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Minimal Lineal Measure in Three Dimensions

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**The Problems of the Minimal Surface and
Minimal Lineal Measure in Three Dimensions**

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Abstract

A solution is given to the classical problem of the minimal surface in three dimensions formed from a repeating cell microstructure under isotropic conditions. The solution is found through a global/local minimization procedure and the resulting basic cell is composed of 14 faces. At the junctions where the intersections between faces meet at a point, half of the junctions involve 4 intersections and half involve 3 intersections. The same general solution also applies to the related minimal lineal measure problem where the measure is that of the length of the intersections connecting the junctions. Some implications and applications for materials science are given.

Background and Formalisms

The problem of the minimal surface in three dimensions (3-D) has intrigued mathematicians over the years. It also has been of considerable scientific interest in the area of liquid foams. But now it is of special importance in low density materials and in many aspects of micro-scale or nano-scale materials behavior where surface effects dominate. Despite great activity, this problem of the minimal surface for cellular forms has remained resistant to solution. However, very recently a bound on the solution has been given, Kusner (1992). The bound is $F \geq 14$ where F is the number of faces in the basic cell. This bound and a re-derivation of it are of importance in consummating the last step, i.e. proving the explicit form of the minimal surface, as done here. The complete procedure to be given involves a global/local minimization.

There actually are two related minimal problems for cellular agglomerations. The two problems are:

- i) Minimal surface area of the faces of the cells (minimal surface problem) and
- ii) Minimal length of the edges of the cells (minimal lineal measure problem)

Problem i) has been of interest since the time of Plateau's elegant experiments on soap films. However, Problem ii) is probably of far more wide ranging technological importance, and the application given here will fall into that class.

The unusual aspect of the method to be given is to relate the characteristics of the intrinsic cell type to a characteristic of the continuum of cells. Take the latter characteristic as the average number of intersections per junction, I , involving the intersections of faces (surfaces) that meet at a point called a junction. The individual cell characteristic of faces, edges and vertices thus carry over to the corresponding cellular continuum characteristics of surfaces, intersections and junctions. To distinguish between the average I of all junctions and cases where a group of junctions have the same number of intersections, take the latter as I_3 , I_4 etc.

The local conditions for minimal surfaces are well known. Three surfaces meet at an intersection, and 4 intersections meet at a junction, all at equal angles. The most recent proof is by Taylor (1976). Thus the local minimal conditions are $I=I_3$ in 2-D, as will be explained, and $I=I_4$ in 3-D. Also as will be seen, this 3-D local minimal condition of $I=I_4$ cannot provide a corresponding global minimum, even though in 2-D $I=I_3$ is minimal both locally and globally. In 3-D it is not possible to construct a cellular

microstructure with all junctions of the I_4 type while preserving isotropy. In the approach taken here, a procedure will be found for maximizing the population of junctions of I_4 type since that represents a coalescence of cells having 3 edges per vertex, which is the class of cell types to be considered. In 3-D consider 4 vertices from 4 different and separated cells, but in close proximity. When ideally merged to form 1 junction, the 12 associated faces meld into 6 surfaces and the 12 emanating edges become 4 intersections. This certainly is a process for forming microstructure with a minimal expenditure of surface, showing the benefit of I_4 type junctions.

The cells are treated as polyhedra which are governed by Euler's fundamental formula. Suggestive terminology will be followed, but with no implication of regularity. Thus for example the term tetrahedron simply means any 4 sided cell having 3 edges at each vertex. In the following developments the terminology of problem i) (minimal surface) will be followed for the most part. There is only required a slight variation in terminology to recast the methodology to problem ii).

It is necessary to begin by stating the well known results for the corresponding minimal 2-D problem. It would not be possible to proceed with the immensely more complicated 3-D case, unless there were complete agreement on the 2-D formalism.

Two Dimensional Case

In considering 2-D conditions, it is necessary to restrict the symmetry class to give isotropy of physical/mechanical effects. Otherwise the global minimization problem is not well posed. Hexagonal symmetry is sufficient to give 2-D isotropy, and all candidate cellular forms must conform with this symmetry. The two common cell types are equilateral triangles and hexagons. It is not difficult to identify the completing microstructure, it is composed of 6 pointed star shaped cells with included angles of 60 degrees. All three cell types are shown in Fig. 1, with lattice spacing $2b$. Table I shows the characteristics of the 3 cellular forms. The units of the surface areas are area of surface per unit thickness (2-D) per unit cross sectional area. The hexagonal cell has the lowest and minimal surface area, while those of the other two cellular microstructures are strongly larger than the minimal value.

It is helpful to view the 2-D cases as a subsets of 3-D constructions. Take the 2-D cells as extending indefinitely in the direction normal to the 2-D plane. Take the number of faces of the basic 2-D cell as F_{2D} , thus for example in the hexagonal case, $F_{2D}=6$. Each face intersects the 2-D plane to create an intersection. The intersections meet at discrete points called junctions. Take the average number of intersections meeting at junctions as I_{2D} . Thus, for the hexagonal cell case $I_{2D}=3$, and $I_{2D}=3$ which is minimal both locally and globally.. In the star cell case of Fig. 1 and Table I there are junctions with 2 and 4 intersections, but the average is still $I_{2D}=3$. This terminology will have a useful counterpart in the 3-D situation to be considered next.

Note that although periodic, the star cell does not pack to fill 2-space. This feature also will emerge in the 3-D case.

Three Dimensional Case

Begin by considering cell characteristics. First there is Euler's relation

$$F - E + V = 2 \quad (1)$$

where F, E & V are numbers of faces, edges and vertices respectively

Now take two restrictions on cell types. Take only cells with 3 edges at a vertex, the same as Kusner (1992). Cells coalesce to form the continuum of minimal surface and in order to form or approach forming a 4 intersection junction which is locally minimal, one must start with 3 edges per vertex. Thus take

$$2E = 3V \quad (2)$$

This for example excludes the octahedral and icosahedral type cells.

The other limitation is required to bring in a characteristic of the continuum of cells as related to intrinsic cell characteristics. First, there will be stated the form of the restriction, then a constant entering it will be evaluated, and then the form will be tested against a known independent case. Take the average of the number of intersections per junction, I, for the continuum as depending on cell characteristics, F, E and V. Eliminate E through (2). Since faces form intersections and vertices form junctions, take the intersections per junction (average) as

$$I = \eta \frac{F}{V} \quad (3)$$

where η is a constant. Determine η from the limiting case $F \rightarrow \infty$. In this limiting case, the cell has infinitely many, infinitesimally small sized faces, but contact with other cells occurs only through a finite number of contacts. This sphere – like system thus has each of the unlimited number of junctions as unmodified from the cell characteristic of 3 edges at a vertex, thus

$$I = 3 \text{ as } F \rightarrow \infty$$

Using this condition in (1) – (3) determines η as

$$\eta = 6 \quad (4)$$

Combining (1) – (4) gives

$$I = \frac{3F}{F-2} \quad (5)$$

Now check the other limiting case. The minimum number of possible faces is $F = 4$, and (5) gives

$$I = 6 \text{ at } F = 4$$

This case of $F = 4$ has tetrahedrons as cells and the result, $I = 6$, is the expected result. The tetrahedrons meet vertex to vertex, each of which brings 3 intersections to the junction. Finally, the result (5) correctly reflects the impossibility of the case $F = 2$.

The two limiting cases $F = 4$ and $F \rightarrow \infty$ are thus revealed as extremely inefficient packaging of cellular microstructure, and the minimal case is sure to be far removed from either of these extremes. Note that in these extreme cases the cells do not fill space completely. The minimal case will be expected to come as close as possible (in some sense) to filling space. Of course it will be necessary to separately justify the existence of particular cellular forms, that is, forms do not necessarily exist for all even integer values of F in (5).

Relation (5) does not by itself provide a mechanism for minimization. For example, at $F = 8$ then (5) gives $I = 4$, but since $I = 4$ is just an average, the junctions could be composed of $2/3$ I_3 's and $1/3$ I_6 's or an unlimited number of other possibilities. The only thing that is known is that $I = 4$ could not be all I_4 's. Clearly there is a constraint on the possible proportion of I_4 as a function of F . The nature of this constraint will emerge as the global minimization procedure develops.

The results thus far are shown in Table II. Other than the 2 limiting cases, these are but particular and rather obvious possible cases from the spectrum of even integer values of F . The two cases $F = 14$ and $F = 32$ are of special interest. $F = 14$ could correspond to the truncated octahedron having 6 quadrilaterals and 8 hexagons. This form was first examined by Lord Kelvin (1887) and more will be said of this later. The form with 32 faces could be a truncated icosahedron composed of 12 pentagons and 20 hexagons. This form, with regular microstructure, is simply the fullerene form of carbon, Kroto (1988), Curi & Smalley (1988) for which there has been much recent activity.

Two more steps are needed to complete the derivation. First, a lower bound will be found for the number of faces in the cell of the minimal surface, corresponding to a global minimization. Then a local minimization will be used to identify the specific minimal form.

Kusner (1992) has derived the bound $F \geq 13.40$ for the average number of faces of the minimizing cell. Virtually the same bound will be derived here by a different method that reveals the connection between the 2-D and 3-D cases. This connection is particularly important in understanding various physical effects.

As has already been mentioned, the global minimization problem is not simply one of setting $I = 4$ and solving for F . Nor, is the problem simply one of seeking the minimal I value, as the 2-D case transparently shows. The results of Tables I & II are shown in Fig. 2. The objective now is to find the minimal 3-D cell corresponding to the minimal 2-D cell of $F_{2D} = 6$, $I_{2D} = 3$.

Postulate the existence of a mapping or transformation between the F values for the 2-D and 3-D cases. Take this as yet to be derived relation between F_{2D} and F as giving the surface area calibrated mapping between the 2-D and 3-D cases. The required conditions for the mapping are those of the limiting cases

$$\begin{aligned} F &= 4 \text{ at } F_{2D} = 3 \\ \text{and} \\ F &\rightarrow \infty \text{ at } F_{2D} = 12 \end{aligned} \tag{6}$$

If the proper mapping function can be found, then the value of F mapped from $F_{2D} = 6$ (hexagonal) in principle will be used to identify the 3-D minimizing cell characteristic from the 2-D minimizing cell.

Based on dimensionality one would expect the mapping between F_{2D} and F to be from

$$(F_{2D})^{\frac{1}{2}} \text{ to } F^{\frac{1}{3}}$$

But, compared with 3-D, the 2-D case is degenerate in that its cells fit together perfectly forming a lattice with periodicity and having disorder in only one independent direction.

Thus the dimensionality of the 2-D case relative to 3-D is reduced further by one, leaving the possible mapping as

$$F_{2D} \text{ to } F^{\frac{1}{3}}$$

The most general bilinear form preserving the relationship between F_{2D} and F through $F^{\frac{1}{3}}$ is

$$F^{\frac{1}{3}} F_{2D} + \alpha F^{\frac{1}{3}} + \beta F_{2D} + \gamma = 0 \quad (7)$$

where 3 parameters α , β and γ are to be found. Using the two conditions in (6) with (7) requires

$$\begin{aligned} \alpha &= -12 \\ \text{and} \\ \gamma &= 9(4)^{\frac{1}{3}} - 3\beta \end{aligned} \quad (8)$$

Inserting (8) back into (7) gives

$$F^{\frac{1}{3}} = \frac{\beta(F_{2D} - 3) + 9(4)^{\frac{1}{3}}}{12 - F_{2D}}, \quad 3 \leq F_{2D} \leq 12, \quad \beta \geq 0 \quad (9)$$

where the restriction $\beta \geq 0$ must be explained. Actually, there must be some restriction on β for regularity reasons. For example, F could be negative if β were left unrestricted. Take a completely smooth functional relationship such that F and its derivatives (viewed as a continuous variable) with respect to F_{2D} are monotonic in F_{2D} . This requires $\beta \geq 0$.

Relation (9) with $\beta \geq 0$ gives the bound

$$F \geq 4 \left(\frac{9}{12 - F_{2D}} \right)^3 \quad (10)$$

Using the 2-D minimal cell value of $F_{2D} = 6$ in (10) gives

$$F \geq \frac{27}{2} \quad (11)$$

This value of $F \geq 13\frac{1}{2}$ is the bound on the minimizing cell characteristic. This bound of $F \geq 13\frac{1}{2}$ is in remarkable agreement with the bound of Kusner (1992) of $F \geq 13.3973 \dots$

Using the bound derived here, or that of Kusner (1992) (or using both) gives

$$F \geq 14 \quad (12)$$

when restricted to a single type of cell.

Unfortunately the global minimization procedure just given only lead to a bound, rather than the explicit solution. However, the combined local minimization requirement will complete the process. Using relation (5), the present status can be written as

$$\begin{aligned} I &= 3\frac{1}{2} \text{ at } F = 14 \\ I &< 3\frac{1}{2} \text{ for } F > 14 \end{aligned} \quad (13)$$

The constraint on the proportion of I_4 as a function of F , mentioned earlier, occurs only for $F < 14$ according to (12). For $F \geq 14$ there is freedom to proportion I_4 & I_3 accordingly. A junction with only 3 intersections shows that the cells haven't been able to combine to form the more efficient 4 intersection type of junction. To have minimal local surface requires the microstructure that has the greatest population of I_4 's and the least population of I_3 's. From (12) and (13) it is seen that the additional local minimization requires

$$F = 14 \text{ with } \left. \begin{array}{l} 50\% I_3 \\ 50\% I_4 \end{array} \right\} I = 3\frac{1}{2} \quad (14)$$

Thus the combined result of the global/local minimization gives a microstructure having 14 faces in the basic cell along with half of the junctions having 3 intersections and half

having 4 intersections. This is the solution to problem i) of the first section and with slightly changed terminology it is also the general solution to problem ii).

It is important to note that Lord Kelvin (1887) proposed a form with 14 faces as being the likely minimal surface. The form is that of a truncated octahedron, which fills space perfectly. With warping of the faces and curvature in the edges, the resulting form is composed entirely of I_4 tetrahedral type junctions. Thus the solution here, (14), and that of Lord Kelvin are incompatible. This difference is quite easy to reconcile. The form of Lord Kelvin does not provide isotropy since it has cubic symmetry. Thus, Lord Kelvin's highly original and imaginative form may very well be the minimal or near minimal form for a cubic symmetry class, but it is not so for the class of forms having isotropy. Lord Kelvin apparently understood the need for isotropy, but he only used what he called isotropy in 3 orthogonal directions, which falls within cubic symmetry. This circumstance is perhaps not too difficult to understand. Although the conditions necessary for isotropy have been loosely understood for a long time, the formal and rigorous sufficiency proof is surprisingly recent, Dresselhaus and Dresselhaus (1992). Even though the present 3-D methodology did not explicitly use a 3-D requirement of isotropy, it is implicit in the results which were synthesized from corresponding 2-D isotropic cases.

The requirement for isotropy can be most clearly appreciated when viewed in the context of problem ii) the minimal lineal measure problem. If cubic symmetry were allowed (with violation of isotropy) then the probable solution is already in existence and quite obvious. The diamond cubic pattern is totally I_4 , and infact it likely is the minimal solution within cubic symmetry. The isotropic case imposes a completely different, and much more difficult class of problems.

Coexter (1963) has made a connection between the present problem and the packing of spheres. References to other related works are given by Kusner (1992). The mathematical solution given by (14) may be difficult to realize in certain physical situations. For example for problem i) involving soap films, it may well be true that a lower surface form can be achieved by "grains" of the Kelvin type cells with grain boundaries between the cubic grains of differing orientation, to give isotropy on some sufficiently large scale. However, for problems of type ii) the solution (14) will be shown in the next section to have the necessary aspects of physical reality.

Materials Science Applications

The mathematical problem and solution from the previous section which is of interest here is that of the minimal lineal measure between junctions. In physical terms imagine a space network of one dimensional material members interconnected at junctions throughout the three dimensional space. For material members of fixed cross sectional area, a minimal material condition corresponds to the minimal lineal dimension problem. The problem is posed appropriate to isotropic conditions, which precludes crystalline type order. The solution from the previous section gives the basic cell as having 14 faces and 1/2 of the material junctions as joining 3 material members, and the other 1/2 of the junctions as joining 4 material members. It is the latter feature of equal amounts of I₃ and I₄ junctions that provides the easily identifiable and fundamental connection to the actual material architecture.

Figure 3 shows a micrograph of reticulated carbon, the combination of which forms a very porous material. It is widely acknowledged that surface tension effects play a dominate role in the material synthesis process. Expecting that the minimal surface energy condition drives the morphology to a corresponding minimal form gives the solution of the previous section. The unexpected feature of the solution found there was the appearance of the I₃ type junctions, which are normally associated only with 2-D conditions. Figure 3 shows ample evidence of the 3-D existence of I₃ junctions along with I₄ forms and possibly I₅'s also. It is likely that a perfect balance of 50% I₃'s and 50% I₄'s, although ideal, is easily perturbed.

In Fig. 4 a biological material form is shown, sea urchin spines. It is probably reasonable to expect that the biological process has evolved into the most efficient, minimal material forms. Certainly I₃ and I₄ junctions are evident in Fig. 4.

These two examples, from previous mechanical properties studies, appear to show the physical viability of the basic solution found here. This solution has as its central result the equal proportions of 3 directional joining at junctions and 4 directional joining at junctions. Applications of this result may be wide ranging, from atomic bonding in amorphous materials to a variety of transport properties in materials science.

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Table I 2-D Microstructures and Surface Areas

Cell Type	Surface Area	Faces F _{2D}	Intersections per Junction I _{2D}
Triangle	$3/b$	3	6
Hexagon (Minimal)	$1/b$	6	3
Star	$2/b$	12	3

Table II 3-D Cell and Continuum Characteristics

Cell Type	Faces F	Intersections per Junction I
Tetrahedron	4	6
Hexahedron (Cube)	6	$4\frac{1}{2}$
Truncated Tetrahedron	8	4
Dodecahedron	12	$3\frac{3}{5}$
Truncated Octahedron, Truncated Hexahedron, etc.	14	$3\frac{1}{2}$
Truncated Icosahedron	32	$3\frac{1}{5}$
Limit	∞	3

Triangle

Hexagon

Star

Fig. 1 2-D Microstructures

Fig. 2 Intersections versus Faces

Fig. 3 Carbon foam.

Fig. 4 Sea urchin spines.

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