R-MODULES FOR THE ALEXANDER COHOMOLOGY THEORY

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The Alexander Wallace Spanier cohomology theory associates with an arbitrary topological space an abelian group. In this paper, an arbitrary topological space is associated with an R-module. The construction of the R-module is similar to the Alexander Wallace Spanier construction of the abelian group.

In Chapter I, the algebraic and topological material necessary to construct the theory is listed. This material is found in most graduate level algebra or topology courses.

The definition of the cohomology R-modules is given in Chapter II. Also, it is verified that this construction does give a cohomology theory. The axioms necessary for this are proved as Theorems 1-7.

In Chapter III, additional theorems extend the theory. One of these is the Mayer-Vietoris Theorem. In this theorem, the Mayer-Vietoris sequence is shown to be exact.

Some applications of the theory are given in Chapter IV. One of the applications shows the existence of floors for each element of $H^n(X,A)$ if $X$ is compact Hausdorff, and if $A$ is a closed subset of $X$. The modules of the n-cell and the n-sphere are also given.
R-MODULES FOR THE ALEXANDER COHOMOLOGY THEORY

THESIS

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CHAPTER I

INTRODUCTION

The Alexander Wallace Spanier cohomology theory associates with an arbitrary topological space an abelian group. In this paper, an arbitrary topological space is associated with an R-module. The construction of the R-module is similar to the Alexander Wallace Spanier construction of the abelian group.

In Chapter I, the algebraic and topological material necessary to construct the theory is listed. This material is found in most graduate level algebra or topology courses.

The definition of the cohomology R-modules is given in Chapter II. Also, it is verified that this construction does give a cohomology theory. The axioms necessary for this are proved as Theorems 1-7.

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Some applications of the theory are given in Chapter IV. One of the applications shows the existence of floors for each element of $H^p(X,A)$ if $X$ is compact Hausdorff, and if $A$ is a closed subset of $X$. The modules of the n-cell and the n-sphere are also given.
Algebraic Preliminaries

The following preliminaries concerning $R$-modules will be used.

**Preliminary 1.** If $A$ and $B$ are submodules of a $R$-module $G$, then $A + B$ is a submodule of $G$ and $A \cap B$ is a submodule of $G$.

**Preliminary 2.** If $G$ and $H$ are $R$-modules, and if $f: G \to H$ and $f(ra + tb) = rf(a) + tf(b)$ for $r,t \in R$, $a,b \in G$, then $f$ is an $R$-homomorphism of $G$ into $H$. The image of a submodule of $G$ is a submodule of $H$. If $f$ is also 1-1 and onto, then $f$ is called an isomorphism of $G$ onto $H$, and we say $G$ and $H$ are isomorphic (denoted $G \cong H$).

**Preliminary 3.** The kernel of $f$ (denoted $\ker f$) is \{ $x \in G | f(x) = 0$ \}. The set $\ker f$ is a submodule of $G$.

**Preliminary 4.** Let $G$ be an $R$-module, and let $H$ be a submodule of $G$. Sets of the form $a + H$ where $a \in G$ are called cosets of $G$ mod $H$. The collection \{ $a + H | a \in G$ \} is an $R$-module under the operations "\$, "$ \$" defined as follows: for $a,b \in G$, $r \in R$, $(a + H) \$ (b + H) = (a + b) + H$ and $r \$ (a + H) = ra + H$. This collection is called the factor module of $G$ mod $H$ (denoted $G/H$).

Define $\eta: G \to G/H$ as follows: $\eta(a) = a + H$ for $a \in G$. The function $\eta$ is an $R$-homomorphism of $G$ onto $G/H$. The function $\eta$ is called the natural map.
Preliminary 5. *Fundamental Homomorphism Theorem.*

If $G$ and $H$ are $R$-modules, and $f: G \to H$ is an onto homomorphism, then there is a submodule $N$ of $G$ such that $G/N \cong H$.

Preliminary 6. If $A, B, C$ are $R$-modules, and $A \xrightarrow{f} B \xrightarrow{g} C$ where $f$ and $g$ are $R$-homomorphisms, and $h = gf$, then (1) $h$ is an $R$-homomorphism; (2) if $h$ is an isomorphism onto, then

(i) $g$ is onto,
(ii) $f$ is 1-1,
(iii) $B = f(A) + \ker g$.

Preliminary 7. *Induced Homomorphism Theorem.*

If $G, H, M, N$ are $R$-modules, $\alpha, \beta, f$ are $R$-homomorphisms, $\alpha$ is onto, and $f(\ker \alpha) \subseteq \ker \beta$, then there is a unique $R$-homomorphism $f^*: G \to M$ such that $f^* \alpha = \beta f^*$.

\[
\begin{array}{ccc}
G & \xrightarrow{f^*} & M \\
\alpha \uparrow & & \uparrow \beta \\
H & \xrightarrow{f} & N
\end{array}
\]

Definition 1.1. Let $\{M_i\}_{i=1}^{k}$ be a sequence of $R$-modules, and $\{h_i\}_{i=1}^{k}$ be a sequence of $R$-homomorphisms such that (1) $k$ is a positive integer or $\infty$; (2) for every $i$, $h_i: M_i \to M_{i+1}$; (3) $h_n$ is a 1-1 $R$-homomorphism into $M_{n+1}$; (4) for every $i$, $h_i(M_i) = \ker h_{i+1}$; and (5) if $k \neq \infty$, then $h_{k-1}$ is onto. Then the sequence $(M_i, h_i)$ is exact.
Notation

The following notation will be used.

(1) Unless confusion arises, the natural map (defined above) will be denoted by \( n \);

(2) The empty set will be denoted by "\( \emptyset \)"

(3) If \( A \subseteq X \), then the inclusion map \( i: A \to X \) defined by \( i(x) = x \) for all \( x \in A \), will be denoted by \( i: A \subseteq X \);

(4) If \( A \subseteq X \), then the closure of \( A \) will be denoted by \( \overline{A} \).

The interior of \( A \) will be denoted by \( \text{Int}(A) \);

(5) If \( f: X \to Y \), then \( \text{Im} f = \{ f(x) \in Y \mid x \in X \} \).

Topological Preliminaries

The following topological theorems will be used.

Preliminary 8. Every compact Hausdorff space is fully normal.

Preliminary 9. Let \( X \) be a compact Hausdorff space. If \( \mathcal{B} \) is a descending family of closed sets, \( U \) is an open subset of \( X \), and \( \bigcap \mathcal{B} \subseteq U \), then there exists \( F \in \mathcal{B} \) such that \( F \subseteq U \).

Preliminary 10. A space \( X \) is connected if and only if for every open cover \( \mathcal{B} \) of \( X \), and \( a, b \in X \), there exists a finite subcollection \( U_1, \ldots, U_n \) of \( \mathcal{B} \) such that \( a \in U_1 \), \( b \in U_n \), and \( U_i \cap U_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \).

Preliminary 11. Let \( X, Y \) be topological spaces. If \( f: X \to Y \), \( X \) is compact, \( Y \) is Hausdorff, \( f \) is continuous, \( f \) is \( 1-1 \), and \( f \) is onto, then \( f \) is a homeomorphism.
CHAPTER II

THE COHOMOLOGY AXIOMS

Let $X$ be a topological space, and let $M$ be an $R$-module. For $p$ a non-negative integer, let $CP(X) = \{f: X^{p+1} \to M\}$.

For $\phi_1, \phi_2 \in CP(X)$, define addition (denoted "\+") as follows:

$$(\phi_1 + \phi_2)(x_0, \ldots, x_p) = \phi_1(x_0, \ldots, x_p) + \phi_2(x_0, \ldots, x_p).$$

For $\phi \in CP(X)$, $r \in R$, define multiplication (denoted "\cdot") as follows:

$$(r \cdot \phi)(x_0, \ldots, x_p) = r\phi(x_0, \ldots, x_p).$$

Define $0 \in CP(X)$ by $0(x_0, \ldots, x_p) = 0_1$ where $0_1$ is the additive identity in $M$. For $\phi \in CP(X)$, define $-\phi \in CP(X)$ by

$$-\phi(x_0, \ldots, x_p) = -(\phi(x_0, \ldots, x_p)).$$

Then $(CP(X), +, \cdot)$ is an $R$-module and is called the $R$-module of cochains of $X$ with dimension $p$.

Define $\delta_p: CP(X) \to CP^{+1}(X)$ as follows: $\delta_p(\phi)$ is that function in $CP^{+1}(X)$ such that

$$\delta_p(\phi)(x_0, \ldots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}).$$

The hat on $x_i$ means to delete $x_i$.

**Note 2.1.** The function $\delta_p$ is an $R$-homomorphism.

**Proof:** Let $r, t \in R$, $\phi_1, \phi_2 \in CP(X)$. Then

$$\delta_p(r\phi_1 + t\phi_2)(x_0, \ldots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i (r\phi_1 + t\phi_2)(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1})$$

$$= \sum_{i=0}^{p+1} (-1)^i (r\phi_1(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) + t\phi_2(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}))$$

$$= \sum_{i=0}^{p+1} (-1)^i \phi_1(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) + \sum_{i=0}^{p+1} (-1)^i \phi_2(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}).$$
= r\delta_p \phi_1(x_0, \ldots, x_{p+1}) + t\delta_p \phi_2(x_0, \ldots, x_{p+1}). \quad \text{Hence, } \delta_p \text{ is an } R\text{-homomorphism.}

**Note 2.2.** The composition \( \delta_{p+1}\delta_p = 0 \).

Proof: Let \( \phi \in CP(X) \), and let \( (x_0, \ldots, x_{p+2}) \in X^{p+3} \).

Then \( \delta_{p+1}\delta_p \phi(x_0, \ldots, x_{p+2}) = \sum_{i=1}^{p+2} (-1)^i \delta_p \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+2}) \)

\( = \sum_{i=1}^{p+2} (-1)^i \delta_p \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+2}) \)

where

\( h_i^j = \begin{cases} 
  x_j & \text{if } j < i \\
  x_{j+1} & \text{if } j \geq i.
\end{cases} \)

If \( j < i \), and \( (-1)^{i+j} = 1 \), then

\[ \phi(h_0^0, \ldots, \hat{h}_i^j, \ldots, h_{p+1}^i) = \phi(x_0, \ldots, \hat{x}_j, \ldots, x_i, \ldots, x_{p+2}) \]

\( = \phi(h_0^{j-1}, \ldots, \hat{h}_i^{j-1}, \ldots, h_{p+1}^{j-1}) = \)

\( (-1)^{i+j-1} \phi(h_0^{j-1}, \ldots, \hat{h}_i^{j-1}, \ldots, h_{p+1}^{j-1}). \)

If \( j > i \), and \( (-1)^{i+j} = 1 \), then

\[ \phi(h_0^0, \ldots, \hat{h}_i^j, \ldots, h_{p+1}^i) = \phi(x_0, \ldots, \hat{x}_j, \ldots, x_i, \ldots, x_{p+2}) \]

\( = \phi(h_0^{j+1}, \ldots, \hat{h}_i^{j+1}, \ldots, h_{p+1}^{j+1}) = \)

\( (-1)^{i+j+1} \phi(h_0^{j+1}, \ldots, \hat{h}_i^{j+1}, \ldots, h_{p+1}^{j+1}). \)

Hence, \( \sum_{i=1}^{p+2} (-1)^i \delta_p \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+2}) = 0 \). Hence,

\( \delta_{p+1}\delta_p = 0. \)
The following notation will be used. If $A \subseteq 2^X$, then

$A^m = \{x \in X^m : \text{there exists } U \in A, \text{and } x \in U^m\}$.

**Definition 2.1.** Let $A \subseteq X$, and let $p$ be a non-negative integer. 

$CP(X,A) = \{\phi \in CP(X) : \text{there exists a cover } \beta \text{ of } A \}

by sets open in $X$ such that $\phi |_{\beta^{p+1} \cap A^{p+1}} = 0\}$.

**Note 2.3.** $CP(X, A)$ is a submodule.

**Proof:** Let $\phi_1, \phi_2 \in CP(X, A)$. Then there exists open covers

$\beta_1, \beta_2$ of $A$ such that $\phi_1 |_{\beta_1^{p+1} \cap A^{p+1}} = 0$, and

$\phi_2 |_{\beta_2^{p+1} \cap A^{p+1}} = 0$. Let $\beta = \{U \cap V \mid U \in \beta_1, V \in \beta_2, \text{and } U \cap V \neq \emptyset\}$. Let $x \in A$. Then there exists $U \in \beta_1$ and

$V \in \beta_2$ such that $x \in U$ and $x \in V$. Therefore, $x \in U \cap V \neq \emptyset$. Hence, $\beta$ is an open cover of $A$.

Let $(x_0, \ldots, x_p) \in \beta^{p+1} \cap A^{p+1}$. Then

$x_0, \ldots, x_p \in U$ for some $U \in \beta$. $U = U_1 \cap U_2$ where

$U_1 \in \beta_1$, and $U_2 \in \beta_2$. Therefore, $x_0, \ldots, x_p \in U_1$, and

$x_0, \ldots, x_p \in U_2$. Therefore, $(x_0, \ldots, x_p) \in \beta_1^{p+1} \cap A^{p+1}$, and $(x_0, \ldots, x_p) \in \beta_2^{p+1} \cap A^{p+1}$. Therefore,

$\phi_1(x_0, \ldots, x_p) = 0$ and $\phi_2(x_0, \ldots, x_p) = 0$. Thus,

$(\phi_1 - \phi_2)(x_0, \ldots, x_p) = \phi_1(x_0, \ldots, x_p) - \phi_2(x_0, \ldots, x_p) = 0$.

Hence, $\phi_1 - \phi_2 \in CP(X, A)$. Therefore, $CP(X, A)$ is an additive subgroup of $CP(X)$.

Now let $r \in R$, and let $\phi \in CP(X, A)$. There exists an

open cover $\beta$ of $A$ such that $\phi |_{\beta^{p+1} \cap A^{p+1}} = 0$. Let

$(x_0, \ldots, x_p) \in \beta^{p+1} \cap A^{p+1}$. $r\phi(x_0, \ldots, x_p) = r(\phi(x_0, \ldots, x_p)) = r(0) = 0$. Therefore, $r\phi \in CP(X, A)$. Hence, $CP(X, A)$ is a sub-

module of $CP(X)$. 

Lemma 2.1. The set $\delta_p [CP(X,A)] \subseteq CP^{+1}(X,A)$.

Proof: Let $\delta_p \phi \in \delta_p [CP(X,A)]$. Then $\phi \in CP(X,A)$. Thus, there exists an open cover $\mathcal{B}$ of $A$ such that $\phi|_{\mathcal{B}^{+1}} \cap A^{+1} = 0$.

Let $(x_0, \ldots, x_{p+1}) \in \beta^{+2} \cap A^{+1}$. Now $\delta_p \phi(x_0, \ldots, x_p) = \sum_{i \neq 0} (\sum_{i \neq 0} (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1})$. Also, $(x_0, \ldots, x_{p+1}) \in \beta^{+2}$ implies there exists $U \in \mathcal{B}$ such that $(x_0, \ldots, x_{p+1}) \in U^{+2}$.

Hence, $(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) \in U^{+1}$. Therefore,

$(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) \in \beta^{+1} \cap A^{+1}$ for $i = 0, \ldots, p+1$.

Hence, $\phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) = 0$ for $i = 0, \ldots, p+1$.

Thus, $\sum_{i \neq 0} (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) = 0$.

Therefore, $\delta_p \phi|_{\beta^{+2} \cap A^{+2}} = 0$. Therefore, $\delta_p \phi \in CP^{+1}(X,A)$.

Therefore, $\delta_p [CP(X,A)] \subseteq CP^{+1}(X,A)$.

Definition 2.2. Let $ZP(X,A) = CP(X,A) \cap \delta^{-1}CP(X,X)$.

Note 2.4. The set $ZP(X,A)$ is a submodule.

Proof: Let $\phi \in \delta^{-1}CP^{+1}(X,X)$, and let $r \in R$. Thus,

$\delta_p (\phi) \in CP^{+1}(X,X)$. Since, by Note 2.3, $CP^{+1}(X,X)$ is a submodule, $r \delta_p (\phi) \in CP^{+1}(X,X)$. By Note 2.1, $\delta_p$ is an $R$-homomorphism. Thus, $r \delta_p (\phi) = \delta_p (r \phi)$. Therefore,

$r \phi \in \delta^{-1}CP^{+1}(X,X)$.

Now let $\phi_1, \phi_2 \in \delta^{-1}CP^{+1}(X,X)$. Then

$\delta_p \phi_1, \delta_p \phi_2 \in CP^{+1}(X,X)$. Since $CP^{+1}(X,X)$ is a submodule,

$\delta_p \phi_1 - \delta_p \phi_2 \in CP^{+1}(X,X)$. Since $\delta_p$ is an $R$-homomorphism,

$\delta_p \phi_1 - \delta_p \phi_2 = \delta_p (\phi_1 - \phi_2)$. Therefore, $\phi_1 - \phi_2 \in \delta^{-1}CP^{+1}(X,X)$.

Hence, $\delta^{-1}CP^{+1}(X,X)$ is a submodule.

By Note 2.3, $CP(X,A)$ is also a submodule. Thus, by Preliminary 1, $CP(X,A) \cap \delta^{-1}CP^{+1}(X,X) = ZP(X,A)$ is a submodule.
ZP(X, A) is called the submodule of $p$-cocycles of $X \mod A$.

**Definition 2.3.** Let

$$BP(X, A) = \begin{cases} 0 & \text{if } p = 0 \\ \delta_{p-1}CP^{-1}(X, A) + CP(X, X) & \text{if } p > 0. \end{cases}$$

**Note 2.5.** $BP(X, A)$ is a submodule.

Proof: Assume $p > 0$. $CP^{-1}(X, A)$ is a submodule, and $\delta_{p-1}$ is an $R$-homomorphism. Therefore, by Preliminary 2, $\delta_{p-1}CP^{-1}(X, A)$ is a submodule. $CP(X, X)$ is a submodule. Thus, by Preliminary 1, $\delta_{p-1}CP^{-1}(X, A) + CP(X, X) = BP(X, A)$ is a submodule. If $p = 0$, then $BP(X, A) = \{0\}$ which is a submodule.

$BP(X, A)$ is called the submodule of cobounding cocycles of dimension $p$.

**Lemma 2.2.** If $B \subseteq A \subseteq X$, then $CP(X, A)$ is a subset of $CP(X, B)$.

Proof: Let $\phi \in CP(X, A)$. Then there exists an open cover $\beta$ of $A$ such that $\phi|\beta A^{p+1} \cap A^{p+1} = 0$. Since $B \subseteq A$, $\beta$ is an open cover of $B$. Let $(x_0, \ldots, x_p) \in \beta A^{p+1} \cap B^{p+1}$. Then $(x_0, \ldots, x_p) \in A^{p+1}$. Therefore, $\phi(x_0, \ldots, x_p) = 0$. Hence, $\phi|\beta A^{p+1} \cap B^{p+1} = 0$. Thus, $\phi \in CP(X, B)$. Therefore, $CP(X, A)$ is a subset of $CP(X, B)$.

**Note 2.6.** $BP(X, A) \subseteq ZP(X, A)$.

Proof: Let $\phi \in BP(X, A)$. Then $\phi = \delta_{p-1}\phi_1 + \phi_2$ where $\phi_1 \in CP^{-1}(X, A)$, and $\phi_2 \in CP(X, X)$. Now $\delta_p(\phi) = \delta_p(\delta_{p-1}\phi_1 + \phi_2)$. Since $\delta_p$ is an $R$-homomorphism,
\[ \delta_p(\delta_p^{-1}\phi_1 + \phi_2) = \delta_p\delta_p^{-1}\phi_2 + \delta_p\phi_2. \] By Note 2.2, \\
\[ \delta_p\delta_p^{-1}\phi_1 = 0. \] Therefore, \( \delta_p\phi = 0 + \delta_p\phi_2 = \delta_p\phi_2. \) Thus, \\
\[ \delta_p\phi_2 \in \delta_p^CP(X, X). \] By Lemma 2.1, \( \delta_p^CP(X, X) \subseteq \delta_p^{-1}CP^+(X, X). \) Therefore, \( \delta_p\phi_2 \in \delta_p^CP^+(X, X). \) Hence, \( \phi \in \delta_p^{-1}CP^+(X, X). \) \\
Now \( \delta_p^{-1}\phi_1 \in \delta_p^{-1}CP^-(X, A) \) which is a subset of \( CP(X, A) \) by Lemma 2.1. Thus, \( \delta_p^{-1}\phi_1 \subseteq CP(X, A). \) By Lemma 2.2, \\
\( CP(X, X) \subseteq CP(X, A). \) Thus, \( \phi_2 \subseteq CP(X, A). \) Therefore, \\
\[ \delta_p^{-1}\phi_1 + \phi_2 = \phi \in CP(X, A). \] Hence, \( \phi \in CP(X, A) \cap \delta_p^{-1}CP^+(X, X) = ZP(X, A). \) Therefore, \( BP(X, A) \subseteq ZP(X, A). \) 

**Definition 2.3.** For \( p \geq 0, HP(X, A) = \frac{ZP(X, A)}{BP(X, A)}, \) and \( HP(X, A) \) is called the \( p \)th cohomology module of \( X \) mod \( A. \) 

**Definition 2.4.** Let \( X, Y \) be topological spaces, and let \( f: X \rightarrow Y. \) Define \( f^\#: CP(Y) \rightarrow CP(X) \) by \( f^\# \phi \) is that function in \( CP(X) \) defined as follows: \\
\[ f^\# \phi(x_0, \ldots, x_p) = \phi(f(x_0), \ldots, f(x_p)). \] 

The following notation will be used. \( f: (X, A) \rightarrow (Y, B) \) means that \( f: X \rightarrow Y, A \subseteq X, B \subseteq Y, \) and \( f(A) \subseteq B. \) 

**Note 2.7.** If \( f: (X, A) \rightarrow (Y, B) \), then 

(1) \( f^\# \) is an \( R \)-homomorphism; 
(2) \( f^\# \delta_p = \delta_p f^\#; \) 
(3) if \( f \) is continuous, then \( f^\#[CP(Y, B)] \subseteq CP(X, A); \) 
(4) if \( f \) is continuous, then \( f^\#[ZP(Y, B)] \subseteq ZP(X, A); \) 
(5) if \( f \) is continuous, then \( f^\#[BP(Y, B)] \subseteq BP(X, A). \) 

**Proof of (1):** Let \( \phi_1, \phi_2 \in CP(Y), \) and let \( r, t \in R. \) \\
\[ f^\#(r\phi_1 + t\phi_2)(x_0, \ldots, x_p) = (r\phi_1 + t\phi_2)(f(x_0), \ldots, f(x_p)) = r\phi_1(f(x_0), \ldots, f(x_p)) + t\phi_2(f(x_0), \ldots, f(x_p)) = \\
r f^\# \phi_1(x_0, \ldots, x_p) + t f^\# \phi_2(x_0, \ldots, x_p). \] Hence,
$$f^\#(r\phi_1 + t\phi_2) = rf^\#\phi_1 + tf^\#\phi_2.$$ Hence, $f^\#$ is an $R$-homomorphism.

**Proof of (2):** Let $\phi \in CP(Y, B)$.

$$f^\# \delta_p \phi(x_0, \ldots, x_{p+1}) = f^\# \left[ \sum_{i=0}^{p+1} (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) \right] =$$

$$\delta_p f^\# \phi(x_0, \ldots, x_{p+1}).$$ Therefore, $f^\# \delta_p = \delta_p f^\#$.

**Proof of (3):** Let $f^\# \phi \in f^\# [CP(Y, B)]$ where

$\phi \in CP(Y, B)$. Then there is an open cover $\beta$ of $B$ such that $\phi|_{\beta^{p+1} \cap B^{p+1}} = 0$. Since $f$ is continuous, $f^{-1}(\beta)$ is an open cover of $A$. Let $(x_0, \ldots, x_p) \in f^{-1}(\beta) \cap A^{p+1}$. Then $\phi(x_0, \ldots, x_p) = \phi(f(x_0), \ldots, f(x_p))$. Since $f(A) \subseteq B$, $(f(x_0), \ldots, f(x_p)) \in f(f^{-1}(\beta)) \cap B^{p+1} = \beta^{p+1} \cap B^{p+1}$. Therefore, $\phi(f(x_0), \ldots, f(x_p)) = 0$. Therefore, $f^\# \phi(x_0, \ldots, x_p) = 0$. Thus, $f^\# \phi \in CP(X, A)$. Hence, $f^\# [CP(Y, B)] \subseteq CP(X, A)$.

**Proof of (4):** Let $f^\# \phi \in f^\# [ZP(Y, B)]$ where

$\phi \in ZP(Y, B)$. Then $\phi \in CP(Y, B)$, and $\phi \in \delta_p^{-1} CP^{p+1}(Y, Y)$. By part (3) of this note, $f^\# \phi \in CP(X, A)$, and $f^\# \delta_p \phi \in CP^{p+1}(X, X)$. By part (2), $f^\# \delta_p \phi = \delta_p f^\# \phi$. Therefore, $f^\# \phi \in \delta_p^{-1} CP^{p+1}(X, X)$. Hence, $f^\# \phi \in CP(X, A) \cap \delta_p^{-1} CP^{p+1}(X, X) = ZP(X, A)$. Therefore, $f^\# ZP(Y, B) \subseteq ZP(X, A)$.

**Proof of (5):** Let $\phi \in BP(Y, B)$. If $p = 0$, then $\phi = 0$. Since $f^\#$ is an $R$-homomorphism, $f^\# \phi = 0$. Therefore, $f^\# \phi \in BP(X, A)$. If $p > 0$, then $\phi = \delta_p^{-1} \phi_1 + \phi_2$ where $\phi_1 \in CP^{-1}(Y, B)$, and $\phi_2 \in CP(Y, Y)$. $f^\# \delta_p^{-1} \phi_1 = \delta_p^{-1} f^\# \phi_1$. Therefore, $f^\# \phi \in BP(X, A)$. If $p > 0$, then $\phi = \delta_p^{-1} \phi_1 + \phi_2$ where $\phi_1 \in CP^{-1}(Y, B)$, and $\phi_2 \in CP(Y, Y)$. $f^\# \delta_p^{-1} \phi_1 = \delta_p^{-1} f^\# \phi_1$. Therefore, $f^\# \phi \in BP(X, A)$.
Since \( f^\#[C^P_1(Y, B)] \subseteq C^P_1(X, A) \), \( f^\# \phi \in C^P_1(X, A) \).

Therefore, \( \delta p^{-1}f^\# \phi \in \delta p^{-1}C^P_1(X, A) \). Since \( \phi_2 \in CP(Y, Y) \), \( f^\# \phi_2 \in CP(X, X) \). Therefore, \( f^\# \delta p^{-1} \phi_1 + f^\# \phi_2 \in \delta p^{-1}C^P_1(X, A) + CP(X, X) \). By part (1), \( f^\# \delta p^{-1} \phi_1 + f^\# \phi_2 = f^\# (\delta p^{-1} \phi_1 + \phi_2) = f^\# \phi \). Therefore, \( f^\# \phi \in BP(X, A) \). Hence, \( f^\# [BP(Y, B)] \subseteq BP(X, A) \).

Applying the Induced Homomorphism Theorem to Note 2.7 shows that \( f^\# \) induces a unique \( R \)-homomorphism \( f^* \):

\[
H^P(X, A) \rightarrow H^P(Y, B). \quad \text{Also, } f^*n = nf^\#.
\]

\[
\begin{array}{ccc}
H^P(X, A) & \xrightarrow{f^*} & H^P(Y, B) \\
\uparrow n & & \uparrow n \\
Z^P(X, A) & \xrightarrow{f^\#} & Z^P(Y, B)
\end{array}
\]

**Theorem 1.** Let \( f: (X, A) \rightarrow (X, A) \) be the identity map. Then \( f^*: H^P(X, A) \rightarrow H^P(X, A) \) is the identity isomorphism.

**Proof:** \( f^*: CP(X, A) \rightarrow CP(X, A) \). For \( \phi \in CP(X, A) \),

\[
 f^\# \phi(x_0, \ldots, x_p) = \phi(f(x_0), \ldots, f(x_p)) = \phi(x_0, \ldots, x_p). \quad \text{Hence, } f^\# \phi = \phi. \quad \text{Thus, } f^\# \text{ is the identity map. \ Let } \phi \in H^P(X, A) .
\]

Since \( n \) is onto, \( \phi = n\phi_1 \) for some \( \phi_1 \in Z^P(X, A) \). Thus,

\[
f^*\phi = f^*n\phi_1 = nf^\#\phi_1 = n\phi_1 = \phi. \quad \text{Hence, } f^* \text{ is the identity isomorphism.}
\]

**Theorem 2.** If \( (X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C) \), and if \( f, g \) are continuous, then \( f^*g^* = (gf)^* \).

**Proof:** First, it will be shown that \( f^#g^# = (gf)^# \). Let \( \phi \in Z^P(Z, C) \). For \( (x_0, \ldots, x_p) \in AP \),
\( f\#^\phi(x_0, \ldots, x_p) = g\#^\phi(f(x_0), \ldots, f(x_p)) \\
= \phi(gf(x_0), \ldots, gf(x_p)) \\
= (gf)\#^\phi(x_0, \ldots, x_p). \)

Hence, \( f\#^g = (gf)^\#. \)

Now to prove the theorem, let \( h \in HP(Z, C). \) Then
\( h = \eta^\phi \) for some \( \phi \in Z^P(Z, C). \) Thus,
\( f^\#^g^\#^h = f^\#^g^\#^\eta^\phi = f^\#^\eta^\phi = \eta^\phi^\#^f^\#^g^\phi = \eta(gf)^\#^\phi = (gf)^\#^\eta^\phi = (gf)^\#^h. \) Hence,
\( f^\#^g^\# = (gf)^\#. \)

**Corollary 2.1.** If \( f: (X, A) \to (Y, B) \) is an onto homeomorphism, then \( f^*: HP(Y, B) \to HP(X, A) \) is an onto isomorphism.

**Proof:** Since \( f \) is a homeomorphism, \( f \cdot f^{-1}: (Y, B) \to (Y, B) \) is the identity map. Therefore, by Theorem 1, \( (f \cdot f^{-1})^* \) is the identity isomorphism. By Theorem 2, \( (f \cdot f^{-1})^* = f^{-1} \cdot f^* \) since \( f \) and \( f^{-1} \) are continuous. Now \( f^{-1} \cdot f^* \) is an onto isomorphism so by Preliminary 6, \( f^* \) is 1-1. Similarly, \( (f^{-1} \cdot f)^* = f^* \cdot f^{-1} \cdot f^* \) is the identity isomorphism. Thus, by Preliminary 6, \( f^* \) is onto. Hence, \( f^*: HP(Y, B) \to HP(X, A) \) is an onto isomorphism.
Let $B \subseteq A \subseteq X$. Let $i: A \to X$. Then $i^\#: C^p(X) \to C^p(A)$.

Consider the following diagram.

$\begin{align*}
H^p(A,B) & \quad H^{p+1}(X,A) \\
\eta^+ & \quad \eta^+ \\
Z^p(A,B) & \quad Z^{p+1}(X,A) \\
\cap & \quad \cap \\
C^p(A) & \quad C^{p+1}(X) \\
i^\# & \quad \delta^+ \\
C^p(X)
\end{align*}$

**Note 2.8.** (1) The function $i^\#$ is onto. (2) If $h \in H^p(A,B)$, and $\phi \in i^# - 1\eta^{-1}(h)$, then $\delta_p\phi \in Z^{p+1}(X,A)$.

Proof of (1): Let $\phi \in C^p(A)$. Define $\phi' \in C^p(X)$ as follows:

$$\phi'(x_0, \ldots, x_p) = \begin{cases} 
\phi(x_0, \ldots, x_p) & \text{if } (x_0, \ldots, x_p) \in \mathbb{A}^{p+1} \\
0 & \text{otherwise.}
\end{cases}$$

$i^#\phi' \in C^p(A)$. Let $(x_0, \ldots, x_p) \in \mathbb{A}^{p+1}$. Then $i^#\phi'(x_0, \ldots, x_p) = \phi'(i(x_0), \ldots, i(x_p)) = \phi'(x_0, \ldots, x_p) = \phi(x_0, \ldots, x_p)$. Therefore, $i^#\phi' = \phi$.

Hence, $i^#$ is onto.

Proof of (2): Let $h \in H^p(A,B)$, and let $\phi \in i^# - 1\eta^{-1}(h)$. Then $i^#\phi \in Z^p(A,B)$. Therefore, $i^#\phi \in \delta^{-1}\mathbb{C}^{p+1}(A,A)$. Therefore, $\delta_p i^#\phi \in C^{p+1}(A,A)$.

Hence, there exists a cover $\beta$ of $A$ by sets open in $A$ such that $\delta_p i^#\phi|_{\beta^{p+2} \cap \mathbb{A}^{p+2}} = 0$.

Let $\beta' = \{ U \subseteq X \mid U \cap A \in \beta, U \text{ is open in } X \}$. $\beta'$ is a cover of $A$ by sets open in $X$. Let $(x_0, \ldots, x_{p+1}) \in \beta'^{p+2} \cap \mathbb{A}^{p+2}$. Then there exists
U ∈ β such that \((x_0, \ldots, x_{p+1}) ∈ \cup B^+\). Thus,

\((x_0, \ldots, x_{p+1}) ∈ (U \cap A)^{p+2}. \ U \cap A ∈ β. \ Therefore,

\((x_0, \ldots, x_{p+1}) ∈ β^{p+2} ∩ A^{p+2}; \ \ δ_p(x_0, \ldots, x_{p+1}) =

δ_p(i(x_0), \ldots, i(x_{p+1}) = i_\#\ δ_p(x_0, \ldots, x_{p+1}) =

δ_{p+1} i_\#(x_0, \ldots, x_{p+1}) = 0. \ Hence, \ δ_p B^{p+2} ∩ A^{p+2} = 0. \ Therefore, \ δ_p ∈ C^{p+1}(X, A).

By Note 2.2, \ δ_{p+1} δ_p = 0. \ Therefore, \ δ_{p+1} δ_p ∈ C^{p+2}(X, X). \ Thus, \ δ_p ∈ \ δ_{p+1} C^{p+2}(X, X).

Therefore, \ δ_p ∈ C^{p+1}(X, A) ∩ \ δ_{p+1} C^{p+2}(X, X) = Z^{p+1}(X, A).

Define \ δ: H^p(A, B) → H^{p+1}(X, A) by \ δ(h) = \ η δ_p φ \ where \ φ ∈ i^\#_{-1} n^\#_{-1}(h).

Note 2.9. The function \ δ is well-defined.

Proof: Let \ h_1, h_2 ∈ H^p(A, B) such that \ h_1 = h_2. \ Since \ η and \ i^\# \ are onto, there exists \ φ_1, φ_2 ∈ C^p(X) such that \ η i^\# φ_1 = h_1,

and \ η i^\# φ_2 = h_2. \ Now \ δ(h_1) = η δ_p φ_1, \ and \ δ(h_2) = η δ_p φ_2. \ Since \ η i^\# φ_1 = η i^\# φ_2, \ i^\# φ_1 - i^\# φ_2 ∈ B^p(A, B). \ Two cases will be considered.

Case I: \ p = 0. \ Then \ i^\# φ_1 - i^\# φ_2 = i^\#(φ_1 - φ_2) = 0.

Now \ i^\#(φ_1 - φ_2) ∈ C^0(A). \ X is an open cover of A. \ Let

\((x_0) ∈ X \cap A. \ Then \ (φ_1 - φ_2)(x_0) = (φ_1 - φ_2)(i(x_0)) =

i^\#(φ_1 - φ_2)(x_0) = 0. \ Therefore, \ φ_1 - φ_2 | X \cap A = 0. \ Therefore,

\ φ_1 - φ_2 ∈ C^0(X, A). \ Hence, \ δ_0(φ_1 - φ_2) ∈ δ_0 C^0(X, A).

Also, \ δ_0(φ_1 - φ_2) = δ_0 φ_1 - δ_0 φ_2 ∈ B^1(X, A). \ Thus,

\ η δ_0 φ_1 = η δ_0 φ_2. \ Hence, \ δ(h_1) = δ(h_2).

Case II: \ p > 0. \ Since \ i^\#(φ_1 - φ_2) ∈ B^p(A, B), \ i^\#(φ_1 - φ_2) = δ_{p-1} α_1 + α_2 \ where \ α_1 ∈ C^{p-1}(A, B), and
\[ \alpha_2 \in CP(A,A). \] Since \( CP^{-1}(A,B) \subseteq CP^{-1}(A), \alpha_1 \in CP^{-1}(A). \)

Therefore, \( \alpha_1 = i^# \alpha_1' \) where \( \alpha_1' \in CP^{-1}(X). \) Thus, \( \delta_{p-1} \alpha_1 = \delta_{p-1} i^# \alpha_1' = i^# \delta_{p-1} \alpha_1'. \) Therefore, \( i^#(\phi_1 - \phi_2) = i^# \delta_{p-1} \alpha_1' + \alpha_2. \) So, \( i^#(\phi_1 - \phi_2) - i^# \delta_{p-1} \alpha_1' = i^#(\phi_1 - \phi_2 - \delta_{p-1} \alpha_1') = \alpha_2 \in CP(A,A). \) Therefore, there exists a cover \( \beta \) of \( A \) by sets open in \( A \) such that \( i^#(\phi_1 - \phi_2 - \delta_{p-1} \alpha_1')(x_0, \ldots, x_p) = 0. \) Let \( \beta' = \{ U \subseteq X | U \cap A \in \beta \}; \beta' \) is a cover of \( A \) by sets open in \( X. \) Let \( (x_0, \ldots, x_p) \in \beta' \cap AP^+1. \) Then \( (x_0, \ldots, x_p) \in UP^+1 \) for some \( U \in \beta'. \) Since \( (x_0, \ldots, x_p) \in AP^+1, (x_0, \ldots, x_p) \in (U \cap A)^{P+1}. \) Now \( U \cap A \in \beta. \) Therefore, \( (x_0, \ldots, x_p) \in \beta' \cap AP^+1. \) Therefore, \( (\phi_1 - \phi_2 - \delta_{p-1} \alpha_1')(x_0, \ldots, x_p) = 0. \) Thus, \( (\phi_1 - \phi_2 - \delta_{p-1} \alpha_1')(\beta' \cap AP^+1) = 0. \) Hence, \( \phi_1 - \phi_2 - \delta_{p-1} \alpha_1' \in CP(X,A). \) So, \( \delta_{p}(\phi_1 - \phi_2 - \delta_{p-1} \alpha_1') \in CP(X,A). \) Since \( \delta_{p}(\phi_1 - \phi_2 - \delta_{p-1} \alpha_1') = \delta_{p} \phi_1 - \delta_{p} \phi_2 - \delta_{p} \delta_{p-1} \alpha_1' = \delta_{p} \phi_1 - \delta_{p} \phi_2 + 0, \delta_{p} \phi_1 - \delta_{p} \phi_2 + 0 \in \delta_{p} CP(X,A) + CP^+1(X,X) = BP^+1(X,A). \) Therefore, \( \delta_{p} \phi_1 = \delta_{p} \phi_2. \) Hence, \( \delta(h_1) = \delta(h_2). \)

In any case, \( \delta(h_1) = \delta(h_2). \) Hence, \( \delta \) is well-defined.

**Note 2.10.** The function \( \delta \) is an R-homomorphism.

**Proof:** Let \( h_1, h_2 \in HP(A,B), \) and let \( r_1, r_2 \in R. \) By definition, \( \delta(h_1) = \eta \delta \phi_1 \) where \( \phi_1 \in i^# \eta^{-1} h_1, \) and \( \delta(h_2) = \eta \delta \phi_2 \) where \( \phi_2 \in i^# \eta^{-1} h_2. \) Then \( r_1 \delta h_1 + r_2 \delta h_2 = r_1 \eta \delta \phi_1 + r_2 \eta \delta \phi_2. \) Since \( \eta \) and \( \delta \) are R-homomorphisms,
\[ r_1 \delta_p \phi_1 + r_2 \delta_p \phi_2 = \delta_p (r_1 \phi_1 + r_2 \phi_2) \]

Since \( i^# \) is also an \( R \)-homomorphism,

\[ \eta \delta_p (r_1 \phi_1 + r_2 \phi_2) = \eta (r_1 i^# \phi_1 + r_2 i^# \phi_2) = r_1 \eta i^# \phi_1 + r_2 \eta i^# \phi_2 = r_1 h_1 + r_2 h_2. \]

Therefore,

\[ r_1 \phi_1 + r_2 \phi_2 \in i^{-1} \eta^{-1} (r_1 h_1 + r_2 h_2). \]

Therefore,

\[ \delta (r_1 h_1 + r_2 h_2) = \delta_p (r_1 \phi_1 + r_2 \phi_2) = r_1 \delta h_1 + r_2 \delta h_2. \]

Hence, \( \delta \) is an \( R \)-homomorphism.

The following notation will be used in Theorem 3.

Let \( f: X \to X' \). Let \( B \subseteq A \subseteq X, f(A) \subseteq A', \) and \( f(B) \subseteq B' \). Then \( f: (X,A,B) \to (X',A',B') \).

**Theorem 3.** In the following diagram, \( \delta f_3 \ast = f_1 \ast \delta \).

\[
\begin{array}{ccc}
P^0(A,B) & \xrightarrow{f_3 \ast} & P^0(X,A) \\
\downarrow{\delta} & & \downarrow{\delta} \\
P^0(A',B') & \xrightarrow{f_1 \ast} & P^0(X',A')
\end{array}
\]

**Proof:** Let \( i_1: A \xrightarrow{\subseteq} X \), and let \( i_2: A' \xrightarrow{\subseteq} X' \). Then

\( i_1^\ast: CP(X) \to CP(A), i_2^\ast: CP(X') \to CP(A') \).

Let \( h \in H^p(A',B') \). Then \( h = \eta i_2^\ast \phi \) for some \( \phi \in CP(X') \).

Now \( \delta(h) = \eta \delta_p \phi \). So, \( f_1^\ast \delta(h) = f_1^\ast \eta \delta_p \phi = \eta f_1^\ast \delta_p \phi \).

Let \( (x_0, \ldots, x_p) \in CP(A) \). Then \( i_1^\ast f_1^\ast \phi(x_0, \ldots, x_p) = f_1^\ast \phi(i_1(x_0), \ldots, f_i(x_p)). \) Since \( x_i \in A \) for \( 0 \leq i \leq p \),

\( f_1(x_i) = f_3(x_1) \) for \( 0 \leq i \leq p \). Therefore,

\( \phi(f_1(x_0), \ldots, f_1(x_p)) = \phi(f_3(x_0), \ldots, f_3(x_p)) \). Since

\( f_3(A) \subseteq A' \), and \( i_2: A' \to X' \),

\( \phi(f_3(x_0), \ldots, f_3(x_p)) = \phi(i_2 f_3(x_0), \ldots, i_2 f_3(x_p)) = i_2^\ast \phi(f_3(x_0), \ldots, f_3(x_p)) = f_3 i_2^\ast \phi(x_0, \ldots, x_p). \) Hence,
Therefore, \( f_3 \circ h = f_3 \circ i_2 \circ \phi = n_3 \circ i_2 \circ \phi = n_1 \circ f_1 \circ \phi \).

Since \( f_1 \circ \phi \in i_1 \Delta^{-1} n^{-1}(f_3 \circ h) \),
\( \delta(f_3 \circ h) = n_3 \circ f_1 \circ \phi = n_1 \circ \delta \circ \phi = f_1 \circ \delta(h) \).
Hence, \( \delta f_3 \circ h = f_1 \circ \delta \).

**Lemma 2.3.** Let \( B \subseteq A \subseteq X \), and let \( j : (X, B) \xrightarrow{i} (X, A) \).
Then \( j^* : C^P(X, A) \rightarrow C^P(X, B) \) is 1-1 for all \( p \).

**Proof:** Let \( j^* \phi_1 = j^* \phi_2 \) for \( \phi_1, \phi_2 \in C^P(X, A) \).
Let \((x_0, \ldots, x_p) \in X^{p+1} \). Then
\( \phi_1(j(x_0), \ldots, j(x_p)) = j^* \phi_1(x_0, \ldots, x_p) = j^* \phi_2(x_0, \ldots, x_p) = \phi_2(j(x_0), \ldots, j(x_p)) = \phi_2(x_0, \ldots, x_p) \).
Therefore, \( \phi_1 = \phi_2 \). Hence, \( j^* \) is 1-1.

**Theorem 4.** Let \( A \subseteq X \), and let \( j : (X, \Box) \xrightarrow{i} (X, A) \),
and let \( i : A \subseteq X \).
The sequence
\[
\begin{array}{ccccccc}
H^0(X, A) & \xrightarrow{i^*} & H^0(X, \Box) & \xrightarrow{j^*} & H^0(A, \Box) & \xrightarrow{\delta} & H^1(X, A) & \xrightarrow{j^*} & H^1(X, \Box) & \xrightarrow{i^*} & H^1(A, \Box) & \xrightarrow{\delta} & \ldots
\end{array}
\]
is exact.

**Proof:** Let \( h \in H^P(X, A) \) for any \( p \geq 0 \). Then \( j^* h \in \text{Im} i^* \).
Now \( h = n \phi \) for some \( \phi \in Z^P(X, A) \). Therefore, \( \phi \in C^P(X, A) \).
Thus, there exists an open cover \( \beta \) of \( A \) such that
\( \phi |_{\beta^{p+1} \cap A^{p+1}} = 0 \). Now \( i^* j^* h = i^* (j^* n \phi) = i^* (n j^* \phi) = n i^* j^* \phi \).
Let \((x_0, \ldots, x_p) \in \beta^{p+1} \cap A^{p+1} \).
Then
\[
i^* j^* \phi(x_0, \ldots, x_p) = i^* \phi(j(x_0), \ldots, j(x_p)) = i^* \phi(x_0, \ldots, x_p) = \phi(i(x_0), \ldots, i(x_p)) = \phi(x_0, \ldots, x_p) = 0.
\]
Hence,
\[
i^* j^* \phi |_{\beta^{p+1} \cap A^{p+1}} = 0.
\]
Therefore, \( i^* j^* \phi \in C^P(A, A) \).
Therefore, \( 0 + i^* j^* \phi \in \delta_{p-1} C^{p-1}(A, \Box) + C^P(A, A) = B^P(A, \Box) \).
Hence, \( \eta(0 + i^* j^* \phi) = \eta i^* j^* \phi = i^* \eta j^* \phi = i^* j^* n \phi = i^* j^*(h) = 0 \).
Therefore, \( j^* h \in \ker i^* \). Hence, \( \text{Im} i^* \subseteq \ker i^* \) for any \( p \).
Let $h \in \ker i^*$ for some $h \in H^p(X,\emptyset)$. Then $h = \eta \phi$ for some $\phi \in Z^p(X,\emptyset)$, and $i^*(h) = 0$. Two cases will be considered.

Case I: $p = 0$. So, $\phi \in C^0(X,\emptyset)$. $X$ is an open cover of $A$. Let $(x_0) \in X \cap A$. Then

$\phi(x_0) = \phi(i(x_0)) = i^\# \phi(x_0)$. Since $i^*(h) = 0$,

$i^\# \phi \in B^0(X,\emptyset)$. Thus, $i^\# \phi = 0$. Hence, $i^\# \phi(x_0) = \phi(x_0) = 0$. Therefore, $\phi|X \cap A = 0$. Hence, $\phi \in C^0(X,A)$.

Since $\phi \in Z^0(X,\emptyset)$, $\delta_0 \phi \in C^1(X,X)$. Therefore,

$\phi \in C^0(X,A) \cap \delta_0^{-1}C^1(X,X) = Z^0(X,A)$. Thus, $\eta \phi \in H^0(X,A)$.

Now $j^*(\eta \phi) = \eta j^\# \phi = \eta \phi = h$. Therefore, $h \in \text{Im } j^*$.

Case II: $p > 0$. Since $i^*(h) = 0$, $i^\# \phi \in B^p(A,\emptyset)$. Therefore, $i^\# \phi = \delta_p^{-1} \alpha_1 + \alpha_2$ where $\alpha_1 \in C_{p-1}^1(A,\emptyset)$, and $\alpha_2 \in C^p(A,A)$. Since $i^\#$ is onto, there exists $\overline{\alpha_1} \in C^p_{p-1}(X,\emptyset)$ such that $i^\# \overline{\alpha_1} = \alpha_1$. Therefore, $i^\# \phi = \delta_p^{-1} \alpha_1 + \alpha_2 = \delta_p^{-1} i^\# \overline{\alpha_1} + \overline{\alpha_2}$. Therefore,

$\alpha_2 = i^\# \phi - i^\# \delta_p^{-1} \alpha_1 = i^\# (\phi - \delta_p^{-1} \alpha_1) \in C^p(A,A)$. Hence, there exists a cover $\beta$ of $A$ by sets open in $A$ such that $i^\# (\phi - \delta_p^{-1} \alpha_1)|\beta^{p+1} \cap A^{p+1} = 0$. Let $\beta_0 = \{U \subseteq X| U \cap A \in \beta\}$. $\beta_0$ is a cover of $A$ by sets open in $X$. Let

$(x_0,\ldots,x_p) \in \beta_0^{p+1} \cap A^{p+1}$. Then for some $U \in \beta_0$,

$(x_0,\ldots,x_p) \in U^{p+1}$. Since $U \cap A \in \beta$, and

$(x_0,\ldots,x_p) \in (U \cap A)^{p+1} \cap A^{p+1}$, $(x_0,\ldots,x_p) \in \beta^{p+1} \cap A^{p+1}$. Therefore, $(\phi - \delta_p^{-1} \alpha_1)(x_0,\ldots,x_p) = (\phi - \delta_p^{-1} \alpha_1)(i(x_0),\ldots,i(x_p)) = i^\# (\phi - \delta_p^{-1} \alpha_1)(x_0,\ldots,x_p) = 0$. Therefore,

$\phi - \delta_p^{-1} \alpha_1|\beta_0^{p+1} \cap A^{p+1} = 0$. Hence, $\phi - \delta_p^{-1} \alpha_1 \in C^p(X,A)$.

Now $\delta_p (\phi - \delta_p^{-1} \alpha_1) = \delta_p \phi - \delta_p \delta_p^{-1} \alpha_1 = \delta_p \phi$. Since
\( \phi \in Z^p(X, \Box) \), \( \delta_p \phi \in C^{p+1}(X, X) \). Therefore,
\[ \phi - \delta_{p-1}a_1 \in \delta_p^{-1}C^{p+1}(X, X) \]. Hence, \( \phi - \delta_{p-1}a_1 \in Z^p(X, A) \).
Since \( \overline{a_1} \in C^{-1}(X, \Box) \), \( j^\# \overline{a_1} \in C^{-1}(X, \Box) \) and
\[ \delta_{p-1}j^\# \overline{a_1} \in \delta_{p-1}C^{-1}(X, \Box) \]. Therefore,
\[ \delta_{p-1}j^\# \overline{a_1} = 0 \in B^p(X, \Box) \]. Thus, \( \eta \delta_{p-1}j^\# \overline{a_1} = 0 \). Now
\[ j^* [\eta (\phi - \delta_{p-1}a_1)] = nj^\# (\phi - \delta_{p-1}a_1) = nj^\# \phi - nj^\# \delta_{p-1}a_1 = nj^\# \phi - 0 = \eta \phi = h. \] Therefore, \( h \in \text{Im } j^* \).

In any case \( h \in \text{Im } j^* \). Therefore, \( \ker i^* \subseteq \text{Im } j^* \).

Hence, \( \ker i^* = \text{Im } j^* \) for all \( p \).

Let \( i^*h \in \text{Im } i^* \) for \( h \in H^p(X, \Box) \). Then \( h = \eta \phi \) for some \( \phi \in Z^p(X, \Box) \). Therefore, \( \delta i^*h = \delta i^* \eta \phi = \delta i^* \phi = \eta \delta_p \phi \). Since \( \phi \in Z^p(X, \Box) \), \( \delta_p \phi \in C^{p+1}(X, X) \). Therefore,
\[ 0 + \delta_p \phi \in B^{p+1}(X, A) \]. So, \( \eta (0 + \delta_p \phi) = \eta \delta_p \phi = \delta i^*h = 0 \).
Therefore, \( i^*h \in \ker \delta \). Hence, \( \text{Im } i^* \subseteq \ker \delta \) for all \( p \geq 0 \).

Let \( h \in \ker \delta \) for some \( h \in H^p(A, \Box) \). Then \( \delta h = \eta \delta_p \phi = 0 \) for some \( \phi \in i^{-#1\Box}h \). Therefore, \( \delta_p \phi \in B^{p+1}(X, A) \).
Thus, \( \delta_p \phi = \delta_p a_1 + a_2 \) where \( a_1 \in C^p(X, A) \), and
\( a_2 \in C^{p+1}(X, X) \). Therefore, \( \delta_p \phi - \delta_p a_1 = \delta_p (\phi - a_1) = a_2 \in C^{p+1}(X, X) \). So, \( \phi - a_1 \in \delta_p^{-1}C^{p+1}(X, X) \). Since
\( \phi \in i^{-#1\Box}h, \phi \in C^p(X, \Box) \). Since \( a_1 \in C^p(X, A) \),
\( a_1 \in C^p(X, \Box) \). Therefore, \( \phi - a_1 \in \text{Im } C^p(X, \Box) \). Hence,
\( \phi - a_1 \in Z^p(X, \Box) \). Thus, \( \eta (\phi - a_1) \in H^p(X, \Box) \).

Consider \( \eta i^# a_1 \). Since \( a_1 \in C^p(X, A) \), there is an open
cover \( \beta \) of \( A \) such that \( a_1 | \beta^{p+1} \cap A^{p+1} = 0 \). If
\( (x_0, \ldots, x_p) \in \beta^{p+1} \cap A^{p+1} \), then
\[ i^#a_1(x_0, \ldots, x_p) = a_1(i(x_0), \ldots, i(x_p)) = a_1(x_0, \ldots, x_p) = 0. \]
Therefore, \( i^#a_1 \mid \beta^{p+1} \cap \alpha^{p+1} = 0 \). Hence, \( i^#a_1 \in CP(A, A) \).
Therefore, \( 0 + i^#a_1 \in BP(A, \square) \). Thus, \( n(0 + i^#a_1) = n^#a_1 = 0 \).

Now \( i^*[n(\phi - a_1)] = n^#(\phi - a_1) = n^#(\phi) - n^#a_1 = n^#\phi - 0 = \phi - 0 = \phi \). Hence, \( h \in Im i^* \). Therefore, \( ker \delta \subseteq Im i^* \). Therefore, \( ker \delta = Im i^* \) for all \( p \geq 0 \).

Let \( \delta h \in Im \delta \) for some \( h \in HP(A, \square) \). Then
\[ \delta h = n^#_{\beta^p} \phi \] for some \( \phi \in i^#-1 \delta^{-1}h \). Then
\[ j^*\delta h = j^*n^#_{\beta^p} \phi = n^#_{\delta^p} j^#\phi. \]
Since \( \phi \in CP(X) \), \( j^#\phi \in CP(X, \square) \).
Therefore, \( \delta^p j^#\phi \in \delta^p CP^{p+1}(X, \square) \). Therefore,
\[ \delta^p j^#\phi + 0 \in BP^{p+1}(X, \square). \]
Hence,
\[ n(\delta^p j^#\phi + 0) = n^#_{\delta^p} j^#\phi = j^*\delta h = 0. \]
Thus, \( \delta h \in ker j^* \). Hence, \( Im \delta \subseteq ker j^* \).

Let \( h \in ker j^* \) for some \( h \in HP(X, A) \). Then \( h = n^#\phi \) for some \( \phi \in ZP(X, A) \), and
\[ j^#h = j^*n^#\phi = n^#j^#\phi = 0. \]
Therefore, \( j^#\phi \in BP(X, \square) \). Thus,
\[ j^#\phi = \delta_{p-1}a_1 + a_2 \]
where \( a_1 \in CP^{-1}(X, \square) \), and \( a_2 \in CP(X, X) \). Let \( (x_0, \ldots, x_p) \in \pi^{p+1} \). Then
\[ \delta_{p-1}a_1(x_0, \ldots, x_p) = \delta_{p-1}a_1(j(x_0), \ldots, j(x_p)) = j^#\delta_{p-1}a_1(x_0, \ldots, x_p), \]
and
\[ a_2(x_0, \ldots, x_p) = a_2(j(x_0), \ldots, j(x_p)) = j^#a_2(x_0, \ldots, x_p). \]
Thus, \( j^#\delta_{p-1}a_1 = \delta_{p-1}a_1 \), and \( j^#a_2 = a_2 \). Therefore,
\[ j^#\phi = j^#\delta_{p-1}a_1 + j^#a_2 = j^#(\delta_{p-1}a_1 + a_2). \]
By Lemma 2.3, \( j^# \) is 1-1. Therefore, \( \phi = \delta_{p-1}a_1 + a_2 \). Thus, \( h = n^#\phi = n(\delta_{p-1}a_1 + a_2) = n^#\delta_{p-1}a_1 + na_2 \).
Since \( a_2 \in CP(X, X) \), \( a_2 \in BP(X, A) \). Therefore, \( na_2 = 0 \). Thus,
\[ h = n^#\delta_{p-1}a_1 = \delta(n^#a_1). \]
Therefore, \( h \in Im \delta \). So, \( ker j^* \subseteq Im \alpha \). Hence, \( ker j^* = Im \delta \), for all \( p \geq 0 \).
To show $j^*: H^0(X,A) \to H^0(X,\square)$ is 1-1, let
\[ j^*(h) = 0 \text{ for some } h \in H^0(X,A). \]
Then $h = \eta \phi$ for some $\phi \in Z^0(X,A)$. Now
\[ j^*(h) = j^*\eta\phi = \eta j^*\phi = 0. \]
Therefore, $j^*\phi \in B^0(X,\square) = 0$. Thus, $j^*\phi = 0$. By Lemma 2.3, $j^*$ is 1-1. Hence, $\phi = 0$. Therefore, $\eta \phi = h = 0$. Hence,
\[ j^*: H^0(X,A) \to H^0(X,\square) \text{ is 1-1}. \]
Therefore, by Definition 1.1, the sequence is exact.

**Corollary 2.2.** If $H^p(X,A) = 0$ for all $p \geq 0$, then $i^*$ is an isomorphism of $H^p(X,\square)$ onto $H^p(A,\square)$.

**Proof:** Since $H^p(X,A) = 0$, $\text{Im } j^* = 0$. By Theorem 4,
\[ \text{Im } j^* = \ker i^*. \]
Therefore, $\ker i^* = 0$. Hence, $i^*$ is 1-1.

Let $\phi \in H^p(A,\square)$. Then $\delta\phi \in H^{p+1}(X,A)$. By hypothesis,
\[ H^{p+1}(X,A) = 0. \]
Thus, $\delta\phi = 0$. Hence, $\phi \in \ker \delta = \text{Im } i^*$.

Hence, $i^*$ is onto. Therefore, $i^*$ is an isomorphism.

**Theorem 4A.** Let $B \subseteq A \subseteq X$. Let $m: (A,B) \xrightarrow{\cong} (X,B)$,
\[ n: (X,B) \xrightarrow{\cong} (X,A), \]
\[ j: (A,\square) \xrightarrow{\cong} (A,B), \]
and $i: A \xrightarrow{\cong} X$. Let
\[ \delta = j^*. \]
Then the sequence 
\[ \ldots H^{p-1}(X,B) \xrightarrow{m^*} H^{p-1}(A,B) \xrightarrow{\delta} \]
\[ H^p(X,A) \xrightarrow{n^*} H^p(X,B) \xrightarrow{m^*} H^p(A,B) \]
\[ \ldots \]
is exact.

**Proof:** Let $m^*h \in \text{Im } m^*$ for some $h \in H^{p-1}(X,B)$. Then
\[ h = \eta \phi \text{ for some } \phi \in Z^{p-1}(X,B). \]
Thus,
\[ \delta m^*(h) = \delta j^*(m^*\eta\phi) = \delta j^*\eta m^*\phi = \delta n j^* m^*\phi. \]
Since $i^* = j^* m^*$,
\[ \delta n j^* m^*\phi = \delta n i^* \phi = \eta \delta_{p-1}\phi. \]
Since $\phi \in Z^{p-1}(X,B)$,
\[ \delta_{p-1}\phi \in C^p(X,X). \]
Therefore, $0 + \delta_{p-1}\phi \in B^p(X,A)$.
Therefore, $\eta(0 + \delta_{p-1}\phi) = \eta \delta_{p-1}\phi = \delta m^*h = 0$. Hence, $m^*h \in \ker \delta$.
Therefore, $\text{Im } m^* \subseteq \ker \delta$.\[ \]
Let $h \in \ker \delta$ for some $h \in H^p(A,B)$. Then $h = \eta \phi$ for some $\phi \in Z^p(A,B)$, and $\delta h = 0$. By Note 2.8, $m^\#$ is onto. So, $\phi = m^\# \phi'$ for some $\phi' \in Z^p(X,B)$. Now, $m^*(\eta \phi') = \eta m^\# \phi' = \eta \phi = h$. Thus, $h \in \text{Im } m^*$. Therefore, $\ker \delta \subseteq \text{Im } m^*$. Hence, $\text{Im } m^* = \ker \delta$.

Let $h \in \ker n^*$ for some $h \in H^p(X,A)$. Then $h = \eta \phi$ for some $\phi \in Z^p(X,A)$, and $n^* h = n^* \eta \phi = \eta n^\# \phi = 0$. Therefore, $n^\# \phi \in B^p(X,B)$. Thus, $n^\# \phi = \delta_{p-1} \alpha_1 + \alpha_2$ where $\alpha_1 \in C^p-1(X,B)$, and $\alpha_2 \in C^p(X,X)$. Since $n^\# \phi = \phi \in C^p(X,B)$, $\phi = \delta_{p-1} \alpha_1 + \alpha_2$. Therefore, $h = \eta \phi = \eta (\delta_{p-1} \alpha_1 + \alpha_2) = \eta \delta_{p-1} \alpha_1 + \alpha_2$. Since $\alpha_2 \in C^p(X,X)$, $0 + \alpha_2 \in B^p(X,A)$. Therefore, $n \alpha_2 = 0$. Thus, $h = n \delta_{p-1} \alpha_1$. Since $\phi \in C^p(X,A)$, and $\alpha_2 \in C^p(X,X)$, then $\phi \in C^p(A,A)$ and $\alpha_2 \in C^p(A,A)$. Therefore, $\phi - \alpha_2 = \delta_{p-1} \alpha_1 \in C^p(A,A)$. Thus, $\alpha_1 \in \delta_{p-1} C^p(A,A)$. Also, $\alpha_1 \in C^p-1(X,B)$ implies $\alpha_1 \in C^p-1(A,B)$. Hence, $\alpha_1 \in Z^p-1(A,B)$. Now $\delta (n \alpha_1) = \delta j^* (n \alpha_1) = \delta (n j^* \alpha_1) = n \delta_{p-1} \alpha_1 = h$. Therefore, $h \in \text{Im } \delta$. Hence, $\ker n^* \subseteq \text{Im } \delta$.

Let $h \in \text{Im } \delta$ for some $h \in H^{p-1}(A,B)$. Let $k: B \cong A$. By Theorem 4, $\ldots H^{p-1}(A,B) \xrightarrow{j^*} H^{p-1}(A) \xrightarrow{k^*} H^{p-1}(B) \xrightarrow{\delta} H^p(A,B) \xrightarrow{\cdots}$ is an exact sequence. Therefore, $\text{Im } j^* = \ker k^*$. Now $n^* (\delta h) = n^* \delta j^* h$. By Theorem 3, $n^* \delta = \delta k^*$. Therefore, $n^* \delta j^* h = \delta k^* j^* h$. Since $\text{Im } j^* = \ker k^*$, $k^* (j^* h) = 0$. Thus, $\delta k^* j^* h = n^* \delta h = 0$. Therefore, $\delta h \in \ker n^*$. Therefore, $\text{Im } \delta \subseteq \ker n^*$. Hence, $\text{Im } \delta = \ker n^*$. 
Let \( n \ast h \in \text{Im } n \ast \) for some \( n \in H^p(X,A) \). Then \( h = n \phi \) for some \( \phi \in Z^p(X,A) \). Thus, \( m \ast n \ast h = m \ast n \ast n \phi = m \ast n \ast n \phi = nm \ast n \phi \). Since \( \phi \in Z^p(X,A) \), \( \phi \in C^p(X,A) \). Therefore, there is an open cover \( \beta \) of \( A \) such that \( \phi |_{\beta P^+1} \cap A^{P+1} = 0 \). Let \( (x_0, \ldots, x_p) \in \beta P^+1 \cap A^{P+1} \). Then \( m \ast n \ast \phi(x_0, \ldots, x_p) = m \ast \phi(n(x_0), \ldots, n(x_p)) = m \ast \phi(x_0, \ldots, x_p) = \phi(m(x_0), \ldots, m(x_p)) = \phi(x_0, \ldots, x_p) = 0 \). Hence, \( m \ast n \ast \phi \in C^p(A,A) \). Therefore, \( 0 + m \ast n \ast \phi \in B^p(A,B) \). Therefore, \( nm \ast n \phi = m \ast n \ast h = 0 \). Thus, \( n \ast h \in \ker m \ast \). Hence, \( \text{Im } n \ast \subseteq \ker m \ast \).

Let \( h \in \ker m \ast \) for some \( h \in H^p(X,A) \). Then \( h = n \phi \) for some \( \phi \in Z^p(X,B) \), and \( m \ast h = m \ast n \phi = n \phi = 0 \). Therefore, \( m \ast h \in B^p(A,B) \). Thus, \( m \ast \phi = \delta_{p-1} a_1 + a_2 \) where \( a_1 \in C^{p-1}(A,B) \), and \( a_2 \in C^p(A,A) \). Now \( a_2 = m \ast \phi - \delta_{p-1} a_1 \in C^p(A,A) \). Hence, there exists a cover \( \beta \) of \( A \) by sets open in \( A \) such that \( m \ast \phi - \delta_{p-1} a_1 |_{\beta P^+1} \cap A^{P+1} = 0 \). Let \( \beta_1 = \{ U \subseteq X | U \cap A \in \beta \} \). Then \( \beta_1 \) is a cover of \( A \) by sets open in \( X \). Let \( (x_0, \ldots, x_p) \in \beta_1 P^+1 \cap A^{P+1} \). Then \( (\phi - \delta_{p-1} a_1)(x_0, \ldots, x_p) = \phi(x_0, \ldots, x_p) - \delta_{p-1} a_1(x_0, \ldots, x_p) = \phi(m(x_0), \ldots, m(x_p)) - \delta_{p-1} a_1(x_0, \ldots, x_p) = m \ast \phi(x_0, \ldots, x_p) - \delta_{p-1} a_1(x_0, \ldots, x_p) = (m \ast \phi - \delta_{p-1} a_1)(x_0, \ldots, x_p) \). Since \( (x_0, \ldots, x_p) \in \beta_1 \), there exists \( U \in \beta_1 \) such that \( (x_0, \ldots, x_p) \in U P^+1 \). Now \( U \cap A \in \beta \). Since \( (x_0, \ldots, x_p) \in A^{P+1} \), \( (x_0, \ldots, x_p) \in \beta P^+1 \). Hence, \( (m \ast \phi - \delta_{p-1} a_1)(x_0, \ldots, x_p) = (\phi - \delta_{p-1} a_1)(x_0, \ldots, x_p) = 0 \). Therefore, \( \phi - \delta_{p-1} a_1 \in C^p(X,A) \). Since \( \phi \in C^p(X,B) \), \( \delta_{p} \phi \in C^{p+1}(X,X) \). Now
\[ \delta_p(\phi - \delta_p^{-1}a_1) = \delta_p\phi - \delta_p\delta_p^{-1}a_1 = \delta_p\phi - e = \delta_p\phi \in CP^{+1}(X,X). \]

Therefore, \( \phi - \delta_p^{-1}a_1 \in CP^{+1}(X,X) \). Hence, \( \phi - \delta_p^{-1}a_1 \in ZP(X,A) \). Since \( a_1 \in CP^{-1}(A,B) \), \( n\#a_1 \in CP^{-1}(X,B) \). Therefore,
\[
\delta_p^{-1}n\#a_1 \in \delta_p^{-1}CP(X,B). \quad \text{Thus,} \quad \delta_p^{-1}n\#a_1 + 0 = \delta_p^{-1}n\#a_1 \in B^p(X,B). \quad \text{Therefore,} \quad n\delta_p^{-1}n\#a_1 = 0. \quad \text{Hence,}
\]
\[
n^*(n(\phi - \delta_p^{-1}a_1)) = nn\#(\phi - \delta_p^{-1}a_1) = nn\#\phi - nn\#\delta_p^{-1}a_1 = nn\#\phi - nn\#a_1 = nn\#\phi - 0 = n\phi = h. \quad \text{Therefore,}
\]
\[ h \in \text{Im } n^*. \quad \text{Therefore, ker } n^* \subseteq \text{Im } n^*. \quad \text{Hence, } \text{Im } n^* = \ker n^*.
\]

To show \( n^*: H^0(X,A) \to H^0(X,B) \) is 1-1, let \( n^*(h) = 0 \) for some \( h \in H^0(X,A) \). Then \( h = n\phi \) for some \( \phi \in ZP(X,A) \).

Now \( n^*(h) = n^*n\phi = nn\#\phi = 0. \quad \text{Therefore,} \quad n\#\phi \in B^0(X,B). \)

Thus, \( n\#\phi = 0. \quad \text{By Lemma 2.3, } n\# \text{ is 1-1.} \quad \text{Hence, } \phi = 0. \)

Therefore, \( n\phi = h = 0. \quad \text{Hence, } n^*: H^0(X,A) \to H^0(X,B) \) is 1-1. Therefore, by Definition 1.1, the sequence is exact.

**Definition 2.5.** Let \( f: X \to Y \), and let \( g: X \to Y \).
Define \( D: CP(X) \to CP^{-1}(X) \) as follows: for \( \phi \in CP(X) \),
\( D\phi \) is that function in \( CP^{-1}(X) \) such that
\[ D\phi(x_0,\ldots,x_{p-1}) = \frac{p-1}{\sum_{i=0}^{p-1}(-1)^i\phi(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_{p-1})).} \]
The function \( D\phi \) is called the deformation cochain of \( \phi \) with respect to \( f \) and \( g \).

**Lemma 2.4.** If \( f: X \to Y \), and \( g: X \to Y \), then
\( D\delta_p\phi = f^{\#}\phi - g^{\#}\phi \), and if \( p > 0 \), \( \delta_p^{-1}D\phi + D\delta_p\phi = f^{\#}\phi - g^{\#}\phi \).

**Proof:** Assume \( p = 0 \). Let \( x_0 \in X \). Then \( D\delta_0\phi(x_0) = \delta_p\phi(g(x_0),f(x_0)) = \phi(f(x_0)) - \phi(g(x_0)) = f^{\#}\phi(x_0) - g^{\#}\phi(x_0). \)
Therefore, \( D\delta_p\phi = f^{\#}\phi - g^{\#}\phi \) for \( p = 0 \).
Assume \( p > 0 \). Let \( (x_0, \ldots, x_p) \in \mathbb{R}^{p+1} \). Then

\[
D \phi_p(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^i \left[ \sum_{j=0}^{i} (-1)^j \phi(h^0_j, \ldots, h^i_j) \right]
\]

where \( h^i_j = \begin{cases} g(x_j) & \text{if } j \leq i \\ f(x_{j-1}) & \text{if } j > i. \end{cases} \)

Also, \( \delta_{p-1} \phi(x_0, \ldots, x_p) = \)

\[
\sum_{i=0}^{p} (-1)^i \left[ \sum_{j=0}^{i} (-1)^j \phi(h^0_j, \ldots, h^i_j) \right]
\]

where \( h^i_j = \begin{cases} x_i & \text{if } i < j \\ x_{i+1} & \text{if } i \geq j. \end{cases} \)

Thus, \( D \phi_p(x_0, \ldots, x_p) = \phi(h^0_0, \ldots, h^0_p) + \cdots + \phi(h^p_0, \ldots, h^p_p) \)

\[
= (-1)^p \left[ \sum_{i=0}^{p} \sum_{j=0}^{i} (-1)^j \phi(h^i_j) \right] + f^\# \phi(x_0, \ldots, x_p) - \sum_{i=0}^{p} (-1)^i \phi(h^i_0, \ldots, h^i_p) \]

For \( i = j \neq 0 \),

\[
(-1)^i \phi(h^i_0, \ldots, h^i_i, \ldots, h^i_p) =
\]

\[
= (-1)^i \phi(g(x_0), \ldots, g(x_{i-1}), f(x_{i-1}), \ldots, f(x_p)) =
\]

\[
= -(-1)^{(i-1)} \phi(g(k^i_0), \ldots, g(k^i_{i-1}), f(k^i_{i-1}), \ldots, f(k^i_p)).
\]

For \( i < j \), \( (-1)^i \phi(h^i_0, \ldots, h^i_j, \ldots, h^i_p) = \)

\[
= (-1)^i \phi(g(x_0), \ldots, g(x_i), f(x_i), \ldots, f(x_{i-2}), f(x_{i-1}), \ldots, f(x_p)) =
\]

\[
= -(-1)^{(i-1)} \phi(g(k^i_0), \ldots, g(k^i_{i-1}), f(k^i_{i-1}), \ldots, f(k^i_{i-2})).
\]

For \( j < i \), \( (-1)^i \phi(h^i_0, \ldots, h^i_j, \ldots, h^i_p) = \)

\[
= (-1)^i \phi(g(x_0), \ldots, g(x_{j-1}), g(x_{j+1}), \ldots, g(x_i), f(x_i), \ldots, f(x_p)) =
\]

\[
= -(-1)^{(i-1)} \phi(g(k^i_0), \ldots, g(k^i_{i-1}), f(k^i_{i-1}), \ldots, f(k^i_p)).
\]

Therefore, \( \delta_{p-1} \phi(x_0, \ldots, x_p) = \)

\[
= \sum_{i=0}^{p} (-1)^i \left[ \sum_{j=0}^{i} (-1)^j \phi(h^i_j) \right] - \sum_{i=0}^{p} (-1)^i \phi(h^i_0, \ldots, h^i_p) \]

\[
= -[\delta_{p-1} \phi(x_0, \ldots, x_p)] = \]
The Fundamental Lemma of Spanier.

1. Let \( f, g: (X, A) \rightarrow (Y, B) \) (not necessarily continuous).
2. Let \( \phi \in Z^p(Y, B) \).
3. There exists an open cover \( \beta \) of \( Y \) such that
   \[ \delta_p \beta \cap B^{p+2} = 0 \text{ and } \phi \cap B^{p+1} \cap B^{p+1} = 0. \]
4. There exists an open cover \( \gamma \) of \( X \) such that \( U \in \gamma \) implies
   there exists \( V \in \beta \) such that \( f(U) \cup g(U) \subseteq V \).

If (1)-(4) are satisfied, then

(a) \( f#\phi, g#\phi \in Z^p(X, A) \), and (b) \( f#\phi - g#\phi \in B^p(X, A) \).

Proof: Let \( (x_0, \ldots, x_p) \in \gamma^{p+1} \cap A^{p+1} \). Then

\[ (f(x_0), \ldots, f(x_p)) \in B^{p+1}, \text{ and } (g(x_0), \ldots, g(x_p)) \in B^{p+1}. \]

Also, \( (x_0, \ldots, x_p) \in U^{p+1} \) for some \( U \in \gamma \). By (4), there is
a \( V \in \beta \) such that \( f(U) \cup g(U) \subseteq V \). Thus,

\[ (f(x_0), \ldots, f(x_p)) \in V^{p+1}, \text{ and } (g(x_0), \ldots, g(x_p)) \in V^{p+1}. \]

Hence, \( (f(x_0), \ldots, f(x_p)), (g(x_0), \ldots, g(x_p)) \in \beta^{p+1} \cap B^{p+1} \).

Therefore, by (3), \( \phi(f(x_0), \ldots, f(x_p)) = f#\phi(x_0, \ldots, x_p) = 0 \),
and \( \phi(g(x_0), \ldots, g(x_p)) = g#\phi(x_0, \ldots, x_p) = 0 \). Hence, \( \gamma \) is
an open cover of \( A \), and \( f#\phi | \gamma^{p+1} \cap A^{p+1} = 0 \), and
\( g#\phi | \gamma^{p+1} \cap A^{p+1} = 0 \). Therefore, \( f#\phi, g#\phi \in CP(X, A) \).
Let \((x_0, \ldots, x_{p+1}) \in \gamma^{p+2}\). Then for some \(U \in \gamma\), 
\((x_0, \ldots, x_{p+1}) \in U^{p+2}\). By (4), there exists \(V \in \beta\) such that 
\((f(x_0), \ldots, f(x_{p+1})) \in V^{p+2}\), and \((g(x_0), \ldots, g(x_{p+1})) \in V^{p+2}\).
Therefore, \((f(x_0), \ldots, f(x_{p+1})), g(x_0), \ldots, g(x_{p+1})) \in \beta^{p+2}\).

Hence, \(\delta_p f(x_0), \ldots, f(x_{p+1})) = \delta_p g(x_0), \ldots, g(x_{p+1})) = 0\), and \(\delta_p f(x_0), \ldots, f(x_{p+1}))) = \delta_p g(x_0), \ldots, g(x_{p+1})) = 0\). Therefore, \(\gamma\) is an open cover of \(X\), and \(\delta_p f, g \in C_p^{p+1}(X,X)\).

To prove (b), it will be shown that \(D\phi \in C_p^{-1}(X,A)\).

Let \((x_0, \ldots, x_{p-1}) \in \gamma^p \cap A^p\). Then \((x_0, \ldots, x_{p-1}) \in U^p\) for some \(U \in \gamma\). By (4), there exists \(V \in \beta\) such that 
\(f(U) \cup g(U) \subseteq V\). Let \(K = \{(g(x_0), \ldots, f(x_i), \ldots, f(x_{p-1})) | 0 \leq i \leq p-1\}\). Thus, \(K\) is a subset of \(V^{p+1}\). Therefore, every element in \(K\) is an element of \(\beta^{p+1}\). Since \(x_i \in A\) for \(0 \leq i \leq p-1\), \(K\) is a subset of \(B^{p+1}\). Therefore, if 
\((g(x_0), \ldots, g(x_i), f(x_i), \ldots, f(x_{p-1})) \in K\), then 
\((g(x_0), \ldots, g(x_i), f(x_i), \ldots, f(x_{p-1})) \in \beta^{p+1} \cap B^{p+1}\). Hence, by (3), \(D\phi(x_0, \ldots, x_{p-1}) = P_{i \neq 0} (-1)^i (g(x_0), \ldots, g(x_i), f(x_i), \ldots, f(x_{p-1})) = 0\). Therefore, \(\gamma\) is an open cover of \(A\), and \(D\phi | \gamma^p \cap A^p = 0\). Thus, \(D\phi \in C_p^{-1}(X,A)\).

Now it will be shown that \(D\delta_p \phi \in CP(X,X)\). Let 
\((x_0, \ldots, x_p) \in \gamma^{p+1} \cap x^{p+1}\). Then \((x_0, \ldots, x_p) \in U\) for some
U \in \gamma. By (4), there is a V \in \beta such that
\begin{align*}
f(U) \cup g(U) \subseteq V. \text{ Let } K = \{(g(x_0), \ldots, g(x_i), f(x_i), \ldots, f(x_p)) | 0 \leq i \leq p\}. \text{ Then } K \text{ is a subset of } V^{p+1}. \text{ Therefore, if }
(g(x_0), \ldots, g(x_i), f(x_i), \ldots, f(x_p)) \in K, \text{ then }
(g(x_0), \ldots, g(x_i), f(x_i), \ldots, f(x_p)) \in \beta. \text{ By (3), }
\delta_\phi|\beta^{p+1} = 0. \text{ Hence, } D\delta_\phi (x_0, \ldots, x_p) = 
\begin{equation}
\sum_{i=0}^{p} (-1)^i \delta_\phi (g(x_0), \ldots, g(x_i), f(x_i), \ldots, f(x_p)) = 0.
\end{equation}
Therefore, \gamma is an open cover of X, and } D\delta_\phi|\gamma^{p+1} = 0.
\text{ Hence, } D\delta_\phi \in C^p(X,x).
\end{align*}

If } p = 0, \text{ then by Lemma 2.4, }
f#_\phi - g#_\phi = D\delta_0 \phi \in C^0(X,X). \text{ Thus, }
0 + f#_\phi - g#_\phi = f#_\phi - g#_\phi \in B^0(X,A). \text{ If } p > 0, \text{ then by Lemma 2.4, } f#_\phi - g#_\phi = \delta_{p-1} D\phi - D\delta_p \phi. \text{ Since }
D\phi \in C^{p-1}(X,A), \text{ then } \delta_{p-1} D\phi \in \delta_{p-1} C^{p-1}(X,A). \text{ Therefore, }
\delta_{p-1} D\phi + D\delta_p \phi \in \delta_{p-1} C^{p-1}(X,A) + C^p(X,X) = B^p(X,A). \text{ Thus, }
f#_\phi - g#_\phi \in B^p(X,A). \text{ In any case, } f#_\phi - g#_\phi \in B^p(X,A).
\end{equation}

The following notation will be used. For
\pi: (X,A) \times T \rightarrow (Y,B), \text{ fix } t \in T. \text{ Define }
\hat{t}: (X,A) \rightarrow (Y,B) \text{ by } \hat{t}(x) = \pi(t,x) \text{ for each } x \in X. \text{ Then }
\hat{t} \text{ induces } \hat{t}^*: H^p(Y,B) \rightarrow H^p(X,A).

**Lemma 2.5.** Let X be compact. Let B be a closed subset of Y. Let \pi: (X,A) \times T \rightarrow (Y,B). \text{ Let } h \in H^p(Y,B), \text{ and } t \in T. \text{ There exists a set } 0 \text{ open in } T \text{ such that } t \in 0, \text{ and if } s \in 0, \text{ then } 8^*(h) = \hat{t}^*(h). \text{ Further, if } T \text{ is connected, then for each } s,t \in T, 8^* = \hat{t}^*.
Proof: Since \( h \in H^p(Y,B) \), \( h = \phi \) for some \( \phi \in Z^p(Y,B) \).

Therefore, there exists an open cover \( \beta_1 \) of \( B \) such that
\[
\phi|_{\beta_1} \cap B^{p+1} = 0.
\]
Also, there exists an open cover \( \beta_2 \) of \( Y \) such that \( \delta_p \phi|_{\beta_2}^{p+2} = 0 \). Then \( \beta_1 \cup (Y\setminus B) \) is an open cover of \( Y \). Let \( \beta = \{ V_1 \cap V_2 \mid V_1 \in \beta_1 \cup (Y\setminus B), V_2 \in \beta_2, V_1 \cap V_2 \neq \emptyset \} \). Then \( \beta \) is an open cover of \( Y \), and
\[
\phi|_{\beta}^{p+1} \cap B^{p+1} = 0, \quad \text{and} \quad \delta_p \phi|_{\beta}^{p+2} = 0.
\]
Since \( \pi \) is continuous, \( \pi^{-1}(\beta) \) is an open cover of \( X \times T \). For each \( x \in X \),
\( (x,t) \in X \times T \). Therefore, \( (x,t) \in \pi^{-1}(V) \) for some \( V \in \beta \). Since \( \pi^{-1}(V) \) is open in \( X \times T \), there exist sets
\( G_{xv}, K_{xv} \) such that \( G_{xv} \) is open in \( X \), \( K_{xv} \) is open in \( T \),
and \( (x,t) \in G_{xv} \times K_{xv} \subseteq \pi^{-1}(V) \). Let \( \beta' = \{ G_{xv} \mid x \in X \} \).

Then \( \beta' \) is an open cover of \( X \). Therefore, there is a finite subcollection \( \beta'' = \{ G_v \mid v \in \{1, \ldots, n\} \} \) of \( \beta' \) that covers \( X \). Now \( t \in K_{vi} \) for \( i = 1, \ldots, n \). Therefore, \( t \in \bigcap_{i=1}^n K_{vi} \).
Let \( 0 = \bigcap_{i=1}^n K_{vi} \). Then \( 0 \) is open in \( T \) and \( t \in 0 \). Let \( s \in 0 \).
Let \( G_{vi} \in \beta'' \). If \( x \in G_{vi} \), then \( \hat{t}(x) = \pi(x,t) \). Since \( t \in K_{vi} \), and \( G_{vi} \times K_{vi} \subseteq \pi^{-1}(V_i) \), then \( \pi(x,t) \in V_i \). Also,
\( \hat{s}(x) = \pi(x,s) \in V_i \).

The above establishes the following:
\( \hat{s}, \hat{t} : (X,A) \to (Y,B) ; \phi \in Z^p(Y,B) \); there is an open
cover \( \beta \) of \( Y \) such that \( \delta_p \phi|_{\beta}^{p+2} = 0 \), and \( \phi|_{\beta}^{p+1} \cap B^{p+1} = 0 \); there is an open cover \( \beta'' \) of \( X \) such that \( G \in \beta'' \) implies there exists \( V \in \beta \) such that \( \hat{s}(G) \cup \hat{t}(G) \subseteq V \). Hence, the hypothesis of the Fundamental Lemma of Spanier is satisfied.
Thus, $\hat{s}^\# \phi - \hat{t}^\# \phi \in BP(X,A)$. Therefore,

$\eta \hat{s}^\# \phi = \eta \hat{t}^\# \phi$. Hence, $\hat{s}^*(h) = \hat{t}^*(h)$.

Now assume also that $T$ is connected. Suppose there exists $t, s \in T$ such that $\hat{t}^*(h) \neq \hat{s}^*(h)$ for some $h \in HP(Y,B)$. From the first part of this proof, there is an open set $U$ such that $r^*(h) = t^*(h)$ for all $r \in U$. Let

$M = \bigcup \{U \mid \text{for every } r \in U, \hat{r}^*(h) = \hat{t}^*(h)\}$. Then $s \in T \setminus M$, and $t \in M$. $M$ is open in $T$. Thus, $T \setminus M$ is closed in $T$. Therefore, $M \cap (T \setminus M) = M \cap (T \setminus M) = \emptyset$. Now suppose $M \cap (T \setminus M) \neq \emptyset$. Let $x \in M \cap (T \setminus M)$. Then every open set about $x$ intersects $M$. There exists an open set $G$ about $x$ such that for every $y \in G$, $\hat{y}^*(h) = \hat{x}^*(h)$. Since $G \cap M \neq \emptyset$, there exists $y \in G \cap M$ such that $\hat{y}^*(h) = \hat{x}^*(h)$. Therefore, $\hat{t}^*(h) = \hat{y}^*(h) = \hat{x}^*(h)$. Therefore, $x \in M$. This contradicts $x \in T \setminus M$. Thus, $M \cap (T \setminus M) = \emptyset$. Therefore, $M \cap (T \setminus M) = M \cap (T \setminus M) = \emptyset$. Since $T = M \cup (T \setminus M)$, this implies $T$ is not connected. This is a contradiction. Hence, for each $t, s \in T$, $\hat{t}^* = \hat{s}^*$.

Theorem 5. (The Generalized Homotopy Theorem). Let $T$ be connected, and $f, g: (X,A) \to (Y,B)$ where $X$ is compact, $B$ is closed, and $f$ and $g$ are continuous. If there exists a continuous function $h: (X,A) \times T \to (Y,B)$, and if there exists $s, t \in T$ such that $h(x,s) = f(x)$, and $h(x,t) = g(x)$ for all $x \in X$, then $f^* = g^*$. 
Proof: Define $\hat{s}, \hat{t}: (X,A) \rightarrow (Y,B)$ by $\hat{s}(x) = h(x,s)$, and $\hat{t}(x) = h(x,t)$ for all $x \in X$. Then $\hat{s} = \hat{f}$, and $\hat{t} = \hat{g}$. By Lemma 2.5, $\hat{s}^* = \hat{t}^*$. Therefore, $f^* = g^*$.

**Definition 2.6.** Let $I$ be the unit interval. Let $f, g: X \rightarrow Y$. Then $f$ is homotopic to $g$ (denoted $f \simeq g$) if there exists $h: X \times I \rightarrow Y$ such that $h$ is continuous, and $h(x,1) = f(x)$ for all $x \in X$, and $h(x,0) = g(x)$ for all $x \in X$.

**Corollary 2.3.** If $X$ is compact, and $B$ is closed, and $f, g: (X,A) \rightarrow (Y,B)$, and $f$ and $g$ are continuous, then $f \simeq g$ implies $f^* = g^*$.

Proof: Since $f \simeq g$, there exists a continuous function $h: X \times I \rightarrow Y$ such that $h(x,1) = f(x)$, and $h(x,0) = g(x)$ for all $x \in X$. Since $I$ is connected, the hypothesis of Theorem 5 is satisfied. Hence, $f^* = g^*$.

**Definition 2.7.** If $f$ is a continuous function from $X$ into $Y$, then $f$ is null-homotopic if $f$ is homotopic to a constant map.

**Definition 2.8.** A space $X$ is contractible if the identity map from $X$ to $X$ is null-homotopic.

**Definition 2.9.** Let $A \subseteq X$. Then $A$ is a retract of $X$ if there exists $r: X \rightarrow A$ such that $r$ is continuous, and $roj = i$ where $i$ is the identity map on $A$, and $j: A \rightarrow X$.

**Definition 2.10.** Let $A \subseteq X$. Then $A$ is a deformation retract of $X$ if there exists $r: X \rightarrow A$ such that $r$ is a retraction of $X$ onto $A$, and $j_{or} = i$ where $i$ is the identity map on $X$, and $j: A \rightarrow X$. 

Corollary 2.4. If A is a deformation retract of a compact space X, then \( H^p(X) \cong H^p(A) \) for all \( p \).

Proof: Let \( i \) be the identity map from \( X \) to \( X \). Since \( A \) is a deformation retract of \( X \), there exists \( r : X \to A \) such that \( r \) is onto and \( jr = i \) where \( j : A \hookrightarrow X \). Now \( r^* : H^p(A) \to H^p(X) \). It will be shown that \( r^* \) is the desired isomorphism. By Corollary 2.3, \( (jr)^* = i^* \). By Theorem 2, \( (jr)^* = r^*j^* = i^* \). Now \( r^* \) is an \( R \)-homomorphism. Let \( h \in H^p(X) \). Then \( j^*(h) \in H^p(A) \). \( r^*(j^*(h)) = i^*(h) = h \).

Thus, \( r^* \) is onto. Let \( r^*(h) = 0 \) for some \( h \in H^p(A) \). Then \( h = \eta \phi \) for some \( \phi \in Z^p(A) \). By Note 2.8, \( j^* \) is onto. Thus, \( \phi = j^*\phi' \) for some \( \phi' \in Z^p(X) \). Therefore, \( \eta \phi' \in H^p(X) \), and \( j^*\eta \phi' = \eta j^*\phi' = \eta \phi = h \). Hence, \( r^*(h) = r^*(j^*\eta \phi') = i^*(\eta \phi') = 0 \). By Theorem 1, \( i^* \) is an isomorphism. Hence, \( i^*(\eta \phi') = 0 \) implies \( \eta \phi' = 0 \). Therefore, \( \phi' \in B^p(X) \). This implies \( j^*\phi' \in B^p(A) \). Thus, \( \eta j^*\phi' = h = 0 \). Therefore, \( r^*(h) = 0 \) implies \( h = 0 \). Hence, \( r^* \) is 1-1. Hence, \( r^* : H^p(A) \to H^p(X) \) is an isomorphism for all \( p \).

Theorem 5. (Weak Excision Theorem).

Let \( A \subseteq X \). Let \( U \) be open such that \( \overline{U} \subseteq \text{Int}(A) \). Let \( j : (X \setminus U, A \setminus U) \to (X, A) \). Then \( j^* : H^p(X, A) \to H^p(X \setminus U, A \setminus U) \) is an onto isomorphism.

Proof: To show \( j^* \) is onto, let \( h \in H^p(X \setminus U, A \setminus U) \). Then \( h = \eta \phi \) for some \( \phi \in Z^p(X \setminus U, A \setminus U) \). Define \( \phi' \in C^p(X) \) as follows:

\[
\phi'(x_0, \ldots, x_p) = \begin{cases} 
\phi(x_0, \ldots, x_p) & \text{if } (x_0, \ldots, x_p) \in (X \setminus U)^{p+1} \\
0 & \text{if } (x_0, \ldots, x_p) \notin (X \setminus U)^{p+1}.
\end{cases}
\]
Since \( \phi \in \mathcal{Z}(X \setminus U, A \setminus U) \), therefore, there exists a covering \( \beta \) of \( A \setminus U \) by sets open in \( X \setminus U \) such that 
\[
\phi|_{\mathcal{B}^{p+1} \cap (A \setminus U)^{p+1}} = 0 \] 
Let 
\[
\beta_1 = \{U\} \cup \{G| \ G \text{ is open in } X, G \cap (X \setminus U) \in \beta\} \] 
Then \( \beta_1 \) is a covering of \( A \) by sets open in \( X \). Let 
\[
(x_0, \ldots, x_p) \in \beta_1^{p+1} \cap A^{p+1} \] 
If \( x_i \in U \) for any \( 0 \leq i \leq p \), then by the definition of \( \phi' \), 
\[
\phi'(x_0, \ldots, x_p) = 0 \] 
Otherwise, 
\[
(x_0, \ldots, x_p) \in [G \cap (X \setminus U)]^{p+1} \] 
for some \( G \cap (X \setminus U) \in \beta \). Therefore, 
\[
(x_0, \ldots, x_p) \in \beta_1^{p+1} \cap (A \setminus U)^{p+1} \] 
Hence, 
\[
\phi'(x_0, \ldots, x_p) = \phi(x_0, \ldots, x_p) = 0 \] 
In any case, 
\[
\phi'(x_0, \ldots, x_p) = 0 \] 
Hence, \( \phi' \in \mathcal{C}(X, A) \).

It should be noted that the above is true for any \( \phi \in \mathcal{C}(X \setminus U, A \setminus U) \). By the definition of \( \phi' \), \( j^\# \phi' = \phi \). Hence, \( j^\# \) is onto \( \mathcal{C}(X \setminus U, A \setminus U) \).

Now since \( \phi \in \mathcal{Z}(X \setminus U, A \setminus U) \), 
\[
\delta \phi \in \mathcal{C}^{p+1}(X \setminus U, A \setminus U) \] 
Therefore, there is a covering \( \gamma \) of \( X \setminus U \) by sets open in \( X \setminus U \) such that 
\[
\delta \phi|_\gamma^{p+2} \cap (X \setminus U)^{p+2} = 0 \] 
Since \( \phi' \in \mathcal{C}(X, A) \), 
\[
\delta \phi' \in \mathcal{C}^{p+1}(X, A) \] 
Therefore, there is a covering \( \gamma_1 \) of \( A \) by sets open in \( X \) such that 
\[
\delta \phi'|_{\gamma_1^{p+2} \cap A^{p+2}} = 0 \] 
Let \( \gamma_2 = \{G| \ G \text{ is open in } X, (X \setminus U) \cap G \in \gamma\} \). Let 
\[
\gamma_3 = \{G \cap (X \setminus U)| \ G \in \gamma_2\}, \text{ and let} \] 
\[
\gamma_4 = \{V \cap \text{Int}(A)| \ V \in \gamma_1\} \] 
Let \( \Gamma = \gamma_3 \cup \gamma_4 \). It will be shown that \( \Gamma \) is an open cover of \( X \). Let \( x \in X \). If \( x \in U \), then \( x \in \text{Int}(A) \). Therefore, \( x \in V \) for some \( V \in \gamma_1 \). Thus, 
\[
x \in V \cap \text{Int}(A) \in \Gamma \] 
If \( x \in X \setminus U \), then \( x \in X \setminus U \). Therefore,
x ∈ G for some G ∈ γ. Then G = (X\U) ∩ G' where G' is open in X. Therefore, x ∈ G' ∩ (X\U) ∈ Γ. In any case, x is covered by a set in Γ. Every set in γ_3 is open, and every set in γ_4 is open. Hence, Γ is an open cover of X. Let

(x_0, ..., x_{p+1}) ∈ P^{p+2}. If (x_0, ..., x_{p+1}) ∈ γ^{p+2}, then

(x_0, ..., x_{p+1}) ∈ [G ∩ (X\U)]P^{p+2} ⊆ [G ∩ (X\U)]P^{p+2}. Since G ∩ (X\U) ∈ γ, (x_0, ..., x_{p+1}) ∈ γ^{p+2}. Therefore,

δ_pφ(x_0, ..., x_p) = 0. For each i, 0 ≤ i ≤ p + 1,

(x_0, ..., x_i, ..., x_{p+1}) ∈ (X\U)^{p+1}. Thus, φ'(x_0, ..., x_{p+1}) = φ(x_0, ..., x_i, ..., x_{p+1}). Therefore,

δ_pφ'(x_0, ..., x_{p+1}) = P^{p+1}_{i=0} (-1)^i φ(x_0, ..., x_i, ..., x_{p+1}) =

P^{p+1}_{i=0} (-1)^i φ'(x_0, ..., x_i, ..., x_{p+1}) = δ_pφ'(x_0, ..., x_{p+1}) = 0.

If (x_0, ..., x_{p+1}) ∈ γ^{p+2}, then

(x_0, ..., x_{p+1}) ∈ (V ∩ Int(A))^P^{p+2} for some V ∈ γ_1.

Thus, (x_0, ..., x_{p+1}) ∈ γ^{p+2} ∩ A^{p+2}. Therefore,

δ_pφ'(x_0, ..., x_{p+1}) = 0. In any case, δ_pφ' = 0. Hence,

δ_pφ'|ΓP^{p+2} = 0. Therefore, δ_pφ' ∈ C^{p+2}(X,X). Therefore,

φ' ∈ δ_p^{-1}C^{p+1}(X,X). By the definition of φ', j#φ' = φ.

Thus, j*(ηφ') = ηj#φ' = ηφ = h. Hence, j* is onto.

To show j* is 1-1, let j*(h) = 0. Then h = ηφ for some φ ∈ ZP(X,A). Since j*(h) = j*(ηφ) = ηj#φ = 0,

j#φ ∈ BP(X\U, A\U). Therefore, j#φ = δ_{p-1}α_1 + α_2 where

α_1 ∈ C^{p-1}(X\U, A\U), and α_2 ∈ C^p(X\U, X\U). It was noted earlier in this proof that j# is onto C^{p-1}(X\U, A\U). Thus,

α_1 = j#α_1 for some α_1 ∈ C^{p-1}(X,A). Therefore,
\( \alpha_2 = j^\# \phi - \delta_{p-1} \alpha_1 = j^\# \phi - \delta_{p-1} j^\# \alpha_1 = j^\# \phi - j^\# \delta_{p-1} \alpha_1 = j^\# (\phi - \delta_{p-1} \alpha_1) \in CP(X \setminus U, X \setminus U). \) Hence, there is a covering \( \beta \) of \( X \setminus U \) by sets open in \( X \setminus U \) such that

\[ j^\# (\phi - \delta_{p-1} \alpha_1) |_{\beta^{p+1}} = 0. \]

Now \( \phi - \delta_{p-1} \alpha_1 \in CP(X, A) \). Thus, there is a covering \( \gamma \) of \( A \) by sets open in \( X \) such that

\[ \phi - \delta_{p-1} \alpha_1 |_{\gamma^{p+1} \cap A^{p+1}} = 0. \]

Let \( \beta_1 = \{ G | G \text{ is open in } X, (X \setminus U) \cap G \in \beta \} \). Let

\[ \beta_2 = \{ G \cap (X \setminus U) | G \in \beta_1 \}. \]

Let \( \gamma_1 = \{ G \cap \text{Int}(A) | G \in \gamma \} \). Let \( \Gamma = \beta_2 \cup \gamma_1 \). Similar to an earlier part of this proof, \( \Gamma \) is an open cover of \( X \).

Let \( (x_0, \ldots, x_p) \in \Gamma^{p+1} \). If \( (x_0, \ldots, x_p) \in \beta_2^{p+1} \), then

\[ (x_0, \ldots, x_p) \in \{ G \cap (X \setminus U) \}^{p+1}. \]

Thus, \( G \cap (X \setminus U) \in \beta \), and

\[ (x_0, \ldots, x_p) \in \beta^{p+1}. \]

Hence, \( j^\# (\phi - \delta_{p-1} \alpha_1)(x_0, \ldots, x_p) = 0 \). Since \( x_i \in X \setminus U \) for \( 0 \leq i \leq p \), \( j^\# (\phi - \delta_{p-1} \alpha_1)(x_0, \ldots, x_p) = (\phi - \delta_{p-1} \alpha_1)(j(x_0), \ldots, j(x_p)) = (\phi - \delta_{p-1} \alpha_1)(x_0, \ldots, x_p) = 0. \)

If \( (x_0, \ldots, x_p) \in \gamma_1^{p+1} \), then \( (x_0, \ldots, x_p) \in (G \cap \text{Int}(A))^{p+1} \) for some \( G \in \gamma \). Hence, \( (x_0, \ldots, x_p) \in \gamma^{p+1} \cap A^{p+1} \). Therefore, \( (\phi - \delta_{p-1} \alpha_1)(x_0, \ldots, x_p) = 0. \) In any case, \( (\phi - \delta_{p-1} \alpha_1)(x_0, \ldots, x_p) = 0. \) Therefore, \( \Gamma \) is an open cover of \( X \) and \( \phi - \delta_{p-1} \alpha_1 |_{\Gamma^{p+1}} = 0. \) Hence,

\[ \phi - \delta_{p-1} \alpha_1 \in CP(X, X). \]

Now \( \delta_{p-1} \alpha_1 \in \delta_{p-1} CP^{-1}(X, A) \). Thus,

\[ \phi = \delta_{p-1} \alpha_1 + (\phi - \delta_{p-1} \alpha_1) \in \delta_{p-1} CP^{-1}(X, A) + CP(X, X) = \beta P(X, A). \]

Therefore, \( \eta \phi = h = 0. \) Hence, \( j^* \) is 1-1. Hence, \( j^* \) is an onto isomorphism.

**Theorem 6A.** (Excision Theorem).

If \( X = \text{Int}(A) \cup \text{Int}(B) \) where \( A \) and \( B \) are closed subsets of \( X \), and if \( j: (B, A \cap B) \xrightarrow{\phi} (X, A) \), then
j*: $H^p(X,A) \to H^p(B,A \cap B)$ is an onto isomorphism.

Proof: Let $U = X \setminus B$. Let $x \in U$. If $x \notin \text{Int}(A)$, then $x \in \text{Int}(B)$. Therefore, since $x \in U$, $\text{Int}(B) \cap U \neq \emptyset$.

But $\text{Int}(B) \cap X \setminus B = \emptyset$. This is a contradiction. Thus, $x \in \text{Int}(A)$. Hence, $U \subseteq \text{Int}(A)$. Since $B = X \setminus (X \setminus B)$, and $A \cap B = A \setminus (X \setminus B)$, $j: (X \setminus U, A \setminus U) \to (X, A)$. Thus, the hypothesis of Theorem 6 is satisfied. Hence, $j^*$ is an onto isomorphism.

Theorem 7. (Dimension Theorem).

If the cardinality of $X$ is 1, then

$$H^p(X) \cong \begin{cases} G & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

where $G$ is the coefficient $R$-module.

Proof: Consider the case $p = 0$. Let $\phi \in C^0(X)$. Then $\delta^0_0 \phi \in C^1(X)$. Let $(x,x) \in X^2$. $\delta^0_0 \phi(x,x) = \frac{1}{i \pi} (-1)^i \phi(x) = \phi(x) - \phi(x) = 0$. Therefore, $\delta^0_0 \phi \in C^1(X,X)$. Thus, $\phi \in \delta^0_0^{-1} C^1(X,X)$. Hence, $\delta^0_0^{-1} C^1(X,X) = C^0(X)$. This implies that $Z^0(X) = C^0(X)$. Now $B^0(X) = 0$. Hence, $H^0(X) = Z^0(X)/B^0(X) = C^0(X)/0 = C^0(X)$.

For each $\phi \in C^0(X)$, $\phi(x) = g$ for some $g \in G$. Define $F: C^0(X) \to G$ by $F(\phi) = g$. To show $F$ is an $R$-homomorphism, let $\phi_1, \phi_2 \in C^0(X)$, and let $r, t \in G$. Then $\phi_1(x) = g_1$, and $\phi_2(x) = g_2$ for some $g_1, g_2 \in G$. Then

$$rF(\phi_1) + tF(\phi_2) = rg_1 + tg_2 = r\phi_1(x) + t\phi_2(x) = (r\phi_1 + t\phi_2)(x) = F(r\phi_1 + t\phi_2).$$

Hence, $F$ is an $R$-homomorphism. $F$ is onto since for each $g \in G$, there is a
function \( \phi \in C^0(X) \) such that \( \phi(x) = g \). If \( F(\phi) = 0 \), then \( \phi(x) = 0 \). Hence, by definition, \( \psi = 0 \). Thus, \( F \) is 1-1.

Hence \( F: H^0(X) \to G \) is an isomorphism.

Now assume that \( p > 0 \). Assume also that \( p \) is odd.

For the remainder of this proof, let \( x_n \) denote the \( p \)-tuple from \( X^n \). Let \( \phi \in C^p(X) \). Since \( p \) is odd,

\[
\delta_p \phi(x_{p+2}) = \sum_{i=0}^{p+2} (-1)^i \phi(x_{p+1}) = 0.
\]

Therefore, \( \phi \in C^{p+1}(X,X) \). Thus, \( C^p(X) = \delta_p^{-1} C^{p+1}(X,X) \).

Hence, \( Z^p(X) = C^p(X) \). Now it will be shown that

\[
C^p(X) = \delta_{p-1} C^{p-1}(X). \quad \text{For } \phi \in C^p(X), \quad \phi(x_{p+1}) = g \text{ for some } g \in G.
\]

Define \( \phi' \in C^{p-1}(X) \) by \( \phi'(x_p) = g \). Then since \( p \) is odd,

\[
\delta_{p-1} \phi'(x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \phi'(x_{p+1}) = \phi'(x_p) + \sum_{i=1}^{p+1} (-1)^i \phi'(x_{p+1}) = 0.
\]

Since \( p \) is even,

\[
\delta_p^{-1} C^{p+1}(X,X) = \{0\}. \quad \text{Thus, } H^p(X) = Z^p(X) \cap C^p(X,X) = C^p(X) + 0 = C^p(X).
\]

Hence, \( H^p(X) = Z^p(X)/B^p(X) = C^p(X)/(C^p(X)) \neq 0 \).

Consider now the case that \( p > 0 \) and \( p \) is even. Let \( \phi \in \delta_p^{-1} C^{p+1}(X,X) \). Then, \( \delta_p \phi \in C^{p+1}(X,X) = \{0\} \). Now

\[
\delta_p \phi(x_{p+2}) = \sum_{i=0}^{p+2} (-1)^i \phi(x_{p+1}) = \phi(x_{p+1}) + \sum_{i=1}^{p+2} (-1)^i \phi(x_{p+1}) = 0.
\]

Since \( p \) is even,

\[
\delta_p^{-1} C^{p+1}(X,X) = \{0\}. \quad \text{Thus, } H^p(X) = Z^p(X)/B^p(X) = \{0\}/\{0\} = 0. \quad \text{In any case, if } p > 0, \text{ then } H^p(X) \neq 0.\]
Corollary 2.5. If $X$ is compact and contractible, then
\[ H^p(X) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases} \]

Proof: If $X$ is compact and contractible, then the identity map $i: X \to X$ is homotopic to a constant map $f: X \to X$. Since $f$ is constant, there exists $k \in X$ such that $f(x) = k$ for every $x \in X$. Now $f$ is a retraction of $X$ onto $\{k\}$, $f: X \to \{k\}$, and $f = i$. Thus, $\{k\}$ is a deformation retract of $X$. Therefore, by Corollary 2.4, $H^p(X) \cong H^p(\{k\})$ for all $p$. By Theorem 7, $H^p(\{k\}) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$

Hence, $H^p(X) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$
CHAPTER III

EXTENDING THE THEORY

Some Topological Theorems

**Definition 3.1.** Let $X$ be a topological space. Let $\beta, \gamma$ be collections of non-empty subsets of $X$. Then $\beta$ refines $\gamma$ (denoted $\beta < \gamma$) if for each $V \in \beta$, there exists $U \in \gamma$ such that $V \subseteq U$.

The following notation will be used. If $\beta$ is an open cover of $X$, $U \subseteq X$, let $U^* = \{ U \in \gamma | U \cap U \neq \emptyset \}$. Let $\beta^* = \{ U^* | U \in \beta \}$.

**Definition 3.2.** The space $X$ is fully normal if for each open cover $\beta$ of $X$, there exists an open cover $\gamma$ of $X$ such that $\gamma^* < \beta$.

**Theorem 8.** A closed subset of a fully normal space is fully normal.

Proof: Let $X$ be fully normal, and let $K$ be a closed subset of $X$. Let $\beta$ be a covering of $K$ by sets open in $K$. Let $\beta' = \{ U \subseteq X | U$ is open in $X, U \cap K \in \beta \}$. Then $\beta' \cup (X \setminus K)$ is an open cover of $X$. Hence, there exists an open cover $\gamma$ of $X$ such that $\gamma^* < \beta' \cup (X \setminus K)$. Let

$\gamma' = \{ V \cap K | V \in \gamma, V \cap K \neq \emptyset \}$. Then $\gamma'$ is a cover of $K$ by sets open in $K$. Let $V^* \in \gamma'^*$. Then $V = V_1 \cap K$ for some $V_1 \in \gamma$. Furthermore, $V^* = \{ H | H \cap V \neq \emptyset, H \in \gamma' \} = \{ H \cap K | (H \cap K) \cap V_1 \neq \emptyset, H \in \gamma \}$. Also,
\[ V_1^* = \{ H \mid H \cap V_1 \neq \emptyset , H \in \gamma \} \]. Therefore, \( V^* \subseteq V_1^* \).

Since \( \gamma' < \beta' U (X \backslash K) \), \( V_1^* \subseteq U \) for some \( U \in \beta' U (X \backslash K) \).

If \( U = X \backslash K \), then \( V \subseteq V^* \subseteq V_1^* \subseteq X \backslash K \). This is a contradiction since \( V \subseteq K \). Therefore, \( U \in \beta' \). Thus, \( U \cap K \in \beta \).

Now \( V^* \subseteq K \), so \( V^* \subseteq U \cap K \). Therefore, for each \( V^* \in \gamma' \), \( V^* \subseteq U \) for some \( U \in \beta \). Hence, \( K \) is fully normal.

**Theorem 9.** A fully normal space is normal.

Proof: Let \( X \) be fully normal. Let \( A, B \) be closed disjoint nonempty subsets of \( X \). Let \( \beta = \{ X \backslash A, X \backslash B \} \). Then \( \beta \) is an open cover of \( X \). Hence, there exists an open cover \( \gamma \) of \( X \) such that \( \gamma^* < \beta \).

Therefore, for every \( U \in \gamma \), \( U^* \subseteq X \backslash A \) or \( U^* \subseteq X \backslash B \). Let \( M = \{ U \in \gamma \mid A \cap U \neq \emptyset \} \), and let \( N = \{ U \in \gamma \mid B \cap U \neq \emptyset \} \). Then since \( \gamma \) covers \( X \), \( M \) and \( N \) are nonempty. If \( U \in M \), then \( U \subseteq U^* \subseteq X \backslash A \) implies \( U \subseteq X \backslash A \). Thus, \( A \cap U = \emptyset \), and \( U \notin M \). Hence, \( U \in M \), implies \( U \subseteq X \backslash B \). Likewise, \( U \in N \), implies \( U \subseteq X \backslash A \). Suppose \( U \in M \), and \( V \in N \), and \( U \cap V \neq \emptyset \). Then \( U \subseteq X \backslash B \). Since \( V \cap U \neq \emptyset \), \( V \subseteq U^* \subseteq X \backslash B \). Therefore, \( V \cap B = \emptyset \). This contradicts \( V \in N \). Therefore, if \( U \in M \), \( V \in N \), then \( U \cap V = \emptyset \). Hence, \( UM \cap UN = \emptyset \). Thus, \( UM \) is an open set containing \( A \), and \( UN \) is an open set containing \( B \), and \( UM \cap UN = \emptyset \). Therefore, \( X \) is normal.

**Reduced Modules**

Let \( A \subseteq X \). Let \( Q \) denote a one point space. Define the following:
Let \( H^0(X, A) = H^0(X,A)/f^*H^0(Q, Q), \) \( H^0(X) = H^0(X)/g^*H^0(Q, Q), \) \( H^0(A) = H^0(A)/h^*H^0(A), \) and \( \tilde{H}^1(X, A) = \tilde{H}^1(X,A)/f^*H^1(Q, Q). \)

Consider the following diagram.

\[
\begin{array}{cccccc}
H^0(Q,Q) & \xrightarrow{m^*} & H^0(Q) & \xrightarrow{q^*} & H^0(Q) & \xrightarrow{\delta} & H^1(Q,Q) \\
& \xrightarrow{f^*} & & \xrightarrow{g^*} & & \xrightarrow{h^*} & \xrightarrow{\delta} \\
H^0(X,A) & \xrightarrow{i^*} & H^0(X) & \xrightarrow{i^*} & H^0(A) & \xrightarrow{\delta} & H^1(X,A) & \xrightarrow{i^*} & H^1(X) \\
& \xrightarrow{n_1^*} & & \xrightarrow{n_2^*} & & \xrightarrow{n_3^*} & & \xrightarrow{n_4^*} & & \xrightarrow{n_5^*} \\
\tilde{H}^0(X,A) & \xrightarrow{j^*} & \tilde{H}^0(X) & \xrightarrow{j^*} & \tilde{H}^0(A) & \xrightarrow{\delta} & \tilde{H}^1(X,A) & \xrightarrow{j^*} & H^1(X) \\
\end{array}
\]

Fig. 1—Reduced sequence

In the above diagram, \( j: (X, \emptyset) \to (X,A), \)
\( i: (X, \emptyset) \to (A, \emptyset). \) Also, \( n_1, n_2, n_3, n_4, \) and \( n_5 \) are natural maps.

**Lemma 3.1.**

In Figure 1, \( g^*m^* = j^*f^*, \) \( h^*q^* = i^*g^*, \) and \( f^*\delta = \delta h^*. \)

Proof: Let \( e \in H^0(Q,Q). \) Then \( e = \eta \phi \) for some \( \phi \in Z^0(Q,Q). \) Thus, \( g^*m^*(e) = g^*m^*(\eta \phi) = g^*\eta m^\# \phi = \eta g^\# m^\# \phi. \) Similarly, \( j^*f^*(e) = nj^\# f^\# \phi. \) Let \( x \in X. \) Then \( g^\# m^\# \phi(x) = m^\# \phi(g(x)) = m^\# \phi(Q) = \phi(m(Q)) = \phi(Q). \) Also, \( j^\# f^\# \phi(x) = f^\# \phi(j(x)) = f^\# \phi(x) = \phi(f(x)) = \phi(Q). \) Therefore, \( g^\# m^\# \phi = j^\# f^\# \phi. \) Hence, \( ng^\# m^\# \phi = nj^\# f^\# \phi. \) Therefore, \( g^*m^*(e) = j^*f^*(e). \) Hence, \( g^*m^* = j^*f^*. \)
In a similar manner, \( h^*q^* = i^*g^* \). By Theorem 3, \( f^*\delta = \delta h^* \).

**Lemma 3.2.** In Figure 1, \( j^* \), \( i^* \), and \( \delta \) induce maps \( \tilde{j}^* \), \( \tilde{i}^* \), and \( \tilde{\delta} \).

**Proof:** All the natural maps are onto. Let \( a \in \ker \eta_1 \). Therefore, \( a \in f^*H^0(Q,Q) \). So, \( a = f^*(a') \) for some \( a' \in H^0(Q,Q) \). By Lemma 3.1, \( j^*(a) = j^*(f^*a') = g^*m^*a' \). Therefore, \( j^*a \in g^*H^0(Q) \). Hence, \( j^*a \in \ker \eta_2 \). Therefore, \( j^*\ker \eta_1 \subseteq \ker \eta_2 \). Hence, by Preliminary 7, there exists an \( R \)-homomorphism \( j^*: H^0(X,A) \rightarrow \tilde{H}^0(X) \).

Now let \( a \in \ker \eta_2 \). Then \( a \in g^*H^0(Q) \). Thus, \( a = g^*a' \) for some \( a' \in H^0(Q) \). By Lemma 3.1, \( i^*(a) = i^*(g^*a') = h^*q^*a' \). Therefore, \( i^*(a) \in h^*H^0(Q) = \ker \eta_3 \). Therefore, \( i^*\ker \eta_2 \subseteq \ker \eta_3 \). Hence, by Preliminary 7, there exists an \( R \)-homomorphism \( i^*: H^0(X) \rightarrow \tilde{H}^0(A) \).

Now let \( a \in \ker \eta_3 \). Then \( a \in h^*H^0(Q) \). Thus, \( a = h^*a' \) for some \( a' \in H^1(Q,Q) \). By Lemma 3.1, \( \delta(a) = \delta(h^*a') = f^*\delta a' \). Therefore, \( \delta a \in f^*H^1(Q,Q) = \ker \eta_4 \). Thus, \( \delta \ker \eta_3 \subseteq \ker \eta_4 \). Hence, by Preliminary 7, there exists an \( R \)-homomorphism \( \delta: \tilde{H}^0(A) \rightarrow \tilde{H}^1(X,A) \).

**Theorem 10.** The bottom line of Figure 1 is an exact sequence.

**Proof:** By Theorem 4, the middle line of Figure 1 is an exact sequence. To show \( \text{Im} \tilde{j}^* = \ker \tilde{i}^* \), let \( j^*(n) \in \tilde{H}^0(X) \) where \( n \in \tilde{H}^0(X,A) \). Then \( n = \eta_1n' \) for some \( n' \in H^0(X,A) \). Thus, \( \tilde{i}^*j^*(n) = \tilde{i}^*j^*\eta_1n' = \tilde{i}^*\eta_2j^*n' = \eta_3i^*j^*n' \). Since
j*n' \in \text{Im } j^*, \ i*j*n' = 0. \ \text{Therefore, } \eta_2 i*n' = i^*j*n = 0.

Hence, j*n \in \ker \tilde{i}^*. \ \text{Therefore, } \text{Im } j^* \subseteq \ker \tilde{i}^*. \ \text{Now let } e \in \ker \tilde{i}^* \text{ where } e \in \tilde{H}^0(X). \ \text{Then } e = \eta_2 e_1 \text{ for some } e_1 \in \tilde{H}^0(X), \text{ and } \tilde{i}^*(e) = 0. \ \text{Thus, } \tilde{i}^* e = \tilde{i}^* \eta_2 e_1 = \eta_3 i^* e_1 = 0.

Therefore, i^*(e_1) \in \ker \eta_3 = \tilde{h}^0(Q). \ \text{Hence, } i^*(e_1) = \tilde{h}^*(e_2) \text{ for some } e_2 \in \tilde{H}^0(Q). \ \text{By Theorem 2, } h^* = i^*g^*. \ \text{Thus, } \tilde{h}^*(e_2) = i^*g^*(e_2) = i^*(e_1). \ \text{Therefore, } i^*(e_1 - g^*e_2) = 0.

Hence, } e_1 - g^*e_2 \in \ker i^* = \text{Im } j^*. \ \text{So } e_1 - g^*e_2 = j^*(a) \text{ for some } a \in \tilde{H}^0(X,A). \ \text{Then, } \eta_1 a \in \tilde{H}^0(X,A). \ \text{Further, } \tilde{j}^*(\eta_1 a) = \eta_2 j^*(a) = \eta_2(e_1 - g^*e_2) = \eta_2 e_1 - \eta_2 g^*e_2 = \eta_2(e_1) = 0 = \eta_2(e_1) = e. \ \text{Therefore, } e \in \text{Im } \tilde{j}^*. \ \text{Therefore, } \ker i^* \subseteq \text{Im } \tilde{j}^*. \ \text{Hence, } \text{Im } \tilde{j}^* = \ker \tilde{i}^*.

To show } \text{Im } \tilde{i}^* = \ker \tilde{\delta}, \text{ let } \tilde{i}^* n \in \tilde{H}^0(A) \text{ where } n \in \tilde{H}^0(X). \ \text{Then } n = \eta_2 n' \text{ for some } n' \in \tilde{H}^0(X). \ \text{Thus, } \tilde{\delta} \tilde{i}^* (n) = \tilde{\delta} \tilde{i}^* \eta_2 n' = \tilde{\delta} \eta_3 i^* n' = \eta_3 \tilde{\delta} i^* n. \ \text{Since } i^* n' \in \text{Im } \tilde{H}^0(X), \ \tilde{\delta} i^* n' = 0. \ \text{Therefore, } \eta_3 \tilde{\delta} i^* n = \tilde{\delta} i^* n = 0. \ \text{Therefore, } \tilde{i}^* n \in \ker \tilde{\delta}. \ \text{Thus, } \text{Im } \tilde{i}^* \subseteq \ker \tilde{\delta}. \ \text{Now let } e \in \ker \tilde{\delta} \text{ where } e \in \tilde{H}^0(A). \ \text{Then } e = \eta_3 e_1 \text{ for some } e_1 \in \tilde{H}^0(A). \ \text{Further, } \tilde{\delta}(e) = \tilde{\delta}(\eta_3 e_1) = \eta_4 \delta e_1. \ \text{By Theorem 7, } \tilde{H}^1(Q,Q) = 0. \ \text{Therefore, } \tilde{H}^1(X,A) = \tilde{H}^1(X,A) / f^* \tilde{H}^1(Q,Q) = \tilde{H}^1(X,A) / 0 = \tilde{H}^1(X,A).

Thus, } \tilde{\delta}(e) = \eta_4 \delta e_1 = \delta(e_1) = 0. \ \text{Therefore, } e_1 \in \ker \delta = \text{Im } i^*. \ \text{Hence, } e_1 = i^* e_2 \text{ for some } e_2 \in \tilde{H}^0(X). \ \text{Then } \eta_2 e_2 \in \tilde{H}^0(X). \ \text{Further, } \tilde{i}^*(\eta_2 e_2) = \eta_3 i^* e_2 = \eta_3 e_1 = e. \ \text{Therefore, } e \in \text{Im } \tilde{i}^*. \ \text{Hence, } \ker \tilde{\delta} \subseteq \text{Im } \tilde{i}^*. \ \text{Hence, } \text{Im } \tilde{i}^* = \ker \tilde{\delta}. \


To show \( \text{Im} \, \tilde{\delta} = \ker \tilde{j}^* \), let \( \tilde{\delta}(n) \in \tilde{H}^1(X,A) \) where \( n \in H^0(A) \). Then \( n = n_3n' \) for some \( n' \in H^0(A) \). Further, \( \tilde{j}^*\tilde{\delta}(n) = \tilde{j}^*\tilde{j}^*\delta(n_3n') = \tilde{j}^*\eta_5\delta n' = \eta_5j^*\delta n' \). Since \( \text{Im} \, \delta = \ker j^* \), \( j^*\delta n' = 0 \). Thus, \( \eta_5j^*\delta n' = \tilde{j}^*\tilde{\delta}(n) = \eta_5(0) = 0 \). Therefore, \( \tilde{\delta}(n) \in \ker \tilde{j}^* \). Thus, \( \text{Im} \, \tilde{\delta} \subseteq \ker \tilde{j}^* \). Now let \( e \in \ker \tilde{j}^* \) where \( e \in \tilde{H}^1(X,A) \). Then \( e = \eta_4e_1 \) for some \( e_1 \in H^1(X,A) \). Also, \( \tilde{j}^*(e) = \tilde{j}^*(\eta_4e_1) = \eta_5j^*e_1 = 0 \). Therefore, \( j^*e_1 = 0 \). Thus, \( e_1 \in \ker j^* = \text{Im} \, \delta \). Therefore, \( e_1 = \delta(e_2) \) for some \( e_2 \in H^0(A) \). Then \( \eta_3e_2 \in H^0(A) \). Further, \( \tilde{\delta}(\eta_3e_2) = \eta_4\delta e_2 = \eta_4e_1 = e \). Therefore, \( e \in \text{Im} \, \tilde{\delta} \). Thus, \( \ker \tilde{j}^* \subseteq \text{Im} \, \tilde{\delta} \). Hence, \( \text{Im} \, \tilde{\delta} = \ker \tilde{j}^* \).

**Theorem 11.** The following are equivalent:

1. \( g^* \) is onto,
2. \( \tilde{H}^0(X) \cong 0 \).

**Proof:** Assume \( g^* \) is onto. Then \( g^*H^0(Q) = H^0(X) \). Thus, \( \tilde{H}^0(X) = H^0(X)/g^*H^0(Q) = H^0(X)/H^0(X) \cong 0 \).

Assume \( \tilde{H}^0(X) \cong 0 \). Then \( H^0(X)/g^*H^0(Q) = 0 \). Thus, \( g^*H^0(Q) = H^0(X) \). Hence, \( g^* \) is onto.

**Lemma 3.3.** (The Modification Lemma).

Let \( A \) and \( B \) be closed subsets of a fully normal space \( X \).

Let \( \beta \) be an open cover of \( X \). Then there exists a function \( f : X \to X \) (not necessarily continuous), an open cover \( \gamma \) of \( X \), and an open set \( N \) such that:

1. \( N \supseteq A \),
2. \( f|_{(X \setminus N) \cup A} \) is the identity,
3. \( f(N \cup B) \subseteq A \cap B \),
(4) \( f(\overline{N} \setminus A) \subseteq A \), and

(5) if \( V \in \gamma \), then there exists \( U \in \beta \) such that \( f(V) \subseteq V^* \subseteq U \).

Proof: Since \( X \) is fully normal, there is an open cover \( \gamma_0 \) of \( X \) such that \( \gamma_0 < \beta \). Let

\[
\gamma_1 = \{ V \cap (X \setminus A) \mid V \in \gamma_0, V \cap (X \setminus A) \neq \emptyset \},
\]

\[
\gamma_2 = \{ V \cap (X \setminus B) \mid V \in \gamma_0, V \cap (X \setminus B) \neq \emptyset \}, \text{ and}
\]

\[
\gamma_3 = \{ V \in \gamma_0 \mid V \cap A \cap B \neq \emptyset \}. \text{ Let } \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3. \text{ Then}
\]

\( \gamma \) is an open cover of \( X \). Let \( V \in \gamma \). If \( V \in \gamma_1 \), then \( V \subseteq V_1 \) for some \( V_1 \in \gamma_0 \). Thus, there exists \( U \in \beta \) such that \( V_1 \subseteq V \subseteq U \). Hence, \( V^* \subseteq U \). Similarly, if \( V \in \gamma_2 \), there exists \( U \in \beta \) such that \( V^* \subseteq U \). If \( V \in \gamma_3 \), then \( V \in \gamma_0 \). Thus, there exists \( U \in \beta \) such that \( V^* \subseteq U \). In any case, \( V^* \subseteq U \) for some \( U \in \beta \).

Hence, \( \gamma^* \subseteq B \). Let \( M = \{ V \in \gamma \mid V \cap A \neq \emptyset \} \). Then \( \cup M \) is an open set containing \( A \). By Theorem 9, \( X \) is normal. Hence, there exists an open set \( N \) such that \( A \subseteq N \subseteq \overline{N} \subseteq \cup M \). For every \( V \in M \), define \( a_V \) as follows: if \( V \in \gamma_1 \) or \( \gamma_2 \), fix \( a_V \in V \cap A \); if \( V \in \gamma_3 \), fix \( a_V \in V \cap A \cap B \). Define \( f: X \rightarrow X \) in the following way. For \( x \in (X \setminus \overline{N}) \cup A \), let \( f(x) = x \). For \( x \in \overline{N} \setminus A \), choose some \( V \in M \) such that \( x \in V \). Let \( f(x) = a_V \).

From the definition of \( f \), \( f \mid (X \setminus \overline{N}) \cup A \) is the identity. Also, from the way the \( a_V \)'s were chosen, \( f(\overline{N} \setminus A) \subseteq A \). Let \( x \in \overline{N} \cap B \). Since \( x \in \overline{N} \), \( x \in V \) for some \( V \in M \). Therefore, \( V \cap A \neq \emptyset \).

Thus, \( V \in \gamma_1 \). Since \( x \in B \), \( V \in \gamma_2 \). Hence, \( V \in \gamma_3 \). Therefore, \( f(x) = a_V \subseteq A \cap B \). Hence, \( f(\overline{N} \cap B) \subseteq A \cap B \). Thus, (1) - (4) are satisfied.
Now let \( V \in \gamma \). Let \( x \in V \). If \( x \in (X \setminus N) \cup A \), then
\[ f(x) = x. \] Therefore, \( f(x) \in V \subseteq V^* \). If \( x \in \overline{N} \setminus A \), then
\[ f(x) \in V_1 \] for some \( V_1 \in \gamma \) such that \( x \in V_1 \). Thus,
\[ V \cap V_1 \neq \emptyset. \] Hence, \( V_1 \subseteq V^* \). Therefore, \( f(x) \in V^* \). In any case, \( f(x) \in V^* \). Hence, \( f(v) \subseteq V^* \). Furthermore, \( V \subseteq V_0 \) for some \( V_0 \in \gamma_0 \). Thus, \( V^* \subseteq V_0^* \). Since \( \gamma^* < \beta \), there exists \( U \in \beta \) such that \( V_0^* \subseteq U \). Therefore, \( V^* \subseteq U \).

Hence, \( f(U) \subseteq V^* \subseteq U \). Hence, (5) is satisfied. Thus, the proof of the Modification Lemma is complete.

**Lemma 3.4.** Let \( A \) be a closed subset of \( X \). Let \( \phi \in Z^p(X,A) \). Then there exists an open cover \( \beta \) of \( X \) such that \( \phi|\beta^{p+1} \cap A^{p+1} = 0 \), and \( \delta_p \phi|\beta^{p+2} \cap X^{p+2} = 0 \).

**Proof:** If \( \phi \in Z^p(X,A) \), then \( \phi \in C^p(X,A) \), and
\[ \delta_p \phi \in C^{p+1}(X,X). \] Thus, there exists an open cover \( \beta_1 \) of \( A \) such that \( \phi|\beta_1^{p+1} \cap A^{p+1} = 0 \). Also, there exists an open cover \( \beta_2 \) of \( X \) such that \( \delta_p \phi|\beta_2^{p+2} \cap X^{p+2} = 0 \). Since \( A \) is closed, \( \beta_1 \cup (X \setminus A) \) is an open cover of \( X \). Let
\[ \beta = \{ U \cap V | U \in \beta_1 \cup (X \setminus A), V \in \beta_2, U \cap V \neq \emptyset \}. \] Let
\[ (x_0, \ldots, x_p) \in \beta^{p+1} \cap A^{p+1}. \] Then \( (x_0, \ldots, x_p) \in (U \cap V)^{p+1} \) for some \( U \cap V \in \beta \). Then \( U \in \beta_1 \cup (X \setminus A) \). Since
\[ (x_0, \ldots, x_p) \in A^{p+1}, U \neq X \setminus A. \] Thus, \( U \in \beta_1 \). Hence,
\[ (x_0, \ldots, x_p) \in U^{p+1}. \] Therefore, \( (x_0, \ldots, x_p) \in \beta_1^{p+1} \cap A^{p+1} \).

Thus, \( \phi(x_0, \ldots, x_p) = 0 \). Now let \( (x_0, \ldots, x_p) \in \beta^{p+2} \cap X^{p+2} \).

Then \( (x_0, \ldots, x_p) \in (U \cap V)^{p+2} \) for some \( U \cap V \in \beta \). Thus,
\[ V \in \beta_2. \] Also, \( (x_0, \ldots, x_p) \in V^{p+2} \). Hence,
Let \( A \) and \( B \) be closed subsets of a fully normal space \( X \).

Let \( h \in H^p(A, A \cap B) \). Then there exists an open set \( N \), and there exists \( e \in H^p(N, N \cap B) \) such that \( A \subseteq N \), and \( i^*(e) = h \) where \( i: (A, A \cap B) \rightarrow (N, N \cap B) \).

**Proof:** Since \( h \in H^p(A, A \cap B) \), then \( h = \eta \phi \) for some \( \phi \in Z^p(A, A \cap B) \). Now \( A \cap B \) is closed in \( A \). Thus, by Lemma 3.4, there exists a cover \( \beta_1 \) of \( A \) by sets open in \( A \) such that \( \phi|\beta_1 \cap (A \cap B) = 0 \), and \( \delta_p \phi|\beta_1 \cap A = 0 \). Let \( \beta = \{ U \subseteq X | U \) is open in \( X \), \( U \cap A \in \beta_1 \} \). Then \( \beta \cup (X \setminus A) \) is an open cover of \( X \). By the Modification Lemma, there exists a function \( f: X \rightarrow X \), an open cover \( \gamma \) of \( X \), and an open set \( N \) such that \( A \subseteq N \), \( f|\gamma \cap (X \setminus A) \) is the identity, \( f(N \cap B) \subseteq A \cap B \), \( f(N \setminus A) \subseteq A \), and if \( V \in \gamma \), then there exists \( U \in \beta \) such that \( f(V) \subseteq V^* \subseteq U \). Define \( \phi_1 \in C^p(N) \) by

\[
\phi_1(x_0, \ldots, x_p) = \phi(f(x_0), \ldots, f(x_p)) \quad \text{for all} \quad (x_0, \ldots, x_p) \in N^{p+1}.
\]

Let \( \gamma_1 = \{ V \cap N | V \in \gamma, V \cap N \neq \emptyset \} \). Then \( \gamma_1 \) is a cover of \( N \cap B \) by sets open in \( N \). Let \( (x_0, \ldots, x_p) \in N^{p+1} \). Then \( \gamma_1 \) is a cover of \( N \cap B \) by sets open in \( N \). Let \( (x_0, \ldots, x_p) \in \gamma_1 \cap (N \cap B)^{p+1} \). Then there exists \( V \in \gamma \) such that \( (x_0, \ldots, x_p) \in V^{p+1} \). By the Modification Lemma, there exists \( U \in \beta \) such that \( f(V) \subseteq U \).

Now \( U \not\subseteq X \setminus A \) since \( f(N \cap B) \subseteq A \cap B \). Therefore, \( f(x_0, \ldots, f(x_p)) \in \beta^{p+1} \cap (N \cap B)^{p+1} \). Since
\[ f(\bar{N} \cap B) \subseteq A \cap B, (f(x_0), \ldots, f(x_p)) \in (A \cap B)^{P+1}. \] Therefore, 
\[ (f(x_0), \ldots, f(x_p)) \in \beta^{P+1} \cap (A \cap B)^{P+1}. \] Therefore, 
\[ \phi_1(x_0, \ldots, x_p) = \phi(f(x_0), \ldots, f(x_p)) = 0. \] Hence, 
\[ \phi_1|_\gamma_1^{P+1} \cap (\bar{N} \cap B)^{P+1} = 0. \] Thus, \( \phi_1 \in CP(\bar{N}, \bar{N} \cap B). \)

Now let \( (x_0, \ldots, x_{P+1}) \in \gamma_1^{P+2} \cap (\bar{N})^{P+2}. \) Then there exists \( V \in \gamma \) such that \( (x_0, \ldots, x_{P+1}) \in V^{P+2}. \) By the Modification Lemma, there exists \( U \in \beta \cup (X \setminus A) \) such that 
\[ f(V) \subseteq U. \] Now \( U \neq X \setminus A \) since \( f(\bar{N}) \subseteq A. \) Also, 
\[ (f(x_0), \ldots, f(x_{P+1})) \in \beta^{P+2} \cap \bar{N}^{P+2}, \] and 
\[ (f(x_0), \ldots, f(x_{P+1})) \in A^{P+2}. \] Therefore, 
\[ (f(x_0), \ldots, f(x_{P+1})) \in \beta^{P+2} \cap A^{P+2}. \] Thus, 
\[ \delta_1\phi_1(x_0, \ldots, x_{P+1}) = \delta \phi(f(x_0), \ldots, f(x_{P+1})) = 0. \] Hence, 
\[ \delta_1\phi_1|_\gamma_1^{P+2} \cap \bar{N}^{P+2} = 0. \] Therefore, \( \delta_1\phi_1 \in CP^{+1}(\bar{N}, \bar{N}). \)

Hence, \( \phi_1 \in ZP(\bar{N}, \bar{N} \cap B). \)

Let \( e = \eta \phi_1 \in H^{P}(\bar{N}, \bar{N} \cap B). \) Then \( i*e = \eta i^\# \phi_1. \) Let 
\( (x_0, \ldots, x_p) \in A^{P+1}. \) Then \( i^\# \phi_1(x_0, \ldots, x_p) = \phi_1(i(x_0), \ldots, i(x_p)) = \phi_1(x_0, \ldots, x_p) = \phi(f(x_0), \ldots, f(x_p)). \) Since \( f|A \) is the identity, \( \phi(f(x_0), \ldots, f(x_p)) = \phi(x_0, \ldots, x_p) = \phi_1(x_0, \ldots, x_p). \) Therefore, \( i^\# \phi_1 = \phi. \) Hence, \( \eta i^\# \phi_1 = n\phi = h. \) Therefore, 
\( i*e = h. \)

Lemma 3.5. Let \( A \) and \( B \) be closed subsets of \( X. \) Let 
\[ t: (A, A \cap B) \xrightarrow{\phi} (X, B). \] Then \( r^\# [BP(X, B)] = BP(A, A \cap B). \)

Proof: By Note 2.7, \( t^\# [BP(X, B)] \subseteq BP(A, A \cap B). \) Let 
\( \phi \in BP(A, A \cap B). \) Then \( \phi = \delta_1^{P-1} \alpha_1 + \alpha_2 \) where 
\( \alpha_1 \in CP^{-1}(A, A \cap B), \) and \( \alpha_2 \in CP(A, A). \) Then there 
exists a cover \( \beta \) of \( A \cap B \) by sets open in \( A \) such that 
\( \alpha_1|_\beta \cap (A \cap B) = 0. \) Also, there exists a cover \( \gamma \) of \( A \)
by sets open in $A$ such that

$$\alpha_2 \gamma \cap A \cap = 0.$$ Let $\beta_1 = \{U \cap A \in \beta\}$, and let $\gamma_1 = \{V \cap A \in \gamma\}$. Define

$$\phi_1 \in \mathcal{C}^{P-1}(X) \text{ by } \phi_1(x_0, \ldots, x_{p-1}) =
\begin{cases}
\alpha_1(x_0, \ldots, x_{p-1}) & \text{if } (x_0, \ldots, x_{p-1}) \in A \cap \\
0 & \text{otherwise.}
\end{cases}
$$

Since $A$ is closed, $\beta_1 \cup (X \setminus A)$ is an open cover of $B$.

Let $(x_0, \ldots, x_{p-1}) \in (\beta_1 \cup (X \setminus A))^P \cap B$. If

$$(x_0, \ldots, x_{p-1}) \in (X \setminus A)^P,$$

then $\phi_1(x_0, \ldots, x_{p-1}) = 0$.

If $(x_0, \ldots, x_{p-1}) \in \beta_1^P$, then $(x_0, \ldots, x_{p-1}) \in U^P$ for

some $U \in \beta_1$. If $x_i \notin A$ for some $0 \leq i \leq p-1$, then

$$\phi_1(x_0, \ldots, x_{p-1}) = 0.$$ If $x_i \in A$ for $0 \leq i \leq p-1$, then

$$(x_0, \ldots, x_{p-1}) \in (U \cap A)^P.$$ Therefore,

$$(x_0, \ldots, x_{p-1}) \in \beta^P \cap (A \cap B)^P.$$ Therefore,

$$\phi_1(x_0, \ldots, x_{p-1}) = \alpha_1(x_0, \ldots, x_{p-1}) = 0.$$ In any case,

$$\phi_1(x_0, \ldots, x_{p-1}) = 0.$$ Hence, $\phi_1|_{\beta_1 \cup (X \setminus A)^P \cap B} = 0$.

Thus, $\phi_1 \in \mathcal{C}^{P-1}(X,B)$.

Define $\phi_2 \in \mathcal{C}^P(X)$ by $\phi_2(x_0, \ldots, x_p) =
\begin{cases}
\alpha_2(x_0, \ldots, x_p) & \text{if } (x_0, \ldots, x_p) \in A \cap \\
0 & \text{otherwise.}
\end{cases}
$$

Since $A$ is closed, $\gamma_1 \cup (X \setminus A)$ is an open cover of $X$.

Let $(x_0, \ldots, x_p) \in (\gamma_1 \cup (X \setminus A))^P$. If $(x_0, \ldots, x_p) \in (X \setminus A)^P$, then

$$(x_0, \ldots, x_p) \in \gamma^P$$ for some $V \in \gamma_1$. If $x_i \in A$ for $0 \leq i \leq p$, then

$$\phi_2(x_0, \ldots, x_p) = 0.$$ If $x_i \notin A$ for $0 \leq i \leq p$, then

$$(x_0, \ldots, x_p) \in (V \cap A)^P.$$ Therefore, $(x_0, \ldots, x_p) \in \gamma^P \cap A^P$. 

Thus, $\phi_2 \in \mathcal{C}^P(X)$.
Thus, $\phi_2(x_0, \ldots, x_p) = \alpha_2(x_0, \ldots, x_p) = 0$. In any case, $\phi_2(x_0, \ldots, x_p) = 0$. Hence, $\phi_2[\gamma_1 \cup (X \setminus A)]^{P+1} = 0$. Therefore, $\phi_2 \in C^P(X, X)$.

Now $\delta_p-1\phi_1 + \phi_2 \in B^P(X, B)$. Let $(x_0, \ldots, x_p) \in A^P$.

Then $t^#(\delta_p-1\phi_1 + \phi_2)(x_0, \ldots, x_p) = (\delta_p-1\phi_1 + \phi_2)(t(x_0), \ldots, t(x_p)) = (\delta_p-1\phi_1 + \phi_2)(x_0, \ldots, x_p) = \delta_p-1\alpha_1(x_0, \ldots, x_p) + \alpha_2(x_0, \ldots, x_p) = \phi(x_0, \ldots, x_p).

Therefore, $t^#(\delta_p-1\phi_1 + \phi_2) = \phi$. Hence, $\phi \in t^#[B^P(X, B)]$.

Therefore, $B^P(A, A \cap B) \subseteq t^#[B^P(X, B)]$. Hence, $t^#[B^P(X, B)] = B^P(A, A \cap B)$.

**Theorem 13.** (Reduction Theorem).

Let $A$ and $B$ be closed subsets of a fully normal space $X$.

Let $j: (A, A \cap B) \xrightarrow{\text{r}} (X, B)$. Let $h \in H^P(X, B)$ such that $j^*(h) = 0$. Then there exists an open set $N$ such that $A \subseteq N$, and $k^*(h) = 0$, where $k: (\overline{N}, \overline{N} \cap B) \xrightarrow{\text{r}} (X, B)$.

Proof: Since $h \in H^P(X, B)$, $h = n\phi$ for some $\phi \in Z^P(X, B)$. Therefore, there exists an open cover $\beta$ of $X$ such that $\delta_p\phi|\beta^{P+2} = 0$. By the Modification Lemma, there exists an open set $N$ containing $A$. Let $k: (\overline{N}, \overline{N} \cap B) \xrightarrow{\text{r}} (X, B)$.

Since $j^*(h) = 0$, $j^# \phi \in B^P(X, B)$. By Lemma 3.5, $B^P(A, A \cap B) = j^#[B^P(X, B)]$. Therefore, $\phi \in B^P(X, B)$.

Also, by Lemma 3.5, $B^P(\overline{N}, \overline{N} \cap B) = k^#[B^P(X, B)]$. Therefore, $k^# \phi \in B^P(\overline{N}, \overline{N} \cap B)$. Thus, $nk^# \phi = k^*(h) = 0$.

**Lemma 3.6.** Let $M$, $M_1$, $M_2$, $L_1$, and $L_2$ be $R$-modules.

Let $f_1$, $f_2$, $g_1$, and $g_2$ be $R$-homomorphisms such that
Im $f_1 = \ker g_1$, Im $f_2 = \ker g_2$, and $g_2f_1$, $g_1f_2$ are onto isomorphisms.

Then (1) $M = \text{Im } f_1 \oplus \text{Im } f_2 = \ker g_1 \oplus \ker g_2$;

(2) $g_1| \ker g_2$ is an isomorphism, and $g_2| \ker g_1$ is an isomorphism;

(3) $x \in \text{Im } f_1$ implies $x = f_1(g_2f_1)^{-1}g_2(x)$, and $x \in \text{Im } f_2$ implies $x = f_2(g_1f_2)^{-1}g_1(x)$;

(4) For every $x \in M$, $x = f_1(g_2f_1)^{-1}g_2(x) + f_2(g_1f_2)^{-1}g_1(x)$.

Proof: Since $g_2f_1$ is an isomorphism, Preliminary 6 gives $M = \text{Im } f_1 \oplus \ker g_2 = \text{Im } f_1 \oplus \text{Im } f_2 = \ker g_1 \oplus \ker g_2$. Thus, (1) is satisfied.

By Preliminary 3, $\ker g_2$ is a submodule of $M$. Therefore, $g_1| \ker g_2$ is an R-homomorphism. Let $a, b \in \ker g_2$. Then $a, b \in \text{Im } f_2$. Thus, $a = f_2(a')$, and $b = f_2(b')$. If $g_1(a) = g_1(b)$, then $g_1f_2(a') = g_1f_2(b')$. Since $g_1f_2$ is 1-1, $a' = b'$. Thus, $f_1(a') = f_1(b')$. Hence, $a = b$.

Therefore, $g_1| \ker g_2$ is 1-1. Let $e \in M_1$. Since $g_1f_2$ is onto $M_1$, there exists $e' \in L_2$ such that $g_1f_2(e') = e$. Thus, $f_2e' \in \text{Im } f_2 = \ker g_2$. Therefore, $g_1(f_2e') = e$. Hence, $g_1| \ker g_2$ is onto. Therefore, $g_1| \ker g_2$ is an isomorphism. Similarly, $g_2| \ker g_1$ is an isomorphism. Thus, (2) is satisfied.
Let $x \in \text{Im } f_1$. Then $x \in \ker g_1$. Now since $g_2 f_1$ is an isomorphism, $g_2(x) = g_2(f_1(g_2 f_1)^{-1} g_2(x))$. Therefore, since $g_2|\ker g_1$ is 1-1, $x = f_1(g_2 f_1)^{-1} g_2(x)$. Similarly, $x \in \text{Im } f_2$, implies $x = f_2(g_1 f_2)^{-1} g_1(x)$. Thus, (3) is satisfied.

Let $x \in M$. By (1) of this lemma, $x = f_1(y) + f_2(z)$, where $y \in L_1$, $z \in L_2$. By (3) of this lemma,

$$f_1(y) = f_1(g_2 f_1)^{-1} g_2(f_1(y)),$$
and

$$f_2(z) = f_2(g_1 f_2)^{-1} g_2(f_2(z)).$$

Since $\text{Im } f_1 = \ker g_1$, and $\text{Im } f_2 = \ker g_2$, $g_2(x) = g_2(f_1(y) + f_2(z)) = g_2(f_1(y)) + g_2(f_2(z)) = g_2 f_1(y) + g_2 f_2(z)$.

Thus, $x = f_1(y) + f_2(z) = f_1(g_2 f_1)^{-1} g_2(f_1(y)) + f_2(g_1 f_2)^{-1} g_1(f_2(z)) = f_1(g_2 f_1)^{-1} f_1(x) + f_2(g_1 f_2)^{-1} g_1(x)$. Thus, (4) is satisfied.

The Mayer-Vietoris Theorem

Consider the following diagram, page 54. In Figure 2, $X$ is a fully normal space, $X_1, X_2$ are closed subsets of $X$ such that $X = X_1 \cup X_2$, and $A = X_1 \cap X_2$. Also,

$$i_2: (X_2, A) \rightarrow (X, A),$$
$$i: A \rightarrow X,$$
$$j_1: (X, A) \rightarrow (X, X_1),$$
$$j_2: (X, A) \rightarrow (X, X_2),$$
$$j: X \rightarrow (X, A),$$
$$l_1: A \rightarrow X_1,$$
$$l_2: A \rightarrow X_2.$$
Fig. 2—M-V Sequence
Using Theorem 4, the following sequences are exact:

(1) $H^q(X, X_1) \xrightarrow{n_1^*} H^q(x) \xrightarrow{P_1^*} H^q(x_1)$,
(2) $H^q(X, X_2) \xrightarrow{n_2^*} H^q(x) \xrightarrow{P_2^*} H^q(x_2)$,
(3) $H^q(X_1) \xrightarrow{i_1^*} H^q(A) \xrightarrow{j_1^*} H^q(x_1, A)$,
(4) $H^q(X_2) \xrightarrow{i_2^*} H^q(A) \xrightarrow{j_2^*} H^q(x_2, A)$,
(5) $H^{q-1}(A) \xrightarrow{\delta} H^q(x, A) \xrightarrow{j^*} H^q(x) \xrightarrow{i^*} H^q(A)$.

Using Theorem 4A, the following sequences are exact:

(6) $H^q(X, X_1) \xrightarrow{j_1^*} H^q(X, A) \xrightarrow{i_1^*} H^q(x_1, A)$,
(7) $H^q(X, X_2) \xrightarrow{j_2^*} H^q(X, A) \xrightarrow{i_2^*} H^q(x_2, A)$,
(8) $H^q(X_1, A) \xrightarrow{i_1^*} H^q(x_1) \xrightarrow{j_1^*} H^q(A)$,
(9) $H^q(X_2, A) \xrightarrow{i_2^*} H^q(x_2) \xrightarrow{j_2^*} H^q(A)$.

The following are true by Theorem 2:

(10) $j^*j_1^* = n_1^*$,
(11) $j^*j_2^* = n_2^*$,
(12) $l_1^*p_1^* = i^*$,
(13) $l_2^*p_2^* = i^*$,
(14) $i_1^*j_2^* = k_1^*$,
(15) $i_2^*j_1^* = k_2^*$. 
The following are true by Theorem 3:

16) \( i_1^* \delta = \delta_1 \),
17) \( i_2^* \delta = \delta_2 \).


Let \( X \) be fully normal. Let \( X_1, X_2 \) be closed subsets of \( X \) such that \( X = X_1 \cup X_2 \). Let \( A = X_1 \cap X_2 \). Define the following:

\[ J: H^q(X) \to H^q(X_1) \oplus H^q(X_2) \]
\[ I: H^q(X_1) \oplus H^q(X_2) \to H^q(A) \]
\[ \Delta: H^q(A) \to H^{q+1}(X) \]

Then \( \cdots \to H^q(X) \to H^q(X_1) \oplus H^q(X_2) \to H^q(A) \to H^{q+1}(X) \to \cdots \) is an exact sequence.

Lemma 3.7. For \( x \in H^q(A) \), \( \Delta(x) = n_2^*k_1^*-1\delta_1(x) \).

Proof: By (6), and (7), \( \text{Im } j_1^* = \ker i_1^* \), and \( \text{Im } j_2^* = \ker i_2^* \). Also, \( i_2^*j_1^* = k_2^* \), and \( i_1^*j_2^* = k_1^* \) are isomorphisms. Thus, the hypothesis of Lemma 3.6 is satisfied. Let \( x \in H^{q-1}(A) \). Then \( \delta(x) \in H^q(X,A) \). Therefore, by Lemma 3.6, \( \delta(x) = j_1^*(i_2^*j_1^*)^{-1}i_2^*(\delta(x)) + j_2^*(i_1^*j_2^*)^{-1}i_1^*(\delta(x)) = j_1^*k_2^*-1i_2^*\delta(x) + j_2^*k_1^*-1i_1^*\delta(x) = j_1^*k_2^*-1\delta_2(x) + j_2^*k_1^*-1\delta_1(x) \). Since \( \text{Im } \delta = \ker j^* \), \( j^*(\delta(x)) = 0 \). Hence, \( j^*(j_1^*k_2^*-1\delta_2(x) + j_2^*k_1^*-1\delta_1(x)) = j^*j_1^*k_2^*-1\delta_2(x) + j^*j_2^*k_1^*-1\delta_1(x) = n_1^*k_2^*-1\delta_2(x) + n_2^*k_1^*-1\delta_1(x) = 0 \). Therefore, \( n_2^*k_1^*-1\delta_1(x) = -n_1^*k_2^*-1\delta_2(x) = \Delta(x) \). Hence, \( \Delta(x) = n_2^*k_1^*-1\delta_1(x) \).

Proof of Theorem 14: To show \( J: H^0(X) \to H^0(X_1) \oplus H^0(X_2) \) is 1-1, let \( J(x) = 0 \). Then
\((p_1^*(x), p_2^*(x)) = 0\). Hence, \(p_1^*(x) = 0\), and \(p_2^*(x) = 0\).

Therefore, \(x \in \ker p_1^* = \Im n_1^*\), and \(x \in \ker p_2^* = \Im n_2^*\).

Therefore, \(x = n_1^*(a) = n_2^*(b)\) for some \(a \in H^q(X_1, X)\),

and \(b \in H^q(X, X_2)\). Since \(n_1^* = j^*j_1^*\), and \(n_2^* = j^*j_2^*\),

\(x = j^*j_1^*(a) = j^*j_2^*(b)\). By (5), \(j^* : H^q(X, A) \to H^q(X)\) is

1-1. Thus, \(j_1^*(a) = j_2^*(b)\). Therefore, \(i_1^*j_1^*(a) = i_1^*j_2^*(b)\).

Since \(\Im j_1^* = \ker i_1^*\), \(i_1^*j_1^*(a) = 0\). Therefore,

\(i_1^*j_2^*(b) = 0\). Since \(i_1^*j_2^*\) is an isomorphism, \(b = 0\).

Hence, \(j^*j_2^*(b) = x = 0\). Hence, \(J : H^q(X) \to H^q(X_1) \oplus H^q(X_2)\)

is 1-1.

To show \(\Im \Delta \subseteq \ker J\), let \(\Delta(x) \in H^q(X)\). Then

\(J(\Delta(x)) = (p_1^*(\Delta(x)), p_2^*(\Delta(x))) = (p_1(-n_1^*k_2^* - \delta_2(x)), p_2(n_2^*k_1^* - \delta_1(x)))\). Since

\(\Im n_1^* = \ker p_1^*\), and \(\Im n_2^* = \ker p_2^*\), then

\(p_1^*(-n_1^*k_2^* - \delta_2(x)) = 0\), and \(p_2^*n_2^*k_1^* - \delta_1(x) = 0\). Therefore, \(J(\Delta(x)) = (0, 0)\). Thus, \(\Delta(x) \in \ker J\). Hence,

\(\Im \Delta \subseteq \ker J\).

To show \(\Im J \subseteq \ker I\), let \(J(x) \in H^q(X_1) \oplus H^q(X_2)\).

Then \(I(J(x)) = I(p_1^*(x), p_2^*(x)) = l_1^*p_1^*(x) - l_2^*p_2^*(x)\).

By (12), and (13), \(l_1^*p_1^*(x) = i^*(x)\), and \(l_2^*p_2^*(x) = i^*(x)\).

Thus, \(I(J(x)) = i^*(x) - i^*(x) = 0\). Therefore, \(J(x) \in \ker I\).

Hence, \(\Im J \subseteq \ker \Delta\).

To show \(\Im I \subseteq \ker \Delta\), let \(I(x, y) \in H^q(A)\). Then

\(\Delta(I(x, y)) = \Delta(l_1^*(x) - l_2^*(y)) = -n_1^*k_2^* - \delta_2(l_1^*(x) - l_2^*(y)) = -n_1^*k_2^* - \delta_2 l_1^*(x) + n_1^*k_2^*\delta_2 l_2^*(y) = n_2^*k_1^* - \delta_1 l_1^*(x) + n_1^*k_2^*\delta_2 l_2^*(y)\). By (3) and (4),
\[ \delta_1 \ast_1 (x) = 0, \text{ and } \delta_2 \ast_2 (x) = 0. \] Thus, \( \Delta(I(x,y)) = 0 \ast 0 = 0. \)

Therefore, \( I(x,y) \in \ker \Delta. \) Hence, \( \text{Im } I \subseteq \ker \Delta. \)

To show \( \ker J \subseteq \text{Im } \Delta, \) let \( y \in \ker J, \) where \( x \in H^q(X). \)

Then \( J(x) = (p_1 \ast(x), p_2 \ast(x)) = 0. \) Therefore, \( p_1 \ast(x) = 0, \)
and \( p_2 \ast(x) = 0. \) Hence, \( x \in \ker p_1 \ast = \text{Im } n_1 \ast, \) and \( x \in \ker p_2 \ast = \text{Im } n_2 \ast. \)

Therefore, \( x = n_1 \ast(x_1) = n_2 \ast(x_2), \) where \( x_1 \in H^q(X,X_1), \)
and \( x_2 \in H^q(X,X_2). \) By (10) and (11), \( x = j \ast_1 \ast(x_1) = j \ast_2 \ast(x_2). \)

Therefore, \( j \ast_1 \ast(x_1) - j \ast_2 \ast(x_2) = j \ast (j \ast_1 \ast(x_1) - j \ast_2 \ast(x_2)) = 0. \)
Thus, \( j \ast_1 \ast(x_1) - j \ast_2 \ast(x_2) \in \ker \ast = \text{Im } \delta. \) Hence, \( j \ast_1 \ast(x_1) - j \ast_2 \ast(x_2) = \delta(y) \) for some \( y \in H^{q-1}(A). \) By (16), (15), and (7), \( \delta_2(y) = i_2 \ast_2 \delta(y) = i_2 \ast(j \ast_1 \ast(x_1) - j \ast_2 \ast(x_2)) = i_2 \ast j \ast_1 \ast(x_1) - i_2 \ast j \ast_2 \ast(x_2) = i_2 \ast j \ast_1 \ast(x_1) - 0 = i_2 \ast j \ast_1 \ast(x_1). \)

Thus, \( \Delta(-y) = -n_1 \ast k_2 \ast^{-1} \delta_2(-y) = -n_1 \ast k_2 \ast^{-1}(-k_2 \ast(x_1)) = n_1 \ast k_2 \ast^{-1}(k_2 \ast(x_1)) = n_1 \ast(x_1) = x. \) Therefore, \( x \in \text{Im } \Delta. \)
Hence, \( \ker J \subseteq \text{Im } \Delta. \)

To show \( \ker \Delta \subseteq \text{Im } I, \) let \( x \in \ker \Delta, \) where \( x \in H^q(A). \)

Then \( \Delta(x) = n_2 \ast k_1 \ast^{-1} \delta_1(x) = j \ast_2 \ast k_1 \ast^{-1} \delta_1(x) = 0. \) Therefore, \( j \ast_2 \ast k_1 \ast^{-1} \delta_1(x) \in \ker \ast = \text{Im } \delta. \) Thus, \( j \ast_2 \ast k_1 \ast^{-1} \delta_1(x) = \delta(a) \)
for some \( a \in H^q(A). \) Hence, \( i_1 \ast(j \ast_2 \ast k_1 \ast^{-1} \delta_1(x)) = i_1 \ast \delta(a). \)

Since \( i_1 \ast j \ast_2 \ast k_1 \ast^{-1} = \text{identity, } \delta_1(x) = i_1 \ast \delta(a). \) By (16), \( i_1 \ast \delta(a) = \delta_1(a). \) Thus, \( \delta_1(x) - \delta_1(a) = \delta_1(x - a) = 0. \)
Therefore, \( x - a \in \ker \delta_1 = \text{Im } l_1 \ast. \) Thus, \( x - a = l_1 \ast(y) \)
for some \( y \in H^{q-1}(X_1). \) Also, \( i_2 \ast(j \ast k_1 \ast^{-1} \delta_1(x)) = i_2 \ast \delta(a). \)
By (17) and (7), \( i_2 \ast \delta(a) = \delta_2(a) = 0. \) Therefore, \( a \in \ker \delta_2 = \text{Im } j_2 \ast. \) Thus, \( a = j_2 \ast(z) \) for some \( z \in H^{q-1}(X_2). \) Now \( (y, -z) \in H^{q-1}(X_1) \ast H^{q-1}(X_2), \) and
I(y, -z) = l_1^*(y) - l_2^*(-z) = x - (\epsilon(x)) = x. Thus, x \in \text{Im } I. Hence, \ker \Lambda \subseteq \text{Im } I.

To show \ker I \subseteq \text{Im } J, let (y, z) \in \ker I, where (y, z) \in H^q(X_1) \otimes H^q(X_2). Then I(y, z) = l_1^*(y) - l_2^*(z) = 0. Thus,
\[ \delta_1(l_1^*(y) - l_2^*(z)) = \delta \ll_1^*(y) - \delta \ll_2^*(z) = 0. \]

By (3),
\[ \delta \ll_1^*(y) = 0. \]
Therefore, \( \delta \ll_2^*(z) = 0. \) By (16),
\[ \delta \ll_2^*(z) = \ll_1^* \delta \ll_2^*(z). \]
Therefore, \( \delta \ll_2^*(z) \in \ker \ll_1^*. \)

By (4) and (17), \( \delta \ll_1^*(z) = \ll_2^* \delta \ll_2^*(z) = 0. \) By Lemma 3.6, \( \ll_2^* \mid \ker \ll_1^* \) is an isomorphism. Therefore, \( \ll_2^*(\delta \ll_2^*(z)) = 0 \)
implies \( \delta \ll_2^*(z) = 0. \) Hence, \( \ll_2^*(z) \in \ker \delta = \text{Im } \ll_1^*. \)
Therefore, \( \ll_2^*(z) = \ll^*(b) \) for some \( b \in H^q(X). \) By (12) and (13),
\[ \ll^*(b) = l_1^* p_1^*(b) = l_2^*(z), \]
and \( \ll^*(b) = l_2^* p_2^*(b) = l_2^*(z). \)
Therefore, \( l_1^*(y) - l_2^* p_1^*(b) = l_1^*(y - p_1^*(b)) = 0, \) and
\[ l_2^* p_2^*(b) - l_2^*(z) = l_2^*(p_2^*(b) - z) = 0. \]
Therefore, by (8) and (9), \( y - p_1^*(b) \in \ker l_1^* = \text{Im } \ll_1^*, \) and
\( p_2^*(b) - z \in \ker l_2^* = \text{Im } \ll_2^*. \) Thus, \( y - p_1^*(b) = \ll_1^*(g) \)
for some \( g \in H^q(X_1, A), \) and \( p_2^*(b) - z = \ll_2^*(h) \) for some
\( h \in H^q(X_2, A). \) Since \( k_1^* \) and \( k_2^* \) are onto isomorphisms,
\( g = k_1^*(g') \) for some \( g' \in H^q(X, X_2), \) and \( h = k_2^*(h') \) for some
\( h' \in H^q(X, X_1). \) Therefore, \( \ll_1^*(g) = \ll_1^* k_1^*(g'), \) and
\( \ll_2^*(h) = \ll_2^* k_2^*(h'). \) Since \( k_1 \ll_1 = n_2 p_1, \) \( (k_1 \ll_1)^* = (n_2 p_1)^*. \)
Thus, by Theorem 2, \( \ll_1^* k_1^* = p_1^* n_2^*. \) Similarly, \( \ll_2^* k_2^* = p_2^* n_1^*. \)
Hence, \( \ll_1^* k_1^*(g') = p_1^* n_2^*(g'), \) and \( \ll_2^* k_2^*(h') = p_2^* n_1^*(h'). \)
Therefore, \( y - p_1^*(b) = p_1^* n_2^*(g'), \) and \( p_2^*(b) - z = p_2^* n_1^*(h'). \) Therefore, \( y = p_1^* n_2^*(g') + p_1^*(b) = p_1^*(n_2^*(g') + b), \) and
z = p_2^*n_1^*(h') + p_2^*(b) = p_2^*(n_1^*(h') + b). Let x =
n_2^*(g') + b + n_1^*(h') \in H^q(X). Then J(x) = (p_1^*(x), p_2^*(x))
= (p_1^*(n_2^*(g') + b + n_1^*(h')), p_2^*(n_2^*(g') + b + n_1^*(h')))
= (p_1^*(n_2^*(g') + b) + p_1^*(n_1^*(h')), p_2^*(h_1''(h') + b) +
p_2^*(n_2^*(g'))) = (y + p_1^*n_1^*(h'), z + p_2^*n_2^*(g')). By
(1) and (2), p_1^*n_1^*(h') = 0, and p_2^*n_2^*(g') = 0. Hence,
J(x) = (y + 0, z + 0) = (y, z). Therefore, (y, z) \in Im J.
Hence, ker I \subseteq Im J.

Therefore, Im J = ker I, Im I = ker \Delta, and Im \Delta = ker J.
Hence, the sequence is exact.
CHAPTER IV

APPLICATIONS

Theorem 15. If X is connected, then $H^0(X) \cong G$, where G is the coefficient module.

Proof: Let X be connected. Since $B^0(X) = 0$, $H^0(X) = Z^0(X)/B^0(X) = Z^0(X) = C^0(X) \cap \delta_0^{-1}C^1(X,X)$. Let $\phi \in H^0(X)$. Then $\phi \in \delta_0^{-1}C^1(X,X)$. Thus, there exists an open cover $\beta$ of X such that $\delta_0 \phi |_{\beta^2} = 0$. Let $a, b \in X$. Then by Preliminary 10, there exists a finite subcollection of $\beta, U_1, \ldots, U_n$, such that $a \in U_1$, $b \in U_n$, and $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Therefore, $U_1 \cap U_2 \neq \emptyset$. Let $k_1 \in U_1 \cap U_2$. Then $(a, k_1) \in \beta^2$. Therefore, $\delta_0 \phi(a, k_1) = \phi(a) - \phi(k_1) = 0$. Thus, $\phi(a) = \phi(k_1)$. Similarly, $\phi(k_1) = \phi(k_2)$ for some $k_2 \in U_2 \cap U_3$. Thus, $\phi(a) = \phi(k_1) = \ldots = \phi(k_i) = \ldots = \phi(k_{n-1}) = \phi(b)$, where $k_i \in U_{i-1} \cap U_i$, and $\phi(k_{i-1}) = \phi(k_i)$. Hence, $\phi(a) = \phi(b)$. Since $a$ and $b$ were arbitrary, for any $a, b \in X$, $\phi(a) = \phi(b)$. Therefore, $\phi$ maps X onto a single point of G. For $x \in G$, define $x\phi \in C^0(X)$ by $x\phi(a) = x$ for each $a \in X$. Now $x\phi \in H^0(X)$, since X is an open cover of X, and $\phi_0 x\phi |_{X^2} = 0$. Define $F: H^0(X) \to G$ by $F(x\phi) = x$ for $x\phi \in H^0(X)$. The above shows that F is well-defined and onto. Let $r, t \in R$, $x\phi, y\phi \in H^0(X)$. Then $rF(x\phi) + tF(y\phi) = rx + ty = F(rx\phi + ty\phi)$. Thus, F is an R-homomorphism. Let
F(x\phi) = 0. Then x = 0. Thus, x\phi = 0. Therefore, F is 1-1. Hence, F: H^0(X) \to G is an isomorphism.

**Corollary 4.1.** If X is connected, then H^0(X) = 0.

**Proof:** Let Q and g be as in Figure 1. By Theorem 15, H^0(X) \cong G. By Theorem 7, H^0(Q) \cong G. Let \phi \in H^0(X) = Z^0(X) \cong G. Then for some a \in G, \phi(x) = a for all x \in X.

Since H^0(Q) \cong G, \phi \in H^0(Q). Thus, \phi(Q) = a. Now g\#\phi = g^\#\phi. Let x \in X. Then g^\#\phi(x) = \phi(g(x)) = \phi(Q) = a = \phi(x). Thus, g^\#\phi = \phi. Hence, g^\# is onto. Therefore, by Theorem 11, H^0(X) = 0.

**Definition 4.1.** Let A \subseteq X. Let h \in H^P(X,A) such that 

If there exists a closed subset F of X such that 
i^*(h) \neq 0, where i: (F, F \cap A) \to (X,A), but i_M^*(h) = 0 for every closed proper subset M of F, where i_M: (M, M \cap A) \to (X,A), then F is called a floor for h.

**Theorem 16.** Let A be a closed subset of a compact Hausdorff space X. Let h \in H^P(X,A) such that h \neq 0. Then there exists a floor for h.

**Proof:** Let K = \{F \subseteq X \mid F is closed, i^*(h) \neq 0, where i: (F, F \cap A) \to (X,A)\}. Let i: (A,A) \to (X,A). Now h = \eta\phi for some \phi \in Z^P(X,A). Thus, i^*(h) = i^*\eta\phi = \eta i^\#\phi. Let 

\((x_0, \ldots, x_p) \in A^{P+1}. Then i^\#\phi(x_0, \ldots, x_p) = \phi(i(x_0), \ldots, i(x_p)) = \phi(x_0, \ldots, x_p). Therefore, i^\#\phi = \phi. Since \eta\phi = h \neq 0, \eta i^\#\phi \neq 0. Therefore, i^\#(h) \neq 0. Hence, A \in K. Thus, K \neq \emptyset. Partially order K by set inclusion. Let P be a linearly ordered subfamily of K. Let F = \cap P. Then F is
closed. Suppose $i_F^*(h) = 0$, where $i_F^*: (F, F \cap A) \rightarrow (X, A)$.

By the Reduction Theorem, there exists an open set $N$ such that $k^*(h) = 0$, where $k: (N, N \cap A) \rightarrow (X, A)$, and $F \subseteq N$.

Since $F$ is a descending family of closed sets, and $N$ is an open set containing $\cap F$, by Preliminary 9, there exists $Q \in P$ such that $Q \subseteq N$. Let $i_Q^*: (Q, Q \cap A) \rightarrow (X, A)$. Now $i_Q^*(h) = i_Q^*\eta \phi = n_i Q^* \phi$. Let $(x_0, \ldots, x_p) \in Q^{p+1}$. Then $i_Q^*\phi(x_0, \ldots, x_p) = \phi(i_Q(x_0), \ldots, i_Q(x_p)) = \phi(k(x_0), \ldots, k(x_p)) = k^*\phi(x_0, \ldots, x_p)$. Therefore, $i_Q^*\phi = k^*\phi$. Thus, $n_i Q^* \phi = n_k^* \phi$. Therefore, $i_Q^*\phi = k^*(h) = 0$. But since $Q \subseteq K$, $i_Q^*(h) \neq 0$. This is a contradiction. Hence, $i_F^*(h) \neq 0$. Therefore, $F \in K$.

Thus, by the Hausdorff Minimal Principle, $K$ has a smallest element, $F_1$. Clearly, $F_1$ is a floor for $h$.

**Theorem 17.** Let $X$ be fully normal, and $h \in H^p(X)$. Let $F$ be a floor for $h$. If $p > 0$, then $F$ is connected.

Proof: Since $F$ is a floor, $F$ is closed. Thus, by Theorem 8, $F$ is fully normal. Suppose $F$ is not connected. Then $F = A \cup B$, where $A$ and $B$ are separated. Thus, $A$ and $B$ are closed in $F$.

Let $i_A: A \subseteq F, i_B: B \subseteq F, i: F \subseteq X, j_A: A \subseteq X, j_B: B \subseteq X$. Since $F$ is a floor, $i^*(h) \neq 0$. Define $J: H^p(F) \rightarrow H^p(A) \oplus H^p(B)$ by $J(x) = (i_A^*(x), i_B^*(x))$. By Theorem 14, ker $J = \text{Im } \Delta$, where $\Delta: H^{p-1}(A \cap B) \rightarrow H^p(F)$. By Theorem 2, $i_A^*i^*(h) = j_A(h)$. Since $F$ is a floor for $h$, and $A$ is a closed subset of $F$, $j_A(h) = 0$. Therefore, $i_A^*(i^*(h)) = 0$. 
Likewise, $i_{B}^*(i^*(h)) = j_{B}^*(h) = 0$. Therefore,

$$j(i^*(h)) = (i_{A}^*(i^*(h)), i_{B}^*(i^*(h))) = (0, 0).$$

Therefore, $i^*(h) \in \ker J = \text{Im } A$. Since $A$ and $B$ are separated, $A \cap B = \emptyset$. Thus, $H_{p-1}(A \cap B) = H_{p-1}(\emptyset) = 0$. Therefore, $\text{Im } A = 0$. Hence, $i^*(h) = 0$. This is a contradiction. Hence, $F$ is connected.

**Definition 4.2.** Let $X$ be a continuum. $X$ is unicoherent if whenever $A$ and $B$ are subcontinua covering $X$, $A \cap B$ is a subcontinuum.

**Definition 4.3.** A continuum $X$ is hereditarily unicoherent if each subcontinua of $X$ is unicoherent.

**Note 4.1.** If $X$ is a hereditarily unicoherent continuum, the minimal subcontinua between two points is unique.

**Proof:** Let $A$ and $B$ be the minimal subcontinua between $a$ and $b$. Then $a, b \in A$, and $a, b \in B$. Hence, $A \cap B \neq \emptyset$. Thus, $A \cup B$ is connected. Also, $A \cup B$ is compact. Therefore, $A \cup B$ is a subcontinuum of $X$. Now $A$ and $B$ are subcontinua covering $A \cup B$. Therefore, since $X$ is hereditarily unicoherent, $A \cap B$ is a subcontinuum. Since $a, b \in A \cap B$, $A \cap B$ is a subcontinuum between $a$ and $b$. Since $A$ and $B$ are minimal, $A \subseteq A \cap B$, and $B \subseteq A \cap B$. Hence, $A = A \cap B = B$. Hence, a minimal subcontinuum between two points is unique.

In Corollary 4.1, it was shown that if $X$ is connected, then $\tilde{H}^0(X) = 0$. If it is assumed that $\tilde{H}^0(X) = 0$ implies that $X$ is connected, then the following can be proved: If $X$ is a continuum and $\tilde{H}^1(X) = 0$, then $X$ is unicoherent. An immediate consequence of this is that the $n$-cell is unicoherent.
Modules of the $n$-cell

Let $S = n$-cell $= \{x \in \mathbb{R}^n | \|x\| \leq 1\}$.

**Lemma 4.1.** The $n$-cell is contractible.

**Proof:** Let $i: S \to S$ by $i(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. Let $k: S \to S$ by $k(x_1, \ldots, x_n) = (0, \ldots, 0)$. Then $k$ is a constant map. Define $h: S \times I \to S$, where $I$ is the unit interval, by $h(x_1, \ldots, x_n, a) = (ax_1, \ldots, ax_n)$. Then $h(x_1, \ldots, x_n, 1) = (x_1, \ldots, x_n) = i(x_1, \ldots, x_n)$, and $h(x_1, \ldots, x_n, 0) = (0, \ldots, 0) = k(x_1, \ldots, x_n)$, for all $(x_1, \ldots, x_n) \in S$. Therefore, $i = k$. Hence, $S$ is contractible.

**Theorem 18.** For $G$ a coefficient module,

$$\mathcal{H}^p(S) \cong \begin{cases} G & \text{if } p = 0, \\ \emptyset & \text{if } p > 0. \end{cases}$$

**Proof:** By Lemma 4.1, $S$ is contractible. Also, $S$ is compact. Therefore, by Corollary 2.5, $\mathcal{H}^p(S) \cong \begin{cases} G & \text{if } p = 0, \\ \emptyset & \text{if } p > 0. \end{cases}$

Modules of the $n$-sphere

Let $X_n = n$-sphere $= \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$. Let $S_n = n$-cell.

**Theorem 19.** Let $G$ be the coefficient module.

(a) If $p = n = 0$, then $\mathcal{H}^p(X_n) \cong G \times G$.

(b) If $p = 0$, and $n > 0$, then $\mathcal{H}^p(X_n) \cong G$.

(c) If $p = n$, and $n > 0$, then $\mathcal{H}^p(X_n) \cong G$.

**Proof of (a):** Now $X_0 = \{1, -1\}$. Since $X_0$ is fully normal, and $X_0 = \{1\} \cup \{-1\}$, by Theorem 14,
$H^0(X_0) \xrightarrow{J} H^0(\{1\}) \oplus H^0(\{-1\}) \xrightarrow{I} H^0(\emptyset)$ is an exact sequence.
Therefore, $J: H^0(X_0) \to H^0(\{1\}) \oplus H^0(\{-1\})$ is 1-1. Since $H^0(\emptyset) = 0$, ker $I = H^0(\{1\}) \oplus H^0(\{-1\})$. Hence, $\text{Im } J = H^0(\{1\}) \oplus H^0(\{-1\})$. Therefore, $J$ is onto. Thus, $J$ is an isomorphism. By the Dimension Theorem, $H^0(\{1\}) \cong G$, and $H^0(\{-1\}) \cong G$. Therefore, $J: H^0(X_0) \to G \times G$ is an isomorphism.

Proof of (b): Now $X_n$ is connected for $n > 0$. Therefore, by Theorem 15, $H^0(X) \cong G$.

**Lemma 4.2.** For any $n$, $X_n = A \cup B$, where

$A = \{(x_0, \ldots, x_n) \in X_n | x_n \geq 0\}$, and $B = \{(x_0, \ldots, x_n) | x_n < 0\}$.

Furthermore, $A$ is homeomorphic to $X_n$, $B$ is homeomorphic to $S^n$, and $A \cap B$ is homeomorphic to $X_{n-1}$.

**Proof:** Clearly, $X_n = A \cup B$. For $(x_0, \ldots, x_n) \in A$,

$x_0^2 + \ldots + x_n^2 = 1$. Thus, $x_0^2 + \ldots + x_{n-1}^2 < 1$. Therefore, $x_0, \ldots, x_{n-1} \in S^n$. Define $f: A \to S^n$ by

$f(x_0, \ldots, x_n) = (x_0, \ldots, x_{n-1})$. Let $\varepsilon > 0$. Let

$N((x_0, \ldots, x_n), \varepsilon) = \{(a_0, \ldots, a_n) \in R^{n+1} | (x_0-a_0)^2 + \ldots + (x_n-a_n)^2 < \varepsilon\}$. Let $(x_0, \ldots, x_n) \in A$. Let $0 < \delta < \varepsilon$.

Let $(a_0, \ldots, a_n) \in N((x_0, \ldots, x_n), \delta)$. Then

$(x_0-a_0)^2 + \ldots + (x_n-a_n)^2 < \delta$. Therefore,

$(x_0-a_0)^2 + \ldots + (x_{n-1}-a_{n-1})^2 < \delta - (x_n-a_n)^2 < \delta < \varepsilon$.

Therefore, $(a_0, \ldots, a_{n-1}) \in N((x_0, \ldots, x_{n-1}), \varepsilon)$. Hence, $f$ is continuous.

Let $f(x_0, \ldots, x_n) = f(a_0, \ldots, a_n)$, where

$(x_0, \ldots, x_n), (a_0, \ldots, a_n) \in A$. Then $x_0^2 + \ldots + x_n^2 = 1 = a_0^2 + \ldots + a_n^2$. Also, $(x_0, \ldots, x_{n-1}) = (a_0, \ldots, a_{n-1})$. 


Therefore, $x_n^2 = a_n^2$. Since $(x_0, \ldots, x_n), (a_0, \ldots, a_n) \in A$, $x_n \geq 0$, and $a_n \geq 0$. Hence, $x_n^2 = a_n^2$ implies $x_n = a_n$.

Hence, $(x_0, \ldots, x_n) = (a_0, \ldots, a_n)$. Therefore, $f$ is 1-1.

Let $(x_0, \ldots, x_{n-1}) \in S_n$. Then $x_0^2 + \ldots + x_{n-1}^2 \leq 1$. Thus, $1 - (x_0^2 + \ldots + x_{n-1}^2) > 0$. Let

$$x_n = \sqrt{1 - (x_0^2 + \ldots + x_{n-1}^2)}.$$ 

Therefore, $(x_1, \ldots, x_n) \in A$, and $f(x_0, \ldots, x_n) = (x_0, \ldots, x_n)$. Hence, $f$ is onto. Therefore, $f: A ightarrow S_n$ is a continuous, 1-1, onto function, and $A$ is compact. Hence, by Preliminary 11, $A$ is homeomorphic to $S_n$.

Similarly, $B$ is homeomorphic to $S_n$.

Now $A \cap B = \{(x_0, \ldots, x_n) \in X_n | x_n = 0\}$. Define $g: X_{n-1} + A \cap B$ by $g(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{n-1}, 0)$.

Clearly, $g$ is a homeomorphism.

Proof of (c): The proof is by induction. First it will be shown that $H^1(X_1) \neq G$. Let $g, h, Q$ be as in Figure 1.

By Theorem 7, $H^0(Q) \neq G$. Thus, there exists an isomorphism $F: H^0(Q) \rightarrow G$. By (a) of this Theorem, and Lemma 4.2,

$H^0(A \cap B) \neq G 	imes G$, where $A = \{(x, y) \in X_1 | y \geq 0\}$, and $B = \{(x, y) \in X_1 | y \leq 0\}$. Then $H^0(A \cap B) = H^0(A \cap B)/h^*H^0(Q) \neq G 	imes G/h^*H^0(Q)$. Let $\phi \in \ker h^*$. Then $h^*\phi = h^#\phi = 0$. Thus, for $x \in A \cap B$, $h^#\phi(x) = \phi(h(x)) = \phi(Q) = 0$. Hence, $\phi = 0$. Therefore, $h^*$ is 1-1. Define $f: h^*H^0(Q) \rightarrow G$ by $f(h^*(a)) = F(a)$ for each $h^*(a) \in h^*H^0(Q)$. Since $h^*$ is 1-1, $f$ is well-defined. If $x \in G$, then $x = F(a)$ for some $a \in H^0(Q)$. Thus, $f(h^*(a)) = F(a) = x$. Therefore, $f$ is onto. If $f(h^*(a)) = 0$, then $F(a) = 0$. Thus, $a = 0$. Therefore,
h*(a) = 0. Hence, f is 1-1. Let r, t ∈ R, h*(a), h*(b) ∈ h*H^0(Q). Then rf(h*(a)) + tf(h*(b)) = rF(a) + tF(b) = F(ra + tb) = f(rh*(a) + th*(b)). Therefore, f is an R-homomorphism. Hence, f: h*H^0(Q) → G is an isomorphism. Therefore, H^0(A ∩ B) ⊆ G × G/h*H^0(Q) ⊆ G × G/G = G.

Now H^0(A) = H^0(A)/g*H^0(Q). By Theorem 18, H^0(A) ⊆ G. Similar to the above, g*H^0(B) ⊆ G. Thus, H^0(A) ⊆ G/G = 0. Also, H^0(B) = 0.

The following sequence is exact:
H^0(X_1) ⊆ H^0(A) ⊆ H^0(B) ⊆ H^0(A ∩ B) ⊆ H^1(X_1) ⊆ H^1(A) ⊆ H^1(B).

By Theorem 18, H^1(A) = 0, and H^1(B) = 0. Therefore, ker J = H^1(X_1). Since ker J = Im A, A is onto. Since H^0(A) = 0, and H^0(B) = 0, H^0(A) ⊆ H^0(B) = 0. Therefore, Im I = 0. Since Im I = ker A = 0, A is 1-1. Therefore, A is an isomorphism. Since H^0(A ∩ B) = G, H^1(X_1) = G.

Now assume that H^{n-1}(X_{n-1}) = G. Let
A = {(x_0, ..., x_n) ∈ X_n | x_n ≥ 0}, and
B = {(x_0, ..., x_n) ∈ X_n | x_n ≤ 0}. By Theorem 14, the following sequence is exact:
H^{n-1}(A) ⊆ H^{n-1}(B) ⊆ H^{n-1}(A ∩ B) ⊆ H_n(X_n) ⊆ H^n(A) ⊆ H^n(B).

By Theorem 18, H^n(A) ⊆ H^n(B) = 0. Therefore, ker J = Im A = H^n(X_n). Thus, A is onto. Also, by Theorem 18, H^{n-1}(A) ⊆ H^{n-1}(B) = 0. Therefore, Im I = ker A = 0.

Thus, A is 1-1. Therefore, A: H^{n-1}(A ∩ B) → H^n(X_n) is an isomorphism. By Lemma 4.2 and the induction hypothesis, H^{n-1}(A ∩ B) = G. Hence, H^n(X_n) = G. Hence, by induction,
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