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Numerical Solution of Volterra Integral
Equations with Weakly Singular Kernels**

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MULTIQUADRIC COLLOCATION METHODS IN THE NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQUATIONS WITH WEAKLY SINGULAR KERNELS

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ABSTRACT. Nonlinear Volterra integral equations with weakly singular kernels are considered and are solved by applying piecewise collocation methods based on multiquadric approximations combined with Gauss quadrature rules.

1. INTRODUCTION

This paper considers Volterra integral equations (VIEs) with weakly-singular kernels of the form

$$(1.1) \quad y(x) = f(x) + \int_0^x \frac{H(x, s, y(s))}{(x-s)^\alpha} ds, \quad 0 < \alpha < 1, \quad 0 \leq x \leq X,$$

where $y(x)$ is the unknown function to be determined, and $f(x)$, $H(x, s, y)$ are given smooth functions. The function H is assumed nonlinear in y .

We apply piecewise collocation methods based on multiquadrics (MQ) augmented with a constant term, combined with Gauss quadrature rules for the evaluation of the weakly-singular integrals.

Existing methods for the numerical solution of weakly-singular equations of the second kind include block-by-block methods (Linz [12], De Hoog and Weiss [9]), collocation methods in certain polynomial spline spaces with equidistant and graded meshes (cf. Brunner [3]), collocation methods in non-polynomial spaces (te Riele

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[16], Brunner and te Riele [5]), fractional linear multistep methods (Lubich [13]), rational basis functions methods (Abelman and Eyre [1]), sinc approximation methods (Riley [17]).

The plan of this paper is as follows. Section 2 gives a description of the application of the MQ approximations to equation (1.1). In §3 are given numerical results, and in §4 conclusions and comments on remaining problems.

2. DESCRIPTION OF THE METHODS

Consider the integral equation (1.1). In a piecewise collocation method we consider mesh points $0 = x_0 < x_1 < \dots < x_N = X$ and seek approximate values of the solution of (1.1) for $0 \leq x_{nj} \leq X$, where $x_{nj} = x_n + c_j h_n$, $h_n = x_{n+1} - x_n$ ($n = 0, 1, \dots, N-1$), and the collocation parameters are such that $0 \leq c_1 < \dots < c_m \leq 1$ (for more details, see for example Brunner and van der Houwen [4]).

The form of a multiquadric augmented by a constant term which interpolates a set of points (x_i, y_i) , $i = 1, 2, \dots, m$, is

$$(2.2) \quad S(x) = \sum_{j=1}^m a_j \bar{u}_j(x),$$

where

$$(2.3) \quad \bar{u}_j(x) = \begin{cases} 1 & \text{if } j = 1, \\ u_j(x) - u_1(x) & \text{if } j > 1, \end{cases}$$

$$(2.4) \quad u_j(x) = \sqrt{(x - x_j)^2 + R^2}$$

and the coefficients a_j are obtained from the solution of the linear system of equations

$$(2.5) \quad S(x_i) = y_i, \quad i = 1, 2, \dots, m.$$

In particular we shall utilize the Lagrange-type expression of a multiquadric approximation (2.2)–(2.4). The Lagrange type expression of a multiquadric approximation for a function $\phi(x)$ is of the form

$$(2.6) \quad S(x) = \sum_{\mu=1}^m l_{\mu}(x) \phi(x_{\mu})$$

where

$$(2.7) \quad l_{\mu}(x) = \sum_{\lambda=1}^m Q_{\mu\lambda} \bar{u}_{\lambda}(x),$$

and the constants $Q_{\mu\lambda}$ can be computed using the cardinality conditions

$$(2.8) \quad l_{\mu}(x_j) = \delta_{\mu j}, \quad j = 1, 2, \dots, m; \mu = 1, 2, \dots, m.$$

As it is shown in Wu Zong-Min and Schaback [18], the values of the cardinal functions $l_\mu(x)$ at a fixed point x , can be also obtained from the solution of a conditional minimization problem. For details on multiquadric interpolation, we refer for example to Hardy [8], Powell [15], Buhmann and Dyn [6], Kansa and Carlson [11] and the references therein.

Discretizing equation (1.1) at points x_{nj} we obtain the equations

$$(2.9) \quad y_{nj} = f(x_{nj}) + \sum_{i=0}^{n-1} h_i \int_0^1 \frac{H(x_{nj}, x_i + h_i s, y(x_i + h_i s))}{(x_{nj} - x_i - h_i s)^\alpha} ds \\ + h_n c_j \int_0^1 \frac{H(x_{nj}, x_n + h_n c_j s, y(x_n + h_n c_j s))}{(x_{nj} - x_n - h_n c_j s)^\alpha} ds,$$

$j = 1, 2, \dots, m; n = 0, 1, \dots, N-1$. Approximating the y values by (2.6)-(2.8) with $x_j = c_j$ we obtain

$$(2.10) \quad y_{nj} = f(x_{nj}) + \sum_{i=0}^{n-1} h_i^{1-\alpha} \int_0^1 \frac{H(x_{nj}, x_i + h_i s, \sum_{k=1}^m l_k(s) y_{ik})}{(A_{nji} - s)^\alpha} ds \\ + (h_n c_j)^{1-\alpha} \int_0^1 \frac{H(x_{nj}, x_n + h_n c_j s, \sum_{k=1}^m l_k(s) \sum_{\lambda=1}^m l_\lambda(c_j c_k) y_{n\lambda})}{(1-s)^\alpha} ds, \\ j = 1, 2, \dots, m; i = 0, 1, \dots, n-1; n = 0, 1, \dots, N-1,$$

where

$$(2.11) \quad A_{nji} = (x_n + h_n c_j - x_i)/h_i.$$

Equations (2.10) are the exact multiquadric collocation methods. The discrete ones are obtained by replacing the integrals by Gauss quadrature rules. One and two point Gauss rules have been derived in Branca [2] for the numerical solution of weakly-singular Volterra integral equations of the first kind. The abscissae and weights for any point Gauss rule with a number of different weight functions can be found in Gautschi [7]. Replacing the first and second sum of equations (2.10) with a p -point Gauss rule with weights and abscissae respectively,

$$(2.12) \quad (w_\mu^{(n,i,j)}, s_\mu^{(n,i,j)}), (\tilde{w}_\mu, \tilde{s}_\mu), \mu = 1, 2, \dots, p,$$

we obtain the discrete multiquadric equations

$$(2.13) \quad y_{nj} = f(x_{nj}) + \sum_{i=0}^{n-1} h_i^{1-\alpha} \sum_{\mu=1}^p w_\mu^{(n,i,j)} H(x_{nj}, x_i + h_i s_\mu^{(n,i,j)}, \sum_{k=1}^m l_k(s_\mu^{(n,i,j)}) y_{ik}) \\ + (h_n c_j)^{1-\alpha} \sum_{\mu=1}^p \tilde{w}_\mu H(x_{nj}, x_n + h_n c_j \tilde{s}_\mu, \sum_{k=1}^m l_k(\tilde{s}_\mu) \sum_{\lambda=1}^m l_\lambda(c_j c_k) y_{n\lambda}),$$

$$j = 1, 2, \dots, m; n = 0, 1, \dots, N - 1.$$

Another possibility which was tried is to start from equation (2.9) and approximate the H function by a multiquadric approximation of the form (2.6)-(2.8) and then use Gauss rules for the weight integrals which now are of the form

$$(2.14) \quad \int_0^1 \frac{l_r(s)}{(A_{nji} - s)^\alpha}, \quad \int_0^1 \frac{l_r(s)}{(1-s)^\alpha} ds,$$

and A_{nji} , $l_r(s)$, $r = 1, 2, \dots, m$ are given by (2.11), (2.7) respectively.

3. NUMERICAL RESULTS

Numerical results are presented for one example:

$$1) \quad y(x) = \ln(1+x) - 2x^{0.5} - \frac{4}{3}x^{1.5} + \int_0^x \frac{e^{y(s)}}{\sqrt{(x-s)}} ds, \text{ with true solution } y(x) = \ln(1+x) \text{ (constructed).}$$

The numerical results are obtained with the Radau II points ($c_1 = 1/3, c_2 = 1$). The value of R was chosen after extensive simulations with a number of different R values.

The resulting nonlinear equations were solved using routine SNSQE of Kahaner, Moler and Nash [10]. A tolerance equal to $1.E-4$ was used. The results with heading "MQ" were obtained using equations (2.13).

Table 1. Example 1, $m=2$, 2-point Gauss rule ($p=2$)

	Polynomial	Spline	Augmented	multiquadric
x	$h=0.05$	$h=0.025$	$h=0.05, R=24$	$h=0.025, R=48$
0.1	0.41 E - 5	0.33 E - 6	0.62 E - 5	0.29 E - 5
0.2	0.27 E - 4	0.39 E - 5	0.32 E - 4	0.31 E - 5
0.3	0.70 E - 4	0.11 E - 4	0.79 E - 4	0.26 E - 5
0.4	0.15 E - 3	0.24 E - 4	0.17 E - 3	0.34 E - 5
0.5	0.34 E - 3	0.54 E - 4	0.38 E - 3	0.27 E - 5
0.6	0.78 E - 3	0.12 E - 3	0.87 E - 3	0.24 E - 5
0.7	0.19 E - 2	0.31 E - 3	0.21 E - 2	0.74 E - 5
0.8	0.52 E - 2	0.84 E - 3	0.58 E - 2	0.13 E - 4
0.9	0.16 E - 1	0.28 E - 2	0.18 E - 1	0.35 E - 4
1.0	0.56 E - 1	0.86 E - 2	0.62 E - 1	0.12 E - 3

4. CONCLUDING REMARKS

In this paper the application of collocation methods based on approximations with multiquadrics to Volterra integral with weakly-singular kernel was considered. The above results show that multiquadric collocation methods can be more accurate than the polynomial spline collocation methods with the same collocation parameters. Results obtained with two more examples though, were almost identical for the two classes of methods.

More tests are needed with more examples and theoretical analysis so that to identify classes of functions where the multiquadrics approximations give considerably more accurate results in the weakly-singular case. Also tests with multiquadrics augmented by a polynomial of higher degree than that considered here. Orders of convergence at least $O(h^2)$ were observed with $m = 2$ for the Radau II points. The accuracy of the results depends on the successful choice of the multiquadric parameter R . We are continuing research on the derivation of a method for choosing R , on an error analysis for obtaining the order of convergence. It will be also interesting to see how the methods perform in solving integro-differential equations with weakly singular kernels.

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