SIMPLICIAL HOMOLOGY

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Chang, Chih-Chen, Simplicial Homology. Master of Science (Mathematics), August, 1973, 60 pp., bibliography, 3 titles.

The purpose of this thesis is to construct the homology groups of a complex over an R-module. The thesis begins with hyperplanes in Euclidean n-space. Simplexes and complexes are defined, and orientations are given to each simplex of a complex. The chains of a complex are defined, and each chain is assigned a boundary. The function which assigns to each chain a boundary defines the set of r-dimensional cycles and the set of r-dimensional bounding cycles. The quotient of those two submodules is the r-dimensional homology group.

In Chapter I, a geometrically independent set in $R^n$ is defined, and some basic properties of a hyperplane, which are useful in Chapter II, are discussed.

Simplexes, complexes, abstract complexes, polytopes, and pseudomanifolds are defined in the second chapter. It is also proved in this chapter that every abstract complex has a realization complex in some $R^n$. Since a point set simplex is a subset of the hyperplane spanned by that simplex, some theorems of Chapter I are useful in this chapter.

An orientation of a simplex is defined in Chapter III. A chain of a complex K over an R-module G is defined by the chain-equivalent class of the oriented complex K over G. It is proved that chains of a complex can always be given an

orientation; thus the representative chains of the oriented complex can be computed. This is important both in Chapter III and Chapter IV. Also found in Chapter III is the proof that the set $C_r(K;G)$ is an R-module.

The boundary of each r-chain ($r > 0$) of a complex K over G, the set of cycles $Z_r(K;G)$, and the set of bounding cycles $B_r(K;G)$ are defined in Chapter IV. The r-dimensional homology group is defined by $H_r(K;G) = Z_r(K;G)/B_r(K;G)$. The 0-dimensional homology group is defined by $H_0(K;G) = C_0(K;G)/B_0(K;G)$. For a simplicially connected K, the theorem that $H_0(K;Z)$ is isomorphic to Z is proved in this chapter.

Some theorems which are proved in this thesis can be found stated, though not proved, in Chapter 1 of John W. Keesee's An _Introduction to Algebraic Topology_. Additional theorems which are proved in this thesis were suggested by Y. W. Lau of the North Texas State University mathematics faculty.

There are results presented in the appendix concerning five problems for computing homology groups.

SIMPLICIAL HOMOLOGY

THESIS

Presented to the Graduate Council of the
North Texas State University in Partial
Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

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Denton, Texas
August, 1973

TABLE OF CONTENTS


| Chapter | | Page |
|---|---|---|
| I. | THE HYPERPLANES IN $R^n$ . . . . . . . . . . . . . . | 1 |
| II. | SIMPLEXES AND COMPLEXES IN $R^n$ . . . . . . . . . . | 14 |
| III. | CHAIN GROUPS. . . . . . . . . . . . . . . . . . | 32 |
| IV. | HOMOLOGY GROUP. . . . . . . . . . . . . . . . . | 39 |
| APPENDIX | . . . . . . . . . . . . . . . . . . . . . . | 53 |
| BIBLIOGRAPHY. | . . . . . . . . . . . . . . . . . . . . | 60 |

# CHAPTER I

## THE HYPERPLANES IN $R^n$

The Euclidean n-space (denoted by $R^n$) associated with the operations of addition and scalar multiplication, which are defined as follows in $R^n$, is an n-dimensional vector space over the field of real numbers.

Let R = Reals; let $x = (x^1, x^2, \ldots, x^n)$ and $y = (y^1, y^2, \ldots, y^n) \in R^n$, where each $x^i$ and each $y^i$ is a real number. Let $c \in R$; define

$x + y \doteq (x^1 + y^1, x^2 + y^2, \ldots, x^n + y^n)$, and

$cx = (cx^1, cx^2, \ldots, cx^n)$.

1.1. Definition. Let $S = \{a_0, a_1, \ldots, a_k\}$ be a finite subset of $R^n$. The hyperplane spanned by S (denoted by $\pi(S)$) is the set $\pi(S) = \{p \in R^n \mid p = \sum_{i=0}^{k} \lambda^i a_i$ where each $\lambda^i \in R$, and $\sum_{i=0}^{k} \lambda^i = 1\}$.

1.2. Definition. A finite subset $S = \{a_0, a_1, \ldots, a_k\}$ of $R^n$ is said to be geometrically independent if S is contained in $\pi(T)$ for no proper subset T of S.

Notation. $V(a_0, a_1, \ldots, a_k)$ stands for the vector space over R spanned by the set $\{a_0, a_1, \ldots, a_k\}$.

1.3. Theorem. Let $S_1 = \{a_0, a_1, \ldots, a_k\}$ and $S_2 = \{b_0, b_1, \ldots, b_s\}$ be finite subsets of $R^n$. Then

(a) $S_1 \subset \pi(S_2)$ implies $\pi(S_1) \subset \pi(S_2)$, and

(b) $\pi(S_1) \subset \pi(S_2)$ implies $V(a_1-a_0,\ldots,a_k-a_0) \subset V(b_1-b_0,\ldots,b_s-b_0)$.

Proof: (a) Let $p = \sum_{j=0}^{k}\lambda^j a_j \in \pi(S_1)$. Since $S_1 \subset \pi(S_2)$, let $a_j = \sum_{i=0}^{s}\mu_j^i b_i \in \pi(S_2)$ for each j. Then

$$p = \sum_{j=0}^{k}\lambda^j a_j = \sum_{j=0}^{k}\lambda^j \sum_{i=0}^{s}\mu_j^i b_i = \sum_{i=0}^{s}\Big(\sum_{j=0}^{k}\lambda^j\mu_j^i\Big)b_i.$$

Since $a_j \in \pi(S_2)$, then $\sum_{i=0}^{s}\mu_j^i = 1$ for each j. Therefore $\sum_{i=0}^{s}\sum_{j=0}^{k}\lambda^j\mu_j^i = \sum_{j=0}^{k}\sum_{i=0}^{s}\lambda^j\mu_j^i = \sum_{j=0}^{k}\lambda^j = 1$. Hence $p \in \pi(S_2)$. Hence $\pi(S_1) \subset \pi(S_2)$.

Proof: (b) Let $t_1(a_1-a_0) + t_2(a_2-a_0) + \ldots + t_k(a_k-a_0) \in V(a_1-a_0,\ldots,a_k-a_0)$, and denote $\sum_{i=1}^{k}t_i$ by T. Either $T \neq 0$, or $T = 0$.

Case 1: $T \neq 0$. Since $\pi(S_1) \subset \pi(S_2)$, then $\sum_{i=1}^{k}t_i/T\, a_i \in \pi(S_1) \subset \pi(S_2)$. Hence there exist real numbers $\mu_0'$, $\mu_1'$, $\ldots,\mu_s'$ such that

$\sum_{i=1}^{k}t_i/T\, a_i = \sum_{j=0}^{s}\mu_j' b_j \in \pi(S_2)$, and

$\mu_0, \mu_1, \ldots,\mu_s$ such that $a_0 = \sum_{j=0}^{s}\mu_j b_j \in \pi(S_2)$. Then

$$\begin{aligned}
\sum_{i=1}^{k}t_i(a_i-a_0) &= -Ta_0 + \sum_{i=1}^{k}t_i a_i \\
&= -T\sum_{i=0}^{s}\mu_i b_i + T\sum_{i=0}^{s}\mu_i' b_i \\
&= T\sum_{i=0}^{s}(\mu_i' - \mu_i)b_i \\
&= T\Big[\Big(1 - \sum_{i=1}^{s}\mu_i' - 1 + \sum_{i=1}^{s}\mu_i\Big)b_0 + \sum_{i=1}^{s}(\mu_i' - \mu_i)b_i\Big] \\
&= T\Big[\sum_{i=1}^{s}(\mu_i - \mu_i')b_0 + \sum_{i=1}^{s}(\mu_i' - \mu_i)b_i\Big] \\
&= T\sum_{i=1}^{s}(\mu_i' - \mu_i)(b_i - b_0) \quad V(b_1-b_0,\ldots,b_s-b_0).
\end{aligned}$$

Case 2: $T = 0$. Suppose $t_i = 0$ for $i = 1,2,\ldots,k$; then $\sum_{i=1}^{k}t_i(a_i - a_0) = 0 \in V(b_1-b_0,\ldots,b_s-b_0)$. Assume $t_0 \neq 0$ for some $1 \leq c \leq k$; then $t_1 + t_2 + \ldots + t_{c-1} + t_{c+1} + \ldots + t_k = -t_c \neq 0$. Hence $t_c(a_c - a_0) \in V(b_1-b_0,\ldots,b_s-b_0)$, and $\sum_{\substack{i=1 \\ i\neq c}}^{k} t_i(a_i - a_0) \in V(b_1-b_0,\ldots,b_s-b_0)$ by the proof of Case 1, $T \neq 0$. Therefore $t_c(a_c - a_0) + \sum_{\substack{i=1 \\ i\neq c}}^{k} t_i(a_i - a_0) = \sum_{i=1}^{k} t_i(a_i - a_0) \in V(b_1-b_0,\ldots,b_s-b_0)$.

1.4. Theorem. Let $S = \{a_0,a_1,\ldots,a_k\}$ be a finite subset of $R^n$. The following properties of S are equivalent.

(a) S is geometrically independent.

(b) If $T = \{b_0,b_1,\ldots,b_t\}$ and $S \subset \pi(T)$, then $t \geq k$.

(c) $\sum_{i=0}^{k} \lambda^i a_i = 0$ and $\sum_{i=0}^{k} \lambda^i = 0$ imply $\lambda^i = 0$ for each $i = 0,1,\ldots,k$.

(d) For each element p of $\pi(S)$, there exist unique real numbers $\lambda^0,\lambda^1,\ldots,\lambda^k$ such that

$$p = \sum_{i=0}^{k} \lambda^i a_i, \text{ and } \sum_{i=0}^{k} \lambda^i = 1.$$

(e) The set $\{a_1-a_0,\ldots a_k-a_0\}$ is linearly independent.

Proof: (a) implies (b). Since $S \subset \pi(T)$, then $V(a_1-a_0,\ldots,a_k-a_0) \subset V(b_1-b_0,\ldots,b_t-b_0)$ by the theorem of 1.3. Suppose there exist real numbers $\lambda^{i_1} \neq 0$, $\lambda^{i_2} \neq 0,\ldots,$ $\lambda^{i_n} \neq 0$ such that $\lambda^{i_1}(a_{i_1} - a_0) + \lambda^{i_2}(a_{i_2} - a_0) + \ldots + \lambda^{i_n}(a_{i_n} - a_0) = 0$, where $1 \leq i_1,i_2,\ldots,i_n \leq k$. Then $\lambda^{i_1}a_{i_1} + \lambda^{i_2}a_{i_2} + \ldots + \lambda^{i_n}a_{i_n} = (\lambda^{i_1} + \lambda^{i_2} + \ldots + \lambda^{i_n})a_0$. Denote $\lambda^{i_1} + \lambda^{i_2} + \ldots + \lambda^{i_n}$ by $\lambda$; either $\lambda \neq 0$, or $\lambda = 0$.

Case 1: $\lambda \neq 0$. Then $\lambda^{i_1}a_{i_1} + \lambda^{i_2}a_{i_2} + \ldots + \lambda^{i_n}a_{i_n} = \lambda a_0$, and $a_0 = \lambda^{i_1}/\lambda\, a_{i_1} + \lambda^{i_2}/\lambda\, a_{i_2} + \ldots + \lambda^{i_n}/\lambda\, a_{i_n} \in \pi(a_1,a_2,\ldots,a_k)$. Hence $S \subset \pi(a_1,a_2,\ldots,a_k)$, a contradiction to S being geometrically independent.

Case 2: $\lambda = 0$. Then $\lambda^{i_1}a_{i_1} + \lambda^{i_2}a_{i_2} + \ldots + \lambda^{i_n}a_{i_n} = 0$, and $\lambda^{i_1} = -\lambda^{i_2} - \ldots - \lambda^{i_n}$. Hence $a_{i_1} = -\lambda^{i_2}/\lambda^{i_1}\, a_{i_2} - \ldots - \lambda^{i_n}/\lambda^{i}\, a_{i_n} \in \pi(S\backslash a_{i_1})$; hence $S \subset \pi(S\backslash a_{i_1})$, which is a contradiction to S being geometrically independent. Therefore $\{a_1-a_0,\ldots,a_k-a_0\}$ is linearly independent. Hence $\{a_1-a_0,\ldots,a_k-a_0\}$ is a base of $V(a_1-a_0,\ldots,a_k-a_0)$. But $V(a_1-a_0,\ldots,a_k-a_0) \subset V(b_1-b_0,\ldots,b_t-b_0)$; hence $\dim V(a_1-a_0,\ldots,a_k-a_0) \leq \dim V(b_1-b_0,\ldots,b_t-b_0)$. This implies $k \leq t$.

(b) implies (c). Let $\sum_{i=0}^{k} \lambda^i a_i = 0$, and $\sum_{i=0}^{k} \lambda^i = 0$. Let $\{\lambda^i \in R \mid \lambda^i \neq 0,\ 0 \leq i \leq k\} = \{\lambda^{i_1}, \lambda^{i_2}, \ldots, \lambda^{i_f}\}$. Since $\sum_{i=0}^{k}\lambda^i a_i = 0$, then $\sum_{j=1}^{k}\lambda^{i_j}a_{i_j} = 0$, and $\lambda^{i_1}a_{i_1} = -\lambda^{i_2}a_{i_2} - \ldots - \lambda^{i_f}a_{i_f}$. Since $\sum_{i=0}^{k}\lambda^i = 0$, then

$\sum_{j=1}^{f}\lambda^{i_j} = 0$, and $\lambda^{i_1} = -\lambda^{i_2} - \lambda^{i_3} - \ldots - \lambda^{i_f}$. Therefore

$\sum_{j=2}^{f}\lambda^{i_j}/-\lambda^{i_1} = 1$, and $a_{i_1} = \sum_{j=2}^{f}\lambda^{i_j}/-\lambda^{i_1}\, a_{i_j} \in \pi(S\ a_{i_1})$. Therefore $S \subset \pi(S\backslash a_{i_1})$. By (b), this implies $k \geq k + 1$. It is impossible. Therefore $\lambda^i = 0$ for $i = 0, \ldots, k$.

(c) implies (d). Let $p = \sum_{i=0}^{k}\lambda^i a_i = \sum_{i=0}^{k}\mu^i a_i \in \pi(S)$. Then $0 = p-p = \sum_{i=0}^{k}(\lambda^i - \mu^i)a_i$, and $\sum_{i=0}^{k}(\lambda^i - \mu^i) = \sum_{i=0}^{k}\lambda^i - \sum_{i=0}^{k}\mu^i = 1 - 1 = 0$. By (c),

$\lambda^i - \mu^i = 0$, for $i = 0,1,\ldots,k$. Therefore $\lambda^i = \mu^i$ for $i = 0,1,\ldots,k$. Hence $\lambda^0,\lambda^1,\ldots,\lambda^k$ are unique.

(d) implies (e). Suppose the set $\{a_1-a_0,\ldots,a_k-a_0\}$ is linearly dependent; then there is $a_r \in S\backslash a_0$ such that

$a_r - a_0 = \sum_{\substack{i=1 \\ i \neq r}}^{k} c^i(a_i - a_0)$, where

$c^i \in R$ for $i = 1,2,\ldots,r-1,r+1,\ldots,k$. Hence

$a_r = \sum_{\substack{i=1 \\ i \neq r}}^{k} c^i a_i + (1 - \sum_{\substack{i=1 \\ i \neq r}}^{k} c^i)a_0$. But $a_r \in \pi(S)$. Therefore the last equation contradicts the uniqueness hypothesis of (d).

(e) implies (a). Suppose there exists a proper subset $S'$ of $S$ such that $S \subset \pi(S')$. Let $a_r \in S \setminus S'$, and let

$a_r = \sum_{a_i \in S'} \lambda^i a_i \in \pi(S')$. Then

$0 = \sum_{a_i \in S'} \lambda^i a_i - a_r = \sum_{a_i \in S'} \lambda^i (a_i - a_r)$. This implies

$\{a_i - a_r \mid a_i \in S'\}$ is linearly dependent; hence $\{a_i - a_r \mid a_i \in S \setminus \{a_r\}\}$ is linearly dependent. To complete the proof of (e) implies (a), it is necessary to show that $\{a_0 - a_r, a_1 - a_r, \ldots, a_{r-1} - a_r, a_{r+1} - a_r, \ldots, a_k - a_r\}$ is linearly dependent implies that $\{a_1 - a_0, \ldots, a_k - a_0\}$ is linearly dependent. Assume

$a_\ell - a_r = \alpha^0(a_0 - a_r) + \ldots + \alpha^{\ell-1}(a_{\ell-1} - a_r) + \alpha^{\ell+1}(a_{\ell+1} - a_r) + \ldots + \alpha^{r-1}(a_{r-1} - a_r) + \alpha^{r+1}(a_{r+1} - a_r) + \ldots + \alpha^k(a_k - a_r)$,

where $\alpha^0,\ldots,\alpha^k$ are real numbers. Then

$(a_\ell - a_0) + (-1)(a_r - a_0) + (\alpha^0 + \alpha^1 + \ldots + \alpha^{\ell-1} + \alpha^{\ell+1} + \ldots + \alpha^{r-1} + \alpha^{r+1} + \ldots + \alpha^k)(a_r - a_0) = \alpha^1(a_1 - a_0) + \ldots + \alpha^{\ell-1}(a_{\ell-1} - a_0) + \alpha^{\ell+1}(a_{\ell+1} - a_0) + \ldots + \alpha^{r-1}(a_{r-1} - a_0) + \alpha^{r+1}(a_{r+1} - a_0) + \ldots + \alpha^k(a_k - a_0)$. This equation implies $\{a_1 - a_0, \ldots, a_k - a_0\}$ is linearly dependent. It is a contradiction to $\{a_1 - a_0, \ldots, a_k - a_0\}$ being linearly independent.

1.5. Definition. Let $S = \{a_0, \ldots, a_k\}$ be a geometrically independent subset of $R^n$. The hyperplane $\pi(S)$ is called a k-dimensional hyperplane. For each element p of $\pi(S)$, the numbers $\lambda^0, \lambda^1, \ldots, \lambda^k$ given in part (d) of 1.4. Theorem are called the barycentric coordinates of the point p relative to the set S.

1.6. Theorem. The hyperplane $\pi(a_0, a_1, \ldots, a_k)$ is a translation of the vector space $V(a_1-a_0, a_2-a_0, \ldots, a_k-a_0)$; $a_0 + V(a_1-a_0, a_2-a_0, \ldots, a_k-a_0) = \pi(a_0, a_1, \ldots, a_k)$.

Proof: Let $p = \sum_{i=0}^{k} \lambda^i a_i \in \pi(a_0, a_1, \ldots, a_k)$. Since $\sum_{i=0}^{k} \lambda^i = 1$, then $\lambda^0 - 1 = -\lambda^1 - \lambda^1 - \ldots - \lambda^k$. Therefore

$$\begin{aligned} p &= \sum_{i=0}^{k} \lambda^i a_i \\ &= a_0 + (\lambda^0 - 1)a_0 + \lambda^1 a_1 + \ldots + \lambda^k a_k \\ &= a_0 + (-\lambda^1 - \lambda^2 - \ldots - \lambda^k)a + \lambda^1 a_1 + \ldots + \lambda^k a_k \\ &= a_0 + \lambda^1(a_1-a_0) + \lambda^2(a_2-a_0) + \ldots + \lambda^k(a_k-a_0), \text{ and} \end{aligned}$$

$\sum_{i=1}^{k} \lambda^i(a_i-a_0) \in V(a_1-a_0, \ldots, a_k-a_0)$. This proves $\pi(a_0, a_1, \ldots, a_k) \subset a_0 + V(a_1-a_0, \ldots, a_k-a_0)$.

Let $q = a_0 + \sum_{i=1}^{k} \alpha^i(a_i-a_0) \in a_0 + V(a_1-a_0, \ldots, a_k-a_0)$. Then

$$\begin{aligned} q &= a_0 + \alpha^1(a_1-a_0) + \ldots + \alpha^k(a_k-a_0) \\ &= (1 - \alpha^1 - \alpha^2 - \ldots - \alpha^k)a_0 + \alpha^1 a_1 + \ldots + \alpha^k a_k. \end{aligned}$$

Since $1 - \alpha^1 - \alpha^2 - \ldots - \alpha^k + \alpha^1 + \ldots + \alpha^k = 1$, then $q \in \pi(a_0, a_1, \ldots, a_k)$. This proves $a_0 + V(a_1-a_0, \ldots a_k-a_0) \subset \pi(a_0, a_1, \ldots, a_k)$.

1.7. Theorem. A subset of a k-dimensional hyperplane containing k+2 points is geometrically dependent.

Proof: Let $\pi(a_0,a_1,\ldots,a_k)$ be a k-dimensional hyperplane, and let $\{b_0,b_1,\ldots,b_k,b_{k+1},b_{k+2}\} \subset \pi(a_0,a_1,\ldots,a_k)$. Then $V(b_1-b_0,b\ -b_0,\ldots,b_{k+2}-b_0) \subset V(a_1-a_0,\ldots,a_k-a_0)$. This implies $\dim V(b_1-b_0,\ldots,b_{k+2}-b_0) \leq \dim V(a_1-a_0,\ldots,a_k-a_0) = k$. Hence $\{b_1-b_0,b_2-b_0,\ldots,b_{k+2}-b_0\}$ is not linearly independent. Hence $\{b_0,\ldots,b_{k+2}\}$ is not geometrically independent.

1.8. Theorem. Let $S = \{a_0,a_1,\ldots,a_n\}$ be a subset of $R^n$, and, for each i, let $a_i = (a_i^1,a_i^2,\ldots,a_i^n)$. Then the set S is geometrically independent if and only if the determinant

$$d(S) = \begin{vmatrix} a_0^1 & a_0^2 & \ldots & a_0^n & 1 \\ a_1^1 & a_1^2 & \ldots & a_1^n & 1 \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ a_n^1 & a_n^2 & \ldots & a_n^n & 1 \end{vmatrix} \neq 0.$$

Proof: By part (c) of 1.4. Theorem, the set S is geometrically dependent if and only if the system of equations

$$\sum_{i=0}^{n} \lambda^i a_i^j = 0,\ j = 1,2,\ldots,n$$

$$\sum_{i=0}^{n} \lambda^i = 0$$

has a nontrivial solution vector $(\lambda^0,\lambda^1,\ldots,\lambda^n)$. Therefore the set S is geometrically independent if and only if every solution of the system of equations is trivial. By Stein (2, p. 123), the set S is geometrically independent if and only if the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} a_0^1 & a_1^1 & \cdots & a_n^1 \\ a_0^2 & a_0^2 & \cdots & a_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_0^n & a_1^n & \cdots & a_n^n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

has rank $n + 1$. That is, $d(S) \neq 0$.

1.9. Theorem. Let $S = \{a_0, a_1, \ldots, a_n\}$ be a geometrically independent subset of $R^n$, and for each $x = (x^1, x^2, \ldots, x^n)$ in $R^n$, denote by $\lambda^i(x)$ or $\lambda^i$ the barycentric coordinates of $x$ with respect to the set $S$ so that

$$x = \sum_{i=0}^{n} \lambda^i(x) a_i \text{ and } \sum_{i=0}^{n} \lambda^i(x) = 1.$$

Then $\lambda^i(x) = d^i(x)/d(S)$ where $d^i(x)$ is the determinant obtained by replacing the $i^{th}$ row of $d(S)$ by the vector $(x^1, x^2, \ldots, x^n, 1)$.

Proof: The two equations $x = \sum_{i=0}^{n} \lambda^i(x) a_i$ and $\sum_{i=0}^{n} \lambda^i(x) = 1$ are equivalent to the system

$$\begin{array}{ccccccccc} \lambda^0 a_0^1 & + & \lambda^1 a_1^1 & + & \cdots & + & \lambda^n a_n^1 & = & x^1 \\ \lambda^0 a_0^2 & + & \lambda^1 a_1^2 & + & \cdots & + & \lambda^n a_n^2 & = & x^2 \\ \cdot & & \cdot & & \cdots & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdots & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdots & & \cdot & & \cdot \\ \lambda^0 a^n & + & \lambda^1 a^n & + & \cdots & + & \lambda^n a_n^n & = & x^n \\ \lambda^0 & + & \lambda^1 & + & \cdots & + & \lambda^n & = & 1. \end{array}$$

By Cramer's Rule, we have $\lambda^i(x) = d^i(x)/d(S)$.

1.10. Theorem. Let $T = \{a_0, a_1, \ldots, a_k\}$ be a geometrically independent subset of $R^n$. Then there exist points

$a_{k+1}, a_{k+2}, \ldots, a_n$ of $R^n$ such that the set $S = \{a_0, a_1, \ldots, a_n\}$ is geometrically independent.

Proof: Let $T = \{a_0, a_1, \ldots, a_k\}$ be a geometrically independent subset of $R^n$. Then the set $\{a_1 - a_0, \ldots, a_k - a_0\}$ is linearly independent by 1.4. Theorem, part (e). Since $\dim R^n = n$, there exist points $b_{k+1}, b_{k+2}, \ldots, b_n$ such that the set $\{a_1 - a_0, a_2 - a_0, \ldots, a_k - a_0, b_{k+1}, b_{k+2}, \ldots, b_n\}$ is a basis of $R^n$. That is, the set $\{a_1 - a_0, a_2 - a_0, \ldots, a_k - a_0, (b_{k+1} + a_0) - a_0, (b_{k+2} + a_0) - a_0, \ldots, (b_n + a_0) - a_0\}$ is linearly independent. Let $a_i = b_i + a_0$ for $i = k+1, k+2, \ldots, n$. Then, again by 1.4. Theorem, part (e), the set $S = \{a_0, a_1, \ldots, a_n\}$ is geometrically independent.

1.11. Theorem. Let $S = \{a_0, a_1, \ldots, a_n\}$ be a geometrically independent subset of $R^n$. Then for $k < n$, $\pi(a_0, a_1, \ldots, a_k) = \bigcap_{i=k+1}^{n} \{x \in R^n | \ d(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = 0\}$.

Proof: Let $x \in \pi(a_0, \ldots, a_k)$. Then the set $\{a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n\}$ is geometrically dependent by 1.2. Definition, for $i = k+1, k+2, \ldots, n$. Hence $d(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = 0$ for $i = k+1, k+2, \ldots, n$ by 1.8. Theorem. Hence $x \in \bigcap_{i=k+1}^{n} \{x \in R^n | \ d(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = 0\}$. Therefore $\pi(a_0, a_1, \ldots, a_k) \subset \bigcap_{i=k+1}^{n} \{x \in R^n | \ d(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = 0\}$.

Let $x \in \bigcap_{i=k+1}^{n} \{x \in R^n | \ d(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = 0\}$. Then the set $\{a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n\}$ is geometrically dependent for $i = k+1, k+2, \ldots, n$ by 1.8. Theorem. Suppose $x \notin \pi(a_0,\ldots,a_k)$. Let $k+1 \leq f \leq n$. Then there exist $\lambda^f \neq 0$ and $x = \sum_{j=0}^{n} \lambda^j a_j \in \pi(a_0,\ldots,a_n)$. Thus

$d(a_0,\ldots,a_k,a_{k+1},\ldots,a_{f-1},x,a_{f+1},\ldots,a_n) =$

$d(a_0,\ldots,a_k,a_{k+1},\ldots,a_{f-1},\sum_{j=0}^{n} \lambda^j a_j,a_{f+1},\ldots,a_n) =$

$d(a_0,\ldots,a_k,a_{k+1},\ldots,a_{f-1},\sum_{j=0}^{n} \lambda^j a_j - \sum_{\substack{j=0\\j\neq f}}^{n} \lambda^j a_j,a_{f+1},\ldots,a_n) =$

$d(a_0,\ldots,a_k,a_{k+1},\ldots,a_{f-1},\lambda^f a_f,a_{f+1},\ldots,a_n) =$

$\lambda^f d(a_0,\ldots,a_n) =$

$\lambda^f d(S)$. But $d(S) \neq 0$, by 1.8. Theorem. Hence

$d(a_0,\ldots,a_k,a_{k+1},\ldots,a_{f-1},x,a_{f+1},\ldots,a_n) \neq 0$, which is a contradiction to the supposition that

$x \in \bigcap_{i=k+1}^{n}\{x \in R^n |\ d(a_0,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n) = 0\}$.

Therefore $x \in \pi(a_0,\ldots,a_k)$. Therefore

$\bigcap_{i=k+1}^{n}\{x \in R^n |\ d(a\ ,\ldots,a_{i-1},x,a_{i+1},\ldots a_n) = 0\} \subset \pi(a_0,\ldots a_n)$.

This completes the proof.

<u>1.12</u>. <u>Theorem</u>. Let $S = \{a_0,a_1,\ldots,a_k\}$ be a geometrically independent subset of $R^n$, and, for each point $x$ of $\pi(S)$, let $\lambda^i(x)$ be the barycentric coordinates of $x$ with respect to the set $S$. Then each $\lambda^i(x)$ is a continuous real-valued function on $\pi(S)$.

<u>Proof</u>: By 1.10. Theorem, there exist $a_{k+1},\ldots,a_n \in R^n$ such that the set $\{a_0,\ldots,a_n\}$ is geometrically independent. By the definition of hyperplane, $x \in \pi(a_0,\ldots,a_k)$ implies $x \in \pi(a_0,\ldots,a_n)$. By 1.9. Theorem, $\lambda^i(x) = d^i(x)/d(a_0,\ldots,a_n)$ for $i = 0,\ldots,n$. By part (d) of 1.4. Theorem, the barycentric

coordinates of x with respect to the set $\{a_0, \ldots, a_n\}$ are unique. Hence the barycentric coordinates $\lambda^i(x)$ of x with respect to the set $\{a_0, \ldots, a_k\}$ are equal to the barycentric coordinates of x with respect to the set $\{a_0, \ldots, a_n\}$ for $i = 0, 1, \ldots, k$, and $\lambda^i(x) = 0$ for $i = k+1, \ldots, n$. Since $\lambda^i(x) = d^i(x)/d(a_0, \ldots, a_n)$, we can let $\lambda^i(x) = c_1^i x^1 + \ldots + c_n^i x^n + c_{n+1}^i$, for every $x \in \pi(a_0, \ldots, a_k)$. Also, $x^1, \ldots, x^n$ are Euclidean coordinates of x, and $c_1^i, \ldots, c_n^i, c_{n+1}^i$ are real numbers.

Let i be an integer; $1 \leq i \leq k$. For $\varepsilon > 0$, let $\delta_i = \varepsilon/n\phi_i$. Where $\phi_i = \max\{|c_1^i|, \ldots, |c_n^i|, 1\}$, let $y \in \pi(a_0, \ldots, a_k)$ and $|x-y| < \delta_i$. That is, $[(x^1-y^1)^2 + \ldots + (x^n-y^n)^2]^{1/2} < \delta_i$. Hence $|x^j-y^j| < \delta_i = \varepsilon/n\phi_i$ for $j = 1, \ldots, n$. Hence

$$\begin{aligned} |\lambda^i(x)-\lambda^i(y)| &= |c_1^i(x^1-y^1) + \ldots + c_n^i(x^n-y^n)| \\ &\leq |c_1^i(x^1-y^1)| + \ldots + |c_n^i(x^n-y^n)| \\ &< |c_1^i\,\varepsilon/n\phi_i| + \ldots + |c_n^i\,\varepsilon/n\phi_i| \\ &\leq \varepsilon/n + \ldots + \varepsilon/n \\ &= \varepsilon. \end{aligned}$$

This proves each $\lambda^i(x)$ is a continuous real-valued function on $\pi(a_0, \ldots, a_k)$.

<u>1.13</u>. <u>Theorem</u>. If S is an n-simplex in $R^n$, then $\pi(S) = R^n$. An n-simplex is a simplex containing n+1 elements.

<u>Proof</u>: Let $S = \{a_0, \ldots, a_n\}$ be the n-simplex in $R^n$. By part (e) of 1.4. Theorem, the set $\{a_1-a_0, \ldots, a_n-a_0\}$ is linearly independent. Since $R^n$ is n-dimensional vector

space over the real numbers, then by Herstein (1, Lemma 4.7), $\{a_1-a_0,\ldots,a_n-a_0\}$ is a basis of $R^n$. Therefore $V(a_1-a_0,\ldots,a_n-a_0) = R^n$. By 1.6. Theorem, $\pi(S) = a_0 + V(a_1-a_0,\ldots,a_n-a_0) = a_0 + R^n = R^n$. This completes the proof.

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## CHAPTER II

## SIMPLEXES AND COMPLEXES IN $R^n$

2.1. Definition. Let S be a finite subset of $R^n$. Then S is called a simplex if and only if S is geometrically independent.

Notation: Let $S = \{a_0, \ldots, a_k\}$. Denote the set $\{\sum_{s \in S} f(s)s \mid f: S \to \text{Reals, such that } \sum_{s \in S} f(s) = 1, f(s) > 0 \text{ for each } s \in S\}$ by $\Delta(S)$.

2.2. Definition. Let $S = \{a_0, \ldots, a_k\}$ be a simplex. The dimension of S, denoted by dim(S), is the integer k, and S is called k-simplex.

2.3. Definition. Let S be a simplex, and let T be a subset of S. Then T is called a face of S and $\Delta(T)$ is a point-set face of $\Delta(S)$.

2.4. Theorem. A hyperplane in $R^n$ is a closed set in $R^n$.

Proof: Let $\pi(a_0, \ldots, a_k)$ be a k-dimensional hyperplane in $R^n$. By 1.10. Theorem, there exist points $a_{k+1}, \ldots, a_n$ such that the set $\{a_0, \ldots, a_n\}$ is geometrically independent. Let x be a limit point of $\pi(a_0, \ldots, a_k)$, and $x \notin \pi(a_0, \ldots, a_k)$, since $x \in \pi(a_0, \ldots, a_n) = R^n$. Let $x = \sum_{i=0}^{n} \lambda^i a_i \in \pi(a_0, \ldots, a_n)$. Then there exists $\lambda^j(x) \neq 0$ for some $k < j \leq n$. By 1.12. Theorem, $\lambda^j(x)$ is a continuous function on $R^n$. Hence for $\varepsilon = |\lambda^j(x)|/2$, there exists $\delta > 0$ such that $|\lambda^j(x) - \lambda^j(y)| < \varepsilon = |\lambda^j(x)|/2$, whenever y belongs to the neighborhood $N_\delta(x)$ of $x$. Since $x$ is a limit point of $\pi(a_0,\ldots,a_k)$, then there exists $y' \in N_\delta(x) \cap \pi(a_0,\ldots,a_k)$. But $y' \in \pi(a_0,\ldots,a_k)$ implies $\lambda^j(y) = 0$ by the fact that $k < j$. Therefore $|\lambda^j(x)-\lambda^j(y')| = |\lambda^j(x)| < \varepsilon = |\lambda^j(x)|/2$. This is impossible since $\lambda^j(x) \neq 0$. Therefore $x \in \pi(a_0,\ldots,a_k)$. Hence $\pi(a_0,\ldots,a_k)$ is a closed set in $R^n$.

2.5. Theorem. Let $S$ be a simplex in $R^n$, and $x \in \Delta(S)$. Then the expression of $x = \sum_{s \in S} f(s)s$ is unique.

Proof: By the definitions of $\Delta(S)$ and $\pi(S)$, $\Delta(S) \subset \pi(S)$. Hence $x \in \Delta(S)$ implies $x \in \pi(S)$. By part (d) of 1.4. Theorem, the uniqueness of the expression $x = \sum_{s \in S} f(s)s$ follows.

2.6. Theorem. If $S$ is a simplex in $R^n$, then $\Delta(S)$ is an open set in the relative topology of the hyperplane $\pi(S)$.

Proof: Let $x = \sum_{s \in S} f(s)s \in \Delta(S)$. Since $\Delta(S) \subset \pi(S)$, then $x \in \Delta(S)$ implies $x \in \pi(S)$. Denote $f(s)$ by $g_s(x)$. By 1.12. Theorem, $g_s(x)$ is a continuous function on $\pi(S)$ for each $s \in S$. For each $s \in S$, define $G_s = \{x \in \pi(S) \mid g_s(x) > 0\}$. Since $g_s(x)$ is a continuous function on $\pi(S)$ for each $s \in S$, then $G_s$ is an open set in $\pi(S)$ for each $s \in S$. By the definition of $\Delta(S)$, $\Delta(S) = \bigcap_{s \in S} G_s$ which is open in $\pi(S)$.

2.7. Theorem. Let $S$ be a simplex in $R^n$, and denote the closure of $\Delta(S)$ in $R^n$ by $\overline{\Delta(S)}^{R^n}$. Then $\overline{\Delta(S)}^{R^n} \subset \pi(S)$.

Proof: If the cardinal number of $S$ is $n+1$, then $\pi(S) = R^n$. This theorem is obviously true. Now, we shall assume that $\pi(S) \neq R^n$.

By 1.10. Theorem, there exists a simplex S' in $R^n$ such that $S \subsetneq S'$, and $\pi(S') = R^n$. Suppose there exists $x \in \overline{\Delta(S)}^{R^n}$, and $x \notin \pi(X)$. Then $x \in \pi(S')\setminus\pi(S)$. Let $x = \sum_{s \in S'} f(s)s \in \pi(S')$. Since $x \notin \pi(S)$, then there exist $s_0 \in S'\setminus S$ such that $f(s_0) \neq 0$. Let $g_s(y)$ be the barycentric coordinates of y relative to the set S'. Define $G_{s_0} = \{y \in R^n | \; g_{s_0}(y) \neq 0\}$. By 1.12 Theorem, $g_{s_0}(y)$ is a continuous function on $R^n$. Hence $G_{s_0}$ is an open set in $R^n$, and $x \in G_{s_0}$. By the definition of $G_{s_0}$, $G_{s_0} \cap \Delta(S) = \phi$. This is a contradiction to the supposition. Therefore $\overline{\Delta(S)}^{R^n} \subset \pi(S)$.

2.8. Theorem. Let S be a simplex in $R^n$. Denote the closure of $\Delta(S)$ in $R^n$ by $\overline{\Delta(S)}^{R^n}$, and denote the closure of $\Delta(S)$ in the hyperplane $\pi(S)$ by $\overline{\Delta(S)}^{\pi(S)}$. Then $\overline{\Delta(S)}^{R^n} = \overline{\Delta(S)}^{\pi(S)}$.

Proof: If $x \in \Delta(S)$, then $x \in \overline{\Delta(S)}^{\pi(S)}$. Let x be a limit point of $\overline{\Delta(S)}^{R^n}$, $x \notin \Delta(S)$, U an open set in $R^n$, and $x \in U$. Then $U \cap \Delta(S) \neq \phi$. Let u' be an open set in $\pi(S)$, and let $x \in u'$. By the definition of relative topology, there exists an open set v in $R^n$ such that $u' = v \cap \pi(S)$. Hence $v \cap \Delta(S) = v \cap (\pi(S) \cap \Delta(S)) = (v \cap \pi(S)) \cap \Delta(S) = u' \cap \Delta(S) \neq \phi$. Since $x \notin \Delta(S)$, then $x \notin u' \cap \Delta(S)$. Therefore $u' \cap \Delta(S) \neq \phi$ implies that x is a limit point of $\overline{\Delta(S)}^{\pi(S)}$. Hence $\overline{\Delta(S)}^{R^n} \subset \overline{\Delta(S)}^{\pi(S)}$.

Let $x \in \overline{\Delta(S)}^{\pi(S)}$. Then $x \in \pi(S)$. For any open set u' in $\pi(S)$ and $x \in u'$, then $u' \cap \Delta(S) \neq \phi$. Let u be an open set in $R^n$, and let $x \in u$. But $x \in \pi(S)$; therefore $x \in u \cap \pi(S)$ is an open set in $\pi(S)$. Hence $u \cap \Delta(S) = u \cap (\Delta(S) \cap \pi(S)) = (u \cap \pi(S)) \cap \Delta(S) \neq \phi$. This implies $x \in \overline{\Delta(S)}^{R^n}$. Hence $\overline{\Delta(S)}^{\pi(S)} \subset \overline{\Delta(S)}^{R^n}$. Both of $\overline{\Delta(S)}^{R^n} \subset \overline{\Delta(S)}^{\pi(S)}$ and $\overline{\Delta(S)}^{\pi(S)} \subset \overline{\Delta(S)}^{R^n}$ imply $\overline{\Delta(S)}^{R^n} = \overline{\Delta(S)}^{\pi(S)}$. This completes the proof of this theorem.

2.9. Theorem. If S is a simplex in $R^n$, then $\overline{\Delta(S)} = \{\sum_{s \in S} f(s)s \mid f: S \to \text{non-negative reals}, \sum_{s \in S} f(s) = 1\}$.

Proof: For the sake of simplicity, denote the set $\{\sum_{s \in S} f(s)s \mid f: S \to \text{non-negative reals}, \sum_{s \in S} f(s) = 1\}$ by A. By 2.4, Theorem, $\overline{\Delta(S)} \subset \pi(S)$. Denote the barycentric coordinates of a point $x \in R^n$ relative to the set S by $g_s(x)$, where $s \in S$. Suppose there exist $y \in R^n$ such that $y \in \overline{\Delta(S)}$ and $y \notin A$. Then $y \in \overline{\Delta(S)} \subset \pi(S)$ and $y \notin A$ imply $y = \sum_{s \in S} g_s(y)s \in \pi(S)$ and $g_{s'}(y) < 0$ for some $s' \in S$. Define $G_{s'} = \{w = \sum_{s \in S} g_s(w)s \in \pi(S) \mid g_{s'}(w) < (g_{s'}(y))/2\}$. By 1.12. Theorem, $g_{s'}(w)$ is a continuous function on $\pi(S)$. Therefore $G_{s'}$ is an open set in $\pi(S)$. By definition of $G_{s'}$, $y \in G_{s'}$, and $G_{s'} \cap \Delta(S) = \phi$. This contradicts $y \in \overline{\Delta(S)}$. Hence the supposition is false. Hence $y \in \overline{\Delta(S)}$ implies that $y \in A$. Hence $\overline{\Delta(S)} \subset A$.

If $\dim(S) = 0$, then S contains only one point. Hence $S = \Delta(S) = \overline{\Delta(S)} = A$. Therefore assume that $\dim(S) = m > 0$. Denote the norm of $x \in R^n$ by $|x|$, and the set $\{z \in R^n \mid |z-y| < \delta$, where $y \in R^n$ and $\delta$ is a positive real number$\}$ by $N_\delta(y)$. For a given $\delta > 0$, let $y \in A$; then

$y = \sum_{s \in S} g(s)s \in \pi(S)$, and $g(s) \geq 0$ for each $s \in S$. Since $y = \sum_{s \in S} g(s)s \in \pi(S)$, then $\sum_{s \in S} g(s) = 1$. But $g(s) \geq 0$ for each $s \in S$; therefore there exists $s' \in S$ such that $0 < g(s') \leq 1$. Denote $g(s')$ by K and $\sum_{s \in S} |s|$ by M. Then $0 < K \leq 1$. Since $\dim(S) > 0$, $M > 0$. Define a function $g'$: $S \to$ reals by

$$g'(s) = g(s) + t/m, \text{ if } s \neq s';$$

$$g'(s') = g(s') - t = K - t, \text{ where}$$

$$t = \min\{K/2,\ \delta/M(1+m)\}.$$

Then $\sum_{s \in S} g'(s) = \sum_{\substack{s \in S,\\ s \neq s'}} (g(s) + t/m) + g(s') - t = \sum_{s \in S} g(s) = 1$,

and $g'(s) > 0$ for each $s \in S$. Hence $\sum_{s \in S} g'(s)s \in \Delta(S)$. Denote $\sum_{s \in S} g'(s)s$ by z. Then

$$\begin{aligned} |z-y| &= \left|\sum_{s \in S} g'(s)s - \sum_{s \in S} g(s)s\right| \\ &= \left|\sum_{\substack{s \in S,\\ s \neq s'}} (t/m)s - ts'\right| \\ &\leq \left|\sum_{\substack{s \in S,\\ s \neq s'}} (t/m)s\right| + t|s'| \\ &\leq t/m \sum_{\substack{s \in S,\\ s \neq s'}} |s| + t|s'| \\ &\leq (t/m)M + tM \\ &= (t(1+m))/m\, M. \end{aligned}$$

If $t = K/2$, then $(t(1+m)/m)M = (K(1+m)/2m)M \leq (\delta/M(1+m))(1+m)/m\, M = \delta/m \leq \delta$.

If $t = \delta/M(1+m)$, then $(t(1+m)/m)M = (\delta/M(1+m))((1+m)/m)M = \delta/m \leq \delta$.

Hence $|z-y| < \delta$. Therefore $z \in N_\delta(y)$, and $z \in \Delta(S)$. Hence $z \in N_\delta(y) \cap \Delta(S)$. Thus $y \in \overline{\Delta(S)}$; therefore $A \subset \overline{\Delta(S)}$.

Taken together, $\overline{\Delta(S)} \subset A$ and $A \subset \overline{\Delta(S)}$ imply that $\overline{\Delta(S)} = A$. This completes the proof of this theorem.

2.10. Theorem. In $R^n$, if a simplex S' is a face of a simplex S, then $\Delta(S')$ is contained in $\overline{\Delta(S)}$. Conversely, each point in $\overline{\Delta(S)}$ is an element of a unique point-set face of $\overline{\Delta(S)}$.

Proof: Since S' is a face of S, then $S' \subset S$. Therefore, by 2.9, $\Delta(S') = \{\sum_{s \in S} f(s)s \mid f: S' \to \text{positive reals}; \sum_{s \in S} f(s) = 1\}$ $\subset \{\sum_{s \in S} g(s)s \mid g: S \to \text{non-negative reals}; \sum_{s \in S} g(s) = 1\} = \overline{\Delta(S)}$.

Let $x \in \overline{\Delta(S)}$. Then $x = \sum_{s \in S} h(s)s$ for some function $h: S \to$ non-negative reals and $\sum_{s \in S} h(s) = 1$. Let $S'' = \{s \in S \mid h(s) \neq 0\}$; then $S'' \subset S$, and $x \in \Delta(S'')$. But $S'' \subset S$ implies $\Delta(S'') \subset \pi(S)$. Hence $x \in \pi(S)$. By part (d) of 1.4, the set S" is unique. This completes the proof of this theorem.

2.11. Definition. Let K be a finite collection of simplexes in $R^n$ such that the following is true:

(1) If $x \in K$ and $\phi \neq y \subset x$, then $y \in K$.

(2) If $t,u \in K$, then $\Delta(t) \cap \Delta(u) = \phi$.

Then K is called a complex in $R^n$.

2.12. Definition. Let K be a complex in $R^n$; the positive integer $\sup\{\dim(x) \mid x \in K\}$ is called the dimension of K. If $\sup\{\dim(x) \mid x \in K\} = m$, then we write $\dim(K) = m$ and say that K is an m-complex.

2.13. Definition. Let K be a complex in $R^n$; let the set $\bigcup\{\Delta(S) \mid S \in K\}$ be denoted by $|K|$. Then a subset P of $R^n$

is called a polytope in $R^n$ if and only if $P = |K|$ for some complex K in $R^n$. A complex K is called a triangulation of the polytope $|K|$.

2.14. Definition. If $S_1$ and $S_2$ are distinct faces of a simplex S in $R^n$, then by 2.10, $\Delta(S_1) \cap \Delta(S_2) = \phi$. It follows that the set $K_1 = \{S' | S' \subset S; S' \neq \phi\}$ is a complex in $R^n$. This complex $K_1$ is called the combinatorial closure of S. The set $K_2 = \{S' | S' \subsetneq S; S' \neq \phi\}$ is also a complex. This complex is called the combinatorial boundary of S.

2.15. Theorem. Let S be a simplex in $R^n$, and let $K_2$ be the combinatorial boundary of S. Then $|K_2|$ is the point-set boundary of $\Delta(S)$ in $\pi(S)$.

Proof: Let B be the point-set boundary of $\Delta(S)$ in $\pi(S)$. Let $x \in |K_2|$; then $x \in \Delta(S')$ for some proper face S' of S. By 2.9, $x \in \overline{\Delta(S)} \setminus \Delta(S)$. By 2.6, $\Delta(S)$ is open in $\pi(S)$. Hence $x \in B$. Hence $|K_2| \subset B$.

Let $x \in B$; then $x \in \overline{\Delta(S)} \setminus \Delta(S)$. Let $x = \sum_{s \in S} g(s)s$, $\sum_{s \in S} g(s) = 1$, and $g(s) \geq 0$ for each $s \in S$. Let $M = \{s \in S | g(s) = 0\}$. Since $x \notin \Delta(S)$, then $M \neq \phi$. Hence $S \setminus M$ is a proper face of S, and $x \in \Delta(S \setminus M)$. Therefore $x \in |K_2|$. Hence $B \subset |K_2|$.

Together, $|K_2| \subset B$ and $B \subset |K_2|$ imply $|K_2| = B$.

2.16. Definition. Let $S = \{a_0, \ldots, a_m\}$ be a finite set. Each element $a_i$ of S is called an abstract vertex. Each non-empty subset of S is called an abstract simplex of S. A collection of subsets K of S is called an abstract complex of S if and only if $K = (\bigcup_{i=0}^{m}\{\{a_i\}\}) \cup (\bigcup_{j=1}^{r} 2^{S_j})$ for some subsets $S_1,\ldots,S_r$ of S, where $2^{S_j}$ is the collection of all non-empty subsets of $S_j$. The set S is called the vertices of K.

2.17. Definition. Two complexes (or abstract complexes) $K_1$ and $K_2$ are said to be isomorphic provided there exists a one-to-one onto function $f: K_1^0 \to K_2^0$, where $K_1^0$ are the vertices of $K_1$, and $K_2^0$ are the vertices of $K_2$, having the property that a subset $\{a_{i_0},a_{i_1},\ldots,a_{i_t}\}$ of $K_1^0$ is the set of vertices of a simplex in $K_1$ if and only if $\{f(a_{i_0}),f(a_{i_1}),\ldots,f(a_{i_t})\}$ is the set of vertices of a simplex in $K_2$. If an abstract complex $K_1$ is isomorphic to a complex $K_2$, then $K_2$ is said to be a realization of $K_1$.

2.18. Theorem. If $R^n$ is an n-dimensional Euclidean space, then there exists an n-simplex in $R^n$.

Proof: Let $S = \{(1,0,0,\ldots 0), (0,1,0,0,\ldots,0),\ldots, (0,0,\ldots,0,1,0), (0,0,\ldots,0,1)\}$. Obviously S is linearly independent. By 1.4, the set $S' = \{(1,0,0,\ldots,0), (2,0,0,\ldots,0), (1,1,0,\ldots,0), (1,0,1,0,0,\ldots,0),\ldots, (1,0,0,\ldots,0,1)\}$ is geometrically independent. Hence S' is an n-simplex in $R^n$.

2.19. Theorem. Every abstract complex $K_1$ has a realization $K_2$ in some Euclidean space $R^m$.

Proof: Let $\{a_0,\ldots,a_m\}$ be the vertices of the abstract complex $K_1$ and $\{b_0,\ldots,b_m\}$ be an m-simplex in $R^m$. By 2.18, such an m-simplex exists. Define the function $f: \{a_0, \ldots, a_m\} \to \{b_0, \ldots, b_m\}$ by $f(a_i) = b_i$ for $i = 0,1,\ldots,m$. Then f is an one-to-one onto function. Let $K_2 = \{f(S) \mid S \in K_1\}$; then $\{b_i\} \in K_2$ for $i = 0,1,\ldots,m$, because $\{a_i\} \in K_1$ for $i = 0,1,\ldots,m$. If $A \in K_2$ and B is a face of A, then $f^{-1}(A) \in K_1$ and $f^{-1}(B) \in f^{-1}(A)$. Since $K_1$ is an abstract complex, then $f^{-1}(B) \in K_1$. Therefore $B \in K_2$. By part (d) of 1.4, if $x,y \in K_2$ and $x \neq y$, then $\Delta(x) \cap \Delta(y) = \phi$. Hence $K_2$ is a complex in $R^m$, and f is an isomorphism. Therefore $K_2$ is a realization of $K_1$.

2.20. Definition. An n-dimensional pseudomanifold is an n-complex with the following properties:

(a) Each simplex is a face of an n-simplex.

(b) Each (n-1)-simplex is a face of exactly two n-simplexes.

(c) For each pair $t_1^n$ and $t_2^n$ of distinct n-simplexes, there exists a finite sequence $S_1^n, S_1^{n-1}, S_2^n, S_2^{n-1}, \ldots, S_{k-1}^{n-1}, S_k^n$ of simplexes such that $S_1^n = t_1^n$, $S_k^n = t_2^n$; also, for $1 \leq i < k$, each $S_i^{n-1}$ is a face of both $S_i^n$ and $S_{i+1}^n$, where $S_i^n$ is n-simplex, and where $S_i^{n-1}$ is (n-1)-simplex, for $i = 1,\ldots,k$.

In order to realize the geometric meaning of a pseudo-manifold, the following are four examples in $R^3$:

Example 1. Let K be constructed as follows:

| | | |
|---|---|---|
| $a_0 = (0,0,0)$ | $a_1 = (1,0,0)$ | $a_2 = (0,1,0)$ |
| $a_3 = (0,0,1)$ | | |
| $S_1^2 = \{a_0, a_1, a_2\}$ | $S_1^1 = \{a_0, a_1\}$ | $S_1^0 = \{a_0\}$ |
| $S_2^2 = \{a_0, a_1, a_3\}$ | $S_2^1 = \{a_0, a_2\}$ | $S_2^0 = \{a_1\}$ |

| | | |
|---|---|---|
| $S_3^2 = \{a_0, a_2, a_3\}$ | $S_3^1 = \{a_0, a_3\}$ | $S_3^0 = \{a_2\}$ |
| $S_4^2 = \{a_1, a_2, a_3\}$ | $S_4^1 = \{a_1, a_2\}$ | $S_4^0 = \{a_3\}$ |
| | $S_5^1 = \{a_1, a_3\}$ | |
| | $S_6^1 = \{a_2, a_3\}$ | |

$K = \{S_i^2 \mid i = 1,2,3,4\} \cup \{S_i^1 \mid i = 1,2,3,4,5,6\} \cup \{S_i^0 \mid i = 1,2,3,4\}$. Then K is a 2-dimensional pseudomanifold.

Example 2. Let K be constructed as follows:

| | | |
|---|---|---|
| $a_0 = (0,0,0)$ | $a_1 = (1,0,0)$ | $a_2 = (0,1,0)$ |
| $a_3 = (0,0,1)$ | $a_4 = (0,2,0)$ | $a_5 = (0,2,1)$ |
| $a_6 = (1,2,0)$ | | |
| $S_1^2 = \{a_0, a_1, a_2\}$ | $S_1^1 = \{a_0, a_1\}$ | $S_1^0 = \{a_0\}$ |
| $S_2^2 = \{a_0, a_1, a_3\}$ | $S_2^1 = \{a_0, a_1\}$ | $S_2^0 = \{a_1\}$ |
| $S_3^2 = \{a_0, a_2, a_3\}$ | $S_3^1 = \{a_0, a_3\}$ | $S_3^0 = \{a_2\}$ |
| $S_4^2 = \{a_1, a_2, a_3\}$ | $S_4^1 = \{a_1, a_2\}$ | $S_4^0 = \{a_3\}$ |
| $S_5^2 = \{a_2, a_4, a_5\}$ | $S_5^1 = \{a_1, a_3\}$ | $S_5^0 = \{a_4\}$ |
| $S_6^2 = \{a_2, a_4, a_6\}$ | $S_6^1 = \{a_2, a_3\}$ | $S_6^0 = \{a_5\}$ |
| $S_7^2 = \{a_2, a_5, a_6\}$ | $S_7^1 = \{a_2, a_4\}$ | $S_7^0 = \{a_6\}$ |
| $S_8^2 = \{a_4, a_5, a_6\}$ | $S_8^1 = \{a_2, a_5\}$ | |
| | $S_9^1 = \{a_2, a_6\}$ | |
| | $S_{10}^1 = \{a_4, a_5\}$ | |
| | $S_{11}^1 = \{a_4, a_6\}$ | |
| | $S_{12}^1 = \{a_5, a_6\}$ | |

$K = \{S_i^2 \mid i = 1,\ldots,8\} \cup \{S_i^1 \mid i = 1,\ldots,12\} \cup \{S_i^0 \mid i = 1,\ldots,7\}$. Then K is a two-dimensional complex satisfying the conditions (a) and (b), but not (c).

Example 3. Let $a_0$, $a_1$, $a_2$, $a_3$ be as in Example 1.

$S_1^2 = \{a_0, a_1, a_3\}$ $S_1^1 = \{a_0, a_1\}$ $S_1^0 = \{a_0\}$

$S_2^2 = \{a_0, a_2, a_3\}$ $S_2^1 = \{a_0, a_2\}$ $S_2^0 = \{a_1\}$

$S_3^1 = \{a_0, a_3\}$ $S_3^0 = \{a_2\}$

$S_4^1 = \{a_1, a_3\}$ $S_4^0 = \{a_3\}$

$S_5^1 = \{a_2, a_3\}$

$K = \{S_1^2, S_2^2\} \cup \{S_i^1 | i = 1,\ldots,5\} \cup \{S_i^0 | i = 1,\ldots,4\}$. Then $K$ is a 2-dimensional complex satisfying the conditions (a) and (c), but not (b).

Example 4. Let $a_5 = (1,1,1)$, and denote the 2-dimensional pseudomanifold of Example 1 by $K'$. Let $K'' = K' \cup \{a_5\}$; then $K''$ is a 2-dimensional complex satisfying the conditions (b) and (c), but not (a).

<u>2.21</u>. <u>Theorem</u>. If $S = \{s_0,\ldots,s_n\}$ is a simplex in $R^m$, then $\Delta(S)$ is connected.

<u>Proof</u>: Suppose $\Delta(S)$ is not connected; let $\Delta(S) = A \cup B$ such that $\overline{A} \cap B = A \cap \overline{B} = \phi$ and $A \neq \phi \neq B$. Let $a = \sum_{s \in S} f(s)s \in A$, $\sum_{s \in S} f(s) = 1$, and $f(s) > 0$ for each $s \in S$; also let $b = \sum_{s \in S} g(s)s \in B$, $\sum_{s \in S} g(s) = 1$, and $g(s) > 0$ for each $s \in S$. Let $D = \{0 \leq k < 1 | \sum_{f(s) \neq g(s)} (f(s) + k(g(s)-f(s)))s + \sum_{f(s) = g(s)} f(s)s \in A\}$. Since

$$\sum_{f(s) \neq g(s)} (f(s) + k(g(s)-f(s))) + \sum_{f(s)=g(s)} f(s)$$
$$= \sum_{f(s) \neq g(s)} f(s) + \sum_{f(s)=g(s)} f(s) + \sum_{f(s) \neq g(s)} k(g(s)-f(s))$$
$$= \sum_{s \in S} f(s) + k[\sum_{f(s) \neq g(s)} g(s) - \sum_{f(s) \neq g(s)} f(s)]$$

$= 1 + k[(1 - \sum_{f(s)=g(s)} g(s)) - (1 - \sum_{f(s)=g(s)} f(s))]$

$= 1 + k[\sum_{f(s)=g(s)} f(s) - \sum_{f(s)=g(s)} g(s)]$

$= 1 + k[0]$

$= 1$, the set D is non-empty and the point p =
$\sum_{f(s)\neq g(s)}[f(s) + L(g(s)-f(s))]s + \sum_{f(s)=g(s)} f(s)s \in \Delta(S)$,
where L = lub D. Since $p \in \Delta(S)$, then either $p \in A$ or $p \in B$.

Case 1: $p \in A$. Let $\max\{|s_0|,\ldots,|s_n|\} = \theta$ where $|s_i|$ is the norm of $s_i$. Since L = lub D, then for $\varepsilon > 0$ there exists a real number L' such that
$L < L' < L(1 + \varepsilon/(1+\varepsilon)(1+n)\theta)$ and the point p' =
$\sum_{f(s)\neq g(s)}[f(s) + L'(g(s)-f(s))]s + \sum_{f(s)=g(s)} f(s)s \in B$.

Then $|p' - p| =$

$|\sum_{f(s)\neq g(s)}[f(s) + L'(g(s)-f(s))]s + \sum_{f(s)=g(s)} f(s)s -$

$\sum_{f(s)\neq g(s)}[f(s) + L(g(s)-f(s))]s - \sum_{f(s)=g(s)} f(s)s| =$
$|\sum_{f(s)\neq g(s)}(L' - L)(g(s)-f(s))s| <$

$\varepsilon/(1+\varepsilon)(1+n)\theta|\sum_{f(s)\neq g(s)}(g(s)-f(s))s| \leq$

$\varepsilon/(1+\varepsilon)(1+n)\theta \sum_{f(s)\neq g(s)}|g(s)-f(s)||s| \leq \varepsilon/(1+\varepsilon) < \varepsilon$.
This implies p is a limit point of B. Hence $A \cap \overline{B} \neq \phi$, which is a contradiction to the supposition.

Case 2: $p \in B$. Since L = lub D, then for $\varepsilon > 0$ there exists a real number L' such that

$L(1 - \varepsilon/(1+\varepsilon)(1+n)\theta) < L' < L$, and the point $p' =$

$$\sum_{f(s)\neq g(s)} [f(s) + L'(g(s)-f(s))]s + \sum_{f(s)=g(s)} f(s)s \in A.$$

Then $|p - p'| =$

$$\Big|\sum_{f(s)\neq g(s)} [f(s) + L(g(s)-f(s))]s + \sum_{f(s)=g(s)} f(s)s -$$

$$\sum_{f(s)\neq g(s)} [f(s) + L'(g(s)-f(s))]s - \sum_{f(s)=g(s)} f(s)s\Big| =$$

$$\Big|\sum_{f(s)\neq g(s)} (L - L')(g(s)-f(s))s\Big| <$$

$$\varepsilon/(1+\varepsilon)(1+n)\theta\Big|\sum_{f(s)\neq g(s)} (g(s)-f(s))s\Big| \leq$$

$$\varepsilon/(1+\varepsilon)(1+n)\theta\sum_{f(s)\neq g(s)} |g(s)-f(s)|\,|s| \leq \varepsilon/(1+\varepsilon) < \varepsilon.$$

This implies $p$ is a limit point of $A$. Hence $\overline{A} \cap B \neq \phi$, which is a contradiction to the supposition. Hence the supposition is false. Hence $\Delta(S)$ is connected.

2.22. Theorem. If $S$ is a simplex in $R^n$, and if $S_1,\ldots,S_m$ are subsets of $S$, then the set $T = \Delta(S) \cup \Delta(S_1) \cup \ldots \cup \Delta(S_m)$ is connected.

Proof: Assume $T$ is not connected; let $T = A \cup B$ such that $\overline{A} \cap B = A \cap \overline{B} = \phi$, and let $A \neq \phi \neq B$. By 2.21, $\Delta(S)$ is connected. The sets $A$ and $B$ are separated. Then either $\Delta(S) \subset A$ or $\Delta(S) \subset B$. Assume $\Delta(S) \subset A$. Since $B \neq \phi$, then there exists a point $y \in B$. That is, $y \in \Delta(S_k)$ for some $1 \leq k \leq m$. By 2.21, $\Delta(S_k)$ is connected. Therefore $y \in B$ and $y \in \Delta(S_k)$ imply $\Delta(S_k) \subset B$. But $S_k$ is a face of $S$. By 2.9, $\Delta(S_k) \subset \overline{\Delta(S)}$. Since $\Delta(S) \subset A$, then $\overline{\Delta(S)} \subset \overline{A}$. Hence

$\Delta(S_k) \subset \overline{A}$. Therefore $\Delta(S_k) \subset \overline{A} \cap B$. That is $\overline{A} \cap B \neq \phi$. This contradicts the supposition that A and B are separated. Similarly, a contradiction can be found if $\Delta(S) \subset A$. Therefore the supposition is false, and T is connected.

2.23. Theorem. If S is an n-simplex, $n \geq 2$, and $K_2$ is the combinatorial boundary of S, then $|K_2|$ is connected.

Proof: Let $S = \{s_0, s_1, \ldots, s_n\}$. Denote the set $S \setminus \{s_i\}$ by $S_i$ for $i = 0,1,\ldots,n$; denote the combinatorial closure of $S_i$ by $K_i'$ for $i = 0,\ldots,n$. Then $|K_2| = \bigcup_{i=0}^{n} |K_i'|$. By 2.22, $|K_i'|$ is connected for $i = 0,1,\ldots,n$. Since $n \geq 2$, then $S_i \cap S_j \neq \phi$ for $0 \leq i, j \leq n$. Hence $|K_i'| \cap |K_j'| \neq \phi$ for $0 \leq i, j \leq n$. Since $\bigcup_{i=0}^{n} |K_i'|$ is connected, then $|K_2|$ is connected.

2.24. Theorem. If K is an n-dimensional pseudomanifold, then $|K|$ is connected.

Proof: Suppose $|K|$ is not connected; let $|K| = A \cup B$ such that $\overline{A} \cap B = A \cap \overline{B} = \phi$, and $A \neq \phi \neq B$. Since $A \neq \phi \neq B$, then there exist $a \in A$ and $b \in B$. Then $a \in \Delta(S_a)$, and $b \in \Delta(S_b)$ for some simplex $S_a$ and $S_b \in K$. Since K is an n-dimensional pseudomanifold, then $S_a$ and $S_b$ are faces of some n-simplexes $S_a^n$ and $S_b^n$ respectively. Hence there exist a finite sequence $S_1^n$, $S_1^{n-1}$, $S_2^n$ $S_2^{n-1}$, ..., $S_{k-1}^{n-1}$, $S_k^n$ of simplexes such that $S_1^n = S_a^n$, $S_k^n = S_b^n$, and for $1 \leq i < k$, each $S_i^{n-1}$ is a face of both $S_i^n$ and $S_{i+1}^n$. Denote this sequence of simplexes by $r_1, r_2, \ldots, r_{2k-1}$. Then

$r_1 = S_1^n = S_a^n$, and $r_{2k-1} = S_k^n = S_b^n$. By 2.21, $\Delta(r_i)$ is connected for $i = 1,\ldots,2k-1$. The sets A and B are separated. Then either $\Delta(r_i) \subset A$ or $\Delta(r_i) \subset B$ for $i = 1,\ldots,2k-1$. Let m be an integer such that $\Delta(r_m) \subset A$ and $\Delta(r_{m+1}) \subset B$.

Case 1: For some $1 \leq t < 2k-1$, $r_m = S_t^n$. Then $\Delta(r_{m+1}) = \Delta(S_t^{n-1}) \subset B$. Since $S_t^{n-1}$ is a face of $S_t^n$, then $\Delta(S_t^{n-1}) \subset \overline{\Delta(S_t^n)} \subset \bar{A}$. Hence $\Delta(S_t^{n-1}) \subset \bar{A} \cap B$. This contradicts A and B being separated.

Case 2: For some $1 \leq t < 2k-1$, $r_m = S_t^{n-1}$. Then $\Delta(r_{m+1}) = \Delta(S_{t+1}^n) \subset B$. Since $S_t^{n-1}$ is a face of $S_{t+1}^n$, then $\Delta(S_t^{n-1}) \subset \overline{\Delta(S_{t+1}^n)} \subset \bar{B}$. Hence $\Delta(S_t^{n-1}) \subset A \subset \bar{B}$. This contradicts A and B being separated. Therefore the supposition is false. Hence $|K|$ is connected. This theorem is proved.

The condition (c) of 2.20 is stronger than the property of connectedness. The following example is a complex K satisfying conditions (a) and (b) of 2.20 only, but $|K|$ is connected.

Example: In $R^3$, let K be constructed as follows:

| | | |
|---|---|---|
| $a_0 = (0,0,0)$ | $a_1 = (1,0,0)$ | $a_2 = (0,1,0)$ |
| $a_3 = (0,0,1)$ | $a_4 = (1,0,2)$ | $a_5 = (0,1,2)$ |
| $a_6 = (0,0,2)$ | | |
| $S_1^2 = \{a_0,a_1,a_2\}$ | $S_1^1 = \{a_0,a_1\}$ | $S_1^0 = \{a_0\}$ |
| $S_2^2 = \{a_0,a_1,a_3\}$ | $S_2^1 = \{a_0,a_2\}$ | $S_2^0 = \{a_1\}$ |
| $S_3^2 = \{a_0,a_2,a_3\}$ | $S_3^1 = \{a_0,a_3\}$ | $S_3^0 = \{a_2\}$ |

| | | |
|---|---|---|
| $S_4^2 = \{a_1, a_2, a_3\}$ | $S_4^1 = \{a_1, a_2\}$ | $S_4^0 = \{a_3\}$ |
| $S_5^2 = \{a_3, a_4, a_5\}$ | $S_5^1 = \{a_2, a_3\}$ | $S_5^0 = \{a_4\}$ |
| $S_6^2 = \{a_3, a_4, a_6\}$ | $S_6^1 = \{a_3, a_1\}$ | $S_6^0 = \{a_5\}$ |
| $S_7^2 = \{a_3, a_5, a_6\}$ | $S_7^1 = \{a_6, a_3\}$ | $S_7^0 = \{a_6\}$ |
| $S_8^2 = \{a_4, a_5, a_6\}$ | $S_8^1 = \{a_6, a_4\}$ | |
| | $S_9^1 = \{a_6, a_5\}$ | |
| | $S_{10}^1 = \{a_3, a_4\}$ | |
| | $S_{11}^1 = \{a_4, a_5\}$ | |
| | $S_{12}^1 = \{a_5, a_3\}$ | |

$K = \{S_i^2 \mid i = 1,\ldots,8\} \cup \{S_i^1 \mid i = 1,\ldots,12\} \cup \{S_i^0 \mid i = 1,\ldots,7\}$. Then K is a complex satisfying the conditions (a) and (b) but not (c) of 2.20. Also $|K|$ is connected.

2.25. Definition. If K is a complex and S is a simplex in K such that S is not a proper face of each simplex in K, then S is called a maximal simplex in K.

2.26. Theorem. Let K be a complex, and let S be a maximal n-simplex in K where $n \geq 2$. If $|K|$ is connected, then $|K| \setminus \Delta(S)$ is connected.

Proof: Denote the combinatorial closure of S by $K_1$ and the combinatorial boundary of S by $K_2$. Suppose $|K| \setminus \Delta(S)$ is not connected. Let $|K| \setminus \Delta(S) = A \cup B$ such that $\overline{A} \cap B = A \cap \overline{B} = \phi$, and $A \neq \phi \neq B$. By 2.23, $|K_2|$ is connected. Hence either $|K_2| \subset A$ or $|K_2| \subset B$. It is no loss of generality if $|K_2| \subset A$ is assumed true. Since $|K|$ is connected and $|K| \setminus \Delta(S)$ is not connected, then either $\Delta(S) \cap \overline{B} \neq \phi$ or $\overline{\Delta(S)} \cap B \neq \phi$. Also for each simplex $S' \in K \setminus \{S\}$, either $\Delta(S') \subset A$ or $\Delta(S') \subset B$.

Case 1: $\Delta(S) \cap \overline{B} \neq \phi$. This inequality means there exists a point $p \in \Delta(S)$ and $p \in \overline{B}$. But $p \in \Delta(S)$ implies $p \notin |K|\backslash\Delta(S) = A \cup B$. Hence $p \notin B$. If p is a limit point of B, then p is a limit point of $\Delta(S_1)$ for some simplex $S_1 \in K$ and $\Delta(S_1) \subset B$. Hence $p \in \overline{\Delta(S_1)} \cap \Delta(S)$. But S is a maximal simplex in K, and $S \neq S_1$; hence $\overline{\Delta(S_1)} \cap \Delta(S) = \phi$. Therefore the supposition of p being a limit point of B is false. Therefore $\Delta(S) \cap \overline{B} \neq \phi$ is impossible.

Case 2: $\overline{\Delta(S)} \cap B \neq \phi$. This inequality means there exists a point $p \in B$ and $p \in \overline{\Delta(S)}$. But $p \in B$ implies $p \in |K|\backslash\Delta(S) = A \cup B$. Hence $p \notin \Delta(S)$. If p is a limit point of $\Delta(S)$, then $p \in |K_2| \subset A$. Hence $p \in A \cap B$. This contradicts $\overline{A} \cap B = \phi$. Therefore $\overline{\Delta(S)} \cap B \neq \phi$ is impossible.

Combining Case 1 and Case 2, we know that the supposition is false. Hence $|K|\backslash\Delta(S)$ is connected.

<u>2.27</u>. <u>Theorem</u>. Let K be an n-complex and $|K|$ be connected. Then the set $T = \{x|\ x \text{ is a vertex in } K\} \cup (\cup\{\Delta(S^1)|\ S^1 \text{ is 1-simplex in } K\})$ is connected.

<u>Proof</u>: If n = 0 or 1, this theorem is automatically true.

Assume $n \geq 2$; then each n-simplex in K is a maximal simplex in K. If $S^n$ is an n-simplex in K, then $K\backslash\{S^n\}$ is also a complex. By 2.26, $|K|\backslash\Delta(S)$ is connected.

Assume K has just $t_i$ i-simplexes denoted by $S^i_1, S^i_2, \ldots, S^i_{t_i}$ for $i = 2,3,\ldots,n$. Let $\sum_{i=2}^{n} t_i = m$, and denote the sequence

$S_1^n$, $S_2^n$, ..., $S_{t_n}^n$, $S_1^{n-1}$, $S_2^{n-1}$, ..., $S_{t_{n-1}}^{n-1}$, ..., $S_1^2$, $S_2^2$, ..., $S_{t_2}^2$ of simplexes by $r_1,\ldots,r_m$. Define $K_1 = K\setminus\{r_1\}$, and define $K_i = K_{i-1}\setminus\{r_i\}$ for $i = 2,3,\ldots,m$. Then $|K_m| = T$. By 2.26, $|K_1|$ is connected. Use 2.26 m times to conclude that $|K_m|$ is connected. That is, the set T is connected.

CHAPTER III

# CHAIN GROUPS

For a complex K and an abelian group G, the first step in constructing the chain group is to define the orientation for each simplex in K. We use the notation $S^n$ to denote an n-simplex.

3.1. Definition. If $S^n$ is an n-simplex where $n \geq 1$, let $F_{S^n} = \{f |\ f$ is a one-to-one onto function $\{0,\ldots,n\} \to S^n\}$, and define a relation ~ in F by the following: Let $f,g \in F$; then $f \sim g$ if and only if $f = g$ or there exist an even number of transpositions $\phi_1,\ldots,\phi_{2m}$ of the set $\{0,1,\ldots,n\}$ such that $f = g\phi_1,\ldots,\phi_{2m}$. Since

[(1) By definition, $f \sim f$ for every $f \in F$.

(2) If $f,g \in F$ and $f \sim g$, then there exist transpositions of $\{0,1,\ldots,n\}$, $\phi_1,\ldots,\phi_{2t}$ such that $f = g\phi_1,\ldots,\phi_{2t}$. But the inverse of a transposition is also a transposition; hence $f\phi_{2t}^{-1},\ldots,\phi_1^{-1} = g\phi_1,\ldots,\phi_{2t}\phi_{2t}^{-1},\ldots,\phi_1 = g$. Therefore $g \sim f$.

(3) If $f,g,h \in F$ and $f \sim g$, $g \sim h$, then there exist transpositions of $\{0,\ldots,n\}$, $\phi_1,\ldots,\phi_{2r},\theta_1,\ldots,\phi_{2t}$ such that $f = g\phi_1,\ldots,\phi_{2r}$ and $g = h\theta_1,\ldots,\theta_{2t}$. Hence $f = g\phi_1,\ldots,\phi_{2r}$ $= h\theta_1,\ldots,\theta_{2t}\phi_1,\ldots,\phi_{2r}$. Therefore $f \sim h$.],

then the relation ~ is an equivalent relation.

By the definition of ~, $F_{S^n}/\sim$ contains only two equivalent classes.

The statement that D is an oriented n-simplex means $D = (S^n, e)$ where $e \in F_{S^n}/\sim$ and $F_{S^n}/\sim$ is called the orientation set of $S^n$, denoted by $N(S^n)$. If $e \in N(S^n)$, then e is called an orientation of $S^n$. If $S^n = \{a_0, \ldots, a_n\}$, we denote D by the notation $\langle a_{i_0}, a_{i_1}, \ldots, a_{i_n} \rangle$ where $i_0, i_1, \ldots, i_n$ is a permutation of $0, 1, \ldots, n$. Let $F_{S^n}/\sim = \{e_1, e_2\}$, then we say $(S^n, e_1) = -(S^n, e_2)$ and $(S^n, e_2) = -(S^n, e_1)$.

For 0-simplex $S^0$, the orientation set of $S^0$ contains just one element. Let $S^0 = \{a_0\}$; the meaning of $-\langle a_0 \rangle$ will be defined later.

3.2. Definition. The symbol $K^\alpha$ is an oriented complex meaning K is a complex, $\alpha$ is a function $\alpha: K \to \bigcup_{S \in K} N(S)$ and $\alpha(S) \in N(S)$ for every $S \in K$, where $N(S)$ is the orientation set of S. The symbol $\alpha$ is called an orientation of K.

Notation: Let $\alpha, \beta$ be orientations of a complex K. Then the symbol $f_{\alpha\beta}$ will be used to express the function $K \to \{1, -1\}$ defined by

$$f_{\alpha\beta}(S) = 1 \text{ if } \alpha(S) = \beta(S), \text{ and}$$
$$f_{\alpha\beta}(S) = -1 \text{ if } \alpha(S) \neq \beta(S), \text{ for each } S \in K.$$

Let $\gamma$ be another orientation of the complex K. Then by the definition of $f_{\alpha\beta}$, the following properties follow immediately:

(1) $f_{\alpha\alpha}(S) = 1$,

(2) $f_{\alpha\beta}(S) = f_{\beta\alpha}(S)$, and

(3) $f_{\alpha\beta}(S) f_{\beta\gamma}(S) = f_{\alpha\gamma}(S)$.

3.3. Definition. Let K be an oriented complex, and let G be an additive abelian group. An r-dimensional chain of $K^\alpha$ over G is a function $C^\alpha$ which assigns to each oriented r-simplex of $K^\alpha$ an element of the group G. For convenience, $C^\alpha((S^r,\alpha(S^r)))$ is denoted by $C^\alpha(S^r)$.

3.4. Definition. The set of r-dimensional chains of an oriented complex $K^\alpha$ over a group G is indicated by the symbol $C_r(K^\alpha;G)$. Let $c_1^\alpha, c_2^\alpha \in C_r(K^\alpha;G)$; define $c_1^\alpha + c_2^\alpha$ by $(c_1^\alpha + c_2^\alpha)(S^r) = c_1^\alpha(S^r) + c_2^\alpha(S^r)$.

3.5. Theorem. If G is a unital R-module, define $(rc^\alpha)(S^t) = r(c^\alpha(S^t))$ for every $r \in R$, $c^\alpha \in C_t(K^\alpha;G)$, and $S^t \in K$. Then $C_t(K^\alpha;G)$ is a unital R-module.

Proof: Define the function $c_0^\alpha$ by $_0^\alpha(S^t) = 0$ for all $S^t \in K$ where 0 is the zero element of G. Then $c_0^\alpha$ is the zero element in $C_t(K^\alpha;G)$.

Let $c^\alpha \in C_t(K^\alpha;G)$; then the inverse element of $c^\alpha$ is $-c^\alpha$ defined by $(-c^\alpha)(S^t) = -(c^\alpha(S^t))$ for all $S^t \in K$. Since G is closed, abelian, and associative, then $C_t(K^\alpha;G)$ is closed, abelian, and associative. Hence $C_t(K^\alpha;G)$ is an abelian group under addition. Furthermore,

$$\begin{aligned}(r(c_1^\alpha + c_2^\alpha))(S^t) &= r((c_1^\alpha + c_2^\alpha)(S^t)) \\ &= r(c_1^\alpha(S^t) + c_2^\alpha(S^t)) \\ &= r(c_1^\alpha(S^t)) + r(c_2^\alpha(S^t)) \\ &= (rc_1^\alpha + rc_2^\alpha)(S^t).\end{aligned}$$

$$\begin{aligned}(r(sc_1^\alpha))(S^t) &= r((sc_1^\alpha)(S^t)) \\ &= r(s(c_1^\alpha(S^t))) \\ &=(rs)(c_1^\alpha(S^t)).\end{aligned}$$

$$((r + s)c_1^\alpha)(S^t) = (r + s)(c_1^\alpha(S^t))$$
$$= r(c_1^\alpha(S^t)) + s(c_1^\alpha(S^t))$$
$$= (rc_1^\alpha)(S^t) + (sc_1^\alpha)(S^t).$$
$$(1c_1^\alpha)(S^t) = 1(c_1^\alpha(S^t)) = c_1^\alpha(S^t)$$

for all $c_1^\alpha, c_2^\alpha \in C_t(K^\alpha;G)$ and $r,s \in R$. Hence $C_t(K^\alpha;G)$ is a unital R-module.

3.6. Definition. If $\alpha,\beta$ are orientations of a complex K and $c^\alpha \in C_r(K^\alpha;G)$, $c^\beta \in C_r(K^\beta;G)$, then $c^\alpha$ is defined to be chain-equivalent to $c^\beta$ provided that for each r-simplex $S^r$ of K it is true that $c^\alpha(S^r) = f_{\alpha\beta}(S^r)c^\beta(S^r)$. From the properties of the function $f_{\alpha\beta}$, the relation so defined is an equivalent relation.

3.7. Definition. Let $K^\alpha$ be an oriented complex and $c^\alpha$ an r-dimensional chain of $K^\alpha$ over a unital R-module G. The chain-equivalent class of $c^\alpha$ is called an r-dimensional chain of K over G. The set of r-dimensional chains of K over G is indicated by the symbol $C_r(K;G)$. If $c^\alpha$ is in the chain-equivalence class c, then $c^\alpha$ is called a representative of the chain c. The notation $[c^\alpha]$ is used to express the chain-equivalence class of $c^\alpha$.

3.8. Theorem. Let c be an r-dimensional chain of a complex K over a module G. For each orientation $\alpha$ of K, there exists exactly one chain $c^\alpha \in C_r(K^\alpha;G)$ such that $c^\alpha$ is a representative of c.

Proof: Let $\alpha$ be an orientation of the complex K. The r-dimensional chain $c \in C_r(K;G)$ means there is an orientation $\beta$ of K, and there is an r-dimensional chain $c^\beta \in C_r(K^\beta;G)$ such that $c^\beta$ is in the chain-equivalence class c. Define an r-dimensional chain $c^\alpha \in C_r(K^\alpha;G)$ by $c^\alpha(S^r) = f_{\alpha\beta}(S^r)c^\beta(S^r)$ for all r-simplexes $S^r \in K$. But this is just the definition of chain equivalence; hence $c^\alpha$ is chain-equivalent to $c^\beta$. Hence both of $c^\alpha$ and $c^\beta$ are in the chain-equivalence class c. Therefore $c^\alpha$ is a representative of c. Let $c_1^\alpha$ be another r-dimensional chain in c; then $c^\alpha$ is chain-equivalent to $c_1^\alpha$. By definition, $c_1^\alpha(S^r) = f_{\alpha\alpha}(S^r)c^\alpha(S^r)$ for all $S^r \in K^\alpha$. But $f_{\alpha\alpha}(S^r) = 1$ for all $S^r \in K^\alpha$. Therefore $c_1^\alpha(S^r) = c^\alpha(S^r)$ for all $S^r \in K^\alpha$. This proves $c\alpha$ is unique.

<u>3.9</u>. <u>Theorem</u>. If $\alpha,\beta$ are orientations of a complex K and G is a unital R-module, let $c_1^\alpha, c_2^\alpha \in C_r(K^\alpha;G)$, and let $c_1^\beta, c_2^\beta \in C_r(K^\beta;G)$ with $c_1^\alpha$ equivalent to $c_1^\beta$ and $c_2^\alpha$ equivalent to $c_2^\beta$. Then $c_1^\alpha + c_2^\alpha$ is equivalent to $c_1^\beta + c_2^\beta$.

<u>Proof</u>: For each simplex $S^r$ of K, we have

$$\begin{aligned}(c_1^\alpha + c_2^\alpha)(S^r) &= c_1^\alpha(S^r) + c_2^\alpha(S^r) \\ &= f_{\alpha\beta}(S^r)c_1^\beta(S^r) + f_{\alpha\beta}(S^r)c_1^\beta(S^r) \\ &= f_{\alpha\beta}(S^r)(c_1^\beta(S^r) + c_2^\beta(S^r)) \\ &= f_{\alpha\beta}(S^r)(c_1^\beta + c_2^\beta)(S^r).\end{aligned}$$

Hence, by definition, $c_1^\alpha + c_2^\alpha$ is chain-equivalent to $c_1^\beta + c_2^\beta$.

With the theorem of 3.7 and the theorem of 3.8, the following important definition can be given.

3.10. Definition. If $c_1, c_2 \in C_r(K;G)$, then the sum $c_1 + c_2$ is defined to be the equivalence class of $c_1^\alpha + c_2^\alpha$ where $\alpha$ is an orientation of the complex K and $c_1^\alpha$ and $c_2^\alpha$ are representatives of $c_1$ and $c_2$ respectively.

Theorem 3.7. states that for any orientation $\alpha$ of K, such representatives $c_1^\alpha$ and $c_2^\alpha$ of $c_1$ and $c_2$ exist. Theorem 3.8 proves that the addition of $c_1$ and $c_2$ defined above is well-defined.

3.11. Theorem. Let G be a unital R-module, and define $rc = [rc^\alpha]$ for every $r \in R$, $c \in C_t(K;G)$ where $\alpha$ is an orientation of K, $c^\alpha$ is the representative of c, and $rc^\alpha$ is defined in 3.5. Then $C_t(K;G)$ is an unital R-module. Also $C_t(K;G)$ is isomorphic to $C_t(K^\alpha;G)$.

Proof: Let $\alpha$ be an orientation of the complex K. By 3.5, it is known that $C_t(K^\alpha;G)$ is a unital R-module. To show $C_t(K;G)$ is a unital R-module, it must be shown that the mapping $R \times C_t(K;G) \to C_t(K;G)$ defined by $rc = [rc^\alpha]$ is well-defined. Let $\beta$ be another orientation of K and $c^\beta \in C_t(K^\beta;G)$ be the representative of c; then $(rc^\alpha)(S^t) = r(c^\beta(S^t)) = r(f_{\alpha\beta}(S^t)c^\beta(S^t)) = f_{\alpha\beta}(S^t)(r(c^\beta(S^t)))$ $= f_{\alpha\beta}(rc^\beta)(S^t)$. Hence $rc^\alpha$ is chain-equivalent to $rc^\beta$ for each $r \in R$. Therefore $rc = [rc^\alpha]$ is well-defined. Therefore the computation in the proof of $C_t(K;G)$ being a unital module is just the computation in $C_t(K^\alpha;G)$. But $C_t(K^\alpha;G)$ is a unital R-module; hence $C_t(K;G)$ is also a unital R-module.

To show $C_t(K;G)$ is isomorphic to $C_t(K^\alpha;G)$, define the function $\phi$: $C_t(K;G) \to C_t(K^\alpha;G)$ by $\phi(c) = c^\alpha$ where $c \in C_t(K;G)$ and $c^\alpha$ is the representative of C. Let $c_1, c_2 \in C_t(K;G)$ with $c_1 \neq c_2$ and $c_1^\alpha, c_2^\alpha \in C_t(K^\alpha;G)$ such that $c_1^\alpha, c_2^\alpha$ are the representatives of $c_1, c_2$ respectively. Then $\phi(c_1) = c_1^\alpha$, and $\phi(c_2) = c_2^\alpha$. Since $c_1 \neq c_2$ implies $[c_1^\alpha] \neq [c_2^\alpha]$, then $c_1^\alpha \neq c_2^\alpha$. Therefore the function $\phi$ is one-to-one. Let $c^\alpha \in C_t(K^\alpha;G)$; then $[c^\alpha] \in C_t(K;G)$, and $\phi([c^\alpha]) = c^\alpha$. Hence $\phi$ is onto. Also $\phi(c_1 + c_2) = \phi([c_1^\alpha] + [c_2^\alpha]) = \phi([c_1^\alpha + c_2^\alpha]) = c_1^\alpha + c_2^\alpha = \phi(c_1) + \phi(c_2)$. This proves $\phi$ is an isomorphism.

CHAPTER IV

HOMOLOGY GROUP

4.1. Definition. Let $K\alpha$ be an oriented complex, $(S^r,\alpha(S^r)) = \langle a_{i_0},a_{i_1},\ldots,a_{i_r}\rangle$ where $r \geq 1$ is an oriented r-simplex in $K^\alpha$, and $S^{r-1} = S^r\setminus\{a_{i_j}\}$ is a face of $S^r$; then $(-1)^j \langle a_{i_0},\ldots,a_{i_{j-1}},a_{i_{j+1}},\ldots,a_{i_r}\rangle$ is called the oriented simplex $S^{r-1}$ inherited from $(S^r,\alpha(S^r))$. We also use the notation $(-1)^j\langle a_{i_0},\ldots,\hat{a}_{i_j},\ldots,a_{i_r}\rangle$ to denote

$$(-1)^j\langle a_{i_0},\ldots,a_{i_{j-1}},a_{i_{j+1}},\ldots,a_{i_r}\rangle.$$

Since the orientation set of a 0-simplex contains only one element, the oriented 0-simplex has not been defined in 3.1. Now, using the idea of 4.1, the following definition is derived.

4.2. Definition. Let K be a complex and $S^0,S^1 \in K$ such that $S^0 \subset S^1$; then x is an oriented 0-simplex $S^0$ if and only if x is an oriented simplex $S^0$ inherited from $(S^1,\alpha(S^1))$ for some orientation $\alpha$ of $K_0$. If $S^0 = \{a\}$, we denote the two types of oriented 0-simplexes $S^0$ by a and -a. The minus sign means the other type of this oriented simplex.

Notation: Let S be a simplex in a complex K and $\alpha$ an orientation of K; then denote $(S,\alpha(S))$ by $W_\alpha(S)$.

Let $S^r, S^{r-1}$ be a simplex in K and $S^{r-1} \subset S$; then denote the oriented simplex $S^{r-1}$ inherited from $(S^r, \alpha(S^r))$ by $W_I(S^r, S^{r-1})\alpha$.

4.3. Definition. Let $S^r, S^{r-1}$ be simplexes of a complex K and $\alpha$ an orientation of K; then the incidence number of $S^r$ and $S^{r-1}$ under the orientation $\alpha$, denoted by $[S^r, S^{r-1}]^\alpha$, is defined as follows:

$$[S^r, S^{r-1}]^\alpha = 0 \text{ if } S^{r-1} \not\subset S^r;$$

$$[S^r, S^{r-1}]^\alpha = 1 \text{ if } S^{r-1} \subset S^r \text{ and } W_\alpha(S^{r-1}) = W_I(S^r, S^{r-1})\alpha;$$

$$[S^r, S^{r-1}]^\alpha = -1 \text{ if } S^{r-1} \subset S^r \text{ and } W_\alpha(S^{r-1}) \neq W_I(S^r, S^{r-1})\alpha.$$

4.4. Theorem. If $S^r$ and $S^{r-1}$ are simplexes of a complex K, and $\alpha$ and $\beta$ are orientations of K, then $[S^r, S^{r-1}]^\alpha = f_{\alpha\beta}(S^r) f_{\alpha\beta}(S^{r-1}) [S^r, S^{r-1}]^\beta$.

Proof: If $S^{r-1}$ is not a face of $S^r$, then both of $[S^r, S^{r-1}]^\alpha$ and $[S^r, S^{r-1}]^\beta$ are zero. Hence the theorem is true. Assume $S^{r-1} \subset S^r$, and discuss the following four cases:

Case 1: $f_{\alpha\beta}(S^r) = 1$, and $f_{\alpha\beta}(S^{r-1}) = 1$.

(a) If $[S^r, S^{r-1}]^\alpha = 1$, then $f_{\alpha\beta}(S^{r-1}) = 1$ implies that $W_\beta(S^{r-1}) = W_\alpha(S^{r-1})$; $[S^r, S^{r-1}]^\alpha = 1$ implies that $W_\alpha(S^{r-1}) = W_I(S^r, S^{r-1})\alpha$; $f_{\alpha\beta}(S^r) = 1$ implies that $W_I(S^r, S^{r-1})\alpha = W_I(S^r, S^{r-1})\beta$. Hence $W_\beta(S^{r-1}) = W_I(S^r, S^{r-1})\beta$. Hence $[S^r, S^{r-1}]^\beta = 1$. Therefore the theorem is true.

(b) If $[S^r, S^{r-1}]^\alpha = -1$, then $W_\alpha(S^{r-1}) \neq W_I(S^r, S^{r-1})^\alpha$. Using part (a), $W_\beta(S^{r-1}) \neq W_I(S^r, S^{r-1})^\beta$. That is, $[S^r, S^{r-1}]^\beta = -1$. The theorem is also true.

Case 2: $f_{\alpha\beta}(S^r) = -1$, and $f_{\alpha\beta}(S^{r-1}) = -1$.

(a) If $[S^r,S^{r-1}]\alpha = 1$, then $f_{\alpha\beta}(S^{r-1}) = -1$ implies that $W_\beta(S^{r-1}) \neq W_\alpha(S^{r-1})$; $f_{\alpha\beta}(S^r) = -1$ implies that $W_I(S^r,S^{r-1})\alpha \neq W_I(S^r,S^{r-1})\beta$. But $W_\beta(S^{r-1}) \neq W_\alpha(S^{r-1})$ implies $W\beta(S^{r-1}) = -W\alpha(S^{r-1})$. Hence $W\beta(S^{r-1}) = -W\alpha(S^{r-1}) = -W_I(S^r,S^{r-1})^\alpha = W_I(S^r,S^{r-1})^\beta$. Hence $W_\beta(S^{r-1}) = W_I(S^r,S^{r-1})^\beta$. That is, $[S^r,S^{r-1}] = 1$. Hence the theorem is true.

(b) If $[S^r,S^{r-1}] = -1$, using the above discussion, we have $W_\beta(S^{r-1}) = -W_\alpha(S^{r-1}) = W_I(S^r,S^{r-1})^\alpha = -W_I(S^r,S^{r-1})^\beta$. Hence $W_\beta(S^{r-1}) = -W_I(S^r,S^{r-1})\beta$, and $[S^r,S^{r-1}]\beta = -1$. The theorem is also true.

Case 3: $f_{\alpha\beta}(S^r) = 1$, and $f_{\alpha\beta}(S^{r-1}) = -1$.

(a) If $[S^r,S^{r-1}]^\alpha = 1$, then $W_\beta(S^{r-1}) = -W_\alpha(S^{r-1}) = -W_I(S^r,S^{r-1})^\alpha = -W_I(S^r,S^{r-1})^\beta$. Hence $[S^r,S^{r-1}]^\beta = -1$ The theorem is true.

(b) If $[S^r,S^{r-1}]^\alpha = -1$, then $W_\beta(S^{r-1}) = -W_\alpha(S^{r-1}) = W_I(S^r,S^{r-1})\alpha = W_I(S^r,S^{r-1})\beta$. Hence $[S^r,S^{r-1}]\beta = 1$. The theorem is true.

Case 4: $f_{\alpha\beta}(S^r) = -1$, and $f_{\alpha\beta}(S^{r-1}) = 1$.

(a) If $[S^r,S^{r-1}]^\alpha = 1$, then $W_\beta(S^{r-1}) = W_\alpha(S^{r-1}) = W_I(S^r,S^{r-1})^\alpha = -W_I(S^r,S^{r-1})^\beta$. Hence $[S^r,S^{r-1}]^\beta = -1$. The theorem is true.

(b) If $[S^r,S^{r-1}]^\alpha = -1$, then $W_\beta(S^{r-1}) = W_\alpha(S^{r-1}) = -W_I(S^r,S^{r-1})^\alpha = W_I(S^r,S^{r-1})^\beta$. Hence $[S^r,S^{r-1}]^\beta = 1$. The theorem is true.

Combining the above cases, the theorem is proved.

4.5. Theorem. Let $S^r$, $S_i^{r-1}$, $S^{r-2}$ be simplexes of a complex K and $\alpha$ an orientation of K; then

$$\sum_i [S^r, S_i^{r-1}]^\alpha [S_i^{r-1}, S^{r-2}]^\alpha = 0.$$

Proof: By the definition of incidence number, $[S^r, S_i^{r-1}]^\alpha [S_i^{r-1}, S^{r-2}]^\alpha \neq 0$ if and only if $S^{r-2} \subset S_i^{r-1} \subset S^r$. Let $\beta$ be another orientation of K; then $\sum_i [S^r, S_i^{r-1}]^\alpha [S_i^{r-1}, S^{r-2}]^\alpha =$

$\sum_i f_{\alpha\beta}(S^r) f_{\alpha\beta}(S_i^{r-1}) [S^r, S_i^{r-1}]^\beta f_{\alpha\beta}(S_i^{r-1}) f_{\alpha\beta}(S^{r-2}) [S_i^{r-1}, S^{r-2}]^\beta =$

$f_{\alpha\beta}(S^r) f_{\alpha\beta}(S^{r-2}) \sum_i [S^r, S_i^{r-1}]^\beta [S_i^{r-1}, S^{r-2}]^\beta$. Therefore

$\sum_i [S^r, S_i^{r-1}]^\alpha [S_i^{r-1}, S^{r-2}]^\alpha = 0$ if and only if

$\sum_i [S^r, S_i^{r-1}]^\beta [S_i^{r-1}, S^{r-2}]^\beta = 0$. Hence this theorem does not depend on the particular orientation $\alpha$. Let $S^r = \{a_0, a_1, \ldots, a_r\}$, $S^{r-2} = \{a_2, \ldots, a_r\}$, $S_1^{r-1} = \{a_0, a_2, \ldots, a_r\}$, $S_2^{r-1} = \{a_1, a_2, \ldots, a_r\}$. Then for any $S_i^{r-1}$ which satisfies the condition $S^{r-2} \subset S_i^{r-1} \subset S^r$, either $S_i^{r-1} = S_1^{r-1}$ or $S_i^{r-1} = S_2^{r-1}$.

Let $\gamma$ be an orientation of K such that

$(S_1^r, \gamma(S^r)) = \langle a_0, a_1, \ldots, a_r \rangle$,

$(S_1^{r-1}, \gamma(S_1^{r-1})) = \langle a_0, a_2, \ldots, a_r \rangle$, $(S_2^{r-1}, \gamma(S_2^{r-1})) = \langle a_1, a_2, \ldots, a_r \rangle$,

$(S^{r-2}, \gamma(S^{r-2})) = \langle a_2, \ldots, a_r \rangle$; then

$\sum_i [S^r, S_i^{r-1}]^\gamma [S_i^{r-1}, S^{r-2}]^\gamma =$

$[S^r, S_1^{r-1}]^\gamma [S_1^{r-1}, S^{r-2}]^\gamma + [S^r, S_2^{r-1}]^\gamma [S_2^{r-1}, S^{r-2}]^\gamma =$

$(-1)(1) + (1)(1) = 0$. Since this theorem does not depend on the orientation of K, then this theorem is proved.

4.6. <u>Definition</u>. A chain of a complex or of an oriented complex over the additive group Z of integers is called an <u>integral chain</u>.

4.7. <u>Definition</u>. Let $K^\alpha$ be an oriented complex and $S_j^r \in K$. Let $(\sigma_j^r)^\alpha \in C_r(K^\alpha;Z)$ defined by

$$(\sigma_j^r)^\alpha(S_i^r,\alpha(S_i^r)) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

Then $(\sigma_j^r)^\alpha$ is called an <u>elementary integral r-chain</u> of $K^\alpha$, and its equivalent class $[(\sigma_j^r)^\alpha]$ is called an elementary integral r-chain of K.

4.8. <u>Definition</u>. Let K be a complex, G a Z-module, c an integral chain of K, g an element of G, $\alpha$ an orientation of K, and $c^\alpha$ the representative of c. Then define the chain $(gc)^\alpha$ of $K^\alpha$ over G by $(gc)^\alpha(S,\alpha(S)) = (c^\alpha(S,\alpha(S)))\cdot g$ for each $S \in K$.

If $\beta$ is another orientation of K, $c^\beta$ the representative of c and $c^\alpha$ is chain-equivalent to $c^\beta$, then $(gc)^\alpha$ is equivalent to $(gc)^\beta$ since $(gc)^\alpha(S,\alpha(S)) = (c^\alpha(S,\alpha(S)))\cdot g = f_{\alpha\beta}(S)c^\beta(S,\beta(S))\cdot g = f_{\alpha\beta}(S)(gc)^\beta(S,\beta(S))$ for each $S \in K$.

4.9. <u>Theorem</u>. Each r-chain $c^\alpha$ of $C_r(K^\alpha;G)$ can be written uniquely in the form

$$c^\alpha = \sum_j g_j^\alpha(\sigma_j^r) .$$

<u>Proof</u>: Since

$$c^\alpha(S_i^r,\alpha(S_i^r)) = (\sum_j g_j^\alpha(\sigma_j^r)^\alpha(S_i^r,\alpha(S_i^r))$$
$$= \sum_j g_j^\alpha((\sigma_j^r)^\alpha(S_i^r,\alpha(S_i^r))) = g_i^\alpha$$ for each oriented r-simplex $(S_i^r,\alpha(S_i^r))$ in $K^\alpha$, then

$c^\alpha = \sum_j g_j^\alpha(\sigma_j^r)^\alpha$ if and only if $g_j^\alpha = c^\alpha(S_j^r, \alpha(S_j^r))$ for each j. Hence the expression of $c^\alpha = \sum_j g_j(\sigma_j^r)^\alpha$ is unique.

4.10. Definition. For each r-chain ($r > 0$) $c^\alpha$ of $K^\alpha$ over G, assign an (r-1)-chain called its boundary (indicated by $\partial c^\alpha$) given by

$$\partial c^\alpha = \sum_k (\sum_j [S_j^r, S_k^{r-1}]^\alpha g_j^\alpha)(\sigma_k^{r-1})^\alpha,$$

or equivalently by

$$\partial c^\alpha(S_k^{r-1}, \alpha(S_k^{r-1})) = \sum_j [S_j^r, S_k^{r-1}]^\alpha g_j^\alpha \text{ where}$$

$c^\alpha = \sum_j g_j^\alpha(\sigma_j^r)^\alpha$ is a r-chain of $K^\alpha$ over G.

If $\langle a_0, a_1, \ldots, a_r \rangle$ is an oriented r-simplex in $K^\alpha$, and if $\langle a_0, a_1, \ldots, a_r \rangle$ is considered an elementary integral r-chain of $K^\alpha$ defined by

$$\langle a_0, a_1, \ldots, a_r \rangle(x) = \begin{cases} 1 \text{ if } x = \langle a_0, \ldots, a_r \rangle \\ 0 \text{ if } x \neq \langle a_0, \ldots, a_r \rangle, \end{cases}$$

then $\partial \langle a_0, a_1, \ldots, a_r \rangle = \sum_i (-1)^i \langle a_0, \ldots, \hat{a}_i, \ldots, a_r \rangle$.

4.11. Theorem. If $c^\alpha$ is chain-equivalent to $c^\beta$, then $\partial c^\alpha$ is chain-equivalent to $\partial c^\beta$.

Proof: Let $c^\alpha = \sum_j g_j^\alpha(\sigma_j^r)^\alpha$ and $c^\beta = \sum_j g_j^\beta(\sigma_j^r)^\beta$. Since $c^\alpha$ is chain-equivalent to $c^\beta$, then

$g_j^\beta = c^\beta(S_j^r, \beta(S_j^r)) = f_{\alpha\beta}(S_j^r) c^\alpha(S_j^r, \alpha(S_j^r)) = f_{\alpha\beta}(S_j^r) g_j^\alpha$.

Hence $(\partial c^\beta)(S_k^{r-1}, \beta(S_k^{r-1})) = \sum_j [S_j^r, S_k^{r-1}]^\beta g_j^\beta =$

$\sum_j f_{\alpha\beta}(S_j^r) f_{\alpha\beta}(S_k^{r-1}) [S_j^r, S_k^{r-1}]^\alpha f_{\alpha\beta}(S_j^r) g_j^\alpha =$

$\sum_j f_{\alpha\beta}(S_k^{r-1}) [S_j^r, S_k^{r-1}]^\alpha g_j^\alpha = f_{\alpha\beta}(S_k^{r-1})(\partial c^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1}))$.

That is, $\partial c^\alpha$ is chain-equivalent to $\partial c^\beta$.

4.12. Definition. The boundary of an r-chain c in $C_r(K;G)$ $(r > 0)$ is defined to be the chain-equivalence class of $\partial c^\alpha$ where $c^\alpha$ is any representative of c. The boundary of c is indicated by the symbol $\partial c$. By 4.11, this definition is well-defined.

4.13. Theorem. The operator $\partial$ is a R-homomorphism $\partial\colon C_r(K;G) \to C_{r-1}(K;G)(r > 0)$.

Proof: Let $\alpha$ be an orientation of K and $c_1, c_2 \in C_r(K;G)$. By 3.8, there exist $c_1^\alpha, c_2^\alpha \in C_r(K^\alpha;G)$ such that $c_1^\alpha$ and $c_2^\alpha$ are the representatives of $c_1$ and $c_2$ respectively. Let $(S_k^{r-1},\alpha(S_k^{r-1}))$ be an oriented (r-1)-simplex in $K^\alpha$; then

$$(\partial(c_1^\alpha + c_2^\alpha))(S_k^{r-1},\alpha(S_k^{r-1})) = \sum_j [S_j^r,S_k^{r-1}]^\alpha(c_1^\alpha + c_2^\alpha)(S_j^r,\alpha(S_j^r))$$
$$= \sum_j [S_j^r,S_k^{r-1}]^\alpha c_1^\alpha(S_j^r,\alpha(S_j^r)) + \sum_j [S_j^r,S_k^{r-1}]^\alpha c_2^\alpha(S_j^r,\alpha(S_j^r))$$
$$= (\partial c_1^\alpha)(S_k^{r-1},\alpha(S_k^{r-1})) + (\partial c_2^\alpha)(S_k^{r-1},\alpha(S_k^{r-1})).$$

Remembering that G is a R-module, let $t \in R$, $c \in C_r(K;G)$, $(S_k^{r-1},\alpha(S_k^{r-1}))$ be an oriented (r-1) simplex in $K^\alpha$, and $c^\alpha$ a representative of c. Then

$$(\partial(tc^\alpha))(S_k^{r-1},\alpha(S_k^{r-1})) = \sum_j [S_j^r,S_k^{r-1}]^\alpha(tc^\alpha)(S_k^{r-1},\alpha(S_k^{r-1}))$$
$$= \sum_j [S_j^r,S_k^{r-1}]^\alpha t((c^\alpha)(S_k^{r-1},\alpha(S_k^{r-1})))$$
$$= t\sum_j [S_j^r,S_k^{r-1}]^\alpha(c^\alpha)(S_k^{r-1},\alpha(S_k^{r-1}))$$
$$= t((\partial c^\alpha)(S_k^{r-1},\alpha(S_k^{r-1})))$$
$$= (t(\partial c^\alpha))(S_k^{r-1},\alpha(S_k^{r-1})).$$

That is, $\partial(tc^\alpha) = t(\partial c^\alpha)$. Hence $\partial$ is an R-homomorphism.

4.14. Theorem. For any r-chain $c \in C_r(K;G)$, $\partial\partial(c) = 0$ $(r > 1)$.

Proof: Let $\alpha$ be an orientation of the complex K, and let $c^\alpha = \sum_j g_j^\alpha (\sigma_j^r)^\alpha \in C_r(K^\alpha;G)$, the representative of c. By definition, $\partial c^\alpha = \sum_k (\sum_j g_j^\alpha [S_j^r, S_k^{r-1}]^\alpha)(\sigma_k^{r-1})^\alpha$. Then

$\partial\partial c^\alpha = \partial(\partial c^\alpha) = \sum_i(\sum_k(\sum_j g_j^\alpha [S_j^r, S_k^{r-1}]^\alpha [S_k^{r-1}, S_i^{r-2}]^\alpha))(\sigma_i^{r-2})^\alpha$.

By 4.5, $\partial\partial c^\alpha = 0$. Hence $\partial\partial c = 0$.

4.15. Definition. If c is an element of $C_r(K;G)$ $(r > 0)$ and $\partial c = 0$, then c is called a cycle. If $c = \partial c'$ for some chain c' in $C_{r+1}(K;G)$, then by 4.14, c is a cycle and is called a bounding cycle of a boundary. The set of cycles in $C_r(K;G)$ is the kernel of the homomorphism $\partial$: $C_r(K;G) \to C_{r-1}(K;G)$ and is denoted by $Z_r(K;G)$. The set of bounding cycles is the image of the homomorphism $\partial$: $C_{r+1}(K;G) \to C_r(K;G)$ and is denoted by $B_r(K;G)$.

4.16. Theorem. The sets $Z_r(K;G)$ and $B_r(K;G)$ are submodules of $C_r(K;G)$, and $B_r(K;G)$ is contained in $Z_r(K;G)$.

Proof: Let $\alpha$ be an orientation of K, $c_1$ and $c_2 \in Z_r(K;G)$, $c_1^\alpha$ and $c_2^\alpha \in C_r(K;G)$ such that $c_1^\alpha$ and $c_2^\alpha$ are representative of $c_1$ and $c_2$ respectively. Let $(S_k^{r-1}, \alpha(S_k^{r-1}))$ be an oriented (r-1)-simplex in $K^\alpha$; then by 4.13, $(\partial(c_1^\alpha - c_2^\alpha))(S_k^{r-1}, \alpha(S_k^{r-1})) = (\partial c_1^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1})) + (\partial(-c_1^\alpha))(S_k^{r-1}, \alpha(S_k^{r-1}))$. Since $c_1, c_2 \in Z_r(K;G)$ and $c_1 = [c_1^\alpha]$, $c_2 = [c_2^\alpha]$, then $\partial c_1^\alpha = 0$ and $\partial c_2^\alpha = 0$. By 3.5, $-c_2^\alpha$ is the inverse element of $c_2^\alpha$ under addition. Hence $(-c_2^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1})) = -((c_2^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1})))$. Hence $(\partial(-c_2^\alpha))(S_k^{r-1}, \alpha(S_k^{r-1})) = \sum_j [S_j^r, S_k^{r-1}]^\alpha (-c_2^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1})) = -(\sum_j [S_j^r, S_k^{r-1}]^\alpha (c_2^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1}))$

$= -((\partial c_2^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1}))) = 0$.

Hence $(\partial(c_1^\alpha - c_2^\alpha))(S_k^{r-1}, \alpha(S_k^{r-1})) = 0$. Therefore $[c_1^\alpha - c_2^\alpha] \in Z_r(K;G)$. That is $(c_1 - c_2) \in Z_r(K;G)$. This proves $Z_r(K;G)$ is a subgroup of $C_r(K;G)$.

Let $t \in R$, $c \in Z_r(K;G)$, and $c\alpha$ be the representative of c. Then

$$
\begin{aligned}
(\partial(tc^\alpha))(S_k^{r-1}, \alpha(S_k^{r-1})) &= \sum_j [S_j^r, S_k^{r-1}]^\alpha (tc^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1})) \\
&= \sum_j [S_j^r, S_k^{r-1}]^\alpha t((c^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1}))) \\
&= t\sum_j [S_j^r, S_k^{r-1}]^\alpha (c^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1})) \\
&= t(\partial c^\alpha)(S_k^{r-1}, \alpha(S_k^{r-1})) = 0.
\end{aligned}
$$

Hence $[tc^\alpha] \in Z_r(K;G)$. That is, $tc \in Z_r(K;G)$. Hence $Z_r(K;G)$ is a submodule of $C_r(K;G)$.

Let $c_1, c_2 \in B_r(K;G)$ and $c_1^\alpha$ and $c_2^\alpha$ be the representatives of $c_1, c_2$ respectively. Since $c_1, c_2 \in B_r(K;G)$, then there exist $c_1'$ and $c_2' \in C_{r+1}(K;G)$ such that $\partial c_1' = c_1$ and $\partial c_2' = c_2$. Let $c_1'^\alpha$ and $c_2'^\alpha$ be the representatives of $c_1'$ and $c_2'$ respectively. By 4.12, $\partial c_1'^\alpha = c_1^\alpha$, and $\partial c_2'^\alpha = c_2^\alpha$. Let $(S_k^r, \alpha(S_k^r))$ be an oriented r-simplex in $K^\alpha$. By 4.13 and the previous proof,

$$
\begin{aligned}
&(\partial(c_1'^\alpha - c_2'^\alpha))(S_k^r, \alpha(S_k^r)) = \\
&\quad (\partial c_1'^\alpha)(S_k^r, \alpha(S_k^r)) + (\partial(-c_2'^\alpha))(S_k^r, \alpha(S_k^r)) = \\
&\quad \sum_j [S_j^{r+1}, S_k^r]^\alpha (c_1'^\alpha)(S_k^r, \alpha(S_k^r)) - \sum_j [S_j^{r+1}, S_k^r]^\alpha (c_2'^\alpha)(S_k^r, \alpha(S_k^r)) \\
&= (\partial c_1'^\alpha)(S_k^r, \alpha(S_k^r)) - (\partial c_1'^\alpha)(S_k^r, \alpha(S_k^r)) \\
&= (c_1^\alpha)(S_k^r, \alpha(S_k^r)) - (c_2^\alpha)(S_k^r, \alpha(S_k^r)) \\
&= (c_1^\alpha)(S_k^r, \alpha(S_k^r)) + (-c_2^\alpha)(S_k^r, \alpha(S_k^r)) \\
&= (c_1^\alpha - c_2^\alpha)(S_k^r, \alpha(S_k^r)).
\end{aligned}
$$

Hence $\partial(c_1'^{\alpha}-c_2'^{\alpha}) = c_1^{\alpha}-c_2^{\alpha}$. By definition, $[c_1^{\alpha}-c_2^{\alpha}] \in B_r(K;G)$. That is, $(c_1-c_2) \in B_r(K;G)$. Hence $B_r(K;G)$ is a subgroup of $C_r(K;G)$.

Let $t \in R$, $c \in B_r(K;G)$, and $c^{\alpha}$ be the representative of c. Since $c \in B_r(K;G)$, then there exist $c_h \in C_{r+1}(K;G)$ such that $\partial c_h = c$. Let $c_h^{\alpha}$ be the representative of $c_h$. By 4.12, $\partial c_h^{\alpha} = c^{\alpha}$. Furthermore,

$$(\partial(tc_h^{\alpha}))(S_k^r,\alpha(S_k^r)) =$$

$$\sum_j [S_j^{r+1},S_k^r]^{\alpha}(tc_h^{\alpha})(S_k^r,\alpha(S_k^r)) =$$

$$\sum_j [S_j^{r+1},S_k^r]^{\alpha} t((c_h^{\alpha})(S_k^r,\alpha(S_k^r))) =$$

$$t\sum_j [S_j^{r+1},S_k^r]^{\alpha}(c_h^{\alpha})(S_k^r,\alpha(S_k^r)) =$$

$$t((\partial c_h^{\alpha})(S_k^r,\alpha(S_k^r))) = t(c^{\alpha})(S_k^r,\alpha(S_k^r)) = (tc^{\alpha})(S_k^r,\alpha(S_k^r)).$$

Hence $\partial(tc_h^{\alpha}) = tc^{\alpha}$. But $tc_h^{\alpha} \in C_{r+1}(K^{\alpha};G)$. Therefore $tc^{\alpha} \in B_r(K^{\alpha};G)$. Hence $tc \in B_r(K;G)$. Hence $B_r(K;G)$ is a submodule of $C_r(K;G)$.

By 4.14 and definitions of $Z_r(K;G)$ and $B_r(K;G)$, $B_r(K;G)$ is contained in $Z_r(K;G)$.

4.17. Definition. The factor group $Z_r(K;G)/B_r(K;G) = H_r(K;G)$ is called the r-dimensional homology group of K over G.

Two cycles $z_1$ and $z_2$ in $Z_r(K;G)$ are said to be homologous if they are in the same coset of $B_r(K;G)$.

4.18. Definition. Since there are no negative dimensional simplexes, then $C_t(K;G)$ is defined as the zero R-module for each negative integer t. Then the R-homomorphism $\partial: C_0(K;G) \to C_{-1}(K;G)$ is the zero R-homomorphism, and its kernel is $C_0(K;G)$. Hence $Z_0(K;G) = C_0(K;G)$. Then the zero dimensional homology groups are defined by $H_0(K;G) = C_0(K;G)/B_0(K;G)$.

4.19. Definition. A complex K is said to be simplicially connected if, for each pair p and q of vertices of K, there exists a sequence $\{a_1^0, S_1^1, a_2^0, S_2^1, \ldots, a_k^0, S_k^1, a_{k+1}^0\}$ where each $a_i^0$ is a vertex of K; each $S_i^1$ is a 1-simplex of K; for each $j = 1,2,\ldots,k$, $a_j^0$ and $a_{j+1}^0$ are vertices of $S_j^1$; finally, $a_1^0 = p$, and $a_{k+1}^0 = q$.

4.20. Definition. If $K_1$ and $K_2$ are complexes with $K_1$ contained in $K_2$, then $K_1$ is called a subcomplex of $K_2$. A component of a complex is a maximal connected subcomplex.

4.21. Theorem. Let $v_1^0$ and $v_2^0$ be vertices of a complex K. Then the integral 0-chain $v_2^0 - v_1^0$ bounds if and only if $v_1^0$ and $v_2^0$ belong to the same component of K.

Proof: Let $K_1$ be a component of K such that $v_1^0, v_2^0 \in K_1$. Since a component is simplicially connected, then there exists a sequence $\{v_1^0, S_1^1, a_2^0, S_2^1, \ldots, a_k^0, S_k^1, v_2^0\}$ satisfying 4.19. Let $\alpha$ be an orientation of K, and let the 1-chain $c_1^\alpha = \sum_{i=1}^{k} t_i(S_i^1, \alpha(S_i^1))$ where

$$t_i = 1 \text{ if } (S_i^1, \alpha(S_i^1)) = \langle a_i^0, a_{i+1}^0 \rangle, \text{ and}$$
$$t_i = -1 \text{ if } (S_i^1, \alpha(S_i^1)) = \langle a_{i+1}^0, a_i^0 \rangle.$$

Then $\partial c_1^\alpha = a_2^0 - v_1^0 + a_3^0 - a_2^0 + \ldots + a_{i+1}^0 - a_i^0 + \ldots + v_2^0 - a_k^0 = v_2^0 - v_1^0$.

Hence $v_2^0-v_1^0$ bounds. This proves $v_1^0$ and $v_2^0$ belong to the same component which implies that $v_2^0-v_1^0$ bounds.

If $v_2^0-v_1^0$ bounds, let $v_2^0-v_1^0 = \partial \sum_{i=1}^{k} t_i(S_i^1,\alpha(S_i^1))$ where $\alpha$ is an orientation of K, $S_i^1$ is a 1-simplex in K, and $t_i$ is an integer for each i. Let $\beta$ be another orientation of K such that for each i, $f_{\alpha\beta}(S_i^1) = -1$ if $t_i$ is negative and $f_{\alpha\beta}(S_i^1) = 1$ if $t_i$ is positive. Then $v_2^0-v_1^0 = \partial \sum_{i=1}^{k} t_i(S_i^1,\alpha(S_i^1)) = \partial \sum_{i=1}^{k} |t_i|(S_i^1,\beta(S_i^1))$. Consider $\sum_{i=1}^{k} |t_i|(S_i^1,\beta(S_i^1))$ as the sum of a sequence Q of oriented 1-simplexes in $K^\beta$ with the coefficient +1 for each term of Q. Since Q is a finite sequence, then there exist a minimal subsequence $Q_m$ of Q such that $\partial(\Sigma Q_m) = v_2^0-v_1^0$. For simplicity, $v_1^0-v_2^0 = \partial(\Sigma Q_m)$ is called the equation A. By equation A, the sequence $Q_m$ must have one term in the form of $\langle x_1,v_2^0\rangle$. Rearrange the sequence $Q_m$ in the following way:

$$Q_m = \langle x_1,v_2^0\rangle, \langle x_2,y_2\rangle, \ldots, \langle x_d,y_d\rangle, \ldots, \langle x_n,y_n\rangle$$

where d is the maximum positive integer such that $x_i = y_{i+1}$ for $i = 1,\ldots,d-1$ and $x_d \neq y_{i+1}$ for $i = d,d+1,\ldots,n$. Since the left hand side of equation A is $v_2^0-v_1^0$, then either $x_d = v_1^0$ or $x_d = v_2^0$.

Case 1: $x_d = v_1^0$. Then $v_2^0-v_1^0 = \partial(\langle x_1,v_2^0\rangle + \ldots + \langle v_1^0,y_d\rangle)$, and $\{v_2^0\}$, $\{x_1,v_2^0\}$, $\{x_1\}$, $\{x_1,x_2\}$, $\{x_2\}$, $\{x_2,x_3\}$, ..., $\{x_{d-1},v_1^0\}$, $\{v_1^0\}$ is a sequence of simplexes satisfying 4.19. Hence $v_1^0$ and $v_2^0$ belong to the same component of K.

Case 2: $x_d = v_2^0$. Then $v_2^0-v_1^0 = \partial(\langle x_1,v_2^0\rangle + \ldots + \langle v_2^0,y_d\rangle + \langle x_{d+1},y_{d+1}\rangle + \ldots + \langle x_n,y_n\rangle) =$

being a minimal subsequence of Q such that

$v_2^0 - v_1^0 = \partial(\Sigma Q_m)$.

Combining Case 1 and Case 2, it must be true that $x_d = v_1^0$, and obviously $d = n$. Therefore the theorem is proved.

4.22. Theorem. Let $v_1^0, v_2^0, \ldots, v_n^0$ be vertices of a simplicially connected complex K. Then the integral 0-chain $\sum_{i=1}^{n} t_i v_i^0$ bounds if and only if $\sum_{i=1}^{n} t_i = 0$.

Proof: Since

$$\sum_{i=1}^{n} t_i v_i^0 = t_1 v_1^0 + \ldots + t_n v_n^0$$
$$= t_1 v_1^0 - t_1 v_2^0 + (t_1+t_2)v_2^0 - (t_1+t_2)v_3^0 + \ldots +$$
$$(t_1+\ldots+t_{n-1})v_{n-1}^0 - (t_1+\ldots+t_{n-1})v_n^0 + (t_1+\ldots+t_n)v_n^0,$$

then by 4.21, $\sum_{i=1}^{n} t_i v_i^0$ bounds if and only if $(t_1+\ldots+t_n)v_n^0$ bounds. But $(t_1+\ldots+t_n)v_n^0$ bounds if and only if $\sum_{i=1}^{n} t_i = 0$. Hence $\sum_{i=1}^{n} t_i v_i^0$ bounds if and only if $\sum_{i=1}^{n} t_i = 0$.

4.23. Theorem. If K is a simplicially connected complex, then $H_0(K;Z)$ is isomorphic to Z.

Proof: Let $\phi$ be a function, $\phi\colon H_0(K;Z) \to Z$, defined by $\phi(B_0(K;Z)+z) = \sum_{j=1}^{k} t_{i_j}$ where $z = \sum_{j=1}^{k} t_{i_j} v_{i_j}^0 \in Z_0(K;Z)$ and $v_{i_1}^0, v_{i_2}^0, \ldots, v_{i_k}^0$ are vertices of K.

Let $z_1, z_2 \in Z_0(K;Z)$ and $z_1 = \sum_{j=1}^{\ell} d_{n_j} v_{n_j}^0$, $z_2 = \sum_{j=1}^{m} b_{n'_j} v_{n'_j}^0$.

Then $\phi((B_0(K;Z)+z\ ) + (B_0(K;Z)+z\ )) =$

$\phi(B_0(K;Z)+z_1+z_2) = \sum_{j=1}^{\ell} d_{n_j} + \sum_{j=1}^{m} b_{n'_j} =$

$\phi(B_0(K;Z)+z_1) + \phi(B_0(K;Z)+z_2)$. Hence $\phi$ is a homomorphism.

If $B_0(K;Z)+z_1 \neq B_0(K;Z)+z_2$, then $z_1-z_2 \notin B_0(K;Z)$. By 4.22, $\sum_{j=1}^{\ell} d_{n_j} \neq \sum_{j=1}^{m} b_{n'_j}$. Hence $\phi(B_0(K;Z)+z_1) \neq \phi(B_0(K;Z)+z_2)$. Therefore $\phi$ is a one-to-one function.

Let N be an integer. Then $\phi(B_0(K;Z) + Nv_1^0) = N$. Hence $\phi$ is onto. Therefore $\phi$ is an isomorphism.

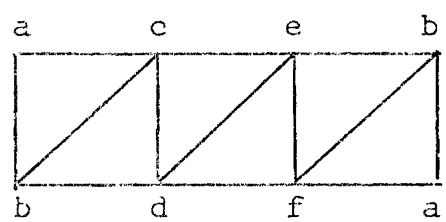
# APPENDIX

## FIVE PROBLEMS

The computation of homology groups is very tedious. The following are five problems computed by the author but not included in the body of the thesis because of their length. The results of the problems are stated in the following examples.

For simplicity, we denote an oriented r-simplex by $(S^r)^\alpha$ where $\alpha$ is the orientation.

1. Möbius band.




The oriented complex $K^\alpha$ is constructed as follows:

| | | |
|---|---|---|
| $(S^2_1)^\alpha = \langle a,c,b \rangle$ | $(S^1_1)^\alpha = \langle b,a \rangle$ | $(S^0_1)^\alpha = a$ |
| $(S^2_2)^\alpha = \langle b,c,d \rangle$ | $(S^1_2)^\alpha = \langle c,d \rangle$ | $(S^0_2)^\alpha = b$ |
| $(S^2_3)^\alpha = \langle d,c,e \rangle$ | $(S^1_3)^\alpha = \langle e,f \rangle$ | $(S^0_3)^\alpha = c$ |
| $(S^2_4)^\alpha = \langle d,e,f \rangle$ | $(S^1_4)^\alpha = \langle a,c \rangle$ | $(S^0_4)^\alpha = d$ |
| $(S^2_5)^\alpha = \langle f,e,b \rangle$ | $(S^1_5)^\alpha = \langle d,b \rangle$ | $(S^0_5)^\alpha = e$ |
| $(S^2_6)^\alpha = \langle f,b,a \rangle$ | $(S^1_6)^\alpha = \langle c,e \rangle$ | $(S^0_6)^\alpha = f$ |
| | $(S^1_7)^\alpha = \langle f,d \rangle$ | |

$$(S^1_8)^\alpha = \langle e,b \rangle$$

$$(S^1_9)^\alpha = \langle a,f \rangle$$

$$(S^1_{10})^\alpha = \langle c,b \rangle$$

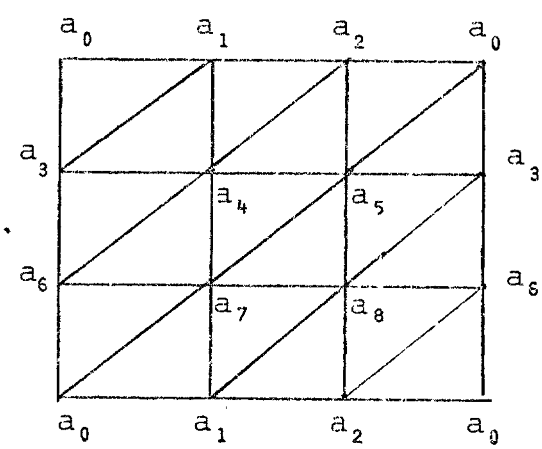
$$(S^1_{11})^\alpha = \langle e,d \rangle$$

$$(S^1_{12})^\alpha = \langle b,f \rangle$$

Then $H_n(K^\alpha;Z) = 0$ for $n \geq 2$

$H_n(K^\alpha;Z) \overset{\text{iso}}{\cong} Z$ for $n = 0,1$

2. Torus.




The oriented complex $K^\alpha$ is constructed as follows:

| | | |
|---|---|---|
| $(S^2_1)^\alpha = \langle a_3,a_0,a_1\rangle$ | $(S^1_1)^\alpha = \langle a_0,a_1\rangle$ | $(S^0_1)^\alpha = a_0$ |
| $(S^2_2)^\alpha = \langle a_3,a_1,a_4\rangle$ | $(S^1_2)^\alpha = \langle a_1,a_2\rangle$ | $(S^0_2)^\alpha = a_1$ |
| $(S^2_3)^\alpha = \langle a_4,a_1,a_2\rangle$ | $(S^1_3)^\alpha = \langle a_2,a_0\rangle$ | $(S^0_3)^\alpha = a_2$ |
| $(S^2_4)^\alpha = \langle a_4,a_2,a_5\rangle$ | $(S^1_4)^\alpha = \langle a_3,a_4\rangle$ | $(S^0_4)^\alpha = a_3$ |
| $(S^2_5)^\alpha = \langle a_5,a_2,a_0\rangle$ | $(S^1_5)^\alpha = \langle a_4,a_5\rangle$ | $(S^0_5)^\alpha = a_4$ |
| $(S^2_6)^\alpha = \langle a_5,a_0,a_3\rangle$ | $(S^1_6)^\alpha = \langle a_5,a_3\rangle$ | $(S^0_6)^\alpha = a_5$ |
| $(S^2_7)^\alpha = \langle a_6,a_3,a_4\rangle$ | $(S^1_7)^\alpha = \langle a_6,a_7\rangle$ | $(S^0_7)^\alpha = a_6$ |
| $(S^2_8)^\alpha = \langle a_6,a_4,a_7\rangle$ | $(S^1_8)^\alpha = \langle a_7,a_8\rangle$ | $(S^0_8)^\alpha = a_7$ |
| $(S^2_9)^\alpha = \langle a_7,a_4,a_5\rangle$ | $(S^1_9)^\alpha = \langle a_8,a_6\rangle$ | $(S^0_9)^\alpha = a_8$ |

$(S_{10}^2)^\alpha = \langle a_7, a_5, a_8 \rangle$ $(S_{10}^1)^\alpha = \langle a_3, a_0 \rangle$

$(S_{11}^2)^\alpha = \langle a_8, a_5, a_3 \rangle$ $(S_{11}^1)^\alpha = \langle a_6, a_3 \rangle$

$(S_{12}^2)^\alpha = \langle a_8, a_3, a_6 \rangle$ $(S_{12}^1)^\alpha = \langle a_0, a_6 \rangle$

$(S_{13}^2)^\alpha = \langle a_0, a_6, a_7 \rangle$ $(S_{13}^1)^\alpha = \langle a_4, a_1 \rangle$

$(S_{14}^2)^\alpha = \langle a_0, a_7, a_1 \rangle$ $(S_{14}^1)^\alpha = \langle a_7, a_4 \rangle$

$(S_{15}^2)^\alpha = \langle a_1, a_7, a_8 \rangle$ $(S_{15}^1)^\alpha = \langle a_1, a_7 \rangle$

$(S_{16}^2)^\alpha = \langle a_1, a_8, a_2 \rangle$ $(S_{16}^1)^\alpha = \langle a_5, a_2 \rangle$

$(S_{17}^2)^\alpha = \langle a_2, a_8, a_6 \rangle$ $(S_{17}^1)^\alpha = \langle a_8, a_5 \rangle$

$(S_{18}^2)^\alpha = \langle a_2, a_6, a_0 \rangle$ $(S_{18}^1)^\alpha = \langle a_2, a_8 \rangle$

$(S_{19}^1)^\alpha = \langle a_3, a_1 \rangle$

$(S_{20}^1)^\alpha = \langle a_6, a_4 \rangle$

$(S_{21}^1)^\alpha = \langle a_4, a_2 \rangle$

$(S_{22}^1)^\alpha = \langle a_0, a_7 \rangle$

$(S_{23}^1)^\alpha = \langle a_7, a_5 \rangle$

$(S_{24}^1)^\alpha = \langle a_5, a_0 \rangle$

$(S_{25}^1)^\alpha = \langle a_1, a_8 \rangle$

$(S_{26}^1)^\alpha = \langle a_8, a_3 \rangle$

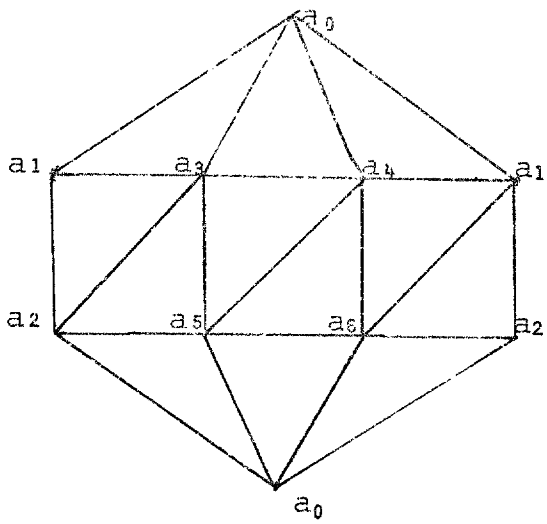
$(S_{27}^1)^\alpha = \langle a_2, a_6 \rangle$

Then $H_n(K^\alpha;Z) = 0$ for $n \geq 3$

$H_n(K^\alpha;Z) \overset{\text{iso}}{\cong} Z$ for $n = 0,2$

$H_1(K^\alpha;Z) \neq 0$.

3. Pinched torus.




The oriented complex $K^\alpha$ is constructed as follows:

$(S_1^2)\alpha = \langle a_1, a_0, a_3 \rangle$ $(S_1^1)\alpha = \langle a_1, a_0 \rangle$ $(S_1^0)\alpha = a_0$

$(S_2^2)\alpha = \langle a_3, a_0, a_4 \rangle$ $(S_2^1)\alpha = \langle a_3, a_0 \rangle$ $(S_2^0)\alpha = a_1$

$(S_3^2)\alpha = \langle a_4, a_0, a_1 \rangle$ $(S_3^1)\alpha = \langle a_4, a_0 \rangle$ $(S_3^0)\alpha = a_2$

$(S_4^2)\alpha = \langle a_2, a_1, a_3 \rangle$ $(S_4^1)\alpha = \langle a_2, a_1 \rangle$ $(S_4^0)\alpha = a_3$

$(S_5^2)\alpha = \langle a_2, a_3, a_5 \rangle$ $(S_5^1)\alpha = \langle a_2, a_3 \rangle$ $(S_5^0)\alpha = a_4$

$(S_6^2)\alpha = \langle a_5, a_3, a_4 \rangle$ $(S_6^1)\alpha = \langle a_5, a_3 \rangle$ $(S_6^0)\alpha = a_5$

$(S_7^2)\alpha = \langle a_5, a_4, a_6 \rangle$ $(S_7^1)\alpha = \langle a_5, a_4 \rangle$ $(S_7^0)\alpha = a_6$

$(S_8^2)\alpha = \langle a_6, a_4, a_1 \rangle$ $(S_8^1)\alpha = \langle a_6, a_4 \rangle$

$(S_9^2)\alpha = \langle a_6, a_1, a_2 \rangle$ $(S_9^1)\alpha = \langle a_6, a_1 \rangle$

$(S_{10}^2)\alpha = \langle a_2, a_5, a_0 \rangle$ $(S_{10}^1)\alpha = \langle a_0, a_2 \rangle$

$(S_{11}^2)\alpha = \langle a_5, a_6, a_0 \rangle$ $(S_{11}^1)\alpha = \langle a_0, a_5 \rangle$

$(S_{12}^2)\alpha = \langle a_6, a_2, a_0 \rangle$ $(S_{12}^1)\alpha = \langle a_0, a_6 \rangle$

$(S_{13}^1)\alpha = \langle a_1, a_3 \rangle$

$(S_{14}^1)\alpha = \langle a_3, a_4 \rangle$

$(S_{15}^1)\alpha = \langle a_4, a_1 \rangle$

$(S_{16}^1)^\alpha = \langle a_2, a_5 \rangle$

$(S_{17}^1)^\alpha = \langle a_5, a_6 \rangle$

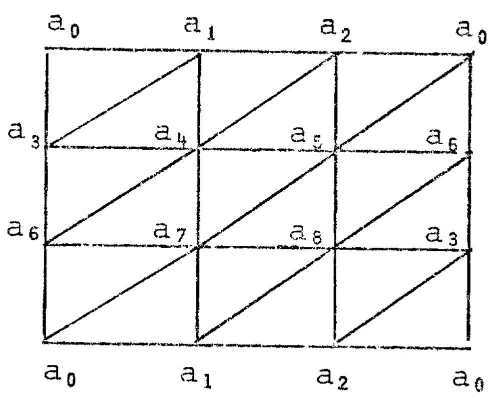
$(S_{18}^1)^\alpha = \langle a_6, a_2 \rangle$

Then $H_n(K\alpha;Z) = 0$ for $n \geq 3$

$H_n(K^\alpha;Z) \overset{iso}{\approx} Z$ for $n = 0,2$

$H_1(K^\alpha;Z) \neq 0.$

4. Klein bottle.




The oriented complex $K\alpha$ is constructed as follows:

$(S_1^2)^\alpha = \langle a_0, a_1, a_3 \rangle$ $(S_1^1)^\alpha = \langle a_0, a_1 \rangle$ $(S_1^0)^\alpha = a_0$

$(S_2^2)^\alpha = \langle a_1, a_4, a_3 \rangle$ $(S_2^1)^\alpha = \langle a_1, a_2 \rangle$ $(S_2^0)^\alpha = a_1$

$(S_3^2)^\alpha = \langle a_1, a_2, a_4 \rangle$ $(S_3^1)^\alpha = \langle a_2, a_0 \rangle$ $(S_3^0)^\alpha = a_2$

$(S_4^2)^\alpha = \langle a_2, a_5, a_4 \rangle$ $(S_4^1)^\alpha = \langle a_3, a_4 \rangle$ $(S_4^0)^\alpha = a_3$

$(S_5^2)^\alpha = \langle a_2, a_0, a_5 \rangle$ $(S_5^1)^\alpha = \langle a_4, a_5 \rangle$ $(S_5^0)^\alpha = a_4$

$(S_6^2)^\alpha = \langle a_0, a_6, a_5 \rangle$ $(S_6^1)^\alpha = \langle a_5, a_6 \rangle$ $(S_6^0)^\alpha = a_5$

$(S_7^2)^\alpha = \langle a_3, a_4, a_6 \rangle$ $(S_7^1)^\alpha = \langle a_6, a_7 \rangle$ $(S_7^0)^\alpha = a_6$

$(S_8^2)^\alpha = \langle a_4, a_7, a_6 \rangle$ $(S_8^1)^\alpha = \langle a_7, a_8 \rangle$ $(S_8^0)^\alpha = a_7$

$(S_9^2)^\alpha = \langle a_4, a_5, a_7 \rangle$ $(S_9^1)^\alpha = \langle a_8, a_3 \rangle$ $(S_9^0)^\alpha = a_8$

$(S_{10}^2)^\alpha = \langle a_5, a_8, a_7 \rangle$ $(S_{10}^1)^\alpha = \langle a_0, a_3 \rangle$

$(S_{11}^2)^\alpha = \langle a_5, a_6, a_8 \rangle$ $(S_{11}^1)^\alpha = \langle a_3, a_6 \rangle$

$(S_{12}^2)^\alpha = \langle a_6, a_3, a_8 \rangle$ $(S_{12}^1)^\alpha = \langle a_6, a_0 \rangle$

$(S^2_{13})^\alpha = \langle a_0, a_6, a_7 \rangle$ $(S^1_{13})^\alpha = \langle a_1, a_4 \rangle$

$(S^2_{14})^\alpha = \langle a_0, a_7, a_1 \rangle$ $(S^1_{14})^\alpha = \langle a_4, a_7 \rangle$

$(S^2_{15})^\alpha = \langle a_1, a_7, a_8 \rangle$ $(S^1_{15})^\alpha = \langle a_7, a_1 \rangle$

$(S^2_{16})^\alpha = \langle a_1, a_8, a_2 \rangle$ $(S^1_{16})^\alpha = \langle a_2, a_5 \rangle$

$(S^2_{17})^\alpha = \langle a_2, a_8, a_3 \rangle$ $(S^1_{17})^\alpha = \langle a_5, a_8 \rangle$

$(S^2_{18})^\alpha = \langle a_0, a_2, a_3 \rangle$ $(S^1_{18})^\alpha = \langle a_8, a_2 \rangle$

$(S^1_{19})^\alpha = \langle a_1, a_3 \rangle$

$(S^1_{20})^\alpha = \langle a_2, a_4 \rangle$

$(S^1_{21})^\alpha = \langle a_4, a_6 \rangle$

$(S^1_{22})^\alpha = \langle a_0, a_5 \rangle$

$(S^1_{23})^\alpha = \langle a_5, a_7 \rangle$

$(S^1_{24})^\alpha = \langle a_7, a_0 \rangle$

$(S^1_{25})^\alpha = \langle a_6, a_8 \rangle$

$(S^1_{26})^\alpha = \langle a_8, a_1 \rangle$

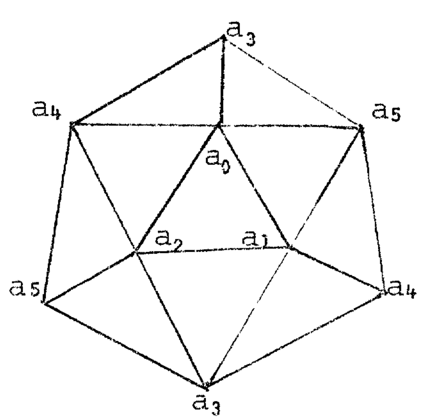
$(S^1_{27})^\alpha = \langle a_3, a_2 \rangle$

Then $H_n(K^\alpha;Z) = 0$ for $n \geq 2$

$H_1(K^\alpha;Z) \neq 0$

$H_0(K^\alpha;Z) \underset{\approx}{\text{iso}} Z.$

5. Projective plane.



The oriented complex $K^{\alpha}$ is constructed as follows:

| | | |
|---|---|---|
| $(S_1^2)^{\alpha} = \langle a_4, a_3, a_0 \rangle$ | $(S_1^1)^{\alpha} = \langle a_4, a_3 \rangle$ | $(S_1^0)^{\alpha} = a_0$ |
| $(S_2^2)^{\alpha} = \langle a_0, a_3, a_5 \rangle$ | $(S_2^1)^{\alpha} = \langle a_0, a_3 \rangle$ | $(S_2^0)^{\alpha} = a_1$ |
| $(S_3^2)^{\alpha} = \langle a_5, a_4, a_2 \rangle$ | $(S_3^1)^{\alpha} = \langle a_5, a_3 \rangle$ | $(S_3^0)^{\alpha} = a_2$ |
| $(S_4^2)^{\alpha} = \langle a_2, a_4, a_0 \rangle$ | $(S_4^1)^{\alpha} = \langle a_5, a_4 \rangle$ | $(S_4^0)^{\alpha} = a_3$ |
| $(S_5^2)^{\alpha} = \langle a_2, a_0, a_1 \rangle$ | $(S_5^1)^{\alpha} = \langle a_2, a_4 \rangle$ | $(S_5^0)^{\alpha} = a_4$ |
| $(S_6^2)^{\alpha} = \langle a_0, a_5, a_1 \rangle$ | $(S_6^1)^{\alpha} = \langle a_2, a_0 \rangle$ | $(S_6^0)^{\alpha} = a_5$ |
| $(S_7^2)^{\alpha} = \langle a_1, a_5, a_4 \rangle$ | $(S_7^1)^{\alpha} = \langle a_1, a_0 \rangle$ | |
| $(S_8^2)^{\alpha} = \langle a_5, a_2, a_3 \rangle$ | $(S_8^1)^{\alpha} = \langle a_1, a_5 \rangle$ | |
| $(S_9^2)^{\alpha} = \langle a_2, a_1, a_3 \rangle$ | $(S_9^1)^{\alpha} = \langle a_3, a_2 \rangle$ | |
| $(S_{10}^2)^{\alpha} = \langle a_3, a_1, a_4 \rangle$ | $(S_{10}^1)^{\alpha} = \langle a_3, a_1 \rangle$ | |
| | $(S_{11}^1)^{\alpha} = \langle a_4, a_0 \rangle$ | |
| | $(S_{12}^1)^{\alpha} = \langle a_0, a_5 \rangle$ | |
| | $(S_{13}^1)^{\alpha} = \langle a_5, a_2 \rangle$ | |
| | $(S_{14}^1)^{\alpha} = \langle a_2, a_1 \rangle$ | |
| | $(S_{15}^1)^{\alpha} = \langle a_1, a_4 \rangle$ | |

Then $H_n(K^{\alpha};Z) = 0$ for $n \geq 2$

$H_1(K^{\alpha};Z) \neq 0$

$H_0(K^{\alpha};Z) \overset{\text{iso}}{\cong} Z.$

# BIBLIOGRAPHY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