The Normal Integral

W. J. Cody*
Mathematics and Computer Science Division
Argonne National Laboratory
Argonne, IL 60439-4801

Key Words: Normal curve.

Language

Fortran 77.

Description and Purpose

ANORM is a reliable, portable Fortran function program, written in the style of the SPECFUN package [2, 3], for computing the normal probability distribution to full machine precision on most contemporary computers. The main computation evaluates near-minimax approximations (derived from those in Cody [1]) that are theoretically accurate to at least 18 significant decimal digits. Special care has been taken in implementation to minimize error contamination in the crucial computation of the exponential and to provide full accuracy in the computation with large negative arguments. ANORM returns 0.0 for arguments smaller than the machine-dependent constant XLOW and returns 1.0 for arguments greater than the machine-dependent constant XUPPR.

Structure

FUNCTION ANORM(ARG)

Formal parameter

ARG Real

input: the argument value.

Related Algorithms

Algorithms AS 2 [4] and AS 66 [5] also calculate the normal integral. The advantages of our algorithm are ease of use, accuracy for the full argument range, and ease of conversion between single and double precision.

*Work supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U. S. Department of Energy, under Contract W-31-109-Eng-38.

DISCLAIMER

1

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

The submitted manuscript has been authored by a contractor of the U. S. Government under contract No. W-31-109-ENG-38. Accordingly, the U. S. Government retains a nonexclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U. S. Government purposes.

MASTER

Constants

The constants **EPS**, **XLOW**, and **XUPPR** must be calculated and initialized in **DATA** statements. An approximate value for **XLOW** is the negative of the solution to the equation

$$\frac{e^{-x^2/2}}{x\sqrt{2\pi}}(1 - 1/x^2) = xmin,$$

where *xmin* is the smallest positive floating-point number. **XUPPR** is estimated as $\sqrt{\ln \text{EPS}}$, where **EPS** is the largest machine number such that $1 + \text{EPS} = 1$. Comments in **ANORM** contain suggested single-precision and double-precision values of these constants for most contemporary machines.

Accuracy

The program will return function values accurate to the minimum of machine precision and 18 significant decimal digits for arguments greater than **XLOW**, and will return 0.0 for smaller arguments.

Precision

A single-precision version of **ANORM** can be prepared for a particular machine by first globally changing all occurrences of **CS** in columns 1 and 2 to blanks, and then specifying the machine-dependent constants **EPS**, **XLOW**, and **XUPPR** in the **DATA** statements. A double-precision version can be obtained in the same manner, except that **CD** in columns 1 and 2 must be globally changed to blanks.

References

- [1] Cody, W. J. (1969) Rational Chebyshev approximations for the error function. *Math. Comp.* 23, 631-637.
- [2] — (1987) SPECFUN – a portable special function package. In *New Computing Environments: Microcomputers in Large-Scale Scientific Computing*, A. Wouk, ed., SIAM, Philadelphia.
- [3] — (1990) SPECFUN – a portable package of special functions and test drivers. Preprint MCS-P179-0990, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Illinois, 1990.
- [4] Cooper, B. E. (1968) Algorithm AS 2. The normal integral. *Appl. Stat.* 17, 186-187.
- [5] Hill, I. D. (1973) Algorithm AS 66. The normal integral. *Appl. Stat.* 22, 424-427.

```
C-----C
C This function evaluates the normal distribution function:
C
C          / x
C      1   |    -t*t/2
C P(x) = ----- | e      dt
C      sqrt(2 pi) |
C                /-oo
C
C The main computation evaluates near-minimax approximations
C derived from those in "Rational Chebyshev approximations for
C the error function" by W. J. Cody, Math. Comp., 1969, 631-637.
C This transportable program uses rational functions that
C theoretically approximate the normal distribution function to
C at least 18 significant decimal digits. The accuracy achieved
C depends on the arithmetic system, the compiler, the intrinsic
C functions, and proper selection of the machine-dependent
C constants.
C*****
C*****
C Explanation of machine-dependent constants
C
C XMIN  = the smallest positive floating-point number.
C EPS   = argument below which anorm(x) may be represented by
C         0.5 and above which x*x will not underflow.
C         A conservative value is the largest machine number X
C         such that 1.0 + X = 1.0 to machine precision.
C XLOW  = the most negative argument for which ANORM does not
C         vanish. This is the negative of the solution to
C           W(x) * (1-1/x**2) = XMIN,
C         where W(x) = exp(-x*x/2)/[x*sqrt(2*pi)].
C XUPPR = positive argument beyond which anorm = 1.0. A
C         conservative value is the solution to the equation
C           exp(-x*x/2) = EPS,
C         i.e., XUPPR = sqrt[-2 ln(eps)].
C
C Approximate values for some important machines are:
C
C              XMIN             EPS             XLOW             XUPPR
C CDC 7600      (S.P.) 3.13E-294  7.11E-15  -36.641  8.072
C CRAY-1        (S.P.) 4.58E-246  7.11E-157 -106.521 26.816
C IEEE (IBM/XT, SUN, etc.) (S.P.) 1.18E-38  5.96E-8  -12.949  5.768
C IEEE (IBM/XT, SUN, etc.) (D.P.) 2.23D-308  1.11D-16 -37.519  8.572
C IBM 195       (D.P.) 5.40D-79   1.39D-17 -18.781  8.811
C VAX D-Format  (D.P.) 2.94D-39   1.39D-17 -13.055  8.811
C VAX G-Format  (D.P.) 5.56D-309  1.11D-16 -37.556  8.572
```

```

C
C*****
C*****
C
C Error returns
C
C The program returns  ANORM = 0      for  ARG .LE. XLOW.
C
C
C Intrinsic functions required are:
C
C     ABS, AINT, EXP
C
C Author: W. J. Cody
C         Mathematics and Computer Science Division
C         Argonne National Laboratory
C         Argonne, IL 60439
C
C Latest modification: September 26, 1990
C
C-----
C      INTEGER I
CS      REAL
CD      DOUBLE PRECISION
C      1      A,ANORM,ARG,B,C,D,DEL,EPS,HALF,P,ONE,Q,RESULT,SIXTEN,
C      2      SQRPI,THRSH,ROOT32,X,XLOW,XDEN,XNUM,Y,XSQ,XUPPR,ZERO
C      DIMENSION A(5),B(4),C(9),D(8),P(6),Q(5)
C-----
C Mathematical constants
C
C  SQRPI = 1 / sqrt(2*pi), ROOT32 = sqrt(32), and
C  THRSH is the argument for which anorm = 0.75.
C-----
CS      DATA ONE,HALF,ZERO,SIXTEN/1.0E0,0.5E0,0.0E0,1.60E0/,
CS      1      SQRPI/3.9894228040143267794E-1/,THRSH/0.66291E0/,
CS      2      ROOT32/5.656854248E0/
CD      DATA ONE,HALF,ZERO,SIXTEN/1.0D0,0.5D0,0.0D0,1.60D0/,
CD      1      SQRPI/3.9894228040143267794D-1/,THRSH/0.66291D0/,
CD      2      ROOT32/5.656854248D0/
C-----
C Machine-dependent constants
C-----
CS      DATA EPS/5.96E-8/,XLOW/-12.949E0/,XUPPR/5.768E0/
CD      DATA EPS/1.11D-16/,XLOW/-37.519D0/,XUPPR/8.572D0/
C-----
C Coefficients for approximation in first interval
C-----
CS      DATA A/2.2352520354606839287E00,1.6102823106855587881E02,
CS      1      1.0676894854603709582E03,1.8154981253343561249E04,
CS      2      6.5682337918207449113E-2/
CS      DATA B/4.7202581904688241870E01,9.7609855173777669322E02,
CS      1      1.0260932208618978205E04,4.5507789335026729956E04/

```

```

CD DATA A/2.2352520354606839287D00,1.6102823106855587881D02,
CD 1 1.0676894854603709582D03,1.8154981253343561249D04,
CD 2 6.5682337918207449113D-2/
CD DATA B/4.7202581904688241870D01,9.7609855173777669322D02,
CD 1 1.0260932208618978205D04,4.5507789335026729956D04/

```

```

C-----
C Coefficients for approximation in second interval

```

```

C-----
CS DATA C/3.9894151208813466764E-1,8.8831497943883759412E00,
CS 1 9.3506656132177855979E01,5.9727027639480026226E02,
CS 2 2.4945375852903726711E03,6.8481904505362823326E03,
CS 3 1.1602651437647350124E04,9.8427148383839780218E03,
CS 4 1.0765576773720192317E-8/
CS DATA D/2.2266688044328115691E01,2.3538790178262499861E02,
CS 1 1.5193775994075548050E03,6.4855582982667607550E03,
CS 2 1.8615571640885098091E04,3.4900952721145977266E04,
CS 3 3.8912003286093271411E04,1.9685429676859990727E04/
CD DATA C/3.9894151208813466764D-1,8.8831497943883759412D00,
CD 1 9.3506656132177855979D01,5.9727027639480026226D02,
CD 2 2.4945375852903726711D03,6.8481904505362823326D03,
CD 3 1.1602651437647350124D04,9.8427148383839780218D03,
CD 4 1.0765576773720192317D-8/
CD DATA D/2.2266688044328115691D01,2.3538790178262499861D02,
CD 1 1.5193775994075548050D03,6.4855582982667607550D03,
CD 2 1.8615571640885098091D04,3.4900952721145977266D04,
CD 3 3.8912003286093271411D04,1.9685429676859990727D04/

```

```

C-----
C Coefficients for approximation in third interval

```

```

C-----
CS DATA P/2.1589853405795699E-1,1.274011611602473639E-1,
CS 1 2.2235277870649807E-2,1.421619193227893466E-3,
CS 2 2.9112874951168792E-5,2.307344176494017303E-2/
CS DATA Q/1.28426009614491121E00,4.68238212480865118E-1,
CS 1 6.59881378689285515E-2,3.78239633202758244E-3,
CS 2 7.29751555083966205E-5/
CD DATA P/2.1589853405795699D-1,1.274011611602473639D-1,
CD 1 2.2235277870649807D-2,1.421619193227893466D-3,
CD 2 2.9112874951168792D-5,2.307344176494017303D-2/
CD DATA Q/1.28426009614491121D00,4.68238212480865118D-1,
CD 1 6.59881378689285515D-2,3.78239633202758244D-3,
CD 2 7.29751555083966205D-5/

```

```

C-----
X = ARG
Y = ABS(X)
IF (Y .LE. THRSH) THEN

```

```

C-----
C Evaluate anorm for |X| <= 0.66291

```

```

C-----
XSQ = ZERO
IF (Y .GT. EPS) XSQ = X * X
XNUM = A(5)*XSQ
XDEN = XSQ
DO 20 I = 1, 3

```

```

      XNUM = (XNUM + A(I)) * XSQ
      XDEN = (XDEN + B(I)) * XSQ
20    CONTINUE
      RESULT = X * (XNUM + A(4)) / (XDEN + B(4))
      RESULT = HALF + RESULT
C-----
C  Evaluate anorm for 0.66291 <= |X| <= sqrt(32)
C-----
      ELSE IF (Y .LE. ROOT32) THEN
        XNUM = C(9)*Y
        XDEN = Y
        DO 120 I = 1, 7
          XNUM = (XNUM + C(I)) * Y
          XDEN = (XDEN + D(I)) * Y
120    CONTINUE
        RESULT = (XNUM + C(8)) / (XDEN + D(8))
        XSQ = AINT(Y*SIXTEN)/SIXTEN
        DEL = (Y-XSQ)*(Y+XSQ)
        RESULT = EXP(-XSQ*XSQ*HALF)*EXP(-DEL*HALF)*RESULT
        IF (X .GT. ZERO) RESULT = ONE - RESULT
C-----
C  Evaluate anorm for |X| > sqrt(32)
C-----
      ELSE
        RESULT = ZERO
        IF ((X .GE. XLOW) .AND. (X .LT. XUPPR)) THEN
          XSQ = ONE / (X * X)
          XNUM = P(6)*XSQ
          XDEN = XSQ
          DO 240 I = 1, 4
            XNUM = (XNUM + P(I)) * XSQ
            XDEN = (XDEN + Q(I)) * XSQ
240    CONTINUE
          RESULT = XSQ *(XNUM + P(5)) / (XDEN + Q(5))
          RESULT = (SQRPI - RESULT) / Y
          XSQ = AINT(X*SIXTEN)/SIXTEN
          DEL = (X-XSQ)*(X+XSQ)
          RESULT = EXP(-XSQ*XSQ*HALF)*EXP(-DEL*HALF)*RESULT
        END IF
        IF (X .GT. ZERO) RESULT = ONE - RESULT
      END IF
C-----
C  Fix up for negative argument, erf, etc.
C-----
      ANORM = RESULT
C----- Last card of ANORM -----
      END

```

DATE

FILMED

5/2/94

END