An Approximation Method for Dynamic Response of Strain-Hardening Structures*

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ABSTRACT

An approximation method is being developed to predict the dynamic plastic response of rigid, strain-hardening structures. This method is analogous to the instantaneous mode techniques used to treat rigid, perfectly plastic structures in that a deflection shape involving a number of arbitrary functions of time is selected, based on static deformation profiles. Two stress fields are associated with the deflection shape: one satisfies the equations of motion with appropriate boundary and continuity conditions, and the other satisfies the strain-hardening constitutive relation with appropriate boundary and smoothness conditions. The method is illustrated using the case of a simply-supported beam with a central plastic region.

INTRODUCTION

The approximation method for dynamic plastic response of strain-hardening structures discussed here is intended to be analogous to the mode approximation method for rigid, perfectly plastic structures introduced by Martin and Symonds [1]. The basic mode for a perfectly plastic structure is the product of a time-dependent amplitude function multiplying a velocity profile made up of rigid links connecting stationary plastic hinges. As a further elaboration of this method, the hinges are allowed to move and the velocity profiles are referred to as "instantaneous modes." These "instantaneous modes" are not true modes in that they are not separable products of functions of position and time, but the methods for determining the amplitude and shape functions are often extensions of those for basic mode solutions, and similar terminology is used.
The deformation shape of a strain-hardening structure always changes significantly during the motion, and the plastic deformation, rather than being concentrated at discrete points, always is spread over time-dependent regions. In the method introduced in Ref. [2], the deformation history of a strain-hardening structure is approximated by a varying mode shape up to the time of maximum deflection. This instantaneous mode is based on quasi-static deformation profiles for general load distributions but has arbitrary time-dependent amplitude coefficients and plastic region size. Two stress fields are associated with the modal shape, one satisfying the dynamic equations and associated boundary conditions and the other satisfying the constitutive equations with their associated boundary and smoothness conditions. The application of suitable matching conditions to the two stress fields results in a set of simultaneous differential and algebraic equations for the time dependence of the plastic region size and the amplitude coefficients of the modal shape. Pulse-shape effects are automatically taken into account, and the motion during the pulse is computed.

In this paper, the problem of a simply-supported beam will be used to illustrate the steps in the procedure. The deflection profile in the plastic region will be modelled as a polynomial in the axial coordinate with time-dependent coefficients. It will be shown that good accuracy is obtained by taking the ratios of the coefficients to be constants, so that the deflection profile just depends on a time-dependent amplitude and a time-dependent plastic region size, and, consequently, is analogous to an instantaneous mode for a perfectly plastic structure. The influence of different types of matching functions on the accuracy of the solution will be investigated. In particular, it will be shown that what appear to be qualitatively quite different matching conditions lead to quantitatively similar relations.

STATEMENT OF PROBLEM

Consider a simply supported beam of length 2L, loaded by a distributed dynamic force \( p(x,t) \). Take \( p(x,t) \) to be symmetric about the center of the beam \( x = 0 \), so that only the half \( 0 \leq x \leq L \) need be considered. The deforming portion of the beam will be assumed to respond as a rigid, linear strain-hardening structure, having yield moment \( m_y \) and strain-hardening coefficient \( \alpha \).
Define dimensionless axial coordinate $X$, time $T$, load $P$, bending moment $M$, deflection $W$, and hardening parameter $\omega$ by

$$X = \frac{x}{L}, \quad T = \frac{t}{t_0}, \quad P(X,T) = \frac{p(x,t)L^2}{m_y},$$

$$M(X,T) = \frac{m(x,t)}{m_y}, \quad W(X,T) = \frac{w(x,t)\mu L^2}{m_y t_0^2},$$

$$\omega^2 = \frac{\alpha t_0^2}{\mu L^4}, \quad (1)$$

where $\mu$ is the mass per unit length and $t_0$ is a measure of the load duration. Let prime and dot superscripts denote derivatives with respect to $X$ and $T$, respectively. The equation of motion can then be written as

$$M'' = -P + W, \quad (2)$$

and the constitutive relation in the plastically deforming region is

$$M = 1 - \omega^2 W'' \quad (3)$$

The instantaneous mode solution to the dynamic problem up to the time of maximum deflection will be taken to be an $N$-th order polynomial in $X$ with time-dependent coefficients to be determined from the equation motion, constitutive equation, and boundary, smoothness, and matching conditions. For plastic deformation occurring in the region $0 < X < \xi(t)$, the $N$-th order approximation to $W$ will be assumed to have the form

$$W_N(X,T) = A(T)(1-X) + \xi(T) \sum_{n=0}^{N} C_n(T) \left( \frac{X}{\xi} \right)^n, 0 \leq X \leq \xi$$

$$W_N(X,T) = A(T)(1-X) \quad \xi \leq X \leq 1, \quad (4)$$

where the amplitude function $A(T)$, plastic region size $\xi(T)$, and polynomial coefficients $C_n(T)$ are to be determined from the solution. The boundary condition $W(1,T) = 0$ is satisfied identically, and $W_N^\prime = 0$ for $\xi \leq X \leq 1$ so this portion of the beam remains rigid.
Two bending moment distributions, $M_D(X,T)$ and $M_C(X,T)$, will be associated with $W_N$ such that $M_D$ satisfies the dynamics part of the problem and $M_C$ satisfies the constitutive relation. Take $M_D$ to be the solution of eq. (2) such that $M_D$ and $0^1$ are continuous at $X = \xi$ and satisfy $M_D = 0$ at $X = 1$ and $M_D$ at $X = 0$. This solution is given by

$$M_D(X,T) = \psi(1,T) - \psi(X,T) - \frac{1}{3}(1-X)(1+X - \frac{1}{2}X^2)\ddot{A}$$

$$+ (1 - \xi)\xi^2 \sum_{n=0}^N F_n(T) + \xi^3 \sum_{n=0}^N \frac{F_n(T)}{(2+n)} \left[ 1 - \left( \frac{X}{\xi} \right)^{2+n} \right],$$

$$0 \leq X \leq \xi;$$

$$M_D(X,T) = \psi(1,T) - \psi(X,T) - \frac{1}{3}(1-X)(1+X - \frac{1}{2}X^2)\ddot{A}$$

$$+ (1 - \xi)\xi^2 \sum_{n=0}^N F_n(T), \xi \leq X \leq 1;$$

with $F_n$ and $\psi$ defined as

$$F_n = \frac{\xi^{n-1}}{1+n} \frac{d^2}{dT^2} \left( \xi^{1-n} C_n \right),$$

$$\psi(X,T) = \int_0^X (X-x)P(X,T)dx.$$  

The bending moment distribution $M_C$ is to satisfy eq. (3) in the plastically deforming region. Therefore,

$$M_C = 1 - \frac{\alpha^2}{\xi^2} \sum_{n=2}^N n(n-1)C_n(T) \left( \frac{X}{\xi} \right)^{n-2}, \quad 0 \leq X \leq \xi.$$  

Kinks can occur in the velocity and deflection profiles at plastic hinges for a perfectly plastic material ($\alpha = 0$). However, the profiles for a strain-hardening beam are smooth; in particular, $W' = 0$ at $X = 0$ and $W'$ is continuous at $X = \xi$. These smoothness conditions, the continuity of $W$ at $X = \xi$, the condition $M_C(\xi, T) = 1$ defining the edge of the plastic region, and
the symmetry condition that $M' = 0$ at $X = 0$ if the loading is sufficiently smooth give five algebraic relations between the $C_n$.

If it were possible to choose $\xi(T)$, $A(T)$, and the $C_n(T)$ such that $M_D$ and $M_C$ were identical for all $X$ and $T$, then $W_N$ would be the exact solution to the problem. However, since $W_N$ is an approximation, we will select $\xi$, $A$, and $C_n$ such that various matching conditions between $M_D$ and $M_C$ are satisfied. The maximum discrepancy between $M_D$ and $M_C$ will then provide a measure of the accuracy involved in the choice of $N$ and the selection of matching conditions. Since there are $N + 3$ unknowns ($A$, $\xi$, $C_0$, $C_1$, ...$C_N$) and five algebraic relations between the coefficients, $N - 2$ matching conditions are needed in order to solve for the remaining unknown functions. The amplitude $A$ and plastic region size $\xi$ are essential elements of the solution, so at least two matching conditions are needed to provide differential equations for $A$ and $\xi$; therefore $N$ must be at least four.

The types of matching conditions that will be considered include putting $M_D = M_C$ at selected points in the plastically deforming region, putting $\partial M_D/\partial X = \partial M_C/\partial X$ at selected points, or making weighted integrals of $M_D - M_C$ vanish. In particular, the following conditions will be used:

$$
\Phi_0(\beta, T) = \xi(T)[M_D(X,T) - M_C(X,T)]_{X=\beta\xi}, \quad 0 \leq \beta \leq 1;
$$

$$
\Phi_1(\beta, T) = (1 + \beta) \int_0^\xi [M_D(X,T) - M_C(X,T)] dX, \quad \beta \geq 0;
$$

$$
\Phi_2(\beta, T) = \xi^2 \left. \frac{\partial(M_D - M_C)}{\partial X} \right|_{X=\beta\xi}, \quad 0 \leq \beta \leq 1,
$$

$$
\Phi_2(\beta, T) = (1 + \beta) \int_0^\xi \left(1 - \frac{X}{\xi}\right)^\beta [M_D(X,T) - M_C(X,T)] dX, \quad \beta \geq 0
$$

A matching condition that will always be used is $\Phi_0(1,T) = 0$, so that the two bending moment distributions agree at the edge of the plastic region. This gives a differential equation for $A$, which as $\alpha \geq 0$ reduces to the basic mode response for a perfectly plastic beam subjected to an arbitrarily distributed dynamic load [2].
COMPARISONS AND RESULTS

Various combinations of sets of coefficients $C_n$ and matching conditions have been tested to determine which coefficients contribute most to the solution, to determine the variation of the accuracy of the results with the number of coefficients, and to determine which matching conditions give the best accuracy for a given number of active coefficients. A number of choices of strain-hardening parameter $\omega$, load distribution, and pulse shape were used in making these determinations.

These parameter studies showed [3] that the maximum discrepancy between $M_D$ and $M_C$ decreases as $N$ increases, as would be expected. Moreover, the maximum discrepancy can be reduced to within the limits of accuracy of the numerical procedure used to solve the differential equations if a sufficient number of coefficients and suitable matching conditions are used.

These studies also showed that the $C_n$ coefficients with $n \leq 4$ are of the same size as $A$ and their ratios to $A$ are almost constant no matter how many coefficients or which matching conditions are used. In contrast, the $C_n$ for $n \geq 5$ are much smaller and have little effect on the time-dependent deflection profile. Consequently, reasonable accuracy is attained by taking $N = 4$ and

$$C_0 = -\frac{3}{8}A, \quad C_1 = A, \quad C_2 = -\frac{3}{4}A, \quad C_3 = 0, \quad C_4 = \frac{1}{8}A. \quad (9)$$

The resulting instantaneous mode, which may be considered the basic mode for this problem, is

$$W_N = A(T)\left[1 - X - \frac{3}{8} \xi(T)\left[1 - \frac{X}{\xi(T)}\right]\left[1 + \frac{X}{3\xi(T)}\right]\right] \quad (10)$$

This is the deflection shape produced by a uniform load distribution applied quasi-statically to a rigid, strain-hardening beam [3].

The matching condition $\Phi_0(1,T) = 0$ becomes
\[
\ddot{A} = \frac{3}{1+\xi-\frac{1}{2}\xi^2}\left\{\frac{1}{1-\xi}[\psi(1,T) - \psi(\xi,T) - 1] + \frac{1}{10} \frac{d^2}{dT^2}(\xi^2A)\right\}
\]

(11)

One additional matching condition is needed to give a second relation between \(A\) and \(x\). Each of the conditions given in eqs. (8) can be put in the form

\[
\omega^2 A = Q[\psi] - \frac{1}{3}(1-a_1\xi)\xi^3 \ddot{A} + a_2\xi^2 \frac{d^2}{dT^2}(\xi^2A) - a_3 \frac{d^2}{dT^2}(\xi^4A)
\]

(12)

where \(Q\) is a functional of the loading. The form of \(Q\) and the values of the constants \(a_1, a_2,\) and \(a_3\) depend on the choice of matching condition.

Equations (11) and (12) are a pair of nonlinear differential equations for the amplitude \(A\) and plastic region size \(\xi\). The numerical solution is more readily effected by replacing these dependent variables by the pair \(A, B\) with \(B = \xi^2A\).

The accuracy of the solution depends on the choice of matching condition used to obtain eq. (12). To illustrate this dependence, consider the particular loading

\[
P(X,T) = P_m(1-X)T, \quad 0 \leq T \leq 1
\]

\[
= 0 \quad T > 1
\]

(13)

and take \(P_m\) such that the maximum load attained is five times the load that initiates yielding. Consider material constants such that \(\Omega = 1\), with \(\Omega\) defined by [2]

\[
\Omega = \sqrt{\frac{3}{2}} \omega I
\]

(14)

where \(I\) is the generalized impulse associated with the applied moment about the supports; i.e.,
\[ I = \int_{T_y}^{T_f} \psi(1,T)dT \] (15)

The times \( T_y \) and \( T_f \) are when deformation begins and reaches its maximum value, respectively. For this loading, the functional \( Q \) in eq. (12) is given by

\[ Q = \frac{1}{3} P_m T^3 \pi (1 - \alpha_4 \xi) \] (16)

with \( \alpha_4 \) depending on the particular choice made from eqs. (8).

Table 1 gives \( \Delta M/M \), defined as the maximum magnitude of \( (M_D - M_C)/M_D \) attained for all \( X \) and \( T \), for a variety of choices of matching conditions \( \Phi_4(\beta,T) = 0 \). The largest discrepancy usually occurs at the instantaneous change in load at \( T = 1 \). The most accurate result is obtained for \( \Phi(0,T) = 0 \), i.e., putting the integral of \( M_D - M_C \) over the plastic region to zero, although some of the other choices are almost as good.

Values of \( \Delta W/W \equiv [W(0,T_f) - W_e(0,T_f)]/W_e(0,T_f) \) and \( \Delta \xi/\xi \equiv [\xi(T_f) - \xi_e(T_f)]/\xi(T_f) \) are listed also in Table 1. The e–subscripts denote more exact results computed using a larger number of coefficients. For every choice of matching condition, the computation of the deflection shape is more accurate than that of \( M_D - M_C \).

Table 2 lists the constants \( a_1, a_2, a_3, \) and \( a_4 \) for each of the matching conditions used, in the order of decreasing accuracy of the solution. The choices that give the best results have quite similar values of the coefficients, even though they belong to different types of matching conditions.

REFERENCES


Table 1. Accuracy of Solution for Various Matching Conditions

<table>
<thead>
<tr>
<th>Matching Condition</th>
<th>$\Delta M$</th>
<th>$\Delta W$</th>
<th>$\Delta \xi$</th>
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<td>$\Phi_0(0)$</td>
<td>2.63%</td>
<td>0.64%</td>
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<td>$\Phi_0(0.25)$</td>
<td>2.27%</td>
<td>0.45%</td>
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<td>$\Phi_0(0.5)$</td>
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<td>$\Phi_1(0)$</td>
<td>-2.39%</td>
<td>-0.08%</td>
<td>0.12%</td>
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<td>-3.54%</td>
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<td>-0.28%</td>
<td>-0.29%</td>
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Table 2. Coefficients in Matching Conditions

<table>
<thead>
<tr>
<th>Matching Condition</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
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<tr>
<td>$\Phi_1(0)$</td>
<td>0.3750</td>
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