam having current density $\langle j \rangle$ and beam energy eV_b is injected, perpendicular to the conducting surface, into

GENERALIZED SCALING LAWS

a PIC simulation. F_R is an arbitrary term that represents a near-resonance function and is multiplied with the resonance $\langle P_r \rangle$, theory equation so that nonresonance data can be analyzed. Again, all equations and the units of all parameters in this paper are in SI, with the exception of $|e|$ which has units of joules per eV, and eV_b and ϕ which have units of eV.

The above equations represent the theory scaling laws with primed unknown constant coefficients and exponents, using the PIC parameter set. Calculation of the unknown constants from the PIC data is best accomplished by using equations that contain the parameters actually used by the PIC simulations, i.e., the PIC parameter set. However, to see how the fitted constants propagate into the same equations when written in terms of laboratory parameters requires applying the following two conversion relations,

$$\beta_{\text{class}} = \sqrt{\frac{2e}{m_e c^2}} \sqrt{\frac{b_1}{\lambda_L} - \phi_s} \quad (266)$$

$$\langle j \rangle = \frac{1}{b_1} \frac{\lambda_L Y \langle P_L \rangle}{A} \quad \text{where} \quad (267)$$

$$b_1 \equiv \frac{hc}{e} \quad \text{and} \quad \phi_s \equiv \frac{|e|\phi}{e} \quad (268)$$

to get the corresponding generalized theory equations converted to lab, or UV, parameters,

$$E_{Z \text{ max}} = a' k_1 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{b'/2} \left(\frac{\lambda_L Y \langle P_L \rangle}{A} \right)^{c'} \quad (269)$$

$$d_{\text{max}} = a' k_1 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{b'/2} \left(\frac{\lambda_L Y \langle P_L \rangle}{A} \right)^{c'} \quad (270)$$

$$\nu_{\text{nat}} = a' k_1 \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{b'/2} \left(\frac{\lambda_L Y \langle P_L \rangle}{A} \right)^{c'} \quad (271)$$

$$\langle P_r \rangle_s = a' k_1 F_D^{d'} \left(\frac{b_1}{\lambda_L} - \phi_s \right)^{b'/2} \lambda_L^{c'} Y^{c'} A^{e'-c'} \langle P_L \rangle^{c'} F_R \quad (272)$$

$$k_1 \equiv \left(\frac{2^{b'} e^{b'+2c'}}{m_e^{b'} h^{2c'} c^{2(b'+c')}} \right)^{1/2} \quad (273)$$

where e is the elementary charge, m_e is the electron mass, h is the Planck constant, and c is the speed of light in vacuum.

Note that the parameters in the photon parameter set cannot all be fitted independently in every parameter when using PIC code *beam emission simulation data* as the beam emission data are described in terms of the PIC parameter set. Consequently, many photon or UV parameters have similar fitted constants because the PIC parameters often represent several photon parameters that scale as a group. For example, there are an infinite number of values for the wavelength of the light source λ_L , the average photoelectron yield Y , the average light source power $\langle P_L \rangle$, and photoelectron emission area A that can represent a single, unique value of the average emitted current density $\langle j \rangle$. Similarly, there are an infinite number of values for λ_L and for the work function of the conducting surface ϕ that can represent a single, unique value of the classical expression for β . As a benefit, by using the generalized theory equations and the conversion relations, scanning a few PIC parameters is an efficient way to search many UV parameters. So using a parameter set developed to model PIC simulations that used *a beam emission model has the virtue of fewer parameters to search, measure and fit*. Of course, the PIC-to-UV conversion relations must realistically capture the basic physics being modeled. If laboratory experiments are conducted that measure variations and take data using UV parameters, then a generalized UV parameter equation can be fitted. However, even then one would hope that the PIC-to-UV converted parameters would be similarly scaled with respect to each other in their fitting constants, so that the smaller PIC parameter space to fit could itself act as additional information with which to constrain the fit. Since using as much information as possible from theoretical, computational, and laboratory research is desirable in any fitting process, one should first try fitting the constants for the generalized PIC parameter set theory equations, *even with laboratory experimental data measured in UV or other parameter sets*. Besides, the theory equations are simpler to work with and easier to understand when using the PIC parameter set.

The values of these generalized constants (coefficients and exponents) were previously derived. Table 2 summarizes the theoretical values for the generalized constants.

Using PIC or laboratory data, several fits in one or more theory constants can be performed to see how the experimental data's fitted constants compare with the theory values. Note that all these equations are for small-spots (spatially and temporally coherent electron motion and no significant radiation source phase differences). Also, all the equations were derived

GENERALIZED SCALING LAWS

Table 2: Theoretical Values of Coefficients and Exponents

Eqn.	$a_1 = a'$	$a_2 = d'$	$a_3 = e'$	$a_4 = b'$	$a_5 = c'$
$E_{Z \max}$	13875	N/A	N/A	0.5	0.5
$d_{\max}$	18.415	N/A	N/A	1.5	-0.5
ν_{nat}	4.0700×10^6	N/A	N/A	-0.5	0.5
$\langle P_r \rangle_s$	24.657	2.0	2.0	2.0	2.0

for nonrelativistic, steady-state, 'resonance' results, i.e., for situations where the resonance condition is $\nu_{\text{mod}} = \nu_{\text{nat}}$. Finally, the model used was one-dimensional with perpendicular, monoenergetic electron beam emission of a modified single, rigid, and 'virtually' thin charge sheet.

$\langle P_r \rangle_s$ data may be obtained from both resonance and nonresonance simulations or laboratory experiments in which for nonresonance data, $\langle j \rangle$ is modulated at $\nu_{\text{mod}} \neq \nu_{\text{nat}}$. In general for various experiments, a lot more nonresonance data could exist than resonance data. If all of the resonance and nonresonance data could be used simultaneously to perform a fit, it could be possible to perform a better fit in more parameters by using the larger combined data set. However, using the larger combined data set requires nonresonance, or approximate near-resonance, generalized theory scaling law equations describing the radiation quantities. For these fits, the nonresonance $\langle P_r \rangle_s$ equation is approximated by the steady-state, 'resonance' $\langle P_r \rangle_s$ theory equation multiplied by F_R , an arbitrarily chosen resonance function. The F_R function chosen has the property that for resonance power data, $F_R = 1$, and the resonance $\langle P_r \rangle_s$ equation is recovered. So at $\nu_{\text{mod}} = \nu_{\text{nat}}$, F_R obtains its maximum or resonance value of one. Also, $0 < F_R \leq 1$, and F_R is equal to $\langle P_r \rangle_s$ at ν_{mod} divided by $\langle P_r \rangle_s$ at ν_{nat} . For nonresonance data, but near-resonance data (i.e. $0.0 \leq \nu_{\text{mod}} \leq 2.3\nu_{\text{nat}}$), a Lorentzian-type resonance function is used to represent F_R .

The arbitrary near-resonance function is defined as:

$$F_R \equiv \frac{a_6^2}{(\nu_{\text{mod}}/\nu_{\text{nat}} - 1)^2 + a_6^2} \quad \text{where} \quad (274)$$

$$\nu_{\text{mod}} \equiv \nu_{\text{nat}} \quad \text{for resonance, and} \quad (275)$$

$$\nu_{\text{nat}} = a_7 \beta^{a_8} \langle j \rangle^{a_9} = a' \beta^{b'} \langle j \rangle^{c'} \quad (276)$$

$$j \equiv \langle j \rangle [1 - F_D \cos(2\pi \nu_{\text{mod}} t)] \quad (277)$$

F_R is symmetric about its maximum value located at $\nu_{\text{mod}} = \nu_{\text{nat}}$, and it is easy to define various Full-Width Half-Maximum (FWHM) parameters. The frequencies defining the FWHM values of F_R are,

$$\nu_{\text{mod}_{\text{FWHM}}} = (1 \pm a_6)\nu_{\text{nat}} \quad (278)$$

so the FWHM of the F_R *near*-resonance curve spans

$$\Delta\nu_{\text{FWHM}} = 2a_6\nu_{\text{nat}} \quad (279)$$

and the F_R *near*-resonance curve widens as ν_{nat} increases.

Also, for $\nu_{\text{mod}} = 0$ and $\nu_{\text{mod}}/\nu_{\text{nat}} = 2$,

$$F_R = \frac{a_6^2}{1 + a_6^2} \quad (280)$$

Other relationships involving F_R are

$$\frac{\nu_{\text{mod}}}{\nu_{\text{nat}}} = a_6 \left(\frac{1}{F_R} - 1 \right)^{1/2} + 1, \text{ or} \quad (281)$$

$$a_6 = \frac{\nu_{\text{mod}}/\nu_{\text{nat}} - 1}{(1/F_R - 1)^{1/2}} \quad (282)$$

In summary, this section presented the generalized theory scaling law equations necessary to test and compare experimental data with the theory equations via curve-fitting techniques.

Conclusions

The generation of radiation via photoelectrons induced off of a conducting surface was explored using a simple model so as to determine fundamental scaling laws, make approximate predictions, understand parameter trade-offs, and optimize predicted radiation. The model was one-dimensional (small-spot) and used monoenergetic, nonrelativistic photoelectrons emitted normal to the illuminated conducting surface. First, the radiation equations were generically derived for the small-spot case. In doing so, a $\sin^2$ type modulation function was used. The generic radiation equations then required knowledge of the photoelectron dynamics. A simple infinite, rigid, thin charge sheet model was used to determine approximate E_z electric fields. The resulting force was then used to determine the photoelectron trajectories. Finally, a very simple steady-state approximation was applied to determine the maximum orbit distances and natural frequency of oscillation for a constant emitted electron beam. These were then substituted into the radiation equations to derive the approximate steady-state equations for the generated radiation. The result was a system of equations in PIC parameters for various scaling laws. Conversions of the PIC parameter equations to photon parameters were then derived and applied. The results were then converted to a compact set of 'natural' parameters and simultaneously solved by graphical techniques using a multidimensional, multiscale nomogram showing most of the UV parameters and their trends. Also, the model constraints were quantified and graphed as well. By using the theory equations and the Fig. 4 nomogram, one can understand the trends in most parameters, and even manually predict and optimize the radiated power with respect to the theory equations and laboratory and model constraints using techniques similar to linear programming.

This research in this paper was performed primarily to generate simple, useful theory equations for comparison, prediction, and optimization purposes. The work had its origin in early efforts to develop simple theoretical scaling laws to use in setting up and testing PIC code simulations of photoelectron-generated radiation. These back-of-the-envelope theory equations evolved into a system of simply modeled equations describing various observables. The theory equations describing the maximum E_z , $d_{\max}$, $t_{\max}$, and ν_{nat} observables should generally work for both small and large spots. And while simple in some of its model approximations, the model approxi-

mations usually have some physical basis. As an example, while the theory modeled self-oscillation of an injected, perpendicular, monoenergetic electron beam may not match actual photoemission with its various emitted photoelectron angles and energies, the electron sheath layer does have characteristic times and distances associated with it that cannot be completely ignored in optimizing the generated radiation, though the actual constraints may be less strict than the simple theory implies. And by making the theory equations easier to understand and apply, the advantages gained from the simple nature of the theory equations to some extent compensate for the limited nature of those equations.

In general, this paper was designed to make it a practical, beneficial, and coherent product not only for making predictions and for doing parameter optimization, but for testing and comparison purposes with other experimental and theory results. In this way, a firmer foundation of inter-supporting computational and laboratory experimental, as well as other theoretical, results can be built that will provide confidence in the basic theory scaling laws and predictions. If these theory equation results are to be compared with other theory quoted independent parameter scaling laws in a meaningful way, they must be quoted in terms of the same *set* of independent parameters. So by rewriting other theory equations in terms of the UV parameters, using the same independent parameter set, the scaling laws of the various theories can be directly compared. Also, it is clearer when writing down various systems of simultaneous equations if all of the dependent parameters are written in terms of the same arbitrary set of independent parameters. In this way, the reader can more easily see the total effect on all the parameters in a system of equations, when one or more of the parameters in that system of equations is changed. The photon or UV set of independent parameters should provide an easy to use set of independent parameters for comparison purposes because they are more general than the PIC code parameter set and are closer to the type of parameters that would be directly adjusted or measured in a laboratory setting. Finally, using the equations should be easy since the equations and parameters were written and expressed in a mostly single system of units, using SI.

In order to make useful predictions and to do valid parameter optimization, the theory system of equations must be simultaneously 'solved' along with the model and experimental or user constraints, to obtain a linear programming type of solution. By writing and graphing the theory equations

CONCLUSIONS

in terms of natural parameters, it becomes easier for a researcher to use the theory equations to make predictions, understand parameter trends, and optimize experiments and generated radiation. For the natural parameter equations and graph, which implement a *special case* of the more general optimization process, it is seen that the parameters D , ν_{nat} , A , Y , and $\langle P_L \rangle$ should be increased, and the parameters ϕ , and λ_L should be decreased (and θ should be set to 90 degrees to maximize $d\langle P \rangle/d\Omega$), to optimize the various photoelectron-generated radiation quantities. However, *the general optimization process is nontrivial*, and the short program listed in **Appendix B** should be used to do general optimization searches.

Table 3 summarizes the results for the previously worked case 1 and **Appendix B** program optimized 15 eV photon energy examples, as well as presenting new examples for case 1, case 2, 5 eV and 15 eV photon energies. Examples in Table 3 are listed in order of increasing photon energies, and

Table 3: Summary of Numerical Small-Spot Theory Predictions

Example	F_D	λ_L (nm)	Y	ϕ (eV)	x (m^2/s^2)	$\langle P_L \rangle$ (W)	η_{rsL}
Case 1	1.0	532.00	10^{-1}	1.0	$10^{15.4934}$	$10^{1.00}$	$10^{-5.63}$
Case 1	1.0	532.00	10^{-1}	1.0	$10^{15.64}$	$10^{1.15}$	$10^{-5.48}$
Case 1	1.0	532.00	10^{-1}	1.0	$10^{17.40}$	$10^{2.91}$	$10^{-3.72}$
Case 2	1.0	266.00	10^{-5}	4.2	$10^{13.4226}$	$10^{3.00}$	$10^{-12.69}$
Case 2	1.0	266.00	10^{-5}	4.2	$10^{15.64}$	$10^{5.22}$	$10^{-10.47}$
5 eV	1.0	247.97	10^{-7}	4.8	$10^{11.5736}$	$10^{3.00}$	$10^{-17.11}$
5 eV	1.0	247.97	10^{-7}	4.8	$10^{15.64}$	$10^{7.07}$	$10^{-13.05}$
5 eV	1.0	247.97	10^{-7}	4.0	$10^{15.64}$	$10^{7.42}$	$10^{-12.00}$
5 eV	1.0	247.97	10^{-5}	4.8	$10^{15.64}$	$10^{5.07}$	$10^{-11.05}$
5 eV	1.0	247.97	10^{-5}	4.0	$10^{15.64}$	$10^{5.42}$	$10^{-10.00}$
15 eV	1.0	82.66	10^{-1}	6.0	$10^{15.64}$	$10^{2.37}$	$10^{-5.04}$
15 eV	1.0	82.66	10^{-1}	4.2	$10^{15.6446}$	$10^{2.414}$	$10^{-4.920}$
15 eV	1.0	82.66	10^{-1}	4.2	$10^{17.00}$	$10^{3.770}$	$10^{-3.564}$

within photon energy levels by order of increasing efficiency. The boldface values of various parameters in Table 3 were calculated or optimized using the given non-boldface parameter values. All examples where $x = x_{\text{max}} = 10^{15.64461}$ are the theoretical maximum small-spot radiation predictions for the other associated parameters, where two examples are presented above

the small-spot constraint $x_{\max}$ to provide trend information. All examples (other than Case 1) use values of photoelectron yields that can occur when *typical* metals are illuminated with photons having the displayed λ_L values. Additionally, examples using 15 eV photons represent optimized theoretical values for typical metals, using published optimum experimental λ_L and Y values, for small-spot limited and nearly small-spot limited solutions. Case 1 examples have low input source power requirements and high efficiencies but use photoelectron yields normally associated with typical metals that are illuminated by photons having energies of 15 to 20 eV. Note that for low-energy (5 eV and less) photons, typical metals: 1) generate very low conversion efficiencies, even with high average source power values for $\langle P_L \rangle$, 2) cannot all be used, since many have photoelectron work functions where approximately $4 \leq \phi \leq 6$, and 3) typical metals that are usable display large, undesirable variations in photoelectron yields and efficiencies. *Photon sources having energies of 15 to 20 eV reduce or eliminate these problems.*

For case 1 and case 2, the small-spot limited theoretical best efficiencies are given as $\eta_{\text{rsL}} = 10^{-5.48}$ and $\eta_{\text{rsL}} = 10^{-10.47}$, respectively. And for low-energy photons and typical metals, the maximum light-to-generated radiation conversion efficiency drops to much less than $\eta_{\text{rsL}} = 10^{-10.47}$ since typical metals that are illuminated by low-energy, 5.0 eV photons have yields that vary as $10^{-7} < Y < 10^{-5}$. Consequently, typical conductors illuminated by low-power sources having low-energy photons (energies less than or equal to 5.0 eV) would have small-spot efficiencies where $\eta_{\text{rsL}} < 10^{-10.5}$. So for the two cases presented involving low-energy photons and typical and very optimistic parameters, the theory predicts that the best total light-to-generated radiation conversion efficiencies for small spots vary from approximately $10^{-10.5}$ to $10^{-5.5}$, respectively.

While both cases 1 and 2 used photons having energies less than 5.0 eV, the optimistic case also used an unrealistically high photoelectron yield value of $Y = 0.1$ that is typical of real metals that are illuminated by 15 to 20 eV photons. As was demonstrated in Table 3, when using typical real conductors that are illuminated by low-power sources having low, less than 5.0 eV photon energies, the optimum purely photoelectron-induced radiated power and radiation conversion efficiencies are usually quite small. Interestingly, the maximum radiation and radiation conversion efficiencies possible for typical conductors illuminated by low-power sources of low-energy photons usually fall within the model limits of this paper's derived small-spot, steady-state analy-

CONCLUSIONS

sis. Consequently, **typical metals illuminated with low-power sources of low-energy photons can be said to be small-spot model limited.** Small-spot model limited experiments will always have a characteristic far-field dipole radiation $\sin^2 \theta$ pattern for $d\langle P \rangle/d\Omega$ and will generally produce small amounts of radiated power at low efficiency when using low-energy photons.

Low efficiency is primarily caused by the low photoelectron yields of typical metals when exposed to low-energy photons. These low photoelectron yields are usually **several orders of magnitude smaller** than the photoelectron yields of typical metals which occur when those metals are exposed to 15 to 20 eV photons. Therefore, since $\eta_{rsL} = F_D^2 Y y_4$, a vast difference in increased photoelectron yields and photoelectron-induced radiation conversion efficiency is made possible by a comparatively slight increase of a few electron volts in the λ_L photon energies. Further, operating at the small-spot upper limit of the theory (and beyond) also helps to reach high efficiencies. Fortunately, high yields also increase the maximum value of $y_1 \equiv Y\langle P_L \rangle$ that it is possible to generate for a given source power $\langle P_L \rangle$, making the production of photoelectron-generated radiation at and beyond the small-spot upper limit more realistic for typical metals. The reverse is also true—high efficiencies are almost impossible in situations where low-power sources add to the difficulty caused by the low photoelectron yields. However, even with high-power sources of low-energy photons, plasma formation could then become a problem in using the potentially large values of $\langle P_L \rangle$ that would be required for high conversion efficiencies. Unfortunately, any high-power sources of low-energy photons causing plasma formation would interfere with the generation of purely photoelectron-induced radiation and might even damage the metal being used. **Consequently, the small-spot theory predicts that η_{rsL} , the total conversion efficiency for incident photon power photoelectron-induced radiated power, can go higher than $10^{-5.5}$**

typical real conductors if photons having energies of 15 eV and higher are used. Significantly, these efficiencies should go even higher still if the small-spot limit of this theory is violated as well.

In summary, the simple theory equations, model constraint equations, and solution techniques presented in this paper provide a foundation for understanding, predicting, and optimizing photoelectron-generated radiation, and the simple theory equations also provide useful scaling laws to compare with computational and laboratory experimental data.

Appendix A

This appendix discusses and presents a very short Fortran program the reader can type in that will automatically find the maximum feasible x and y_i solutions for constant λ_L and ϕ_s inputs. In other words, this program automates the process discussed for using the Fig. 4 nomogram, subject to model constraints and user input maximum constraints on x and y_1 , to find the maximum feasible x , y_1 , y_2 , y_3 , and y_4 values. The program is included in this appendix for several reasons. One, because the optimization and constraint process may not be clearly understood, the program is itself a concise and instructive summary of the optimization process. Secondly, because the program provides a simple and quick way to get optimum feasible solutions, the reader may want to simply use it to explore other cases using different λ_L and ϕ_s constant values. Finally, a computer program solution is flexible and general, and it provides a foundation for doing more complex mathematical programming, where more parameters can be searched and where more model constraints and equations can be implemented in a fairly simple and direct manner.

The program concisely summarizes the process of finding a maximal feasible solution because the program is very short and because the program implements the optimization process of using Fig. 4 subject to model constraints and user parameter constraints. First, the program asks for four natural parameter input values for λ_L , ϕ_s , x , and y_1 . Next, the program basically checks the inputs for photoemission, for nonrelativistic photoemission, and for satisfying the lower limit imposed by the primary model constraints $x_{\min} \leq x \leq x_{\max}$. If the inputs do not support nonrelativistic photoemission, or fall below $x_{\min}$, then the program prints an appropriate error message and waits for new inputs. Then the smallest value among the x , $x_{\text{calc}} = f(y_1)$, and $x_{\max}$ inputs is determined and used as the largest possible value of x for calculating the maximum feasible values of y_1 , y_2 , y_3 , and y_4 . As such, the program enforces model constraints 1, 10, and 12, which effectively enforces model constraints 2, 3, 9, and 11 as well. Note that 'maximal analysis' model constraints 4, 5, 7, and 8 are inherent in the system of theory equations that were developed. This leaves model constraint 6, requiring no plasma formation, to be enforced by the user *when applying* the feasible x and y_1 solutions. Finally, the program displays the input λ_L and ϕ_s constants along with the resulting maximum feasible y_i quantities based on the maximum feasible x

APPENDIX A

possible, and asks if you would like to calculate another problem case.

Because the program is trying to maximize the feasible value of the parameter x , the user should input the maximum x and y_1 experimental parameters possible for the user input λ_L and ϕ_s constants. The program will assume you can also generate smaller x and y_1 values, which should definitely be true for y_1 , and generally true for x . Then the program will determine the best, mathematical programming type, feasible solution for maximizing x , y_1 , y_2 , y_3 , and y_4 , and in the process may need to reduce the maximum input values of x , y_1 , or both x and y_1 . So, the user will input the constant quantities λ_L and ϕ_s , as well as the desired or maximum values for x and y_1 , and the program will calculate and display the maximum feasible parameters for x , y_1 , y_2 , y_3 , and y_4 . The maximum theoretical feasible solution for a given λ_L and ϕ_s occurs at $x_{\max}$, the small-spot constraint limit. Simply input $x \geq 10^{15.65}$ and a large value of y above the small-spot limit, say $y \geq 10^{10}$, and the program will generate the maximum theoretical feasible solution for a specific pair of constant λ_L and ϕ_s values.

Now for some final miscellaneous notes and considerations. First note that like the nomogram, the values of x and y_i are all input and output as common logarithms. Besides being similar to the figures and examples used in the paper, common logarithms make it easy to calculate the required $\langle P_L \rangle$ for a given Y and y_1 , as well as other relationships, by simple addition, subtraction, multiplication, and division of the logarithms. Also note that the $d_{\max}$ equation was not included in the constraints because it was not assumed to be a problem needing constraint. While the $d_{\max}$ equation was not included as a constraint and is *very unlikely* to present a problem, the reader can include $d_{\max}$ as a constraint to be checked by the program as well. In fact, if there are any other constraints the user may think necessary, those new constraints should be incorporated into the programs presented in these appendices, or even into new programs. Still, the desire to add new user constraints would probably be more of a benefit to the more general optimization process presented in the program in **Appendix B**, where the user may wish to explore the model limits past typical experimental parameter regions. Regardless, both programs work well for typical metals and visible or UV photon sources, so problems similar to those used in the examples presented in this paper are correctly handled by the programs. However, since natural parameters are processed by the programs, and since $x \equiv \nu_{\text{nat}}^2 A$ and $y_1 \equiv Y(P_L)$, it should be noted that for feasible x and y_1 program results,

one should not take those results and then proceed to pick values of $\langle P_L \rangle$ and A that cause a plasma to form, as the analysis in this paper is for a pure photoelectric effect without plasma formation. Considering the $\langle P_L \rangle$ and ν_{nat} used in the paper's examples, along with the steady-state, 'resonance' condition $\nu_{\text{mod}} = \nu_{\text{nat}}$, plasma formation appears to be avoidable in many typical to optimistic cases, especially since increasing $\langle P_L \rangle$ while maintaining a constant ν_{mod} photon source modulation frequency generally requires increasing A , the area of the illuminated spot. So in conclusion, even when feasible solutions are generated for natural parameters, those solutions should not be applied in a manner that may violate model constraint 6 or any user defined constraints.

```

program natcalc
implicit double precision (a-h,o-z)
double precision lambda
common/const1/lambda,phis,xx,y1,y2,y3,y4
10 continue
   write(6,*) 'Enter lambda, phis, xx, y1 inputs:'
   read(5,*) lambda,phis,xx,y1
   xx=10.0D0**xx
   y1=10.0D0**y1
   CALL y2y3y4
   write(6,*) ' '
   write(6,*) 'Do you wish to continue (enter 1 for yes)?'
   read(5,*) ncontinue
   if (ncontinue.EQ.1) GOTO 10
stop
end

cc SEND IN (lambda,phis,xx,y1) TO GET OUT CONSTRAINED (xx,y1,y2,y3,y4)
subroutine y2y3y4
   parameter (pii=3.141592654D0, a14=1.4807D-22)
   parameter (a11=1.6431D-37, a12=1.3765D-36, a13=9.2960D-15)
   parameter (b1=1.2398D-6, c1=6.9069D10, c2=1.7880D15)
   implicit double precision (a-h,o-z)
   double precision lambda
   common/const1/lambda,phis,xx,y1,y2,y3,y4
   deltamax=pii/2.0D0
   s1=b1/lambda-phis
   if (s1.LE.0.0D0) then
      write(6,*) 'NO PHOTOEMISSION'

```

APPENDIX A

```

else
  gamma=s1/511000.0D0+1.0D0
  beta=DSQRT(1.0D0-1.0D0/gamma**2.0D0)
  if (beta.GT.0.3D0) then
    write(6,*) 'RELATIVISTIC PHOTOEMISSION'
  else
    xmin=c1*s1
    xcalc=1.0D0/a14/DSQRT(s1)*lambda*y1
    if ((xx.LT.xmin).OR.(xcalc.LT.xmin)) then
      write(6,*) 'VIOLATED CONSTANT Ez MODEL CONSTRAINT'
    else
      xmax=c2*deltamax*deltamax
      xx=DMIN1(xx,xcalc,xmax)
      y1=a14*DSQRT(s1)/lambda*xx
      y2=a11*s1*s1*xx*xx
      y3=a12*s1*s1*xx*xx
      y4=a13*s1**1.5D0*lambda*xx
      write(6,*) '
      write(6,*) 'FEASIBLE NATURAL PARAMETERS:'
      write(6,*) 'lambda =' ,lambda
      write(6,*) 'phis  =' ,phis
      write(6,*) 'xx    =' ,DLOG10(xx)
      write(6,*) 'y1    =' ,DLOG10(y1)
      write(6,*) 'y2    =' ,DLOG10(y2)
      write(6,*) 'y3    =' ,DLOG10(y3)
      write(6,*) 'y4    =' ,DLOG10(y4)
    endif
  endif
endif
return
end

```

Appendix B

This appendix discusses and presents a short Fortran program the reader can type in that will automatically find the maximum feasible y_4 solution for various input λ_L , ϕ_s , x and y_1 constants and constrained ranges. In other words, this program *further* automates the program previously discussed in **Appendix A** so that this program applies the model constraints and also *searches user constrained λ_L , ϕ_s , x , and y_1 ranges* to find the maximum possible y_4 value. This program has been included because it solves the need for a more complete and complex mathematical programming optimization solution, and because an equally general, flexible, and simple to use analytic solution could not be derived. Therefore, a simple computer program was designed and included in this appendix to allow a user to input numerical constraints on the values of the natural parameters λ_L , ϕ_s , x , and y_1 , as *constants or ranges*, which the program will then use to search for the resulting maximum possible y_4 value, subject to important model constraints.

Essentially then, the program is the same as in **Appendix A**, with the addition of a simple Monte-Carlo calling program that takes the place of the user in typing in various combinations of λ_L , ϕ_s , x , and y_1 while trying to find the maximum feasible value for y_4 . While more complex programs have been developed, the simple Monte-Carlo program presented in this appendix is more practical for the reader to type in, is much clearer to understand and to change, and on a reasonably fast workstation will give useful approximate results. Note that the use of the natural parameter set has reduced the parameter searching to only a four-dimensional search, making the simple Monte-Carlo approach practical. This is especially true if the program user elects to set one or more of the dimensions being searched to constants, such as might arise in an experiment or when trying to refine the solution of a four-dimensional search, because this will further decrease the number of natural parameter dimensions that need to be searched.

There are a few notable differences from the discussion and program presented in **Appendix A**. One major difference is that the program asks the user to input a lower and upper search limit for each of the four natural parameter inputs. The user can set any of the search ranges to a single, constant value by simply setting the lower and upper search limit to the same, constant value. Next the program, using a simple random number generator [Ref. 5, pp. 21-22], generates random selections for the four natu-

APPENDIX B

ral parameters based on the user input constants and range limits for those natural parameters. These random selections are then sent to essentially the subroutine listed in **Appendix A**, and the corresponding y_4 (or **yytest**) value and other natural parameter values are returned to the calling program. During the Monte-Carlo iterations, when a new and larger value of y_4 is found, all of the new natural parameters are displayed. Also, counts of the total iterations and feasible 'shots' into the searched parameter space are displayed as diagnostics. And note that f_λ is also displayed as the diagnostic program output **flambda**, so that the program user can easily see if and when a given set of four input parameter constraints results in an 'optimum' $f_\lambda = 1/2$ solution. This process repeats itself as successively larger and larger y_4 values are found during the Monte-Carlo iterative searching. For this simple Monte-Carlo process, it quickly becomes harder to find a larger y_4 maximum from random selections of the four natural parameter inputs, and soon the program essentially stops displaying new and better y_4 constrained optimums. Finally, the user should wait a while and then kill the program once it becomes apparent that no better solution can be found in a reasonable amount of additional time by random searching.

The simple nature of the program should make any desired program changes relatively easy. While not necessary, readers are encouraged to use their Fortran's native random number generator commands. Also, the user may want to change the program so that it maximizes either y_2 or y_3 instead of y_4 , which is easily accomplished by setting **yytest=y2** or **yytest=y3** in the **y2y3y4** subroutine. A comment line in the **y2y3y4** Fortran subroutine indicates where to make the change involving **yytest=y4**. Other cost functions involving any parameters searched and calculated can be designed as needed, so that any function or algorithm of natural parameters and other derived and input quantities can be used for the **yytest** cost function. If generating large y_1 values is difficult, one can try setting **yytest=y4/y1** to find the maximum y_4 to least y_1 ratio possible for the constraints. Also, try using 15 eV photons ($\lambda_L = 82.656$ nm) at the small-spot limit or slightly beyond. Going beyond the small-spot limit requires setting $x_{\max}$ in the program to larger values, such as 10^{18} , but one should then add a test to keep $y_4 \leq 0.1$, and remember that this violates the theory model constraint x_{12} so that the predictions are only useful for seeing the trends relatively near constraint x_{12} . Further, since this program does general optimization, one can also extend the program to include other ways of automating how the

feasible experimental natural parameters are selected in the main program for subsequent processing by the **y2y3y4** subroutine. For instance, one might want to do this so that some discrete feasible combinations of various input parameter settings can be searched. Also, relationships governing continuous coupled feasible sets of input parameters can be programmed into the automatic searching as well, such as automatically setting various permissible relationships among the λ_L , ϕ_s , ν_{mod} , Y , and $\langle P_L \rangle$ combinations for an experiment. For instance, $Y = f(\lambda_L)$ for a specific conductor can be programmed into the **y2y3y4** subroutine and used to calculate $\langle P_L \rangle$ and η_{rsL} for use in the general optimization process. In addition, one can curve fit the a_i theory equation coefficients to experimental data, and use the fitted coefficients in the programs to try and improve the simple theory predictions. One can even set up the program to do UV parameter to natural parameter input conversions and natural parameter to UV parameter output conversions, if desired, perhaps using a more powerful search algorithm. In essence, with such a simple yet powerful program, you should be able to readily modify it to match your unique optimization and experimental requirements.

One final recommendation is that the user do several optimization runs using several different seeds, and even look at the effect of using several different sets of the four input natural parameter constraint limits to confirm the optimizations results. For typical input limits, this Monte-Carlo program will have no difficulty in finding the approximate optimum solutions. The program's approximate results for the feasible parameters that give the best or maximum value of **yytest** subject to the various search constraints can now be used for analysis or in setting up an optimal experiment.

```

program natopt
parameter (pii=3.141592654D0, b1=1.2398D-6)
implicit double precision (a-h,o-z)
double precision lambda,lambd1,lambd2
common/const1/lambda,phis,xx,y1,y2,y3,y4,yytest,count2
data yymax/-1.0D30/, count1/0.0D0/, count2/0.0D0/
write(6,*) ' '
write(6,*) 'Enter seed to random generator (0. to 10.):'
read(5,*) seed
write(6,*) 'Enter ranges to optimize natural parameters over'
write(6,*) '(you may set min and max equal to set a constant):'
write(6,*) 'Enter minimum and maximum lambda range:'
read(5,*) lambd1, lambd2

```

APPENDIX B

```

write(6,*) 'Enter minimum and maximum phis range:'
read(5,*) phis1, phis2
write(6,*) 'Enter minimum and maximum xx range:'
read(5,*) xx1, xx2
write(6,*) 'Enter minimum and maximum y1 range:'
read(5,*) y1a, y1b
xx1=10.0D0**xx1
xx2=10.0D0**xx2
y1a=10.0D0**y1a
y1b=10.0D0**y1b
dlambda=lambd2-lambd1
dphis=phis2-phis1
dxx=xx2-xx1
dy1=y1b-y1a
10 continue
   count1=count1+1.0D0
   seed=(seed+pii)**5.04D0
   seed=seed-DINT(seed)
   lambda=seed*dlambda+lambd1
   seed=(seed+pii)**5.04D0
   seed=seed-DINT(seed)
   phis=seed*dphis+phis1
   seed=(seed+pii)**5.04D0
   seed=seed-DINT(seed)
   xx=seed*dxx+xx1
   seed=(seed+pii)**5.04D0
   seed=seed-DINT(seed)
   y1=seed*dy1+y1a
   CALL y2y3y4
   if (yytest.GT.yymax) then
      yymax=yytest
      write(6,*) '
      write(6,*) 'FEASIBLE NATURAL PARAMETERS'
      write(6,*) 'OPTIMIZED FOR MAX PARAMETER = ',DLOG10(yymax)
      write(6,*) 'total      shots taken =',count1
      write(6,*) 'feasible shots taken =',count2
      write(6,*) 'flambda =',lambda*phis/b1
      write(6,*) 'lambda  =',lambda
      write(6,*) 'phis    =',phis
      write(6,*) 'xx      =',DLOG10(xx)
      write(6,*) 'y1      =',DLOG10(y1)
      write(6,*) 'y2      =',DLOG10(y2)
      write(6,*) 'y3      =',DLOG10(y3)
      write(6,*) 'y4      =',DLOG10(y4)

```

```

        endif
GOTO 10
stop
end

cc SEND IN (lambda,phis,xx,y1) TO GET OUT CONSTRAINED (xx,y1,y2,y3,y4)
subroutine y2y3y4
    parameter (pii=3.141592654D0, a14=1.4807D-22)
    parameter (a11=1.6431D-37, a12=1.3765D-36, a13=9.2960D-15)
    parameter (b1=1.2398D-6, c1=6.9069D10, c2=1.7880D15)
    implicit double precision (a-h,o-z)
    double precision lambda
    common/const1/lambda,phis,xx,y1,y2,y3,y4,yytest,count2
    deltamax=pii/2.0D0
    s1=b1/lambda-phis
    if (s1.LE.0.0D0) then
        yytest=-1.0D30
    else
        gamma=s1/511000.0D0+1.0D0
        beta=DSQRT(1.0D0-1.0D0/gamma**2.0D0)
        if (beta.GT.0.3D0) then
            yytest=-1.0D30
        else
            xmin=c1*s1
            xcalc=1.0D0/a14/DSQRT(s1)*lambda*y1
            if ((xx.LT.xmin).OR.(xcalc.LT.xmin)) then
                yytest=-1.0D30
            else
                count2=count2+1.0D0
                xmax=c2*deltamax*deltamax
                xx=DMIN1(xx,xcalc,xmax)
                y1=a14*DSQRT(s1)/lambda*xx
                y2=a11*s1*s1*xx*xx
                y3=a12*s1*s1*xx*xx
                y4=a13*s1**1.5D0*lambda*xx
cc      TO MAXIMIZE y3 OR y4/y1, USE y3 OR y4/y1 FOR y4 IN NEXT LINE
                yytest=y4
            endif
        endif
    endif
    return
end

```

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