PROPERTIES OF AN INTEGRAL OF E. J. McSHANE

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The problem with which this paper is concerned is that of investigating the properties of an integral which was first defined by E.J. McShane in lecture notes presented at the Conference on Modern Theories of Integration, held at the University of Oklahoma in June, 1969.

The paper is divided into three chapters. In the first chapter, the integral in question is defined and compared with the Riemann integral.

In Chapter II, the fundamental properties of the integral are investigated. A lemma is stated and proved showing the equivalence of a useful Cauchy-type definition for integrability. Finally, the main theorem of this chapter, which states that the absolute value of an integrable function is integrable, is proved.

In Chapter III, convergence properties of the integral are investigated. The first theorem of this chapter shows that uniform convergence of a sequence of integrable functions is a sufficient condition for the limit function to be integrable. Examples are given showing that uniform convergence is not necessary, and in the final theorem of the paper, it is proved that uniform convergence can be replaced
with dominated convergence. Also included in this chapter is an example showing that the product of two integrable functions is not necessarily integrable.
PROPERTIES OF AN INTEGRAL OF E. J. McSHANE

THESIS

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CHAPTER I

INTRODUCTION

The purpose of this paper is to define an integral for real valued functions of a single real variable, which was first introduced by E. J. McShane (2,3), and to investigate the properties of that integral. In Chapter I, the integral is defined and compared to the well known Riemann integral. In Chapter II, some fundamental properties of this form of integration are developed. Finally, Chapter III is concerned with convergence theorems, including a monotonic convergence theorem and a dominated convergence theorem.

Definition 1.1. A gauge, denoted throughout this paper by $\delta$ or $\delta(x)$, is simply an arbitrary function defined on the set $\mathbb{R}$ of all real numbers, such that $\delta(x) > 0$ for all $x \in \mathbb{R}$.

Definition 1.2. A finite collection $P$ of ordered pairs $(x_i, A_i)$ for $i = 1, 2, \cdots, n$ is called a partition of the left-open, right-closed interval $[a, b]$ if and only if the following conditions hold:

1) Each $x_i$ is an element of the closed interval $[a, b]$;

2) Each $A_i$ is either empty or a left-open, right-closed subinterval $(a_i, b_i]$ of $(a, b]$;

3) $A_i \cap A_j$ is empty for $i \neq j$;

4) $\bigcup_{i=1}^{n} A_i = (a, b]$.
Definition 1.3. A finite collection $P$ of ordered pairs $(x_i,A_i)$ for $i = 1,2,\cdots,n$ which satisfies properties 1), 2), and 3) of Definition 1.2, but does not satisfy the last property is called a partial partition of $(a,b]$. 

Definition 1.4. The points $x_1,x_2,\cdots,x_n$ and sets $A_1,A_2,\cdots,A_n$ associated with a full or partial partition $P$ of $(a,b]$ are called the evaluation points and component intervals of the partition $P$, respectively.

Definition 1.5. If $\delta$ is a gauge, and $P = \{(x_1,A_1),(x_2,A_2),\cdots,(x_n,A_n)\}$ is a full or partial partition of $(a,b]$, then the statement that $P$ is $\delta$-fine means that each component interval $A_i$ is contained in the open interval $(x_i - \delta(x_i),x_i + \delta(x_i))$.

Definition 1.6. If $A$ is a left-open, right-closed interval, $A = (a,b]$, then let $m(A)$ denote the quantity $b-a$, called the measure or length of $A$. Also, for convenience define $m(\phi) = 0$.

Definition 1.7. A real valued function $f$, whose domain includes the closed interval $[a,b]$, is said to be integrable from $a$ to $b$ if and only if there is a real number $J$, so that for each positive number $\varepsilon$, there exists a gauge $\delta$, so that if $P$ is any $\delta$-fine partition of $(a,b]$, $P = \{(x_1,A_1),(x_2,A_2),\cdots,(x_n,A_n)\}$, then the inequality $\left| \sum_{i=1}^{n} f(x_i)m(A_i) - J \right| < \varepsilon$ holds. Any such number $J$, when one exists, will be called the integral from $a$ to $b$ of $f$ and will be denoted by the symbol $\int_{a}^{b} f \, dm$. The sum $\sum_{i=1}^{n} f(x_i)m(A_i)$ is called an approximating sum and will be denoted by $\Sigma[P,f]$ to shorten the notation.
To show that this integral is well defined, two lemmas must be proved; the first to show that the hypothesis of Definition 1.7 cannot be satisfied vacuously, and the second to show uniqueness.

Lemma 1.8. If \( \delta \) is an arbitrary gauge, and \([a,b]\) is an arbitrary left-open, right-closed interval, then there exists at least one \( \delta \)-fine partition of \([a,b]\).

Proof. Suppose to the contrary that there is some gauge \( \delta \), and a left-open, right-closed interval \([a,b]\), so that there is no \( \delta \)-fine partition of \([a,b]\). Now, if \([a,\frac{a+b}{2}]\) and \([\frac{a+b}{2},b]\) both have \( \delta \)-fine partitions, say \( P^{(1)} = \{(x_{11},A_{11}), (x_{12},A_{12}), \ldots, (x_{1n},A_{1n})\}\) and \( P^{(2)} = \{(x_{21},A_{21}), (x_{22},A_{22}), \ldots, (x_{2m},A_{2m})\}\), respectively, then \( P = \{(x_{11},A_{11}), \ldots, (x_{1n},A_{1n}), (x_{21},A_{21}), \ldots, (x_{2m},A_{2m})\}\) would be a \( \delta \)-fine partition of \([a,b]\).
Thus, under the assumption, one of the two subintervals \([a,\frac{a+b}{2}]\) and \([\frac{a+b}{2},b]\) cannot have a \( \delta \)-fine partition. Let \([a_1,b_1]\) denote one of these two subintervals which has no \( \delta \)-fine partition.

In the same manner, either \( [a_1,\frac{a_1+b_1}{2}] \) or \( [\frac{a_1+b_1}{2},b_1]\) has no \( \delta \)-fine partition, so let \([a_2,b_2]\) denote one of these subintervals which has no \( \delta \)-fine partition.

Now in general, where \([a_n,b_n]\) is a subinterval of \([a,b]\) which has no \( \delta \)-fine partition, let \([a_{n+1},b_{n+1}]\) denote one of the two subintervals \( [a_n,\frac{a_n+b_n}{2}] \) and \( [\frac{a_n+b_n}{2},b_n]\) which has no \( \delta \)-fine partition.
Then \((a,b) \supseteq (a_1,b_1) \supseteq \cdots \supseteq (a_n,b_n) \supseteq (a_{n+1},b_{n+1}) \supseteq \cdots\) and thus \(b \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots\) with each \(b_i \geq a\) so that \(\{b_i\}_i^{\infty}\) is a non-increasing sequence of real numbers bounded below, and therefore converges to its greatest lower bound \(b_0\). Now, \(b_0\) must be in each of the intervals \([a_n,b_n]\) for \(n = 1,2,\ldots\), for if \(b_0\) failed to be in some interval \([a_j,b_j]\), that would mean that \(b_0 < a_j\), but then \(a_j\) would be a lower bound for the sequence \(\{b_i\}_i^{\infty}\) which cannot be possible since \(b_0\) is its greatest lower bound.

Since \(\delta\) is a gauge, \(\delta(b_0) > 0\), and thus there is a positive integer \(N\), so that \(m((a_N,b_N]) = \frac{b-a}{2^N} < \delta(b_0)\). But then \(P = \{(b_0,(a_N,b_N])\}\) is a \(\delta\)-fine partition of \((a_N,b_N]\), which had no \(\delta\)-fine partition, and therefore the assumption that there is no \(\delta\)-fine partition of \((a,b]\) is false.

Lemma 1.9. If \(f\) is a function such that \(a^{\int_b f dm}\) exists, then \(a^{\int_b f dm}\) is unique.

Proof. Suppose that \(f\) is a real valued function whose domain includes the closed interval \([a,b]\), and that there are two distinct real numbers, \(J_1\) and \(J_2\), which by Definition 1.7 are integrals of \(f\) from \(a\) to \(b\). Then let \(\epsilon\) denote the positive number \(\frac{|J_1-J_2|}{2}\). By definition there must exist gauges \(\delta_1\) and \(\delta_2\), so that if \(P_1\) and \(P_2\) are \(\delta_1\)-fine and \(\delta_2\)-fine partitions of \((a,b]\), respectively, then \(|\Sigma[P_1,f]-J_1|<\epsilon\) and \(|\Sigma[P_2,f]-J_2|<\epsilon\). Let \(\delta_3\) denote the gauge whose value at any point \(x\) is the minimum of \(\{\delta_1(x),\delta_2(x)\}\). By the previous lemma, there exists at least one \(\delta_3\)-fine partition, say \(P_3\),
of \( (a,b] \). Then \( P_3 \) is both \( \delta_1 \)-fine and \( \delta_2 \)-fine because each component interval \( A_i \) is contained in \((x_i-\delta_3(x_i),x_i+\delta_3(x_i))\) which is in turn contained in \((x_i-\delta_1(x_i),x_i+\delta_1(x_i))\) and \((x_i-\delta_2(x_i),x_i+\delta_2(x_i))\). However, this implies that \(|J_1-J_2| \leq |J_1-\Sigma P_3,f|+|\Sigma P_3,f|-J_2|<2\varepsilon\). But this is not possible since \(2\varepsilon = |J_1-J_2|\), and hence \(a^{b}\) is unique when it exists.

To compare this integral with the commonly used Riemann integral, it is necessary to recall from real analysis a few basic definitions and a resulting property of the Riemann integral.

**Definition 1.10.** A subdivision of the closed interval \([a,b]\) is a finite set of points \(\Delta = \{x_0,x_1,x_2,\ldots,x_n\}\) in the order \(a = x_0 < x_1 < \cdots < x_n = b\).

**Definition 1.11.** A lower Darboux sum for a function \(f\) over the subdivision \(\Delta\) of \([a,b]\) is obtained by the formula \(L(f,\Delta) = \sum_{i=1}^{n} \inf\{f(x): x \in [x_{i-1},x_i]\} \cdot (x_i-x_{i-1})\), and likewise an upper Darboux sum for \(f\) on \(\Delta\) is defined by \(U(f,\Delta) = \sum_{i=1}^{n} \sup\{f(x): x \in [x_{i-1},x_i]\} \cdot (x_i-x_{i-1})\).

**Definition 1.12.** A real valued function \(f\), whose domain includes the closed interval \([a,b]\), is Riemann integrable on \([a,b]\) if and only if for each positive number \(\varepsilon\), there exists a subdivision \(\Delta_{\varepsilon}\) of \([a,b]\), so that for any subdivision \(\Delta\) containing \(\Delta_{\varepsilon}\), it follows that \(U(f,\Delta)-L(f,\Delta)<\varepsilon\).

**Property 1.13.** For any function \(f\), which is Riemann integrable on \([a,b]\), there exists a unique real number, denoted here by \(S(f,[a,b])\) and called the Riemann integral of \(f\) from
a to \( b \), such that for any subdivision \( \Delta \) of \([a,b]\), the inequality \( L(f,\Delta) \leq S(f,[a,b]) \leq U(f,\Delta) \) holds.

Theorem 1.14. The integral defined in Definition 1.7 includes the Riemann integral.

Proof. Suppose that \( f \) is a function which is Riemann integrable on the closed interval \([a,b]\), and let \( \varepsilon \) be a positive number.

Since \( f \) is Riemann integrable on \([a,b]\), there exists a subdivision \( \Delta_\varepsilon \) of \([a,b]\), so that for any subdivision \( \Delta \) which contains \( \Delta_\varepsilon \), \( U(f,\Delta)-L(f,\Delta)<\varepsilon \), and in particular \( U(f,\Delta_\varepsilon)-L(f,\Delta_\varepsilon)<\varepsilon \). It is also true that \( L(f,\Delta_\varepsilon) \leq S(f,[a,b]) \leq U(f,\Delta_\varepsilon) \), so that the inequalities \( U(f,\Delta_\varepsilon)-S(f,[a,b])<\varepsilon \) and \( S(f,[a,b])-L(f,\Delta_\varepsilon)<\varepsilon \) both hold simultaneously. By definition, \( U(f,\Delta_\varepsilon) = \sum_{i=1}^{n} \sup \{ f(x) : x \in [x_{i-1},x_i] \} \bullet (x_i-x_{i-1}) \) and \( L(f,\Delta_\varepsilon) = \sum_{i=1}^{n} \inf \{ f(x) : x \in [x_{i-1},x_i] \} \bullet (x_i-x_{i-1}) \), where \( \Delta_\varepsilon = \{ x_0,x_1,\ldots,x_n \} \).

Now, define a gauge \( \delta \) as follows:

\[
\delta(x) = \begin{cases} 
1 & \text{if } x \notin [a,b]; \\
\min \{ x-x_{i-1}, x_i-x \} & \text{when } x \in (x_{i-1},x_i), x_{i-1}, x_i \in \Delta_\varepsilon; \\
\min \{ x_i-x_{i-1}, x_{i+1}-x_i \} & \text{when } x=x_i \in \Delta_\varepsilon, i=1,2,\ldots,n-1; \\
x_{i-1}-a & \text{if } x=a; \\
b-x_{n-1} & \text{if } x=b.
\end{cases}
\]

For any \( \delta \)-fine partition \( P = \{(x_1,A_1),\ldots,(x_m,A_m)\} \) of \([a,b]\), the component intervals \( A_i \) can intersect at most two adjacent subintervals between points of the subdivision \( \Delta_\varepsilon \), and if one of the component intervals, say \( A_k \), intersects two of these adjacent subintervals \([x_{j-1},x_j]\) and \([x_j,x_{j+1}]\),
then $x_k=x_j$ and $f(x_k)m(A_k) = f(x_k)m(A_k \cap [x_{j-1}, x_j]) + f(x_k)m(A_k \cap (x_j, x_{j+1}])$.

It follows therefore that $\Sigma[P, f] = \sum_{i=1}^{m} \left( f(x_i)m(A_i) \right) = \sum_{i=1}^{m} \left( f(x_i)m(A_i) + (x_i \notin \Delta_\epsilon) \right)

\sum_{i=1}^{m} \left( f(x_i)m(A_i \cap [x_{j-1}, x_j]) + f(x_i)m(A_i \cap (x_j, x_{j+1}]) \right) = (x_i=x_j \in \Delta_\epsilon)

\sum_{i=1}^{m} \left( f(x_i)m(A_i \cap [x_{j-1}, x_j]) + f(x_i)m(A_i \cap (x_j, x_{j+1}]) \right) + (x_i \in (x_{j-1}, x_j)) (x_i=x_{j-1})

\sum_{i=1}^{m} f(x_i)m(A_i \cap [x_{j-1}, x_j])).

Let $A$ denote this last summation. Then,

$A \leq \frac{1}{n} \left( \sum_{i=1}^{m} f(x_i)m(A_i) + \sum_{i=1}^{m} f(x_i)m(A_i \cap (x_j, x_{j+1}]) \right)

\sum_{i=1}^{m} \left( f(x_i)m(A_i \cap [x_{j-1}, x_j]) + f(x_i)m(A_i \cap (x_j, x_{j+1}]) \right) = (x_i=x_j)

\sum_{i=1}^{m} \left( f(x_i)m(A_i \cap [x_{j-1}, x_j]) + f(x_i)m(A_i \cap (x_j, x_{j+1}]) \right) = (x_j-x_{j-1}) = U(f, \Delta_\epsilon).

Similarly, $A \leq L(f, \Delta_\epsilon)$, and thus $L(f, \Delta_\epsilon) \leq \Sigma[P, f] \leq U(f, \Delta_\epsilon)$.

Hence $|\Sigma[P, f] - S(f, [a, b])| \leq U(f, \Delta_\epsilon) - L(f, \Delta_\epsilon) < \epsilon$, which shows that $f$ is integrable from $a$ to $b$ and that $\int_{a}^{b} f dm = S(f, [a, b])$.

To show that this inclusion is proper, consider the function $f$ defined on the set of all real numbers as follows:

\[
f(x) = \begin{cases} 
  1 & \text{if } x \text{ is a rational number;} \\
  0 & \text{if } x \text{ is an irrational number.}
\end{cases}
\]
The function $f$ is not Riemann integrable on any closed interval $[a,b]$ since for any subdivision $\Delta$ of $[a,b]$, $U(f,\Delta) = b-a$ and $L(f,\Delta) = 0$, which means that there does not exist a subdivision $\Delta_{b-a}$, so that any subdivision $\Delta$ containing $\Delta_{b-a}$ has the property $U(f,\Delta) - L(f,\Delta) < b-a$.

To show that $f$ is integrable (in the sense of Definition 1.7) from $a$ to $b$, let $\{x_p\}_{p=1}^\infty$ be a sequence consisting of all rational numbers in $[a,b]$. For any positive number $\varepsilon$, define a gauge $\delta$ by:

$$\delta(x) = \begin{cases} 1 & \text{if } x \text{ is irrational or } x \notin [a,b]; \\
\frac{\varepsilon}{2^{p+2}} & \text{if } x = x_p \in \{x_p\}_{p=1}^\infty 
\end{cases}$$

Then, if $P$ is any $\delta$-fine partition of $(a,b]$, $|\Sigma[P,f] - 0| = \Sigma[P,f] = \frac{\varepsilon}{2^{p+1}} \sum_{i=1}^n f(x_i) m(A_i) < \varepsilon \sum_{i=1}^\infty 2^{p+1} = \varepsilon < \varepsilon$, and hence $\int_a^b f \, dm = 0$.

As a result of Theorem 1.14, the following properties are inherited from identical theorems on Riemann integration:

Theorem 1.15. If $f$ is a continuous real valued function on $[a,b]$, then $\int_a^b f \, dm$ exists;

Theorem 1.16. If $f$ is a continuous real valued function on $[a,b]$ and $F$ is any anti-derivative of $f$, then $\int_a^b f \, dm = F(b) - F(a)$;

Theorem 1.17. If $f$ is a function having bounded variation on $[a,b]$, then $\int_a^b f \, dm$ exists.
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CHAPTER II

BASIC PROPERTIES OF THE INTEGRAL

In this chapter, the basic properties of the integral defined in Definition 1.7 are stated and proved. Chief among these are the equivalence of a Cauchy-type definition for integrability and a theorem on the integrability of the absolute value of an integrable function.

Theorem 2.1. If \( f \) is a function whose domain includes the interval \([a,b]\), \( c \) is any real number, and \( \int_a^b f \, dm \) exists, then \( \int_a^c f \, dm \) exists and \( \int_a^c f \, dm = c \cdot \int_a^b f \, dm \).

Proof. If \( c = 0 \), then for any positive \( \varepsilon \), let \( \delta \) denote the gauge whose value is 1 everywhere. For any \( \delta \)-fine partition \( P \) of \((a,b]\), \(|\sum [P,c \cdot f] - 0| = 0 \leq \varepsilon \) and the theorem follows.

In the case \( c \neq 0 \), then for any positive \( \varepsilon \), there exists a gauge \( \delta \) so that if \( P \) is a \( \delta \)-fine partition of \((a,b]\), then

\[
|\sum [P,f] - a \int_a^b f \, dm| < |\frac{\varepsilon}{|c|}.
\]

Then it follows that \(|\sum [P,c \cdot f] - c \cdot a \int_a^b f \, dm| = |c| \cdot |\sum [P,f] - a \int_a^b f \, dm| < |\frac{\varepsilon}{|c|} \cdot |c| \cdot |c| = \varepsilon \), which proves the theorem.

Theorem 2.2. If \( f \) and \( g \) are integrable from \( a \) to \( b \), then so is \( f + g \), and \( \int_a^b f + g \, dm = \int_a^b f \, dm + \int_a^b g \, dm \).

Proof. For each positive \( \varepsilon \), there exist gauges \( \delta_1 \) and \( \delta_2 \), so that if \( P_1 \) and \( P_2 \) are \( \delta_1 \)-fine and \( \delta_2 \)-fine partitions
of \((a,b]\), respectively, then \(\left| \int_{a}^{b} f \, dm \right| < \frac{\epsilon}{2} \) and \(\left| \int_{a}^{b} g \, dm \right| < \frac{\epsilon}{2} \).

Let \(\delta(x)\) denote \(\min\{\delta_1(x), \delta_2(x)\}\) for each \(x\). Then if \(P\) is any \(\delta\)-fine partition of \((a,b]\), then \(\left| \int_{P,f}^{} - \int_{P,g}^{} \right| < \epsilon\), since \(P\) is necessarily both \(\delta_1\)-fine and \(\delta_2\)-fine.

Lemma 2.3. A function \(f\), whose domain includes the closed interval \([a,b]\), is integrable from \(a\) to \(b\) if and only if for each positive number \(\epsilon\), there exists a gauge \(\delta\), so that if \(P_1\) and \(P_2\) are \(\delta\)-fine partitions of \((a,b]\), then \(\left| \int_{P_1,f}^{} - \int_{P_2,f}^{} \right| < \epsilon\).

Proof. First, suppose that \(f\) is integrable from \(a\) to \(b\). Then for any positive \(\epsilon\), there exists a gauge \(\delta\), so that if \(P\) is a \(\delta\)-fine partition of \((a,b]\), then \(\left| \int_{a}^{b} f \, dm \right| < \frac{\epsilon}{2}\). Hence, if \(P_1\) and \(P_2\) are \(\delta\)-fine partitions of \((a,b]\), then

\[
\left| \int_{P_1,f}^{} - \int_{P_2,f}^{} \right| = \left| \int_{a}^{b} f \, dm + \int_{a}^{b} g \, dm - \int_{a}^{b} g \, dm - \int_{a}^{b} f \, dm \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

To show the converse, suppose it is true that for any positive \(\epsilon\), there is a gauge \(\delta\), so that if \(P_1\) and \(P_2\) are arbitrary \(\delta\)-fine partitions of \((a,b]\), then \(\left| \int_{P_1}^{} - \int_{P_2}^{} \right| < \epsilon\). Then let \(\epsilon_1 = 1\). There is a gauge \(\delta_1\), so that if \(P_1\) and \(P_2\) are \(\delta_1\)-fine partitions of \((a,b]\), then \(\left| \int_{P_1,f}^{} - \int_{P_2,f}^{} \right| < 1\). Thus, for any \(\delta_1\)-fine partition \(P\) of \((a,b]\), \(\left| \int_{P}^{} - \int_{P_1}^{} \right| < 1\). Using the triangle inequality, \(\left| \int_{P}^{} - \int_{P_1}^{} \right| < \left| \int_{P}^{} - \int_{P_1}^{} \right| + \left| \int_{P_1}^{} - \int_{P_2}^{} \right| < 1\), so that \(\left| \int_{P}^{} \right| < 1 + \left| \int_{P_1}^{} \right|\) for any \(\delta_1\)-fine partition \(P\) of \((a,b]\).
For each integer $n>1$, there exists a gauge $\delta_n'$, so that if $P_1$ and $P_2$ are $\delta_n'$-fine partitions of $(a,b)$, then $|\Sigma[P_1,f]-\Sigma[P_2,f]|<\frac{1}{n}$. Let $\delta_n(x)=\min\{\delta_1(x),\ldots,\delta_{n-1}(x),\delta_n'(x)\}$. Now, for each $n>1$, there exists a $\delta_n$-fine partition $P_n$ of $(a,b)$, and by the construction of $\delta_n$, each $P_n$ is $\delta_1$-fine, and thus $|\Sigma[P_n,f]|<1+|\Sigma[P_1,f]|$. This shows that $\{\Sigma[P_n,f]\}_{n=1}^\infty$ is a bounded sequence of real numbers and therefore has a cluster point $J$ to which some subsequence $\{\Sigma[P_n,f]\}_{k=1}^\infty$ converges.

Let $\varepsilon>0$. There exists a positive integer $N_1$, so that if $k>N_1$, then $|\Sigma[P_{n_k},f]-J|<\frac{\varepsilon}{2}$, and there exists a corresponding gauge $\delta_N$, where $N=nN_1$. There is also a positive integer $N_2$, so that $\frac{1}{N_2}<\frac{\varepsilon}{2}$ and a corresponding gauge $\delta_{N_2}$.

Now, let $\delta(x)=\min\{\delta_N(x),\delta_{N_2}(x)\}$. Then for any $\delta$-fine partition $P$ of $(a,b)$, pick $n_k>N+N_2$ and then $|\Sigma[P,f]-J|=|\Sigma[P,f]-\Sigma[P_{n_k},f]+\Sigma[P_{n_k},f]-J|$, where $P_{n_k}$ is $\delta_{N}-$fine, $\delta_{N_2}$-fine, and $\delta_{N_2}$-fine, so that $|\Sigma[P,f]-J|\leq|\Sigma[P,f]-\Sigma[P_{n_k},f]|+|\Sigma[P_{n_k},f]-J|<\frac{1}{N_2}+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. Therefore, $a^b\text{fdm}$ exists and the proof is complete.

Theorem 2.4. If $f$ is a function so that $a^b\text{fdm}$ exists, and $a<c<b$, then $a^c\text{fdm}$ and $c^b\text{fdm}$ both exist, and $a^c\text{fdm}+c^b\text{fdm}=a^b\text{fdm}$.

Proof. By Lemma 2.3, for any positive $\varepsilon$, there is a gauge $\delta$, so that if $P_1$ and $P_2$ are $\delta$-fine partitions of $(a,b)$, then $|\Sigma[P_1,f]-\Sigma[P_2,f]|<\varepsilon$. Also, there is at least one $\delta$-fine partition $P_0$ of $(c,b)$. Then if $P_1'$ and $P_2'$ are $\delta$-fine partitions of $(a,c)$, $P_1''=P_1'\cup P_0$ and $P_2''=P_2'\cup P_0$ are $\delta$-fine partitions of $(a,b)$,
and thus \( |\Sigma [P_1, f] - \Sigma [P_2, f]| = |\Sigma [P_1, f] + \Sigma [P_0, f] - (\Sigma [P_2, f] + \Sigma [P_0, f])| = |\Sigma [P_1, f] - \Sigma [P_2, f]| < \varepsilon \), and hence \( a_f^{c \text{fdm}} \) exists. A similar argument will show that \( c_r^{b \text{fdm}} \) also exists.

To obtain the last part of the conclusion, let \( \varepsilon \) be any positive number. There exist gauges \( \delta_1, \delta_2, \) and \( \delta_3 \), so that if \( P_1, P_2, \) and \( P_3 \) are \( \delta_1 \)-fine, \( \delta_2 \)-fine, and \( \delta_3 \)-fine partitions of \( (a, b) \), \( (a, c) \), and \( (c, b) \), respectively, then \( |\Sigma [P_1, f] - a_f^{b \text{fdm}}| < \frac{\varepsilon}{3} \), \( |\Sigma [P_2, f] - a_f^{c \text{fdm}}| < \frac{\varepsilon}{3} \), and \( |\Sigma [P_3, f] - c_f^{b \text{fdm}}| < \frac{\varepsilon}{3} \). For \( \delta(x) = \min \{ \delta_1(x), \delta_2(x), \delta_3(x) \} \) and any \( \delta \)-fine partitions \( P_1 \) and \( P_2 \) of \( (a, c) \) and \( (c, b) \), respectively, \( P = P_1 \cup P_2 \) is a \( \delta \)-fine partition of \( (a, b) \), and \( |a_f^{b \text{fdm}} - (a_f^{c \text{fdm}} + c_f^{b \text{fdm}})| = |a_f^{b \text{fdm}} - \Sigma [P, f] + \Sigma [P, f] - (a_f^{c \text{fdm}} + c_f^{b \text{fdm}})| < |a_f^{b \text{fdm}} - \Sigma [P, f]| + |\Sigma [P_1, f] - \Sigma [P_2, f] - (a_f^{c \text{fdm}} + c_f^{b \text{fdm}})| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \). It follows that \( a_f^{b \text{fdm}} = a_f^{c \text{fdm}} + c_f^{b \text{fdm}} \).

Theorem 2.5. If \( a_f^{b \text{fdm}} \) and \( a_f^{b \text{gdm}} \) exist, and \( f(x) \geq g(x) \) for each \( x \in [a, b] \), then \( a_f^{b \text{gdm}} \geq a_f^{b \text{fdm}} \).

Proof. If \( a_f^{b \text{gdm}} > a_f^{b \text{fdm}} \), let \( \varepsilon = a_f^{b \text{gdm}} - a_f^{b \text{fdm}} \). There must exist gauges \( \delta_1 \) and \( \delta_2 \), so that for any \( \delta_1 \)-fine and \( \delta_2 \)-fine partitions \( P_1 \) and \( P_2 \), respectively, \( |\Sigma [P_1, f] - a_f^{b \text{fdm}}| < \frac{\varepsilon}{2} \) and \( |\Sigma [P_2, g] - a_f^{b \text{gdm}}| < \frac{\varepsilon}{2} \). Then let \( \delta_3(x) = \min \{ \delta_1(x), \delta_2(x) \} \) for all \( x \), and let \( P_3 \) be any \( \delta_3 \)-fine partition of \( (a, b) \). Then \( \varepsilon = a_f^{b \text{gdm}} - \Sigma [P_3, g] + \Sigma [P_3, g] - a_f^{b \text{fdm}} \leq a_f^{b \text{gdm}} - \Sigma [P_3, g] + \Sigma [P_3, f] - a_f^{b \text{fdm}} \leq a_f^{b \text{gdm}} - \Sigma [P_3, g] + \Sigma [P_3, f] - a_f^{b \text{fdm}} | \Sigma [P_3, f] - a_f^{b \text{fdm}} | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \), which shows by contradiction that \( a_f^{b \text{gdm}} \leq a_f^{b \text{fdm}} \).

Corollary 2.6. If \( a_f^{b \text{fdm}} \) exists, and \( f(x) \geq 0 \) for all \( x \in [a, b] \), then \( a_f^{b \text{fdm}} \geq 0 \).
Proof. By Theorem 2.1, \( a^b \cdot f dm = 0 \). So applying Theorem 2.5, it follows that \( a^b f dm \geq a^b \cdot f dm = 0 \).

Theorem 2.7. If \( f \) is a function whose domain includes the interval \([a,b]\) and \( a^b f dm \) exists, then \( a^b |f| dm \) exists, and \( |a^b f dm| \leq a^b |f| dm \).

Proof. Define a function \( f^+(x) = \max\{f(x), 0\} \), and suppose that \( a^b f^+ dm \) does not exist. Then, the conditions of Lemma 2.3 cannot hold. So there must exist some positive number \( \varepsilon \), so that for any gauge \( \delta \), there are \( \delta \)-fine partitions \( P_1 \) and \( P_2 \) of \([a,b]\), so that \( |\Sigma[P_1, f^+] - \Sigma[P_2, f^+]| \geq \varepsilon \). Let \( \delta \) be any gauge, and consider the \( \delta \)-fine partitions \( P_1 = \{(x_1, A_1), \ldots, (x_n, A_n)\} \) and \( P_2 = \{(x_1', B_1), \ldots, (x_m', B_m)\} \) guaranteed by the above.

Let \( P_3 = \{(x_1, A_1 \cap B_1), \ldots, (x_n, A_n \cap B_1), (x_1, A_1 \cap B_2), \ldots, (x_n, A_n \cap B_2), \ldots, (x_n, A_n \cap B_m)\} \) and \( P_4 = \{(x_1', A_1 \cap B_1), \ldots, (x_n', A_n \cap B_1), (x_1', A_1 \cap B_2), \ldots, (x_n', A_n \cap B_2), \ldots, (x_n', A_n \cap B_m)\} \). Then \( P_3 \) and \( P_4 \) are \( \delta \)-fine partitions of \([a,b]\), \( \Sigma[P_3, f^+] = \Sigma[P_1, f^+] \), and \( \Sigma[P_4, f^+] = \Sigma[P_2, f^+] \), so that \( \sum_{p \neq 1}^{m} \sum_{q \neq 1}^{m} \left( f^+(x_p) - f^+(x_q') \right) m(A_p \cap B_q) = \left| \Sigma[P_3, f^+] - \Sigma[P_4, f^+] \right| \geq \varepsilon \).

Dividing this sum into three parts, and using the triangle inequality, it follows that (I)

\[
\varepsilon \leq \sum_{p=1}^{n} \sum_{q=1}^{m} \left( f^+(x_p) - f^+(x_q') \right) m(A_p \cap B_q) + (f^+(x_p) > f^+(x_q'))
\]

\[
\sum_{p=1}^{n} \sum_{q=1}^{m} \left( f^+(x_p) - f^+(x_q') \right) m(A_p \cap B_q) + (f^+(x_p) = f^+(x_q'))
\]

\[
\sum_{p=1}^{n} \sum_{q=1}^{m} \left( f^+(x_p) - f^+(x_q') \right) m(A_p \cap B_q) + (f^+(x_p) < f^+(x_q'))
\]

\[
\sum_{p=1}^{n} \sum_{q=1}^{m} \left( f^+(x_p) - f^+(x_q') \right) m(A_p \cap B_q) + (f^+(x_p) = f^+(x_q'))
\]
and since the middle term is identically zero, it follows that either the first or last term must be greater than or equal to $\frac{\varepsilon}{2}$.

If it were the first term, then

$$\frac{\varepsilon}{2} \leq \left| \sum_{p=1}^{n} \sum_{q=1}^{m} (f^+(x_p) - f^+(x_q')) m(A_p \cap B_q) \right| = \left( f^+(x_p) > f^+(x_q') \right)$$

$$\leq \sum_{p=1}^{n} \sum_{q=1}^{m} (f^+(x_p) - f^+(x_q')) m(A_p \cap B_q) = \left( f^+(x_p) > f^+(x_q') \right)$$

$$\leq \left| \sum_{p=1}^{n} \sum_{q=1}^{m} (f(x_p) - f(x_q')) m(A_p \cap B_q) \right| = \left( f^+(x_p) > f^+(x_q') \right)$$

$$\leq \left| \sum_{p=1}^{n} \sum_{q=1}^{m} (f(x_p) - f(x_q')) m(A_p \cap B_q) \right| + \left( f^+(x_p) > f^+(x_q') \right)$$

$$\leq \left| \sum_{p=1}^{n} \sum_{q=1}^{m} (f(x_p) - f(x_q')) m(A_p \cap B_q) \right| = \left( f^+(x_p) \leq f^+(x_q') \right)$$

$$\left| \Sigma[P_5,f] - \Sigma[P_6,f] \right|,$$ where $P_5$ and $P_6$ are $\delta$-fine partitions of $(a,b]$. This implies that there is a positive number $\frac{\varepsilon}{2}$, so that for any gauge $\delta$, there are $\delta$-fine partitions $P_5$ and $P_6$ of $(a,b]$, so that $\left| \Sigma[P_5,f] - \Sigma[P_6,f] \right| \geq \frac{\varepsilon}{2}$, or in other words that $\int_a^b f dm$ does not exist.
Supposing that the last term of (I) were greater than or equal to $\frac{e}{2}$ and following a similar argument would lead to the same contradiction, and therefore $\int_a^b f^+ dm$ must exist.

Now, define $f^-(x) = f^+(x) - f(x)$. Then, $f^-$ and $f^+$ are non-negative for all $x \in [a, b]$, and by Theorems 2.1 and 2.2, $\int_a^b f^- dm = \int_a^b f^+ dm - \int_a^b f^- dm$.

Furthermore, $|f(x)| = f^+(x) + f^-(x)$, so again using Theorem 2.2, $\int_a^b |f| dm$ exists and $\int_a^b |f| dm = \int_a^b f^+ dm + \int_a^b f^- dm = |\int_a^b f^+ dm| + |\int_a^b f^- dm| = |\int_a^b f^- dm| + |\int_a^b f^+ dm|.$

Corollary 2.8. If $\{f_1, f_2, \ldots, f_n\}$ is a finite collection of functions such that $\int_a^b f_i dm$ exists for $i=1, 2, \ldots, n$, then both of the functions $g_{1,n}(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$ and $g_{2,n}(x) = \min\{f_1(x), f_2(x), \ldots, f_n(x)\}$ are integrable from $a$ to $b$.

Proof. If $n=2$, then $g_{1,n} = \max\{f_1, f_2\} = \frac{1}{2}(f_1 + f_2 - |f_1 - f_2|)$ and $g_{2,n} = \min\{f_1, f_2\} = \frac{1}{2}(f_1 + f_2 + |f_1 - f_2|)$, so that by Theorems 2.1, 2.2, and 2.7, both are integrable from $a$ to $b$ and the corollary holds for $n=2$.

Assume that the corollary is true for $n=k$, and suppose that $f_1, f_2, \ldots, f_{k+1}$ are all integrable functions from $a$ to $b$.

Then $g_{1,k+1} = \max\{f_1, \ldots, f_{k+1}\} = \max\{\max\{f_1, f_2, \ldots, f_k\}, f_{k+1}\}$ is the maximum of two functions, each integrable from $a$ to $b$, and thus is integrable from $a$ to $b$. By the same argument, $g_{2,k+1}$ is also integrable, and thus by induction the corollary is proved.

Theorem 2.9. If $\int_a^b f dm$ exists, and $g$ is any function, whose domain includes $[a, b]$, with the property that $g(x) = f(x)$
for all \( x \in [a,b] \) except those of some countable sequence 
\( \{x_n\}_{n=1}^{\infty} \) of distinct points in \([a,b]\), then \( a^b \text{gdm} \) exists
and \( a^b \text{gdm} = a^b \text{fdm} \).

Proof. Assume that \( g(x_i) \neq f(x_i) \), for \( i = 1, 2, \cdots \) and
let \( \varepsilon \) be a positive number. Then define a gauge \( \delta_1 \) as follows:

\[
\delta_1(x) = \begin{cases} 
1 & \text{if } x \notin [a,b], \text{ or if } x \in [a,b] \text{ and } f(x) = g(x); \\
\frac{\varepsilon}{2^{n+2}|f(x)|} & \text{if } x = x_n \in \{x_n\}_{n=1}^{\infty}.
\end{cases}
\]

For any \( \delta_1 \)-fine partition \( P \) of \((a,b]\), let \( q \) denote the
maximum sequence subscript for the evaluation points of \( P 
\) which belong to \( \{x_n\}_{n=1}^{\infty} \), or \( q = 1 \) if no evaluation points are
in the sequence.

Then it follows that
\[
|\Sigma(P,f) - \Sigma(P,g)| = \left| \sum_{p=1}^{m} (f(x'_p) - g(x'_p))m(A_p) \right|
\leq \sum_{p=1}^{m} |f(x'_p) - g(x'_p)|m(A_p) = \sum_{p=1}^{m} |f(x'_p) - g(x'_p)|m(A_p) \leq \sum_{p=1}^{m} |f(x'_p) - g(x'_p)|m(A_p).
\]

Now there is also a gauge \( \delta_2 \), so that if \( P \) is any
\( \delta_2 \)-fine partition of \((a,b]\), then 
\( |\Sigma(P,f) - \Sigma(P,g)| < \frac{\varepsilon}{2} \). Let
\( \delta(x) = \min\{\delta_1(x), \delta_2(x)\} \). Then if \( P \) is any \( \delta \)-fine partition
of \((a,b]\), \( |\Sigma(P,g) - \Sigma(P,f)| < |\Sigma(P,f) - \Sigma(P,g)| + |\Sigma(P,f) - \Sigma(P,g)| < \frac{\varepsilon}{2} \).

CHAPTER BIBLIOGRAPHY


CHAPTER III

CONVERGENCE PROPERTIES

In this chapter, some general convergence theorems are proved. Examples are given, including an unbounded integrable function and an integrable function whose square is not integrable. Finally, a monotone convergence theorem and a dominated convergence theorem are stated and proved.

Theorem 3.1. If \( \{f_n\}_{n=1}^{\infty} \) is a sequence of functions, each integrable from \( a \) to \( b \), and \( \{f_n\}_{n=1}^{\infty} \) converges uniformly to a function \( F \) on \([a,b]\), then \( \int_a^b F \, dm \) exists and is

\[
\lim_{n \to \infty} \int_a^b f_n \, dm.
\]

Proof. Let \( \epsilon \) be any positive number. Then since \( \{f_n\}_{n=1}^{\infty} \) converges uniformly to \( F \), there must exist a positive integer \( N \), so that

\[
|f_n(x) - F(x)| < \frac{\epsilon}{3(b-a)}
\]

for all \( x \in [a,b] \) and for all \( n \geq N \). Also, for each integer \( n \) there is a gauge \( \delta_n \), so that for any two \( \delta_n \)-fine partitions \( P_1 \) and \( P_2 \) of \([a,b]\),

\[
|\sum [P_1, f_n] - \sum [P_2, f_n]| < \epsilon.
\]

Now, if \( P_1 \) and \( P_2 \) are arbitrary \( \delta_n \)-fine partitions of \([a,b]\), it follows that

\[
|\Sigma[P_1,F]-\Sigma[P_2,F]| = |\Sigma[P_1,F]-\Sigma[P_1,f_N]| + \\
|\Sigma[P_1,f_N]-\Sigma[P_2,f_N]| + |\Sigma[P_2,f_N]-\Sigma[P_2,F]| + |\Sigma[P_2,F]-\Sigma[P_1,F]| + \\
|\Sigma[P_1,f_N]-\Sigma[P_2,f_N]| + |\sum_{q=1}^{m} (f_N(x'_q)-F(x'_q))m(B_q)| + \\
|\sum_{p=1}^{n} (F(x_p)-f_N(x_p))m(A_p)| + \\
|\Sigma[P_1,f_N]-\Sigma[P_2,f_N]| + |\sum_{q=1}^{m} (f_N(x'_q)-F(x'_q))m(B_q)| < \epsilon.
\]

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\[
\begin{align*}
\sum_{p=1}^{\infty} \frac{\varepsilon}{3(b-a)} m(A_p) + \sum_{q=1}^{\infty} \frac{\varepsilon}{3(b-a)} m(B_q) &= \frac{\varepsilon}{3(b-a)} \sum_{p=1}^{\infty} m(A_p) + \\
&= \varepsilon, \text{ and therefore by Lemma 2.3,}
\end{align*}
\]

Next, to show that \(a^{\beta} f dm \) exists.

There may be cases where simple convergence of a sequence \(\{f_n\}_{n=1}^{\infty}\) will produce the same results. An interesting example of this is the following.

Example 3.2. Take any closed interval \([a,b]\) and put the rational numbers in \([a,b]\) into a sequence \(\{x_n\}_{n=1}^{\infty}\) and define a sequence of functions by defining \(f_n\) for each positive \(n\) as follows:

\[f_n(x) = \begin{cases} 
1 & \text{if } x = x_1, x_2, \ldots, x_n; \\
0 & \text{otherwise.}
\end{cases}\]
Next, let \( F(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number in } [a,b]; \\ 0 & \text{otherwise}. \end{cases} \)

Now, since all of the functions \( F, f_1, f_2, \ldots \) differ from the constant function 0 at only a countable sequence of points in \([a,b]\), then each is integrable from \( a \) to \( b \) and the integral of each is 0.

Obviously, \( \{f_n\}_{n=1}^\infty \) does not converge uniformly to \( F \), but it does converge pointwise, and still \( \lim_{n \to \infty} \int_a^b f_n \, dm = \int_a^b F \, dm \).

In this particular example, each of the functions \( f_n \) is Riemann integrable, yet the limit function \( F \) is not.

Although this example appears to imply that it may be possible to replace uniform convergence with simple convergence in the hypothesis of Theorem 3.1, this is not the case and the next theorem is helpful in constructing an example to show this.

Theorem 3.3. If \( f \) is a function whose domain includes \([a,b]\), and \( f \) is bounded over \([a,b]\), and has no more than a countable number of discontinuities in \([a,b]\), then \( \int_a^b f \, dm \) exists.

Proof. Let \( \varepsilon \) be a positive number. Then since \( f \) is bounded, there exists a positive number \( K \), so that \( |f(x)| < K \) for all \( x \in [a,b] \). Now, according to the hypothesis, the points in \([a,b]\) at which \( f \) is not continuous can be put into a sequence \( \{x_p\}_{p=1}^\infty \).
For any point $x_0 \in [a,b]$ where $f$ is continuous, there is a positive number $\delta_{x_0}$, so that if $x \in [a,b]$ with $|x-x_0| < \delta_{x_0}$, then $|f(x) - f(x_0)| < \frac{\varepsilon}{4(b-a)}$.

Now define a gauge $\delta$ as follows:

$$
\delta(x) = \begin{cases} 
1 & \text{if } x \notin [a,b]; \\
\frac{\delta_x}{2} & \text{if } f \text{ is continuous at } x \in [a,b]; \\
\frac{\varepsilon}{8K^2 + 1} & \text{if } x = x_i \in \{x_p\}_{p=1}^\infty.
\end{cases}
$$

If $P_1$ and $P_2$ are $\delta$-fine partitions of $(a,b]$, then there exist partitions $P_3$ and $P_4$, as defined in the proof of Theorem 2.7, which are also $\delta$-fine, both having the same component intervals, so that $|\Sigma[P_1,f] - \Sigma[P_2,f]| = |\Sigma[P_3,f] - \Sigma[P_4,f]| = |\Sigma[P_4,f]| = |

\begin{align*}
&\sum_{p=1}^{n} \sum_{q=1}^{m} (f(x_p) - f(x'_q)) m(A_p \cap B_q) | + \\
&\sum_{p=1}^{n} \sum_{q=1}^{m} (f(x'_p) - f(x_q)) m(A_p \cap B_q) | \\
&\sum_{p=1}^{n} \sum_{q=1}^{m} (f(x_p) - f(x'_q)) m(A_p \cap B_q) | + \\
&\sum_{p=1}^{n} \sum_{q=1}^{m} (f(x'_p) - f(x_q)) m(A_p \cap B_q) |
\end{align*}

(f continuous at both $x_p$ and $x'_q$) 

(f discontinuous at $x_p$) 

(f discontinuous at $x'_q$, but continuous at $x_p$) 

(f continuous at both $x_p$ and $x'_q$) 

(f discontinuous at $x_p$)
\[
\sum_{p=1}^{n} \sum_{q=1}^{m} |f(x_p \_ f(x'_q)| m(A_p \cap B_q) > (I)
\]

(f discontinuous at \(x'_q\), but continuous at \(x_p\))

In the first group of terms, if \(A_p \cap B_q \neq \emptyset\) then pick \(x_p, q \in A_p \cap B_q\), and then \(|f(x_p) - f(x'_q)| m(A_p \cap B_q) \leq \)

\[
|f(x_p) - f(x'_q)| m(A_p \cap B_q) + |f(x_p, q) - f(x'_q)| m(A_p \cap B_q) <
\]

\[
\frac{\varepsilon}{4(b-a)} m(A_p \cap B_q) + \frac{\varepsilon}{4(b-a)} m(A_p \cap B_q) = \frac{\varepsilon}{2(b-a)} m(A_p \cap B_q). \quad \text{If}
\]

\(A_p \cap B_q = \emptyset\), then \(|f(x_p) - f(x'_q)| m(A_p \cap B_q) = 0\). So, the first group of terms

\[
\sum_{p=1}^{n} \sum_{q=1}^{m} |f(x_p) - f(x'_q)| m(A_p \cap B_q) <
\]

(f continuous at both \(x_p\) and \(x'_q\))

\[
\sum_{p=1}^{n} \sum_{q=1}^{m} \frac{\varepsilon}{2(b-a)} m(A_p \cap B_q) = \frac{\varepsilon}{2(b-a)} \sum_{p=1}^{n} \sum_{q=1}^{m} m(A_p \cap B_q) = \frac{\varepsilon}{2}.
\]

In the second group of terms,

\[
\sum_{p=1}^{n} \sum_{q=1}^{m} |f(x_p) - f(x'_q)| m(A_p \cap B_q) \leq
\]

(f discontinuous at \(x_p\))

\[
\sum_{p=1}^{n} \sum_{q=1}^{m} (|f(x_p)| + |f(x'_q)|) m(A_p \cap B_q) <
\]

(f discontinuous at \(x_p\))

\[
\sum_{p=1}^{n} \sum_{q=1}^{m} 2K m(A_p \cap B_q) = \sum_{p=1}^{n} 2K \sum_{q=1}^{m} m(A_p \cap B_q) = \sum_{p=1}^{n} 2K m(A_p).
\]

(f discontinuous at \(x_p\)) (f discontinuous at \(x_p\)) (f discontinuous at \(x_p\))
Now, let \( r \) denote the maximum sequence subscript of the evaluation points \( \{x_1, x_2, \ldots, x_n\} \) in the sequence \( \{x_p\}_{p=1}^{\infty} \) of discontinuities of \( f \). Then

\[
\sum_{p=1}^{n} 2Km(A_p) < \sum_{i=1}^{r} 2K \frac{\varepsilon}{4K2i+1} < \sum_{i=1}^{\infty} \frac{\varepsilon}{4 \cdot 2^i} = \frac{\varepsilon}{4}.
\]

Similarly, the third group of terms in (I),

\[
\sum_{p=1}^{n} \sum_{q=1}^{m} |f(x_p) - f(x_q)| m(A_p \cap B_q) < \frac{\varepsilon}{4},
\]

(\( f \) discontinuous at \( x'_q \), but continuous at \( x_p \))

and it follows that \( |\Sigma [P_1, f] - \Sigma [P_2, f]| < \varepsilon \) for any two \( \delta \)-fine partitions \( P_1 \) and \( P_2 \) of \( (a, b] \), and thus by Lemma 2.3, \( \alpha_{b \text{fim}} \) exists.

It is now possible to construct a sequence of functions \( \{f_n\}_{n=1}^{\infty} \), each integrable on a closed interval such that \( \{f_n\}_{n=1}^{\infty} \) converges pointwise to a function \( F \) on the interval and yet \( F \) is not integrable.

Example 3.4. Consider the closed interval \([0,1]\) and define a sequence of functions \( \{f_n\}_{n=1}^{\infty} \) by

\[
f_n(x) = \begin{cases} 
2 & \text{if } 0 \leq x < \frac{1}{2} ; \\
2^2 & \text{if } \frac{1}{2} \leq x < \frac{3}{4} ; \\
2^n & \text{if } \frac{2^{n-1} - 1}{2n-1} \leq x < 1 ; \\
0 & \text{if } x = 1 ,
\end{cases}
\]
and define $F(x)$ by

$$F(x) = \begin{cases} 
2 & \text{if } 0 < x < \frac{1}{2}; \\
2^2 & \text{if } \frac{1}{2} \leq x < \frac{3}{4}; \\
& \vdots \\
2^n & \text{if } \frac{2^{n-1}-1}{2^n} \leq x < \frac{2^n-1}{2n}; \\
& \vdots \\
0 & \text{if } x = 1.
\end{cases}$$

Each of the functions in the sequence $\{f_n\}_{n=1}^\infty$ is integrable from 0 to 1, since $|f_n(x)| < 2^n$ and $f_n(x)$ is continuous at all points in $[0,1]$ except

$$\left\{\frac{1}{2}, \frac{3}{4}, \ldots, \frac{2^n-1-1}{2n}, 1\right\}.$$ 

Also, for each $x \in [0,1]$, $\lim_{n \to \infty} f_n(x) = F(x)$. But, $F$ cannot be integrable, for otherwise, pick a positive integer $K > \int_0^1 F \, dm$ and then

$$0 \int_0^1 f_K = \int_0^{2K-1} f_K \, dm + \int_{2K-1}^{2K} f_K \, dm > 0 \int_{2K}^{2K-1} f_K \, dm = \int_0^1 f_K \, dm > 0 \int_0^1 F \, dm,$$

but $F(x) > f_K(x)$ for all $x \in [0,1]$, and thus $0 \int_0^1 F \, dm > 0 \int_0^1 f_K \, dm > 0 \int_0^1 F \, dm$, which cannot be possible.
In this last example, \( \lim_{n \to \infty} \int_0^1 f_n \, dm \) does not exist.

However, adding this condition to the pointwise convergence of \( \{f_n\}_{n=1}^\infty \) will still not yield the conclusions of Theorem 3.1.

Example 3.5. Let \( \xi_A \) denote the characteristic function on the arbitrary set \( A \) and define a sequence of functions \( \{f_n\}_{n=1}^\infty \) by

\[
f_n(x) = 2^n \xi_{\left[\frac{2^n-1}{2^n}, \frac{2^n}{2^n-1}\right]}(x).
\]

Again, each function \( f_n \) is bounded and has no more than two points of discontinuity, so that \( \int_0^1 f_n \, dm \) exists for each \( n \). Also, \( \{f_n\}_{n=1}^\infty \) converges pointwise to the constant function 0 on \([0,1]\), and thus \( \int_0^1 \lim_{n \to \infty} f_n \, dm = 0 \), yet \( \lim_{n \to \infty} \int_0^1 f_n \, dm = 1 \), since for each positive integer \( n \),

\[
\int_0^1 f_n \, dm = \int_{2^n-1}^{2^n-1} f_n \, dm + \int_{2^n}^{2^n-1} f_n \, dm + \int_{2^n-1}^1 f_n \, dm = \frac{1}{2^n-1} \int_{2^n-1}^{2^n} f_n \, dm + \frac{1}{2^n} \int_{2^n}^{2^n-1} f_n \, dm + \frac{1}{2^n} \int_{2^n-1}^1 f_n \, dm
\]

\[
= 2^n \left( \frac{2^n-1}{2^n} \cdot \frac{2^n-1}{2^n-1} \right) = 2^n \left( \frac{2^n-1}{2^n} \cdot \frac{2^n-2}{2^n} \right) = 2^n \left( \frac{1}{2^n} \right) = 1.
\]

In the last theorem in this chapter it will be shown that the hypothesis of Theorem 3.1 can be weakened to pointwise convergence of the sequence \( \{f_n\}_{n=1}^\infty \) if the sequence is bounded or dominated by an integrable function, which is not the case in Examples 3.4 and 3.5.

The next theorem will make it possible to give an example to show that the product of two integrable functions is not necessarily integrable.
Theorem 3.6. If \( f \) is a function whose domain includes \( [a,b] \), \( \{a_n\}_{n=1}^{\infty} \) is a strictly increasing sequence in \( [a,b) \) which converges to \( b \), \( a_1 = a, a_n/a_{n+1}f \text{d}m \) exists for \( n=1,2,\ldots \), and the series \( \sum_{p=1}^{\infty} a_p^p+1|f| \text{d}m \) is convergent, then \( \int_{a}^{b}f \text{d}m \) exists and is equal to \( \sum_{p=1}^{\infty} a_p^p+1 \text{d}m \).

Proof. Let \( \varepsilon \) be a positive number. For each positive integer \( n \), there exists a gauge \( \delta_n \), so that if \( P_n \) is a \( \delta_n \)-fine partition of \( (a_n, a_{n+1}] \), then \( |\Sigma[P_n, f] - a_n^p a_{p+1}f \text{d}m| < \frac{\varepsilon}{5 \cdot 2^n} \), and there is a gauge \( \delta'_n \), so that if \( P_n \) is \( \delta'_n \)-fine, then \( |\Sigma[P_n, f] - a_n^p a_{p+1}f \text{d}m| < \frac{\varepsilon}{5 \cdot 2^n} \). Also, there is a positive integer \( N \), so that if \( n \geq N \), then

\[
|P_{n+1}^p a_p^p a_{p+1}f \text{d}m| < \sum_{p=n}^{\infty} a_p^p a_{p+1}f \text{d}m < |P_{n}^p a_p^p a_{p+1}f \text{d}m| < \frac{\varepsilon}{5}.
\]

Now, define a gauge \( \delta \) as follows:

\[
\delta(x) = \begin{cases} 
1 & \text{if } x \notin [a,b]; \\
\min(\delta_1(x), \delta_1'(x), a_2-x) & \text{if } x \in [a,a_2); \\
\min(\delta_2(x), \delta_1'(x), a_2-x) & \text{if } x = a_2; \\
\min(\delta_2(x), \delta_3(x), a_3-x) & \text{if } x \in (a_2, a_3); \\
\min(\delta_2(x), \delta_3(x), a_3-x) & \text{if } x = a_3; \\
\vdots & \\
\min(\delta_n(x), \delta_n'(x), x-a_n, a_{n+1}-x) & \text{if } x \in (a_n, a_{n+1}); \\
\min(\delta_n(x), \delta_n'(x), a_{n+1}(x), a_{n+2}a_{n+1} \text{d}m) & \text{if } x = a_{n+1}; \\
\vdots & \\
\min\left(\frac{c}{5|f(b)|+1}, b-a_N\right) & \text{if } x = b.
\end{cases}
\]
Then if \( P = \{(x_1, A_1), (x_2, A_2), \ldots, (x_m, A_m)\} \) is any \( \delta \)-fine partition of \((a, b]\), then each of the points \( \{a_2, a_3, \ldots, a_N, b\} \) must be an evaluation point of \( P \) because of the construction of the gauge \( \delta \). Also, since there are only a finite number of evaluation points \( \{x_1, x_2, \ldots, x_m\} \), there is a positive integer \( M \), so that if \( A_i \neq \emptyset \), \( x_i \neq b \), then \( x_i < a_M \). Let \( J \) denote the least such positive integer. Then \( J-1 > N \), since \( a_N < a_J \) and \( \{a_n\}_{n=1}^\infty \) is a strictly increasing sequence. This also implies that if \( x \in [a_J, b) \), then \( x \) could not be an evaluation point of \( P \) with a non-empty component interval, and therefore

\[
|\sum_{P} [P, f] - \sum_{p=1}^{\infty} \int_{a_n}^{a_n+1} f dm| = \left| \sum_{p=1}^{m} f(x_p) m(A_p) + (x_p < a_2) \right|
\]

\[
= \sum_{p=1}^{m} f(x_p) m(A_p) \left( a_2 < x_p < a_3 \right) + \left( x_p = a_2 \right)
\]

\[
+ \sum_{p \neq 1}^{m} f(x_p) m(A_p) + \sum_{p=1}^{m} f(x_p) m(A_p \cap (a_2, a_3]) + \ldots + \left( x_p = a_3 \right)
\]

\[
= \sum_{p=1}^{m} f(x_p) m(A_p \cap (a, a_2]) + \sum_{p=1}^{m} f(x_p) m(A_p) + \left( x_p = a_2 \right) \quad \left( a_2 < x_p < a_3 \right)
\]

\[
- \left( \sum_{p=1}^{J-1} a_2 f dm + \sum_{K=J-1}^{\infty} a_K f dm \right) < 0
\]

\[
|\sum_{P} [P, f] - \sum_{p=1}^{\infty} \int_{a_1}^{a_2} f dm| = \left| \sum_{p=1}^{m} f(x_p) m(A_p) + \sum_{p=1}^{m} f(x_p) m(A_p \cap (a, a_2]) - \int_{a_1}^{a_2} f dm \right| + \ldots + \left( x_p = a_2 \right)
\]

\[
= \sum_{p=1}^{m} f(x_p) m(A_p) + \sum_{p=1}^{m} f(x_p) m(A_p \cap (a, a_2]) - \int_{a_1}^{a_2} f dm \quad \left( x_p = a_2 \right)
\]
\[
\begin{align*}
&\sum_{p=1}^{m} f(x_p) m(A_p \cap (a_{N-1}, a_N]) + \sum_{p=1}^{m} f(x_p) m(A_p) + \\
&(x_p = a_{N-1}) \quad (a_{N-1} < x_p < a_N) \\
- \int_{a_{N-1}}^{a_N} f dm \quad + \\
&\sum_{p=1}^{m} f(x_p) m(A_p \cap (a_{N-1}, a_N]) - \int_{a_{N-1}}^{a_N} f dm \quad + \\
&(x_p = a_N) \\
- \sum_{p=1}^{m} f(x_p) m(A_p \cap (a_M, a_{N+1}]) + \sum_{p=1}^{m} f(x_p) m(A_p) + \\
&(x_p = a_N) \quad (a_N < x_p < a_{N+1}) \\
\sum_{p=1}^{m} f(x_p) m(A_p \cap (a_{N}, a_{N+1}]) + \cdots + \sum_{p=1}^{m} f(x_p) m(A_p \cap (a_{j-1}, a_j]) + \\
&(x_p = a_{N+1}) \quad (x_p = a_{j-1}) \\
\sum_{p=1}^{m} f(x_p) m(A_p) + \sum_{p=1}^{m} f(x_p) m(A_p) + \cdots + \sum_{p=1}^{m} f(x_p) m(A_p) + \\
&\int_{a_{j-1}}^{a_j} f dm. \quad (x_p = b) \\
\end{align*}
\]

Each group of terms,

\[
\begin{align*}
&\sum_{p=1}^{m} f(x_p) m(A_p \cap (a_t, a_{t+1}]) + \sum_{p=1}^{m} f(x_p) m(A_p) + \\
&(x_p = a_t) \quad (a_t < x_p < a_{t+1}) \\
\sum_{p=1}^{m} f(x_p) m(A_p \cap (a_t, a_{t+1}]) + \\
&(x_p = a_{t+1})
\end{align*}
\]

starting at \( t = N \) and ending with

\[
\begin{align*}
&\sum_{p=1}^{m} f(x_p) m(A_p \cap (a_{j-1}, a_j]) + \sum_{p=1}^{m} f(x_p) m(A_p), \\
&(x_p = a_{j-1}) \quad (a_{j-1} < x_p < a_j)
\end{align*}
\]

is a partial approximating sum \( \int_{a_t}^{a_{t+1}} f dm \) for

\[ a_t \int_{a_t}^{a_{t+1}} f dm. \]

Since there exists at least one \( \delta_t \)-fine partition \( P_t = \{(x_1', C_1), \ldots, (x_s', C_s)\} \) of \( (a_t, a_{t+1}] \), the \( \delta_t \)-fine partial partition \( \{(x_1, B_1), \ldots, (x_t, B_t)\} \) can be combined with
$P_t$ to get a $\delta_t^i$-fine partition of $(a_t, a_{t+1}]$, namely $P_t^i = \{(x_1, B_1 C_1), (x_2, B_2 C_1), \ldots, (x_r, B_r C_1), (x_1, D_{11}), \ldots, (x_r, D_{1K}), (x_1, B_1 C_2), \ldots, (x_r, B_r C_2), (x_1, D_{21}), \ldots, (x_r, D_{21}), \ldots, (x_s, D_{su})\}$, where $D_{j1}, \ldots, D_{jv}$ are disjoint left-open, right-closed component intervals of $C_j \cap (\bigcup_{i=1}^k B_i)^c$. If follows that

$$\sum_{p=1}^r |f(x_p)| m(B_p) \leq \varepsilon |P_t^i, |f||$ and $\varepsilon |P_t^i, |f|| - \alpha_t^{a_t+1} |f|dm \leq$$

$$|\varepsilon |P_t^i, |f|| - \alpha_t^{a_t+1} |f|dm| < \frac{\varepsilon}{5 \cdot 2^{-t}}$$

Finally, this shows that

$$|\varepsilon |P, f| - \sum_{n=1}^{\infty} a_n \int_t f dm| < |\varepsilon |P_1, f| - \alpha_1^{a_1} |f|dm + \cdots +$$

$$|\varepsilon |P_{N-1}, f| - a_{N-1} \int_t f dm| + \sum_{t=N}^{J-1} \left( \frac{\varepsilon}{5 \cdot 2^{-t}} + \alpha_t^{a_t+1} |f|dm \right) +$$

$$\sum_{p=1}^m |f(b)| m(A_p) + \sum_{k=N}^{\infty} a_k \int_t f dm| < \sum_{p=1}^m \frac{\varepsilon}{5 \cdot 2^{-t}} +$$

$$\sum_{t=N}^{\infty} \frac{\varepsilon}{5 \cdot 2^{-t}} \sum_{t=N}^{\infty} a_t^{a_t+1} |f|dm + \frac{\varepsilon}{5} \leq \varepsilon,$$

for any $\delta$-fine partition $P$ of $(a,b]$, which completes the proof.

Example 3.7. Consider the closed interval $[0,1]$, the

sequence $\{a_n \frac{2^{n-1} - 1}{2^{n-1}} \}_{n=1}^{\infty}$, and the function $f(x) = \sum_{n=1}^{\infty} \sqrt{2^n} \xi[a_n, a_{n+1}](x)$. 
For \( x \in [a_n, a_{n+1}) \), \( f(x) = \sqrt{2^n} \), so that
\[
\int_{a_n}^{a_{n+1}} |f|dm = a_n \int_{a_n}^{a_{n+1}} fdm = \sqrt{2^{n+1}} \frac{1}{2^n}.
\]
Thus the series \( \sum_{p=1}^{\infty} a_p^{p+1} |f|dm \) converges since
\[
\sum_{p=1}^{\infty} a_p^{p+1} |f|dm = \sum_{p=1}^{\infty} a_p^{p+1} fdm = \sum_{p=1}^{\infty} \sqrt{2^n} \frac{1}{2^n} = 1 + \sqrt{2}.
\]
Therefore by Theorem 3.6, \( \int_0^1 f dm = 1 + \sqrt{2} \), but \( f^2(x) = \)
\[
\sum_{n=1}^{\infty} 2^n \xi_{[a_n, a_{n+1})}(x)
\]
is not integrable from 0 to one as shown in Example 3.4.

The next lemma is necessary for the proof of the monotone convergence theorem.

Lemma 3.8. If \( \int_a^b f dm \) exists, \( \delta \) is a gauge so that
\[
|\Sigma[P, f] - \int_a^b f dm| < \varepsilon \quad \text{for all } \delta \text{-fine partitions } P \text{ of } (a, b], \text{ and if }\{(x_1, A_1), \ldots, (x_n, A_n)\} \text{ is a } \delta \text{-fine partial partition of } (a, b], \text{ with } A_i = (a_i, b_i], \text{ then } \left| \sum_{i=1}^{n} f(x_i)m(A_i) - \sum_{i=1}^{n} a_i \int_{a_i}^{b_i} f dm \right| < 2\varepsilon.
\]

Proof. Suppose that
\[
\left| \sum_{i=1}^{n} f(x_i)m(A_i) - \sum_{i=1}^{n} a_i \int_{a_i}^{b_i} f dm \right| \geq 2\varepsilon
\]
and let \( \{B_1, B_2, \ldots, B_r\} \), \( B_i = (a'_i, b'_i] \), denote the left-open, right-closed component intervals of \( (a, b]\cap\bigcup_{i=1}^{n} A_i)^C \).

Then for each \( i, 1 \leq i \leq r \), \( i \) is a gauge \( \delta_i \), so that if \( P_i \) is a \( \delta_i \)-fine partition of \( (a'_i, b'_i] \), then
\[
|\Sigma[P_i, f] - a_i \int_{a_i}^{b_i} f dm| < \varepsilon.
\]
Let \( \delta'(x) = \min \{ \delta(x), \delta_1(x), \ldots, \delta_r(x) \} \). Then for each \( i, 1 \leq i \leq r \), there is a \( \delta'_i \)-fine partition \( P_i = \{(x_{1i}, C_{1i}), (x_{2i}, C_{2i}), \ldots, (x_{si}, C_{si})\} \) of \( B_i \), and thus \( P_0 = \{(x_1, A_1), (x_2, A_2), \ldots, (x_n, A_n), (x_{11}, C_{11}), \ldots, (x_{s_1}, C_{s_1}), (x_{12}, C_{12}), \ldots, (x_{s_r}, C_{s_r})\} \) is a \( \delta \)-fine partition of \([a, b]\).

Then
\[
\varepsilon > |\sum_{i=1}^n f(x_i)m(A_i) + \sum_{j=1}^r \sum_{i=1}^{s_i} f(x_{ji})m(C_{ji}) - \sum_{i=1}^n \int_{a_i}^{b_i} f \, dm| + |\sum_{j=1}^r \int_{a_j}^{b_j} f \, dm| > |\sum_{i=1}^n f(x_i)m(A_i) - \sum_{i=1}^n \int_{a_i}^{b_i} f \, dm| > 2 \varepsilon - r \cdot \varepsilon = \varepsilon,
\]
which is a contradiction, and hence
\[
\sum_{i=1}^n f(x_i)m(A_i) - \sum_{i=1}^n \int_{a_i}^{b_i} f \, dm < 2 \varepsilon.
\]

Theorem 3.9. (Monotone Convergence Theorem). If \( \{f_n\}_{n=1}^\infty \) is a monotonic sequence of functions such that \( a \int_{a}^{b} f_n \, dm \) exists for \( n=1, 2, \ldots \) and \( n \to \infty f_n(x) = F(x) \) for all \( x \in [a, b] \), then \( a \int_{a}^{b} F \, dm \) exists if and only if \( n \to \infty a \int_{a}^{b} f_n \, dm \) exists, and when both exist then are equal.

Proof. Suppose first that \( n \to \infty a \int_{a}^{b} f_n \, dm \) exists. It can also be assumed that \( \{f_n\}_{n=1}^\infty \) is non-decreasing, for in the non-increasing case the sequence \( \{-f_n\}_{n=1}^\infty \) will satisfy the hypothesis and yield the same results.
Now, let \( \epsilon \) be a positive number. Then if \( xe[a,b] \), there is a positive integer \( n(x) \), so that \( f_{n(x)}(x) \geq \frac{e}{3(b-a)} \). Next, there is a positive integer \( N \), so that if \( i \geq N \), then \( \left| \int_a^b f dm - \int_a^b f_{n(x)} dm \right| < \frac{\epsilon}{3} \), and for each positive integer \( i \), there is a gauge \( \delta_i \), so that if \( P \) is any \( \delta_i \)-fine partition of \( (a,b) \), then \( \left| \Sigma [P,f_{i-1}] - \int_a^b f dm \right| < \frac{\epsilon}{3 \cdot 2^{i+1}} \).

Let \( \delta(x) = \min \{ \delta_1(x), \delta_2(x), \ldots, \delta_{N+n(x)}(x) \} \), and let \( P = \{(x_1,A_1), \ldots, (x_m,A_m)\} \) denote an arbitrary \( \delta \)-fine partition of \( (a,b) \), with \( A_i = (a_i,b_i) \). Then \( \lim_{n \to \infty} \int_a^b f dm \to \Sigma [P,f_N] - \int_a^b f dm + \int_a^b f dm - \int_a^b f dm \to \frac{\epsilon}{3 \cdot N+1} \).

\( \epsilon \to \epsilon. \) (I)

Now, let \( M = \max \{ n(x_i) : x_i \text{ is an evaluation point of } P \} \) and \( m_0 = \min \{ n(x_i) : x_i \text{ is an evaluation point of } P \} \). Then using Lemma 3.8, it follows that \( \lim_{n \to \infty} \int_a^b f dm = \)

\[
\sum_{i=1}^{m} F(x_i)m(A_i) + \sum_{i=1}^{m} F(x_i)m(A_i) + \cdots + \sum_{i=1}^{m} F(x_i)m(A_i) - \left( \begin{array}{c}
(n(x_i) = M) \\
(n(x_i) = M-1) \\
(n(x_i) = m_0)
\end{array} \right).
\]

\[
\lim_{n \to \infty} \int_{A_i} f dm \leq \sum_{i=1}^{m} \left( f_{M}(x_i) + \frac{\epsilon}{3(b-a)} \right) m(A_i) + \cdots + \sum_{i=1}^{m} \left( f_{m_0}(x_i) + \frac{\epsilon}{3(b-a)} \right) m(A_i) - \left( \begin{array}{c}
(n(x_i) = M) \\
(n(x_i) = m_0)
\end{array} \right).
\]

\[
\frac{\epsilon}{3(b-a)} m(a_i) - \int_{A_i} f dm + \int_{A_i} f dm - \int_{A_i} f dm = \]

\[
\sum_{i=1}^{m} f_{M}(x_i)m(A_i) + \cdots + \sum_{i=1}^{m} f_{m_0}(x_i)m(A_i) + \frac{\epsilon}{3(b-a)} \sum_{i=1}^{m} m(A_i) - \left( \begin{array}{c}
(n(x_i) = M) \\
(n(x_i) = m_0)
\end{array} \right).
\]
\[
\sum_{i=1}^{m} a_i \int_{M}^{m+N} f_{M+N} \, dm + a \int_{M+N}^{b} f_{M+N} \, dm + \lim_{n \to \infty} a \int_{n}^{b} f_{n} \, dm < (\sum_{i=1}^{m} f_{M}(x_i) m(A_i) - (n(x_i) = M)
\]

\[
\sum_{i=1}^{m} a_i \int_{M}^{M+N} f_{M} \, dm + \cdots + (\sum_{i=1}^{m} f_{M}(x_i) m(A_i) - \sum_{i=1}^{m} a_i \int_{M}^{M+N} f_{M} \, dm) + (n(x_i) = M) \quad (n(x_i) = m) \quad (n(x_i) = M - 1)
\]

\[
\frac{\varepsilon}{3} + \frac{\varepsilon}{3} < (\sum_{i=1}^{m} f_{M}(x_i) m(A_i) - \sum_{i=1}^{m} a_i \int_{M}^{M+N} f_{M} \, dm) + (\sum_{i=1}^{m} f_{M-1}(x_i) m(A_i) - \sum_{i=1}^{m} a_i \int_{M-1}^{M} f_{M-1} \, dm) + (n(x_i) = M) \quad (n(x_i) = m) \quad (n(x_i) = M - 1)
\]

\[
\frac{2\varepsilon}{3} < \frac{2\varepsilon}{3 \cdot 2M + 1} + \frac{2\varepsilon}{3 \cdot 2M - 1 + 1} + \cdots + \frac{2\varepsilon}{3 \cdot 2m + 1} + \frac{2\varepsilon}{3} < \varepsilon. \quad (II)
\]

Combining inequalities (I) and (II) gives that \(|\Sigma [P, f] - \lim_{n \to \infty} a \int_{n}^{b} f_{n} \, dm| < \varepsilon\) for all \(\delta\)-fine partitions \(P\) of \((a, b)\) which concludes the first part of the proof.

For the converse, if \(a \int_{a}^{b} F \, dm\) exists, then for each positive integer \(n\), \(F(x) \geq f_{n}(x)\) for all \(x \in [a, b]\). Hence \(a \int_{a}^{b} F \, dm \geq a \int_{a}^{b} f_{n} \, dm\) and thus the sequence \(\{a \int_{a}^{b} f_{n} \, dm\}_{n=1}^{\infty}\) is monotonic, non-decreasing, and bounded above, which implies that \(\lim_{n \to \infty} a \int_{a}^{b} f_{n} \, dm\) exists, and it follows from the first part of the proof that \(\lim_{n \to \infty} a \int_{a}^{b} F \, dm = \lim_{n \to \infty} a \int_{a}^{b} f_{n} \, dm\).

Theorem 3.10. (Dominated Convergence Theorem). If \(\{f_{n}\}_{n=1}^{\infty}\) is a sequence of functions with the property that \(a \int_{a}^{b} f_{n} \, dm\) exists for \(n=1, 2, \cdots\) and \(\lim_{n \to \infty} f_{n}(x) = F(x)\) for all \(x \in [a, b]\), if \(g\) is a function so that \(a \int_{a}^{b} g \, dm\) exists and \(|f_{n}(x)| \leq g(x)\) for all \(x \in [a, b]\), and for \(n=1, 2, \cdots\), then \(a \int_{a}^{b} F \, dm\) and \(\lim_{n \to \infty} a \int_{a}^{b} f_{n} \, dm\) both exist, and are equal.
Proof. Let

\[ g_{n,p}(x) = \begin{cases} 
\min\{f_n(x), f_{n+1}(x), \ldots, f_p(x)\} & \text{if } p \geq n; \\
 f_n(x) & \text{if } p < n,
\end{cases} \]

and let

\[ h_{n,p}(x) = \begin{cases} 
\max\{f_n(x), f_{n+1}(x), \ldots, f_p(x)\} & \text{if } p \geq n; \\
 f_n(x) & \text{if } p < n.
\end{cases} \]

By Corollary 2.8, both \( g_{n,p} \) and \( h_{n,p} \) are integrable from \( a \) to \( b \) for all positive integers \( n \) and \( p \).

For a fixed \( n \), \( g_{n,p} \geq g_{n,p+1} \) and \( |g_{n,p}(x)| = |f_n(x)| \) if \( p < n \) or \( |g_{n,p}(x)| = |\min\{f_n(x), \ldots, f_p(x)\}| = |f_i(x)| \) if \( p \geq n \), but in either case, \( |g_{n,p}(x)| \leq g(x) \) for all \( x \in [a, b] \). Hence \( \{g_{n,p}\}_{p=1}^{\infty} \) is a bounded, monotonic sequence which converges pointwise to a function \( g_n \) with \( |g_n(x)| \leq g(x) \) for all \( x \in [a, b] \). Then

\[ \int_a^b g_{n,p} \, dm \leq \int_a^b g_n \, dm \leq \int_a^b g_{n,p} \, dm \]

and therefore \( \{\int_a^b g_{n,p} \, dm\}_{p=1}^{\infty} \) is also a bounded, monotonic sequence and thus \( \lim_{p \to \infty} \int_a^b g_{n,p} \, dm \) exists. Then applying Theorem 3.9, \( \int_a^b g_n \, dm = \lim_{p \to \infty} \int_a^b g_{n,p} \, dm \).

Likewise, for a fixed \( n \), \( h_{n,p} \leq h_{n,p+1} \) and \( |h_{n,p}| \leq g \), so \( \{h_{n,p}\}_{p=1}^{\infty} \) converges to a function \( h_n \) with \( |h_n(x)| \leq g(x) \) for all \( x \in [a, b] \), and again by using Theorem 3.9, \( \int_a^b h_n \, dm = \lim_{p \to \infty} \int_a^b h_{n,p} \, dm \).
Now for any \( p > n \), \( g_n, p \leq g_{n+1}, p \) and \( h_n, p \geq h_{n+1}, p \), so that \( g_n \leq g_{n+1} \) and \( h_n \geq h_{n+1} \). But \( |g_n| \leq \sigma \) and \( |h_n| \leq \sigma \) for all positive integers \( n \), and hence \( \{g_n\}^\infty_{n=1} \) and \( \{h_n\}^\infty_{n=1} \) are bounded, monotonic sequences and therefore converge to functions \( G \) and \( H \), respectively, for all \( x \in [a, b] \), and \( |G| \leq \sigma \) and \( |H| \leq \sigma \). The sequences \( \{a \int_{b}^{g_n dm}\}^\infty_{n=1} \) and \( \{a \int_{b}^{h_n dm}\}^\infty_{n=1} \) then are also monotonic, bounded sequences, so that \( \lim_{n \to \infty} a \int_{b}^{g_n dm} \) and \( \lim_{n \to \infty} a \int_{b}^{h_n dm} \) both exist, and again making use of the monotonic convergence theorem, \( a \int_{b}^{G dm} \) and \( a \int_{b}^{H dm} \) both exist and are equal to \( \lim_{n \to \infty} a \int_{b}^{g_n dm} \) and \( \lim_{n \to \infty} a \int_{b}^{h_n dm} \), respectively.

Now, if \( x \in [a, b] \) and \( \epsilon \) is a positive number, then there is a positive integer \( N_1 \), so that if \( n \geq N_1 \), then \( |f_n(x) - F(x)| < \frac{\epsilon}{3} \). Next, there is a positive integer \( N_2 \), so that if \( n \geq N_2 \), then \( |G(x) - g_n(x)| < \frac{\epsilon}{3} \), and there is also a positive integer \( N_3 \), so that if \( n \geq N_3 \), then \( |g_{N_1 + N_2}^{N_1 + N_2} n(x) - g_{N_1 + N_2}^{N_1 + N_2} (x)| < \frac{\epsilon}{3} \).

Then

\[
|G(x) - F(x)| \leq |G(x) - g_{N_1 + N_2}^{N_1 + N_2} (x)| + |g_{N_1 + N_2}^{N_1 + N_2} n(x) - g_{N_1 + N_2}^{N_1 + N_2} (x)| + |g_{N_1 + N_2}^{N_1 + N_2} (x) - F(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\]

and since \( \epsilon \) and \( x \) were arbitrary, it follows that \( G(x) = F(x) \) for all \( x \in [a, b] \). Therefore \( a \int_{b}^{F dm} \) exists and \( a \int_{b}^{F dm} = \lim_{n \to \infty} a \int_{b}^{g_n dm} \).

By the same argument, it follows that \( H(x) = F(x) \) for all \( x \in [a, b] \), and thus \( \lim_{n \to \infty} a \int_{b}^{h_n dm} = a \int_{b}^{F dm} = \lim_{n \to \infty} a \int_{b}^{g_n dm} \).
Now, for any fixed $n$, $g_n, p < f_n$ and $h_n, p > f_n$. Hence
\[ a \int_a^b g_n \, dm \leq a \int_a^b g_n \, dm + a \int_a^b h_n \, dm. \]
Then if $\varepsilon$ is a positive number, there is a positive integer $N_1'$, so
that if $n \geq N_1'$, then $|a \int_a^b F dm - a \int_a^b g_n \, dm| < \varepsilon$, and there is a positive
integer $N_2'$, so that if $n \geq N_2'$, then $|a \int_a^b F dm - a \int_a^b h_n \, dm| < \varepsilon$. Now if
$n > N_1' + N_2'$, then $a \int_a^b F dm - a \int_a^b h_n \, dm < \varepsilon$ and $a \int_a^b F dm - a \int_a^b g_n \, dm > -\varepsilon$, so that $|a \int_a^b F dm - a \int_a^b F dm| < \varepsilon$, which
means that $\lim_{n \to \infty} a \int_a^b F dm$ exists and that $\lim_{n \to \infty} a \int_a^b F dm = a \int_a^b F dm$.

The proof is complete.
CHAPTER BIBLIOGRAPHY


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