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BUCKLING OF ELLIPTICAL RINGS UNDER UNIFORM EXTERNAL PRESSURE*

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Buckling of Elliptical Rings under Uniform External Pressure

ABSTRACT

A thin, elastic elliptical ring is subjected to uniform external pressure. The lowest critical pressure is computed and presented for various ratio of the major axis to the minor axis of the elliptical ring. It is found that the critical pressure for an elliptical ring is higher than that for the circular ring whose diameter is equal to the major axis of the elliptical ring. It can be shown that under the same external pressure, the axial force developed in the elliptical ring is less than that developed in the corresponding circular ring. Thus, a higher pressure is required to buckle the elliptical rings. Therefore, by changing the shape of the ring from circular to elliptical, the capability of the ring to sustain the external pressure can be increased substantially. The results of this study can be useful in the design of elliptical reinforcing rings and thin-walled tubes subjected to external pressure.

INTRODUCTION

The study of the buckling behavior of thin rings subjected to uniform external pressure is of importance and of interest to many mechanical and civil engineers, not only because thin rings are used as reinforcing elements for shell type structures, under uniform external pressure, but the results obtained for rings can also be used in the design of thin-walled tubes also subjected to uniform external pressure. Thin-walled tubes are often used as boiler tubes, oil-well casings, vacuum tanks, and large penstocks.

The buckling of thin rings subjected to uniform pressure was investigated by Bresse in 1866, and the detailed analysis of this problem can be found in Timoshenko and Gere (1961). However, the specific case of buckling of thin rings with elliptical shape, to the author's best knowledge, has not been studied to date.

The objective of the study presented herein is to investigate the buckling behavior of thin elliptical rings subjected to uniform external pressure.

In this paper the governing differential equations for the buckling behavior of elliptical rings subjected to uniform external pressure are derived. These are coupled linear ordinary

differential equations with non-linear functions of the dependent variable as their coefficients. Finding the exact solutions to these equations is a formidable task; therefore, approximate solutions which rely on numerical methods are sought instead. The equations are transformed into an optimal control problem first, and then they are solved by an available algorithm for optimal control problems. The solutions so obtained are believed to be of high accuracy.

DERIVATION OF GOVERNING EQUATIONS

In this section, the Bernoulli beam theory is adopted for its simplicity in deriving the governing differential equations. It has been shown that for a circular ring, the critical pressure obtained from Bernoulli beam theory differs slightly from that obtained by a more elaborate theory (Boresi, 1955).

The thin elliptical ring of uniform thickness considered is shown in Figure 1. The origin of the coordinate system, o , is placed at the center of the ellipse whose major axis is equal to $2a$, and minor axis equal to $2b$. Any point on the neutral axis of the ring can be specified by the vector (see Figure 1)

$$\underline{r}(\theta) = a \cos(\theta) \underline{i} + b \sin(\theta) \underline{j}$$

in which θ is the circumferential coordinate measured counterclockwise from positive x -axis, and $\underline{i}, \underline{j}$ = unit base vectors in rectangular coordinates, x and y , respectively.

- Let
- o' = center of circle of curvature for point B
 - ψ = circumferential coordinate with respect to o'
 - $\underline{e}_t$ = unit tangential vector at point B,
 - $\underline{e}_n$ = unit normal vector at point B,
 - $\underline{e}_r$ = unit radial vector at point B pointed to o ,
 - ρ = radius of curvature,
 - s = undeformed arc length along the neutral axis.

then, from the Frenet formulas (Carmo, 1976), one obtains the following equations

$$\underline{e}_t = \frac{1}{R} (-a \sin\theta \underline{i} + b \cos\theta \underline{j}) \quad (1)$$

$$\underline{e}_n = \frac{1}{R} (-b \cos\theta \underline{i} - a \sin\theta \underline{j}) \quad (2)$$

$$\underline{e}_r = \frac{1}{R_o} (-a \cos\theta \underline{i} - b \sin\theta \underline{j}) \quad (3)$$

$$ds = R d\theta = \rho d\varphi \quad (4)$$

where

$$R = (a^2 \sin^2\theta + b^2 \cos^2\theta)^{1/2} \quad (5)$$

$$R_o = (a^2 \cos^2\theta + b^2 \sin^2\theta)^{1/2} \quad (6)$$

and

$$\rho = \frac{(a^2 \sin^2\theta + b^2 \cos^2\theta)^{3/2}}{ab} = \frac{R^3}{ab} \quad (7)$$

Let M be the bending moment at the cross section of point B, and ρ' be the radius of curvature at point B after deformation. This moment is related to the change in curvature by the equation

$$\frac{1}{\rho'} - \frac{1}{\rho} = - \frac{M}{EI} \quad (8)$$

in which EI is the flexural rigidity of the ring. The moment is taken to be positive when it produces a decrease in the initial curvature of the ring. Let w and v be the displacements along the directions of the vectors $\underline{e}_n$ and $\underline{e}_t$, respectively. Following a similar derivation given in Timoshenko and Gere (1961), one obtains

$$\frac{1}{\rho'} - \frac{1}{\rho} = \frac{w}{\rho^2} + \frac{d^2w}{ds^2} \quad (9)$$

From equations (8) and (9), it follows that

$$\frac{d^2w}{ds^2} + \frac{w}{\rho^2} = - \frac{M}{EI} \quad (10)$$

and from equations (4) and (7), equation (10) can be expressed by

$$\frac{d^2w}{d\theta^2} + \frac{a^2b^2}{R^4} w = - \frac{MR^2}{EI} \quad (11)$$

In order to relate the moment M to the deformations, w and v , and the external pressure, q , a similar figure used in Timoshenko and Gere (1961) is employed herein. It is shown in

Figure 2. Assume that the buckled shape is an elliptical curve shown by the dotted line in Figure 2; then, it can be proved that M is given by

$$M = M_0 - \frac{1}{2} q (\overline{BO}^2 - \overline{AO}^2) \quad (12)$$

where M_0 is the bending moment at cross section A; $\overline{BO}$ and $\overline{AO}$ are the lengths of the line segments between the two points denoted by the notation. Observing that

$$\overline{AO} = a - w_0 \quad (13)$$

where w_0 is the displacement for cross section A, and

$$\begin{aligned} \overline{BO} &= R_0 - (w \underline{e}_n + v \underline{e}_t) \cdot \underline{e}_t \\ &= R_0 - \left(\frac{ab}{RR_0} \right) w - \left(\frac{(a^2 - b^2)}{2RR_0} \sin 2\theta \right) v \end{aligned} \quad (14)$$

in which $\cdot$ denotes the scalar product of two vectors.

Neglecting the higher order terms, one obtains

$$\overline{AO}^2 = a^2 - 2aw_0 \quad (15)$$

and

$$\overline{BO}^2 = R_o^2 - \left(\frac{2ab}{R} \right) w - \left(\frac{(a^2 - b^2)}{R} \sin 2\theta \right) v \quad (16)$$

Substituting equations (15) and (16) in equation (12) results in

$$\begin{aligned} M = M_o - \frac{1}{2} q (R_o^2 - a^2) + \left(\frac{ab}{R} \right) q w - a q w_o \\ + \left(\frac{(a^2 - b^2) \sin 2\theta}{2R} \right) q v \end{aligned} \quad (17)$$

and equation (11) becomes

$$\begin{aligned} \frac{d^2 w}{d\theta^2} + \left(\frac{a^2 b^2}{R^4} + \frac{abR}{EI} q \right) w + \left(\frac{R(a^2 - b^2) \sin 2\theta}{2EI} q \right) v \\ = - \frac{M_o R^2}{EI} + \left(\frac{aR^2}{EI} q \right) w_o + \frac{1}{2} \frac{R^2 (R_o^2 - a^2)}{EI} q \end{aligned} \quad (18)$$

Equation (18) shows that the deformations, w and v , are coupled; therefore, one more equation that relates w and v is needed in order to solve the equations for w and v . This additional equation can be obtained from the condition of inextensibility of the ring, and it is given by (Timoshenko and Gere, 1961)

$$\frac{dv}{ds} - \frac{w}{\rho} = 0 \quad (19)$$

Introducing equations (4) and (7) into equation (19), one obtains

$$\frac{dv}{d\theta} - \frac{ab}{R^2} w = 0 \quad (20)$$

The boundary conditions for equations (18) and (20) are

$$\frac{dw}{d\theta} \Big|_{\theta=0} = 0 \quad \frac{dw}{d\theta} \Big|_{\theta=\pi/2} = 0 \quad (21)$$

for w , and

$$v \Big|_{\theta=0} = 0 \quad v \Big|_{\theta=\pi/2} = 0 \quad (22)$$

for v . Equations (21) and (22) are the results of the symmetry of the elliptical ring.

Note that equations (18) and (20) reduce to the equations for circular rings given in Timoshenko and Gere (1961).

METHOD OF SOLUTION

Two approaches may be used to compute the critical pressure, q_{cr} , from equations (18) and (20). One is to seek q_{cr} such that the solutions, w and v , to equations (18) and (20) become unbound. The other is to seek q_{cr} as the eigenvalue of the associated homogeneous equations. The associated homogeneous equations can be obtained by dropping the forcing terms in the equations (18) and (20).

Using the second approach cited above, the only forcing term -- the last term in equation (18) -- is dropped. Note that the moment M_0 is a function of the deformation, and therefore it is retained in the equation.

For convenience of performing the computation and presenting the results, equations (18) and (20), and the boundary conditions equations (21) and (22) are nondimensionlized as follows:

Define

$$w^* = \frac{w}{w_0} \quad v^* = \frac{v}{w_0} \quad (23)$$

$$\varepsilon = \frac{b}{a}, \quad (0 \leq \varepsilon \leq 1) \quad (24)$$

$$\alpha = \frac{qa^3}{EI} \quad (25)$$

$$\beta = \frac{M_0 a^2}{EI w_0} \quad (26)$$

and

$$f(\varepsilon, \theta) = (\sin^2 \theta + \varepsilon^2 \cos^2 \theta)^{1/2} \quad (27)$$

Equations (18) and (20) can now be expressed in the following forms

$$\begin{aligned}
& \frac{d^2 w^*}{d\theta^2} + \left[\frac{\varepsilon^2}{f^4(\varepsilon, \theta)} + \alpha \varepsilon f(\varepsilon, \theta) \right] w^* \\
& + \left[\frac{1}{2} f(\varepsilon, \theta) \cdot (1 - \varepsilon^2) \sin 2\theta \right] \alpha v^* \\
& = \alpha f^2(\varepsilon, \theta) - \beta f^2(\varepsilon, \theta)
\end{aligned} \tag{28}$$

$$\frac{dv^*}{d\theta} - \frac{\varepsilon}{f^2(\varepsilon, \theta)} w^* = 0 \tag{29}$$

with the boundary conditions

$$\frac{dw^*}{d\theta} \Big|_{\theta=0} = 0 \tag{30a}$$

$$\frac{dw^*}{d\theta} \Big|_{\theta=\pi/2} = 0 \tag{30b}$$

$$v^* \Big|_{\theta=0} = 0 \tag{30c}$$

$$v^* \Big|_{\theta=\pi/2} = 0 \tag{30d}$$

Note that the forcing term has been deleted from equation (28).

Summarizing, we seek the smallest eigenvalue of α such that equations (28) and (29) have non-trivial solutions that satisfy the boundary conditions, equations (30a-30d). This problem can be cast into the Bolza Problem in classical optimal control theory.

The Bolza problem seeks the minimization of a functional defined by

$$J = \int_0^1 f(\underline{x}, \underline{u}, \underline{\pi}, t) dt + [h(\underline{x}, \underline{\pi})]_{t=0} + [g(\underline{x}, \underline{\pi})]_{t=1} \quad (31)$$

with respect to the state $\underline{x}(t)$, the control $\underline{u}(t)$ and the parameters $\underline{\pi}$ subjected to the differential constraints

$$\frac{d\underline{x}}{dt} - \underline{\phi}(\underline{x}, \underline{u}, \underline{\pi}, t) = 0 \quad 0 \leq t \leq 1 \quad (32)$$

the nondifferential constraints

$$\underline{S}(\underline{x}, \underline{u}, \underline{\pi}, t) = 0 \quad 0 \leq t \leq 1 \quad (33)$$

and the boundary conditions

$$[\underline{\omega}(\underline{x}, \underline{\pi})]_{t=0} = 0 \quad (34)$$

$$[\underline{\eta}(\underline{x}, \underline{\pi})]_{t=1} = 0 \quad (35)$$

In the above equations, the functions f , h , and g are scalar, and the functions $\underline{\phi}$, $\underline{S}$, $\underline{\omega}$ and $\underline{\eta}$ are vectors. The independent variable t is a scalar, and the dependent variable $\underline{x}$, $\underline{u}$ and $\underline{\pi}$ are vectors. $t=0$ is the initial point, and $t=1$ is the final point. An algorithm, Sequential Gradient Restoration Algorithm (SGRA), for solving the Bolza problem based on calculus of variations was developed by Miele, Pritchard and Damoulakis (1970) and extended for more general boundary conditions by Gonzalez and Miele (1978). SGRA is an iterative technique which consists of a sequence of two-phase cycles. Each cycle includes a gradient phase and a restoration phase. This algorithm is designed to achieve a decrease in the functional, J , at the end of each cycle while the

constraints are satisfied to a predetermined accuracy. This algorithm has been successfully applied by Lam and Bayazitoglu (1985) to solve an eigenvalue problem related to the thermal stability of a fluid layer. The procedure used by Lam and Bayazitoglu (1985) to recast the eigenvalue problem into an optimal control problem is employed herein; for details and more references on SGRA, the reader is referred to that reference.

NUMERICAL RESULTS

In SGRA, two error functions are defined. One is associated with the constraint error, denoted by P , and the other, denoted by Q , is associated with the optimality conditions. For the exact solutions, one must have

$$P = 0 \quad \text{and} \quad Q = 0$$

For an approximation to the optimal solution, the following relations are to be satisfied

$$P < E_1 \quad \text{and} \quad Q < E_2$$

where E_1 and E_2 are small, preselected positive numbers. In this study, E_1 and E_2 were set at the levels

$$E_1 = 10^{-12} \quad \text{and} \quad E_2 = 10^{-12}$$

To start the iteration of SGRA, the following nominal functions were employed as the initial phase of iteration

$$w = \cos\theta$$

$$\frac{dw}{d\theta} = \theta \left(\frac{\pi}{2} - \theta \right)$$

$$v = \theta^3 \left(\frac{\pi}{2} - \theta \right)$$

$$\alpha = 10$$

and

$$\beta=30$$

also the following normalization condition is imposed to the solutions

$$w^*|_{\theta=0} = 1$$

Note that w and $\frac{dw}{d\theta}$ are treated as independent state variables in SGRA; then, they are

forced to be equal as one of the constraints to the optimal solutions.

The convergence history for the case of $\epsilon=1.0$ (circular ring) is presented in Table 1; clearly, the exact solutions presented in Timoshenko and Gere (1961), $\alpha=3$ and $\beta=3$, are obtained. In addition, the convergence history for $\epsilon=0.75$ and $\epsilon=0.5$ are also presented in Table 1 to show the accuracy of the solutions and efficiency of the algorithm. It shows that the solutions converge to the prescribed accuracy in a few cycles.

The numerical results for α and β for the values of ϵ various from 0.5 to 1.0 are presented in Table 2, and the associated buckling modes for $\epsilon=1.0$, 0.75 and 0.5 are presented in Figure 3. As expected, the buckling mode is in elliptical shape.

It can be seen from Table 2 that both the bending moment and the critical pressure are fairly sensitive to the variation of the shape of the ring considered. Both quantities increase as the value of ϵ decrease. So, by changing the shape of the ring from circular to elliptical, the capability of the ring to sustain external pressure may be increased substantially. For example, an elliptical ring with $\epsilon=0.75$ can sustain 52 percent more pressure than the circular ring can. The reason for this increase may be understood by comparing the axial forces developed in each ring when both rings are subjected to the same pressure. For the circular ring subjected to the uniform external pressure, q , the axial force, denoted by $(N_\theta)_c$, developed at any cross section is given by

$$(N_\theta)_c = qa \quad (36)$$

where a is the radius of the circular ring. As for the elliptical ring, the corresponding axial force, denoted by $(N_\theta)_\epsilon$, is given by

$$(N_\theta)_\epsilon = qa \frac{\epsilon}{f(\epsilon, \theta)} \quad (37)$$

Obviously, equation (36) differs from equation (37) by the function $\epsilon/f(\epsilon, \theta)$. It can be shown that the function $\epsilon/f(\epsilon, \theta)$ has the following properties

$$\frac{\epsilon}{f(\epsilon, \theta)} \leq 1 \quad (38)$$

$$\frac{\partial}{\partial \epsilon} \left(\frac{\epsilon}{f(\epsilon, \theta)} \right) > 0 \quad (39)$$

$$\frac{\epsilon}{f(\epsilon, \theta)} \Big|_{\theta=0} = 1 \quad (40)$$

and

$$\frac{\epsilon}{f(\epsilon, \theta)} \Big|_{\theta=\pi/2} = \epsilon \quad (41)$$

From equation (38), one concludes that

$$(N_{\theta})_{\epsilon} \leq (N_{\theta})_c \quad (42)$$

therefore, the critical pressure for elliptical ring is higher since a less axial force is developed. From equation (39), one concludes that

$$(N_{\theta})_{\epsilon_1} < (N_{\theta})_{\epsilon_2} \quad \text{if } \epsilon_1 < \epsilon_2 \quad (43)$$

therefore, the critical pressure increases as the value of ϵ decreases.

The results presented in Table 2 for elliptical rings can be applied to the buckling of long elliptical tubes subjected to uniform external pressure. To obtain q_{cr} for a given tube it is only necessary to replace E by $\frac{E}{(1-\nu^2)}$ and I by $h^3/12$ in equation (25) where ν is Poisson's ratio of the tube material and h denotes the thickness of the wall of the tube (Timoshenko and Gere, 1961).

CONCLUSIONS

In this paper, it is shown that an elliptical ring can sustain higher external pressure than an equivalent circular ring, and the difference is substantial. One may utilize this result and take the advantage of the more effective elliptical shape to design the tubes and the reinforcing rings which are subjected to uniform external pressure.

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Table 1. Convergence History

No. of Cycle	Phase	P	Q	α	β
$\epsilon = 1.0$					
0	Nominal	0.452141E04	0.0	10.00	30.00
1	REST	0.544511E03	-	-3.083	5.246
2	REST	0.510693E01	-	3.947	3.947
3	REST	0.570184E-01	-	2.906	2.906
4	REST	0.342598E-05	-	2.999	2.999
5	GRAD	0.133066E-13	0.682981E-28	3.0	3.0
$\epsilon = 0.75$					
0	Nominal	0.256306E04	0	10.00	30.00
1	REST	0.365196E03	-	-2.022	5.414
2	REST	0.167906E00	-	4.731	5.162
3	REST	0.171540E-04	-	4.559	4.991
4	GRAD	0.202279E-12	0.642948E-16	4.561	4.992
$\epsilon = 0.50$					
0	Nominal	0.161524E04	0	10.00	30.00
1	REST	0.603689E02	-	4.676	8.931
2	REST	0.687488E-03	-	7.789	9.385
3	REST	0.743975E-10	-	7.807	9.404
4	GRAD	0.147124E-23	0.862817E-16	7.807	9.404

Table 2. Nondimensionalized Critical Pressure and Bending
Moment for Elliptical Rings

ϵ	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.0
α	7.81	6.93	6.19	5.56	5.02	4.56	4.16	3.81	3.51	3.24	3.00
β	9.40	8.19	7.18	6.33	5.60	4.99	4.47	4.02	3.63	3.29	3.00

FIGURE CAPTIONS

- FIG. 1. Geometry and Coordinate System of Elliptical Ring Subjected to Uniform External Pressure
- FIG. 2. Configuration of Assumed Buckling Displacement and Associated Forces
- FIG. 3. Inextensional Buckling Mode of Elliptical Rings

Fig. 1. Geometry and Coordinate System of Elliptical Ring Subjected to Uniform External Pressure

Fig. 2. Configuration of Assumed Buckling Displacement and Associated Forces

Fig. 3. Inextensional Buckling Mode of Elliptical Rings

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