THE FUNDAMENTAL GROUP OF CERTAIN TOPOLOGICAL SPACES

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The problem confronted in this thesis is that of determining direct calculations of the fundamental group of certain topological spaces. The first chapter contains the development of the fundamental group, in general, of any topological space. The theory deals only with loops, avoiding the need for developing the fundamental groupoid.

In order to develop the group, a relation between p-based loops is established, and this relation is shown to be an equivalence relation. Hence, the p-based loops in a space are partitioned into equivalence classes. In the collection of equivalence classes of p-based loops, an identity is discovered, an inverse for each element is discovered, multiplication is defined and shown to be closed, and the associative law is shown valid. Thus, the collection of equivalence classes of p-based loops is shown to be a group.

In the first chapter it is established that if two spaces are homeomorphic, their fundamental groups relative to a certain basepoint are isomorphic. It is also shown that if a space is pathwise connected, the fundamental groups of the space relative to different basepoints are isomorphic. The thesis deals only with pathwise connected spaces, so flexibility of the choice of basepoint is allowed.

The topological spaces selected for direct calculation of fundamental groups are E^n, the surface of a sphere, $E^3-(0,0,0)$, a plane annular region, $E^2-(0,0)$, and E^3 minus the

z-axis. The fundamental group of E^n is the group consisting of only one element, called the trivial group. The method of proof used to establish this fact is to show that every loop based at the origin in E^n is equivalent to the identity loop at the origin. Hence, there is only one class of origin-based loops in E^n, the identity class.

The fundamental group of the surface of a sphere is also the trivial group. The method of proof used is to divide the sphere into three pathwise connected, open sets R, R_1 and R_2 such that $R \cup R_1 \cup R_2$ is the sphere and $R_1 \cap R_2 = R$. A loop in the sphere based at a point p in R is shown to be equivalent to the identity loop in the sphere based at p. This fact is established by constructing a new loop, equivalent to the original loop, such that the new loop is equivalent to the identity.

The fundamental group of the space $E^3-(0,0,0)$ is shown to be the trivial group by showing that it is isomorphic to the fundamental group of the sphere.

The plane annular region, $E^2-(0,0)$, and E^3 minus the z-axis are all shown to have infinite cyclic fundamental groups. All of these proofs rely on the fact that the fundamental group of the circle is infinite cyclic, the proof of which may be found in Crowell and Fox's Knot Theory.

The annulus problem is solved by directly showing that the fundamental group of the annulus is isomorphic to the fundamental group of the circle. A somewhat different approach is taken in finding the fundamental group of $E^2-(0,0)$, for $E^2-(0,0)$ is shown to be homeomorphic to the open annulus.

Finally, the space E^3 minus the z-axis is shown to have a fundamental group isomorphic to the fundamental group of $E^2-(0,0)$.

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CHAPTER I

DEVELOPMENT OF THE FUNDAMENTAL GROUP

This chapter is devoted to the definition and theoretical development of the fundamental group of a topological space. In all definitions and theorems, X will denote a topological space.

Definition 1.1. For any two real numbers x and y, $[x,y]$ is the set of all real numbers t satisfying $x \leq t \leq y$.

Definition 1.2. A p-based loop A in X is a continuous mapping $A: [0,t] \to X$, where $A(0) = A(t) = p$. The number t represents the stopping time of A, and it will be denoted $\|A\|$. The point $A(0)$ is the initial point, and the point $A(\|A\|)$ is the terminal point of the loop A.

Definition 1.3. A collection $\{H_s\}$, $0 \leq s \leq 1$, of p-based loops in X is a continuous family of p-based loops if and only if

(1) The stopping time $\|H_s\|$ depends continuously on s.

(2) The function H defined by the formula $H(s,t) = H_s(t)$ maps the closed region $0 \leq s \leq 1$, $0 \leq t \leq \|H_s\|$, denoted by R, continuously into X.

Definition 1.4. Let p be a point in X and let A and B be two p-based loops in X. A is equivalent to B, denoted

$A \simeq B$, if there exists a continuous family $\{H_s\}$, $0 \leq s \leq 1$, of p-based loops in X such that $A = H_o$ and $B = H_1$.

Theorem 1.5. The relation $\simeq$ is reflexive, symmetric, and transitive.

Proof: Reflexive: Let A be a p-based loop in X. Define a collection of paths $\{H_s\}$ as $H_s(t) = A(t)$, $0 \leq s \leq 1$. For any s, $H_s(t) = A(t)$, so $\{H_s\}$ is a collection of p-based loops, and $H_o = A$ and $H_1 = A$. Since for $0 \leq s \leq 1$, $\|H_s\| = \|A\|$, a constant, the stopping time $\|H_s\|$ depends continuously on s.

Let R be the closed region $0 \leq s \leq 1$, $0 \leq t \leq \|H_s\|$. Choose $(s,t) \in R$. Consider $H(s,t) = H_s(t) = A(t)$. Let N be a neighborhood about $A(t)$. Then, since $A:[0, \|A\|] \to X$ is continuous, there exists an open interval I containing t such that for all $t' \in W = I \cap [0, \|A\|]$, $A(t') \in N$. Now for all $s' \in [0,1]$, $A(t) \in N$. Let J be any open interval about s and let $K = J \cap [0,1]$. Let $D = K \times W$. Then K is an open neighborhood about (s,t). Choose $(s',t') \in D$. Then $H(s',t') = H_{s'}(t') = A(t')$. But $t' \in W$, so $A(t') \in N$. Hence $H(s',t') \in N$, so $H(D) \subset N$. Therefore, the function H maps the closed region R continuously into X.

Symmetric: Let A and B be p-based loops in X such that $A \simeq B$. Suppose $\{H_s\}$ is the continuous family exhibiting $A \simeq B$. Define a family $\{K_s\}$ by $K_s(t) = H_{1-s}(t)$. Then $K_o = H_1 = B$ and $K_1 = H_o = A$. Every element of $\{K_s\}$ is an element of $\{H_s\}$, so $\{K_s\}$ is a collection of p-based loops.

Choose $s_i \in [0,1]$ and $\epsilon > 0$. Consider $\| K_{s_i} \|$. Now $\| K_{s_i} \| = \| H_{1-s_i} \|$ and since $\| H_s \|$ is continuous with respect to s, there exists $\delta > 0$, such that for all $s \in [1-s_i-\delta, 1-s_i+\delta]$, $\| H_s \| \in [\| H_{1-s_i} \| -\epsilon, \| H_{1-s_i} \| + \epsilon] = [\| K_{s_i} \| -\epsilon, \| K_{s_i} \| + \epsilon]$. Hence, for all $s \in [s_i-\delta, s_i+\delta]$, $\| H_{1-s} \| = \| K_s \| \in [\| K_{s_i} \| - \epsilon, \| K_{s_i} \| + \epsilon]$. Therefore, the stopping time $\| K_s \|$ depends continuously on s.

Choose a point (s_i,t) from R', where R' is the closed region $0 \le s_i \le 1$, $0 \le t \le \| K_s \|$, and consider $K_{s_i}(t) = H_{1-s_i}(t)$. Let M be a neighborhood of $K_{s_i}(t)$. Since $\{H_s\}$ is a continuous family, there exists a neighborhood D about $(1-s_i,t)$ such that $H(D) \subset M$. Let $D' = \{(1-s,t):(s,t) \in D\}$. Then D' is a neighborhood of (s_i,t). Pick $(s',t') \in D'$. $K(s',t') = K_{s'}(t') = H_{1-s'}(t') \in M$, since $(1-s',t') \in D$. Hence $K(D') \subset M$. Therefore, the function K maps R' continuously into X. Thus, $\{K_s\}$ is a continuous family and so $B \simeq A$. Therefore, the symmetric property holds.

Transitive: Let A, B, and C be p-based loops in X such that $A \simeq B$ and $B \simeq C$. Let $\{H_s\}$ and $\{K_s\}$ be the continuous families of p-based loops which provide the equivalences $A \simeq B$ and $B \simeq C$, respectively. Define a collection of p-based loops $\{J_s\}$ by the formula

$$J_s(t) = \left\{ \begin{matrix} H_{2s}(t), & 0 \le s \le \frac{1}{2} \\ K_{2s-1}(t), & \frac{1}{2} \le s \le 1 \end{matrix} \right\}.$$

Clearly $J_o = H_o = A$ and $J_1 = K_1 = C$.

By definition $\| H_s \|$ and $\| K_s \|$ depend continuously on s. Then for $0 \le s \le \frac{1}{2}$, $\| H_{2s} \|$ depends continuously on s and for $\frac{1}{2} \le s \le 1$, $\| K_{2s-1} \|$ depends continuously on s. Hence, for $0 \le s \le 1$, $\| J_s \|$ depends continuously on s.

Let (s,t) be a point in R'', where R'' is the region $0 \le s \le 1$, $0 \le t \le \| J_s \|$. If $0 \le s < \frac{1}{2}$, then $0 \le t \le \| H_{2s} \|$. Let N be a neighborhood of $J_s(t) = H_{2s}(t)$. Since H is a continuous family, there exists a neighborhood D about (2s,t) such that $H(D) \subset N$. Let $D' = \{(s,t):(2s,t) \in D\}$. Then D' is a neighborhood of (s,t). Pick $(s',t') \in D'$. Then $(2s',t') \in D$ and $H_{2s'}(t') \in N$. But $J_{s'}(t') = H_{2s'}(t') \in N$, so $J(D') \subset N$. A similar argument can be used for K_s if $\frac{1}{2} < s \le 1$.

If $s = \frac{1}{2}$, there exists a neighborhood U about (s,t) such that for each point (s',t') in U, $s' \le \frac{1}{2}$, and such that $H(U) \subset N$. Also, there exists a neighborhood V about (s,t) such that for each point (s',t') in V, $s' \ge \frac{1}{2}$ and such that $K(V) \subset N$. Let $W = \{(\frac{1}{2}, t):(\frac{1}{2}, t)$ is in U or V but not both$\}$. Let $D = (U \cup V) \setminus W$, i.e. if $(s',t') \in D$, $(s',t') \in U \cup V$, but not in W. Then D is a neighborhood of (s,t), where $s = \frac{1}{2}$, and $J(D) \subset N$.

The function J, then, maps R'' continuously into X. Hence, $\{J_s\}$ is a continuous family, and $A \approx C$.

The proof of the above theorem demonstrates that the relation $\simeq$ is an equivalence relation. The set of all p-based loops in the space X is therefore partitioned into equivalence classes. The equivalence class of an arbitrary p-based loop A will be denoted by [A]. It is upon the set of equivalence classes of p-based loops in X, that a group structure will be presented.

Definition 1.6. The product of two p-based loops A and B in X is given by the formula

$$(A \cdot B)(t) = \begin{cases} A(t), & 0 \le t \le \|A\| \\ B(t - \|A\|), & \|A\| \le t \le \|A\| + \|B\| \end{cases}.$$

Theorem 1.7. If A and B are p-based loops, then $A \cdot B$ is a p-based loop.

Proof: Let A and B be p-based loops. Then $A(0) = A(\|A\|) = B(0) = B(\|B\|) = p$. By Definition 1.6,

$$(A \cdot B)(t) = \begin{cases} A(t), & 0 \le t \le \|A\| \\ B(t - \|A\|), & \|A\| \le t \le \|A\| + \|B\|. \end{cases}$$

Then $(A \cdot B)(0) = A(0) = p$, and $(A \cdot B)(\|A \cdot B\|) = (A \cdot B)(\|A\| + \|B\|)$ $= B(\|A\| + \|B\| - \|A\|) = B(\|B\|) = p$.

Hence, the product of two p-based loops is a p-based loop.

Theorem 1.8. If A, B, and C are p-based loops, then $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

Proof: By Definition 1.6,

$$[A \cdot (B \cdot C)](t) = \begin{cases} A(t), & 0 \le t \le \|A\| \\ (B \cdot C)(t - \|A\|), & \|A\| \le t \le \|A\| + \|B\| + \|C\|. \end{cases}$$

Then

$$[A \cdot (B \cdot C)](t) = \begin{cases} A(t), & 0 \le t \le \|A\| \\ B(t - \|A\|), & \|A\| \le t \le \|A\| + \|B\| \\ C[t - (\|A\| + \|B\|)], & \|A\| + \|B\| \le t \le \|A\| + \|B\| + \|C\| \end{cases}$$

$= [(A \cdot B) \cdot C](t)$, $0 \le t \le \|A\| + \|B\| + \|C\|$.

Hence, $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

Definition 1.8. A p-based loop E is called an identity loop at p, or simply an identity, if it has stopping time $\|E\| = 0$.

Theorem 1.9. The identity loop E at p has the property that if A is any p-based loop, then $E\cdot A = A\cdot E = A$.

Proof: Assume E is an identity at p, and let A be any p-based loop. By Definition 1.8, $\|E\| = 0$, so

$$E\cdot A(t) = \begin{cases} E(t), & t = 0 \\ A(t-0), & 0 \leq t \leq \|A\| + 0 \end{cases} = A(t),\ 0 \leq t \leq \|A\| .$$

Hence $E\cdot A = A$.

Similarly, $A\cdot E(t) = \begin{cases} A(t), & 0 \leq t \leq \|A\| \\ E(t-\|A\|), & t = \|A\| \end{cases}$

$$= \begin{cases} A(t), & 0 \leq t \leq \|A\| \\ E(0), & t = \|A\| \end{cases} = A(t),\ 0 \leq t \leq \|A\| .$$

Hence $A\cdot E = A$, so $E\cdot A = A\cdot E = A$.

Theorem 1.10. For any p-based loops A, A', B, B' in X, if $A \simeq A'$ and $B \simeq B'$, then $A\cdot B \simeq A'\cdot B'$.

Proof: Let $\{H_s\}$ and $\{K_s\}$ be the continuous families of p-based loops which exhibit the equivalences $A \simeq A'$ and $B \simeq B'$ respectively.

Consider the family $\{H_s\cdot K_s\}$. This is again a family of p-based loops. Also $H_0 = A$, $K_0 = B$, so $H_0\cdot K_0 = A\cdot B$, and $H_1 = A'$, $K_1 = B'$, so $H_1\cdot K_1 = A'\cdot B'$.

Since $\{H_s\}$ and $\{K_s\}$ are continuous families, $\|H_s\|$ and $\|K_s\|$ is a continuous function of s. Then $\|H_s\cdot K_s\| = \|H_s\| + \|K_s\|$ is a continuous function of s.

Let R_1 be the closed region $0 \le s \le 1$, $0 \le t \le \| H_s \|$, let R_2 be the closed region $0 \le s \le 1$, $0 \le t \le \| K_s \|$, and let R_3 be the closed region $0 \le s \le 1$, $0 \le t \le \| H_s \| + \| K_s \|$. Let (s,t) be a point in R_3. Consider $(H \cdot K)(s,t) =$

$$(H_s \cdot K_s)(t) = \begin{cases} H_s(t) \ , & 0 \le t \le \| H_s \| \\ K_s(t - \| H_s \|), & \| H_s \| \le t \le \| H_s \| + \| K_s \|. \end{cases}$$

Let N be a neighborhood about $(H_s \cdot K_s(t)$. If $0 \le t < \|H_s\|$, then since $\{H_s\}$ is a continuous collection, there exists a neighborhood M_1 about (s,t) in R_1 such that $H \cdot K(M_1) = H(M_1) \subset N$. If $\| H_s \| < t \le \| H_s \| + \| K_s \|$, then since $\{K_s\}$ is a continuous family, there exists a neighborhood M_2 about $(s, t - \| H_s \|)$ in R_2 such that $K(M_2) \subset N$. Let $M_3 = \{(s,t) : (s, t - \| H_s \|) \in M_2\}$. Then M_3 is a neighborhood about (s,t) in R_3. Clearly, $H \cdot K(M_3) \subset N$, since for any $(s_i, t_i) \in M_3$, $(H_{s_i} \cdot K_{s_i})(t_i) = K_{s_i}(t - \| H_{s_i} \|) \in N$, because $(s_i, t_i - \| H_{s_i} \|) \in M_2$.

If $t = \| H_s \|$, then there is a neighborhood D_1 about (s,t) in R_1 such that $H(D_1) \subset N$. Note that $D_1 \subset R_3$. Also, there is a neighborhood D_2 about $(s, t - \| H_s \|$ in R_2 such that $K(D_2) \subset N$. Let $D_3 = \{(s', t' + \| H_s \|) : (s', t') \in D_2\}$. Then $D_3 \subset R_3$ and $(s,t) \in D_3$. Let $D_4 = D_1 \cup D_3$. Then D_4 is a neighborhood of (s,t) in R_3 and $H \cdot K(D_4) \subset N$.

Therefore, the function $H \cdot K$ maps the region R_3 continuously into X. Hence, $\{H_s \cdot K_s\}$ is a continuous family of p-based loops exhibiting $A \cdot B \simeq A' \cdot B'$.

Definition 1.11. Denote by $\pi(X,p)$, the set G of all equivalence classes of p-based loops in X, together with the operation "$\cdot$", defined by the formula $[A]\cdot[B] = [A\cdot B]$, where A and B are p-based loops in X.

Multiplication in $\pi(X,p)$ is well-defined as a result of Theorem 1.10.

Theorem 1.12. In $\pi(X,p)$, multiplication is closed, the associative law is valid, and there exists an identity.

Proof: Let [A] and [B] be elements of $\pi(X,p)$. By Definition 1.11, $[A]\cdot[B] = [A\cdot B]$. Since A and B are p-based loops, $A\cdot B$ is a p-based loop by Theorem 1.7. Then $[A\cdot B]$ is the equivalence class containing the p-based loop $A\cdot B$. Hence, $[A\cdot B] \in \pi(X,p)$, so multiplication is closed.

Let [A], [B], and [C] be elements of $\pi(X,p)$. $[A]\cdot([B]\cdot[C]) = [A]\cdot[B\cdot C] = [A\cdot(B\cdot C)] = [(A\cdot B)\cdot C] = [A\cdot B]\cdot[C] = ([A]\cdot[B])\cdot[C]$. Hence, the associative law is valid.

Let [E] be the equivalence class of the identity loop E. Let [A] be any element of $\pi(X,p)$. Then $[E]\cdot[A] = [E\cdot A] = [A]$, by Theorem 1.9, and $[A]\cdot[E] = [A\cdot E] = [A]$, by Theorem 1.9. Therefore, [E] is an identity in $\pi(X,p)$.

Definition 1.13. For any p-based loop A in X, the inverse loop, denoted by A^{-1}, is given by the formula

$$A^{-1}(t) = A(\|A\| - t),\ 0 \le t \le \|A\| .$$

Theorem 1.14. If E is the identity loop at p, then for any p-based loop A in X, $A\cdot A^{-1} \simeq E$ and $A^{-1}\cdot A \simeq E$.

Proof: Let E be the identity at p, and let A be a p-based loop. Define a collection of paths $\{H_s\}$, $0 \le s \le 1$, by the formula

$$H_s(t) = \left\{\begin{matrix} A(t), & 0 \le t \le s\,\|A\| \\ A(2s\|A\| - t), & s\|A\| \le t \le 2s\|A\| \end{matrix}\right\}.$$

Then

$$H_o = A(0), \quad \|H_o\| = 0, \text{ so } H_o = E. \quad \text{Also,}$$

$$H_1 = \begin{cases} A(t), & 0 \le t \le \|A\| \\ A(2\|A\| - t), & \|A\| \le t \le 2\|A\| \end{cases} = A \cdot A^{-1}.$$

$\|A\| \ge 0$ is a constant, and the stopping time $2s\|A\|$ is a continuous function of s.

Let R be the closed region $0 \le s \le 1$, $0 \le t \le 2s\|A\|$, and let (s,t) be a point in R. Let N be a neighborhood in X about the point $H(s,t) = H_s(t)$. Assume $\|A\| > 0$.

Case 1: $0 \le t < s\|A\|$: In this case $H_s(t) = A(t)$. Since $A:[0,\|A\|] \to X$ is continuous, $A:[0,s\|A\|] \to X$ is continuous, because $0 \le s \le 1$. Then there exists a number $\delta > 0$ such that for all $t' \in I = (t-\delta, t+\delta)$, $A(t') \in N$. Let $\epsilon_1 = s\|A\| - t$, and let $\xi = \min\{\delta, \frac{\epsilon_1}{2}\}$. Let $y = \max\{0, t-\xi\}$. Then if $I' = [y, t+\xi)$, $I' \subset I$ and $A(I) \subset N$.

Let $\epsilon_2 = \frac{\epsilon_1}{2\|A\|}$ and let $x = \max\{0, s-\epsilon_2\}$.
Now let $J = [x, s+\epsilon_2)$ and $D = J \times I$. Then D is a neighborhood of (s,t). Choose $(s',t') \in D$. Then $t' \le t + \frac{\epsilon_1}{2}$. Also, $s' \cdot \|A\| \ge (s-\epsilon_2)\|A\| =$

$$\left(s - \frac{(s\|A\| - t)}{2\|A\|}\right)\|A\| = s\|A\| - \frac{s\|A\| - t}{2} = \frac{s\|A\|}{2} + \frac{t}{2} = t + \frac{\epsilon_1}{2}.$$

Hence, $0 \leq t' \leq s' \|A\|$, so $H_{s'}(t') = A(t') \in N$, since $t' \in I'$. Therefore, $H(D) \subset N$.

Case 2: $s\|A\| < t \leq 2s\|A\|$. Then $H_s(t) = A(2s\|A\| - t)$. There exists a number $\delta > 0$ such that for all $t_o \in I_1 = [(2s\|A\| - t) - \delta, (2s\|A\| - t) + \delta]$, $A(t_o) \in N$. Let $\epsilon_1 = t - s\|A\|$, and let $\xi = \min\{\frac{\epsilon_1}{2}, \delta\}$. Let $I_2 = [(2s\|A\| - t) - \xi, (2s\|A\| - t) + \xi]$. Now let $I_3 = \{t' : (2s\|A\| - t') \in I_2\}$. Then I_3 is a neighborhood of t.

Let $\epsilon_2 = \frac{\epsilon_1}{2\|A\|}$, and let $x = \max\{0, s - \epsilon_2\}$. Let $J = [x, s + \epsilon_2]$, and let $D = J \times I_3$. Then D is a neighborhood of (s,t).

Choose $(s', t') \in D$. Then $t' \in I_3$, so $2s\|A\| - t' \leq (2s\|A\| - t) + \frac{\epsilon_1}{2}$ and $t' \geq t - \frac{\epsilon_1}{2}$. Also,

$$s'\|A\| \leq (s + \epsilon_2)\|A\| + \epsilon_2(\|A\|) = s\|A\| + \frac{\epsilon_1}{2} = t - \epsilon_1 + \frac{\epsilon_1}{2} = t - \frac{\epsilon_1}{2}.$$

Hence, $s'\|A\| \leq t' \leq 2s'\|A\|$, so $H_{s'}(t') = A(2s'\|A\| - t') \in N$, since $2s'\|A\| - t' \in I_3$. Therefore, $H(D) \subset N$.

Case 3: $t = s\|A\|$. Then there exists a neighborhood M_1 about (s,t), as in Case 1, such that $A(M_1) \subset N$. Also, there exists a neighborhood M_2 about (s,t), as in Case 2, such that for all $(s',t') \in M_2$, $s'\|A\| \leq t' \leq 2s'\|A\|$, and $A(2s'\|A\| - t') \in N$. Let $D_1 = M_1 \cup M_2$. Let $W = \{(s,t) : (s,t) \in (M_1 \cup M_2) \setminus (M_1 \cap M_2)\}$. Let $D_2 = D_1 \setminus W$. Then D_2 is a neighborhood about (s,t) and $H(D_2) \subset N$.

Thus, the function H, defined by $H(s,t) = H_s(t)$, maps the region R continuously into X. Therefore, $\{H_s\}$ is a continuous family of p-based loops, and $A \cdot A^{-1} \approx E$.

One can use a similar proof to show $A^{-1} \cdot A \simeq E$.

Definition 1.15. The inverse of an arbitrary element $[A]$ in $\pi(X,p)$ is given by the formula $[A]^{-1} = [A^{-1}]$.

From Definition 1.15, it is easy to see that for any element $[A]$ in $\pi(X,p)$, there exists $[A^{-1}]$ in $\pi(X,p)$ such that $[A]\cdot[A^{-1}] = [A^{-1}]\cdot[A] = [E]$, where $[E]$ is the equivalence class containing the identity loop E. Hence, it has been established that $\pi(X,p)$ is a group.

Definition 1.16. $\pi(X,p)$ will denote the fundamental group of X relative to the basepoint p.

Theorem 1.17. If $f: X \to Y$ is a homeomorphism of X onto Y, then $\pi(X,p)$ is isomorphic to $\pi(Y,f(p))$, for any basepoint p in X.

Proof: Let $f: X \to Y$ be a homeomorphism of X onto Y. Let p be a point in X, and consider $\pi(X,p)$. Any p-based loop A in X determines an $f(p)$ - based loop $f(A)$ in Y given by the composition $f \circ A(t) = f(A(t))$, $0 \leq t \leq \|A\|$. The stopping time of $f(A)$ is clearly the same as that of A.

Suppose that A and B are p-based loops in X such that $A \simeq B$. Let $\{H_s\}$, $0 \leq s \leq 1$, be a continuous family of p-based loops demonstrating $A \simeq B$. Consider the family $\{f(H_s)\}$, $0 \leq s \leq 1$. $H_0 = A$ and $H_1 = B$, so $f(H_0) = f(A)$ and $f(H_1) = f(B)$. The stopping time $\|H_s\|$ is continuous with respect to s. Then, since $\|H_s\| = \|f(H_s)\|$, $\|f(H_s)\|$ is continuous with respect to s.

Let R be the closed region $0 \le s \le 1$, $0 \le t \le \| H_s \|$. Since $\{H_s\}$ is a continuous family, the function H defined by the formula $H(s,t) = H_s(t)$ maps R continuously into X.

Let R' be the region $0 \le s \le 1$, $0 \le t \le \| f(H_s) \|$. Note that $R' = R$. Let (s,t) be a point in R'. Let N be a neighborhood in Y about $f[H(s,t)]$. Then $f^{-1}(N)$ is a neighborhood about $H(s,t)$ in X. Also, there exists a neighborhood D about (s,t) in $R = R'$ such that $H(D) \subset f^{-1}(N)$. Then $f[H(D)] \subset N$. Hence, the function $f \circ H$, defined by the formula $f \circ H(s,t) = f(H(s,t))$ maps R' continuously into Y. Therefore, $\{f(H_s)\}$, $0 \le s \le 1$, is a continuous family, and $f(A) \simeq f(B)$.

Then f determines a mapping f_* of the fundamental group $\pi(X,p)$ into the fundamental group $\pi(Y,f(p))$ given by the formula $f_*([A]) = [f(A)]$.

Note that $f(A \cdot B) = f(A) \cdot f(B)$, since

$$f(A \cdot B)(t) = f((A \cdot B)(t)) = \begin{cases} f(A(t)), & 0 \le t \le \|A\|, \\ f(B(t - \|A\|)), & \|A\| \le t \le \|A\| + \|B\|, \end{cases}$$

$$= \begin{cases} f(A(t)), & 0 \le t \le \|f(A)\|, \\ f(B(t - \|f(A)\|)), & \|f(A)\| \le t \le \|f(A)\| + \|f(B)\|, \end{cases}$$

$$= (f(A) \cdot f(B))\,(t).$$

Now $f_*([A]) \cdot f_*([B]) = [f(A)] \cdot [f(B)] =$

$[f(A) \cdot f(B)] = [f(A \cdot B)] = f_*([A \cdot B]) = f_*([A] \cdot [B])$.

Hence, the mapping f_* is product preserving. It is clear that $f_*(\pi(X,p)) \subset \pi(Y,f(p))$. Thus, the mapping $f_*\colon \pi(X,p) \to \pi(Y,f(p))$ is a homomorphism. Denote f_* as the homomorphism induced by f.

The mappings $f: X \to Y$ and $f^{-1}: Y \to X$ induce homomorphisms $f_*: \pi(X,p) \to \pi(Y,f(p))$ and $(f^{-1})_*: \pi(Y,f(p)) \to \pi(X,p)$. The composition functions $f^{-1} \circ f$ and $f \circ f^{-1}$ are identity maps. Hence, the compositions $(f^{-1} \circ f)_* = f^{-1}_* \circ f_*$ and $(f \circ f^{-1})_* = f_* \circ f^{-1}_*$ are identity maps. It follows, then, that f_* is a one-to-one, onto mapping.

Therefore, $\pi(X,p)$ is isomorphic to $\pi(Y,f(p))$.

Definition 1.18. A path A in X is a continuous mapping $A:[0,t] \to X$.

Definition 1.19. X is pathwise connected if any two points of X can be joined by a path lying in X.

If X is a pathwise connected space, the fundamental groups of X, defined for different basepoints, are all isomorphic. (1,p. 21). Then it is evident that the fundamental group of a pathwise connected space is independent of the choice of basepoint. In the remainder of this thesis, all spaces considered will be pathwise connected, and the fundamental group of these spaces will be denoted $\pi(X)$.

CHAPTER II

FUNDAMENTAL GROUPS OF SELECTED SPACES

In this chapter specific topological spaces will be chosen, and the fundamental group of each space will be identified. The spaces to be examined are E^n, the surface of a sphere, $E^3-(0,0,0)$, a plane annular region, $E^2-(0,0)$, and E^3 minus the z-axis.

Theorem 2.1. Any constant p-based loop is equivalent to the identity loop at p.

Proof: Let $A:[0, \| A \|] \to X$ be a p-based loop defined by the formula $A(t) = p$. Then A is a constant p-based loop. Let E be the identity loop at p. Define a collection $\{H_s\}$, $0 \leq s \leq 1$, by the formula $H_s(t) = A(t)$, $0 \leq t \leq s \cdot \| A \|$. Since $A(t) = p$, $0 \leq t \leq \| A \|$, and $0 \leq s \leq 1$, $\{H_s\}$ is a collection of p-based loops. Also, $H_o(t) = A(t)$, $0 \leq t \leq 0 = E$ and $H_1(t) = A(t)$, $0 \leq t \leq \| A \| = A$.

For $0 \leq s \leq 1$, $\| H \| = s \cdot \| A \|$, which is s times a constant term. Hence, the stopping time $\| H_s \|$ depends continuously on s.

Now pick a point (s,t) in the region $0 \leq s \leq 1$, $0 \leq t \leq \| H_s \|$. $H(s,t) = H_s(t) = H_s(t) = p$. Let N be a neighborhood about $H_s(t) = p$. Let M be any neighborhood of (s,t) in the region $0 \leq s \leq 1$, $0 \leq t \leq \| H_s \|$. Let (s',t') be a point in M. Then $H_{s'}(t') = p \in N$. Hence, $H(M) \subset N$. Therefore, the function H, defined by the formula $H(s,t) = H_s(t)$ maps the closed region $0 \leq s \leq 1$, $0 \leq t \leq \| H_s \|$, continuously into X. Hence, $\{H_s\}$, $0 \leq s \leq 1$, is a continuous family, and $E \simeq A$.

(2.2) The space E^n. Let A be a p-based loop in E^n. Then A is a continuous mapping $A:[0, \| A \|] \to E^n$, and A can be written by the formula $A(t) = (f_1(t), f_2(t), \cdots, f_n(t))$, $0 \leq t \leq \| A \|$, where $f_1, f_2, \cdots, f_n$ are continuous functions of t.

Suppose, without loss of generality, that p is the origin. This proof will show $E \simeq A$, where E is the identity loop at the origin. Define a collection $\{H_s\}$, $0 \leq s \leq 1$, by the formula $H_s(t) = (s \cdot f_1(t), s \cdot f_2(t), \cdots, s \cdot f_n(t))$, $0 \leq t \leq \| A \|$. Then $\{H_s\}$ is a collection of loops based at the origin.

$H_o(t)$ is the origin, $0 \leq t \leq \| A \|$, and

$H_1(t) = (f_1(t), f_2(t), \cdots, f_n(t))$, $0 \leq t \leq \| A \|$.

Then H_o is a constant loop at the origin and $H_1 = A$. For every $s \in [0,1]$, $\| H_s \| = \| A \|$. Hence, the stopping time $\| H_s \|$ is a constant function of s and thus, continuous.

Let R be the region $0 \leq s \leq 1$, $0 \leq t \leq \| A \|$. Let (s_o, t_o) be a point in R. Consider $H(s_o, t_o) = H_{s_o}(t_o)$ $= (s_o \cdot f_1(t_o), s_o \cdot f_2(t_o), \cdots, s_o \cdot f_n(t_o))$. For points Q and W in E^n, let the distance between Q and W be $d(Q,W)$. Let $Q = H_{s_o}(t_o)$, and let N be a neighborhood of Q. Then there exists a number $r > 0$ such that the set $N' = \{W : d(Q,W) < r\} \subset N$.

By the continuity of A, there exists a neighborhood I' of t_o such that for all $t \in I \cap [0, \|A\|]$, the distance between $(f_1(t), f_2(t), \cdots, f_n(t))$ and $(f_1(t_o), f_2(t_o), \cdots, f_n(t_o))$ is less than $\frac{r}{2(s_o+1)}$. Then

$$\{[f_1(t)-f_1(t_o)]^2+[f_2(t)-f_2(t_o)]^2+\cdots+[f_n(t)-f_n(t_o)]^2\}^{\frac{1}{2}} < \frac{r}{2(s_o+1)}$$

and

$$s_o \cdot \{[f_1(t)-f_1(t_o)]^2+[f_2(t)-f_2(t_o)]^2+\cdots+[f_n(t)-f_n(t_o)]^2\}^{\frac{1}{2}} < \frac{r}{2}.$$

The loop A, based at the origin, is a closed path in E^n. Then there exists a number $M > 0$ such that for $0 \le t \le \|A\|$, the distance between $A(t) = (f_1(t), f_2(t), \cdots, f_n(t))$ and the origin is less than M. That is, $[f_1^2(t) + f_2^2(t) + \cdots + f_n^2(t)]^{\frac{1}{2}} < M$. Let $J = (s_o - \frac{r}{2M}, s_o + \frac{r}{2M}) \cap [0,1]$. Then for all

$$s \in J,\ |s-s_o| < \frac{r}{2M}.$$

Let $D = J \times I$. Then D is a neighborhood of (s_o, t_o). Choose $(s,t) \in D$. Consider also the point (s_o, t). Let

$Q = (s_o \cdot f_1(t_o), s_o \cdot f_2(t_o), \cdots, s_o \cdot f_n(t_o))$,

$W = (s \cdot f_1(t), s \cdot f_2(t), \cdots, s \cdot f_n(t))$, and

$Y = (s_o \cdot f_1(t), s_o \cdot f_2(t), \cdots, s_o \cdot f_n(t))$. Then, by the triangle inequality, $d(Q,W) \le d(W,Y) + d(Y,Q)$,

$$d(W,Y) = [(s-s_o)^2 \cdot f_1^2(t) + (s-s_o)^2 \cdot f_2^2(t) + \cdots + (s-s_o)^2 \cdot f_n^2(t)]^{\frac{1}{2}}$$

$$= |s-s_o| \cdot [f_1^2(t) + f_2^2(t) + \cdots + f_n^2(t)]^{\frac{1}{2}} < |s-s_o| \cdot M < \frac{r}{2}.$$

$$d(Y,Q) = \{s_o^2[f_1(t)-f_1(t_o)]^2 + s_o^2[f_2(t)-f_2(t_o)]^2 + \cdots + s_o^2[f_n(t)-f_n(t_o)]^2\}^{\frac{1}{2}}$$

$$= s_o \cdot \{[f_1(t)-f_1(t_o)]^2 + [f_2(t)-f_2(t_o)]^2 + \cdots + [f_n(t)-f_n(t_o)]^2\}^{\frac{1}{2}} < \frac{r}{2}.$$

Hence, $d(Q,W) < r$. Therefore, $H(s,t) = H_s(t)$
$= (s \cdot f_1(t), s \cdot f_2(t), \cdots, s \cdot f_n(t)) \in N'$, so $H(D) \subset N$.

Then the function H, defined by $H(s,t) = H_s(t)$, maps the region R continuously into X. This fact indicates that the collection of p-based loops $\{H_s\}$, $0 \leq s \leq 1$, is a continuous family exhibiting $H_o \simeq A$. By Theorem 2.1, $E \simeq H_o$. Therefore, by the transitive property of loops, $E \simeq A$.

Hence, $\pi(E^n,p)$ is the trivial group, where p is the origin. Since E^n is pathwise connected $\pi(E^n)$ is the trivial group.

The proof for showing that the fundamental group of E^n is trivial will also hold for many subspaces of E^n. Let Y be a subspace of E^n for which there exists a point p in Y such that if q is any other point in Y, p and q can be connected by a straight line in Y. Then the fundamental group of Y is trivial, and the proof is similar to the proof for E^n. An example of such a subspace Y is the space I^n, where I^n is the closed interval $[0,1]$ crossed with itself n times.

(2.3) The sphere. Let $X = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$. Then X is the unit sphere in E^3. Let
$R_1 = \{(x,y,z) \in X : z > \frac{1}{2}\}$, $R_2 = \{(x,y,z) \in X : z < -\frac{1}{2}\}$,
and
$R = \{(x,y,z) \in X : -\frac{1}{2} < z < \frac{1}{2}\}$. Let p be a point in R, and let $A:[0,\|A\|] \to X$ be a p-based loop in X. For any set M, denote by $\overline{M}$ the closure of the set M.

Lemma 2.3.1. If $t_1, t_2 \in [0,\|A\|]$ such that $A(t_1) \in \overline{R}_1$ and $A(t_2) \in \overline{R}_2$, then, assuming $t_1 < t_2$, there exist points

t_1' and t_2' such that $t_1 \leq t_1' < t_2' \leq t_2$, $A(t_1') \in \overline{R_1}$, $A(t_2') \in \overline{R_2}$, and if $t_1' < t < t_2'$, then $A(t) \in R$.

Proof: Let t_1, t_2 be points in $[0, \|A\|]$ such that $A(t_1) \in \overline{R_1}$ and $A(t_2) \in \overline{R_2}$. Assume, without loss of generality, that $t_1 < t_2$. Let $S_1 = \{t: t < t_2 \text{ and } A(t) \in \overline{R_1}\}$. $\overline{R_1}$ is a closed and compact set, and A is a continuous mapping of $[0, \|A\|] \to X$. Hence, $A^{-1}(\overline{R_1})$ is closed. Then $A^{-1}(\overline{R_1}) \cap [0, t_2]$ is a closed set. But $S_1 = A^{-1}(\overline{R_1}) \cap [0, t_2]$, so S_1 is closed. Hence, S_1 has a largest element. Denote this element t_1'.

Let $S_2 = \{t: t_1' < t \leq t_2 \text{ and } A(t) \in \overline{R_2}\}$. $\overline{R_2}$ is closed and compact, so $A^{-1}(\overline{R_2})$ is closed. Then $A^{-1}(\overline{R_2}) \cap [t_1', t_2]$ is closed. Therefore, S_2 is closed, and S_2 has a smallest element. Denote this element t_2'. Then $t_1 < t_1' < t_2' \leq t_2$. Also, $A(t_1') \in \overline{R}$ and $A(t_2') \in \overline{R_2}$. Let t be such that $t_1' < t < t_2'$. Then $A(t) \notin \overline{R_1}$ and $A(t) \notin \overline{R_2}$. Hence $A(t) \in R$. The points t_1' and t_2' will be called a prime pair in the domain of A.

Lemma 2.3.2. For any loop in X, there exists only a finite number of prime pairs.

Proof. Suppose there exists an infinite number of prime pairs for a given loop A. Then there exists a countable subset of these pairs. Denote the countable sequence of pairs as $t_1, t_1', t_2, t_2', t_3, t_3', \cdots, t_n, t_n', \cdots$. Consider the sequence $t_1, t_2, t_3, \cdots, t_n, \cdots$. For $i \in (0, \infty)$, t_i can be in at most two prime pairs. Hence, the sequence $t_1, t_2, \cdots, t_n, \cdots$ is infinite. Then this sequence has a sequential limit, which will be denoted T. Since $t_1, t_2, \cdots$ is an infinite sequence of real numbers, there exists a monotone subsequence $t_{i_1}, t_{i_2}, t_{i_3}, \cdots$ which has a limit point T. Now $t_{i_{n-1}} < t'_{i_n} < t_{i_{n+1}}$, for $n = 2,3\cdots$. Thus there exists a monotone subsequence $t'_{k_1}, t'_{k_2}, \cdots$. Then T is the limit of $t'_{k_1}, t'_{k_2}, \cdots$. But, for this to be true, A(T) would have to be in both $\bar{R}_1$ and $\bar{R}_2$, which is impossible. Therefore, there can exist only a finite number of prime pairs in A.

Lemma 2.3.3. Let r be a point in the interval $[0, \|A\|]$ such that $A(r) \in R$. Let $B':[0,t] \to R$ be a path in R such that $B'(0) = A(r)$ and $B'(t) = p$. Let $B:[0,2t] \to R$ be defined by the formula

$$B(t') = \begin{cases} B'(t'), & 0 \le t' \le t \\ B'(2t-t), & t \le t' \le 2t. \end{cases}$$

Then B is equivalent to the identity loop at A(r).

Proof: $B(0) = B'(0) = A(r)$, and $B(2t) = B'(0) = A(r)$. Hence B is an A(r)-based loop in R. Define a collection $\{H_s\}$, $0 \le s \le 1$, of A(r)-based loops by the formula

$$H_s(t') = \begin{cases} B'(t'), & 0 \le t' \le s \cdot t \\ B'(2st-t'), & st \le t' \le 2st. \end{cases}$$

Now $H_o(t') = B(0) = A(r)$, $0 \le t \le 0$, so H_o is the identity loop at A(r). $H_1 = B$.

The proof that $\{H_s\}$, $0 \le s \le 1$, is a continuous family is exactly like the proof in Theorem 1.14 that $\{H_s\}$ is a continuous family, changing A to B' and $\|A\|$ to t. Then B is equivalent to the identity loop at A(r).

Lemma 2.3.4. If $A:[0,\|A\| \to X$ is a p-based loop, $t_o \in [0, \|A\|]$, and $B:[0, \|B\|] \to X$ is any $A(t_o)$-based loop equivalent to the identity at $A(t_o)$, then the p-based loop A' defined by the formula

$$A'(t) = \begin{cases} A(t), & 0 \leq t \leq t_o \\ B(t-t_o), & t_o \leq t \leq t_o + \|B\| \\ A(t-\|B\|-t_o), & t_o + \|B\| \leq t \leq t_o + \|A\| + \|B\| \end{cases}$$

is equivalent to A.

Proof: Let $\{H_s\}$, $0 \leq s \leq 1$, be the collection of $A(t_o)$-based loops which establishes B is equivalent to the identity at $A(t_o)$. Define a collection $\{K_s\}$, $0 \leq s \leq 1$, by the formula

$$K_s(t) = \begin{cases} A(t), & 0 \leq t \leq t_o \\ H_s(t-t_o), & t_o \leq t \leq t_o + s\cdot\|B\| \\ A(t-s\cdot\|B\|), & t_o + s \cdot \|B\| \leq t \leq s\cdot\|B\| + \|A\|. \end{cases}$$

The proof that $\{K_s\}$, $0 \leq s \leq 1$, is a continuous collection of p-based loops is similar to the proof in Theorem 1.10. The collection $\{K_s\}$, $0 \leq s \leq 1$, establishes $A \simeq A'$.

With these four lemmas established, the following proof will show that the fundamental group of a sphere is the trivial group. Let $R_1' = R \cup \bar{R}_1$, and let $R_2' = R \cup \bar{R}_2$. Note that R_1' and R_2' are both homeomorphic to the interior of a circle. Then any loop in R_1' or R_2' is equivalent to the identity.

As stated previously, $A: [0, \|A\|] \to X$ is a p-based loop in X. If A lies entirely in R_1' or entirely in R_2', then A is equivalent to the identity loop at p. Suppose, then, that A does not lie entirely in R_1' or entirely in R_2'. Then there exist points $t_1, t_2 \in [0, \|A\|]$ such that $A(t_1) \in \bar{R}_1$ and $A(t_2) \in \bar{R}_2$. By Lemma 2.3.1, there exists at least one prime pair in $[0, \|A\|]$. By Lemma 2.3.2, there exists at most a finite number of prime pairs in $[0, \|A\|]$. Denote the prime pairs in $[0, \|A\|]$ by $t_1, t_1', t_2, t_2' \cdots, t_n, t_n'$.

Suppose, without loss of generality, that $A(t_1) \in \bar{R}_1$. Then $A(t_1') \in \bar{R}_2$, $A(t_2) \in \bar{R}_2$, $A(t_2' \in \bar{R}_1$, $A(t_3) \in \bar{R}_1$, etc. Let $T_o, T_1, T_2, \cdots, T_n$ be points in $[0, \|A\|]$ such that $T_o = 0$, and $t_{i-1} \leq T_i \leq t_{i+1}$, $i=1,2,\cdots, n$. Then, for $i = 1,2, \cdots,n$, For T_i, $i = 1,2,\cdots, n$, let $B_i':[0,t] \to R$ be a path in R such that $B_i'(0) = A(T_i)$ and $B_i'(t) = p$. Let $B_i:[0,2t] \to R$ be defined by the formula

$$B_i(t') = \begin{cases} B'(t'), & 0 \le t' \le t \\ B'(2t-t'), & t \le t' \le 2t. \end{cases}$$

By Lemma 2.3.3, B_i is an $A(T_i)$-based loop in R such that B_i is equivalent to the identity loop at $A(T_i)$.

Define a path $A':[0,\|A'\|] \to X$ as follows:

$$A'(t) = \begin{cases} A(t), & T_0 = 0 \le t \le T_1 \\ B_1(t-T_1), & T_1 \le t \le T_1 + \|B_1\| \\ A(t-\|B_1\|), & T_1 + \|B_1\| \le t \le \|B_1\| + T_2 \\ B_2(t-[T_2 + \sum_{i=1}^{1} \|B_i\|]), & T_2 + \sum_{i=1}^{1} \|B_i\| \le t \le T_2 + \sum_{i=1}^{2} \|B_i\| \\ A(t-\sum_{i=1}^{2} \|B_i\|), & T_2 + \sum_{i=1}^{2} \|B_i\| \le t \le T_3 + \sum_{i=1}^{2} \|B_i\| \\ B_3(t-[T_3 + \sum_{i=1}^{2} \|B_i\|]), & T_3 + \sum_{i=1}^{2} \|B_i\| \le t \le T_3 + \sum_{i=1}^{3} \|B_i\| \\ \vdots & \vdots \\ B_n(t-[T_n + \sum_{i=1}^{n-1} \|B_i\|]), & T_n + \sum_{i=1}^{n-1} \|B_i\| \le t \le T_n + \sum_{i=1}^{n} \|B_i\| \\ A(t-\sum_{i=1}^{n} \|B_i\|), & T_n + \sum_{i=1}^{n} \|B_i\| \le t \le \|A\| + \sum_{i=1}^{n} \|B_i\|. \end{cases}$$

By applying Lemma 2.3.3 n times, $A' \simeq A$.

Let $S_0 = T_0 = 0$, $S_1 = T_1 + \dfrac{\|B_1\|}{2}$, and for $i = 2,3,\cdots,n$, $S_i = T_i + \sum_{j=1}^{i-1} \|B_j\| + \dfrac{\|B_i\|}{2}$. Let $S_{n+1} = \|A'\|$.

Then for $i = 0, 1, 2, \cdots, n$, the path $A'(t)$, $t \in [S_i, S_{i+1}]$, is equivalent to $A'(t+S_i)$, $t \in [0, S_{i+1}-S_i]$, and $A':[0,S_{i+1}-S_i] \to R$ is a p-based loop in R. Since $R \subset R_1'$, $A':[0,S_{i+1}-S_i] \to R$ is a p-based loop in R_1', and is therefore equivalent to the identity loop at p.

For $i = 0, 1, 2, \cdots, n$, let $A_i' = A':[0, S_{i+1} - S_i] \to R$. Then $A':[0, \| A \|] \to X$ is equivalent to $A_0' \cdot A_1' \cdot A_2' \cdot \cdots \cdot A_n'$. Since A_i', $i = 0, 1, 2, \cdots, n$, is equivalent to the identity at p, then A' is equivalent to the identity at p. But $A' \simeq A$, so A is equivalent to the identity at p. Therefore, since A was an arbitrary p-based loop in X, and X is pathwise connected, the fundamental group $\pi(X)$ is the trivial group.

(2.4) $E^3 - (0,0,0)$. The fundamental group of the space $E^3 - (0,0,0)$ is the trivial group. This will be established by showing that the fundamental group of $E^3 - (0,0,0)$ is isomorphic to the fundamental group of the sphere.

In spherical coordinates let $X = \{(\rho, \theta, \varphi) : \rho > 0, 0 \leq \theta < 2\pi, 0 \leq \varphi \leq \pi\}$, and let

$$Y = \{(\rho, \theta, \varphi) : \rho = 1, 0 \leq \theta < 2\pi, 0 \leq \varphi \leq \pi\}.$$

Define a mapping f by the formula $f(\rho, \theta, \varphi) = (1, \theta, \varphi)$, for $(\rho, \theta, \varphi) \in X$. Then $f: X \to Y$.

Let N be an open set in Y. Let $P = (\rho, \theta, \varphi)$ be a point in $f^{-1}(N)$. Since N is open, there exist $\theta_1, \theta_2, \varphi_1$, and φ_2 such that the set $D = \{1, \theta', \varphi') : \theta_1 < \theta' < \theta_2$ and $\varphi_1 < \varphi' < \varphi_2\}$ contains $(1, \theta, \varphi)$, and $D \subset N$. Note that D is a basis element for the topology on Y and that $(1, \theta, \varphi) = f(\rho, \theta, \varphi)$. Let $D' = \{(\rho_0, \theta_0, \varphi_0) : 0 < \rho_0 < \rho + 1, \theta_1 < \theta_0 < \theta_2$, and $\varphi_1 < \varphi_0 < \varphi_2\}$. Then D' is an open set about (ρ, θ, φ) in X. Let $(\rho'', \theta'', \varphi'')$ be an element of D'. Then $f(\rho'', \theta'', \varphi'') = (1, \theta'', \varphi'') \in D$, since $\theta_1 < \theta'' < \theta_2$ and $\varphi_1 < \varphi'' < \varphi_2$. Therefore, $D' \in f^{-1}(N)$, and $f^{-1}(N)$ is an open set. Hence f is a continuous mapping.

As observed in Theorem 1.17, f induces a homomorphism $f_*:\pi(X) \to \pi(Y)$, given by the formula $f_*([A]) = [f(A)]$, where $[A] \in \pi(X)$. Let p be any point in Y. Let $[A]$ be an element in $\pi(Y,p)$. The mapping $j:Y \to X$ is the inclusion mapping, or the identity. Then $j_*([A]) \in \pi(X,p)$, where $j_*:\pi(Y)\to \pi(X)$, and $f_*(j_*([A])) = [A]$. Hence, the composition $f_*(j_*)$ is the identity map, and f_* is onto.

Lemma 2.4.1. For any point p in Y, any p-based loop A in X is equivalent to its projection $f(A)$ in Y.

Proof: Let p be a point in Y. Let $A:[0,\|A\|] \to X$ be a p-based loop in X. The loop A can be described as $(\mathfrak{p}(t), \theta(t), \varphi(t))$, where $\mathfrak{p}$, θ and φ are continuous functions of $t \in [0,\|A\|]$. Define a collection $\{H_s\}$, $0 \le s \le 1$, of p-based loops by the formula

$$H_s(t) = (\mathfrak{p}(t)[s + \frac{1-s}{\mathfrak{p}(t)}],\ \theta(t),\ \varphi(t)),\ \text{for } 0 \le t \le \|H_s\| = \|A\|.$$

$$H_o(t) = (1, \theta(t),\ \varphi(t)),\ \text{so } H_o = f(A).$$

$$H_1(t) = (\mathfrak{p}(t),\ \theta(t),\ \varphi(t)),\ \text{so } H_1 = A.$$

For every $s \in [0,1]$, $\|H_s\| = \|A\|$. Then the stopping time $\|H_s\|$ is a constant function of s. Therefore, the stopping time $\|H_s\|$ is a continuous function of s.

Let R be the closed region $0 \le s \le 1$, $0 \le t \le \|H_s\|$. Let (s,t) be a point in R. Let N be a neighborhood in X about $H(s,t) = H_s(t)$. Let $\mathfrak{p}(t)\cdot(s + \frac{1-s}{\mathfrak{p}(t)}) = \mathfrak{p}^*$. Then there exist a $\delta > 0$ and $\theta_1, \theta_2, \varphi_1$, and φ_2 such that the set $D = \{(p, \theta, \varphi) : p^* - \delta < p < p^* + \delta,\ \theta_1 < \theta < \theta_2,\ \varphi_1 < \varphi < \varphi_2\}$ contains $H_s(t)$ and $D \subset N$.

Since $H_1 = A$ and A is continuous, there exists an interval I' about t such that for $t_1, t_2 \in I = I' \cap [0, \| A \|]$, $A(t_1)$ and $A(t_2)$ are in the set

$K = \{p, \theta, \varphi) : | p(t) - p | < \frac{\delta}{2s},\ \theta_1 < \theta < \theta_2,\ \varphi_1 < \varphi < \varphi_2\}$.

Then $H_1(t_1)$ and $H_1(t_2)$ are in K. Now A is a closed loop in X, based at p. Hence, there exists a number $M > 0$ such that for $t \in [0, \| A \|]$, $p(t) < M$. Let

$J' = \{s' : |s - s'| < \frac{\delta}{2|M-1|}\}$. Then $J = J' \cap [0,1]$ is a neighborhood of s. Let $G = J \times I$. Then G is a neighborhood of (s,t) in R.

Let (s', t') be a point in G. Then

$$H(s', t') = H_{s'}(t') = (p(t')[s' + \frac{1-s}{p(t')}],\ \theta(t'),\ \varphi(t')),$$

and

$$H(s,t) = H_s(t) = (p(t)[s + \frac{1-s}{p(t)}],\ \theta(t),\ \varphi(t)).$$

Since t and $t' \in I$, $H_1(t)$ and $H_1(t')$ are in K. Hence, $\theta_1 < \theta(t') < \theta_2$, $\theta_1 < \theta(t) < \theta_2$, $\varphi_1 < \varphi(t') < \varphi_2$, $\varphi_1 < \varphi(t) < \varphi_2$, and $| p(t) - p(t') | < \frac{\delta}{2s}$.

Now

$$|[p(t')(s' + \frac{1-s'}{p(t')})] - [p(t)(s + \frac{1-s}{p(t)})]|$$
$$\leq |[p(t')(s + \frac{1-s}{p(t')})] - [p(t)(s + \frac{1-s}{p(t)})]|$$
$$+ |[p(t')(s' + \frac{1-s'}{p(t')})] - [p(t')(s + \frac{1-s}{p(t')})]|$$
$$= s \cdot | p(t') - p(t) | + | s' - s | \cdot | p(t') - 1 | < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Then $H(s', t') \in D \subset N$. Therefore, $H(G) \subset N$, and H maps R continuously into X. Hence, $\{H_s\}$, $0 \leq s \leq 1$, is a continuous family, and $f(A) \simeq A$.

Lemma 2.4.2. If A and B are p-based loops in Y such that $A \simeq B$ in X, then $A \simeq B$ in Y.

Proof: Let A and B be p-based loops in Y such that $A \simeq B$ in X. Let $\{H_s\}$, $0 \leq s \leq 1$, be the continuous family of p-based loops in X which establish $A \simeq B$. By Lemma 2.4.1, for all $s \in [0,1]$, $H_s \simeq f(H_s)$, and $f(H_s) \in Y$.

Consider the family $\{f(H_s)\}$, $0 \leq s \leq 1$. Assume $H_o = A$ and $H_1 = B$. Then $f(H_o) = f(A)$ and $f(H_1) = f(B)$. But A and B are contained in Y, so $f(A) = A$ and $f(B) = B$. Hence, $f(H_o) = A$ and $f(H_1) = B$. For every $s \in [0,1]$, $\| H_s \| = \| f(H_s) \|$. Hence, since $\{H_s\}$ is a continuous family, the stopping time $\| H_s \| = \| f(H_s) \|$ is a continuous function of s.

Let R be the closed region $0 \leq s \leq 1$, $0 \leq t \leq \| f(H_s) \| = \| H_s \|$. Choose $(s,t) \in R$. $f[H(s,t)] = f(H_s(t))$. Let N be a neighborhood of $f(H_s(t))$ in Y. Then there exist $\theta_1, \theta_2, \varphi_1$, and φ_2 such that the set $D = \{(1, \theta, \varphi) : \theta_1 < \theta < \theta_2, \ \varphi_1 < \varphi < \varphi_2\}$ contains $f(H_s(t))$ and $D \subset N$. Let $D_1 = \{(\rho, \theta, \varphi) : 0 < \rho < \infty, \ \theta_1 < \theta < \theta_2, \varphi_1 < \varphi < \varphi_2\}$. Then D_1 is a neighborhood of $H_s(t)$ in X. Hence, there exists a neighborhood D_2 of (s,t) in R such that $H(D_2) \subset D_1$. Then $f(H(D_2)) \subset D \subset N$. Hence, $f(H)$ maps the closed region R continuously into Y. Therefore, $A \simeq B$ in Y.

Lemma 2.4.3. If A and B are p-based loops in X, then $A \simeq B$ if and only if $f(A) \simeq f(B)$ in Y.

Proof: Let A and B be p-based loops in X such that $A \simeq B$. By Lemma 2.4.1, $f(A) \simeq A$ and $B \simeq f(B)$. Then by the transitive property of $\simeq$, $f(A) \simeq f(B)$ in X. Hence, by Lemma 2.4.2, $f(A) \simeq f(B)$ in Y.

Now suppose $f(A) \simeq f(B)$ in Y. Then, since $Y \subset X$, $f(A) \simeq f(B)$ in X. By Lemma 2.4.1, $A \simeq f(A)$ and $f(B) \simeq B$. Therefore, by the transitive property of $\simeq$, $A \simeq B$.

Lemma 2.4.4. The induced homomorphism f_* maps an equivalence class in X to an equivalence class in Y if and only if they have an element in common.

Proof: $f_*:\pi(X,p) \to \pi(Y,p)$. Let [A] be an equivalence class in X and suppose $f_*([A])=[B]$, where [B] is an equivalence class in Y. By Lemmas 2.4.1 and 2.4.3, $f_*:[A] \to [f(A)]$. Hence, $[f(A)] = [B]$. Note that $f(A) \in Y$ and $f(A) \in X$, since $Y \subset X$. By Lemma 2.4.1, $A \simeq f(A)$. Hence, $f(A) \in [A]$. Since $f(A) \in [f(A)] = [B]$, f(A) is common to [A] and [B].

Let [A] be an equivalence class in X and [B] be an equivalence class in Y such that there exists a p-based loop C such that $C \in [A]$ and $C \in [B]$. As seen above $f_*([A]) = [f(A)]$ in Y. $C \in [A]$, so $f(C) \in [f(A)]$ by Lemma 2.4.3. But $C \in [B]$, so $C \in Y$ and $f(C) = C$. Hence $C \in [f(A)]$. Therefore, since C cannot be in two different equivalent classes in Y, $[f(A)] = [B]$. Hence $f_*[A] = [B]$.

Therefore, with Lemma 2.4.4 established, it is clear that $f_*:\pi(X,p) \to \pi(Y,p)$ is one-to-one. Then f_* is an isomorphism. Thus, the fundamental group $\pi(X)$ is isomorphic to the fundamental group $\pi(Y)$. In (2.3) it was observed that $\pi(Y)$ was the trivial group. Hence $\pi(X)$ is the trivial group.

Theorem 2.5. The fundamental group of the circle is infinite cyclic. (1,pp. 24-29).

(2.6) The annulus. In the following proof, the fundamental group of the annulus will be shown to be infinite cyclic by establishing that it is isomorphic to the fundamental group of the circle.

In E^2 let $G^i = \{x,y) : x^2+y^2 = 1\}$, $G^b = \{(x,y) : x^2+y^2 = 4\}$, and let G be the annular region between and including G^i and G^b. Define a mapping $p : G \to G^i$ as follows:

For

$$(x,y) \in G,\ p(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \quad \frac{y}{\sqrt{x^2+y^2}} \right);$$

then if (x_1,y_1) were on G^i and L were the line containing (0,0) and (x_1,y_1), then p maps $L \cap G$ to (x_1,y_1).

Let N be an open set in G^i. Then $N = \bigcup_{n=1}^{\infty} M_n$, where M_n is an open arc on G^i, $n = 1,2,\cdots$. For some n, let the end-points of the arc M_n be a_n, b_n. Let $L_{n,1}$ be the line containing (0,0) and a_n, and let $L_{n,2}$ be the line containing (0,0) and b_n. Let $L'_{n,1}$ and $L'_{n,2}$ be that part of $L_{n,1}$ and $L_{n,2}$, respectively, which intersects G. Let K_n be that part of G which lies between, but does not include $L'_{n,1}$ and $L'_{n,2}$. Then $K_n = p^{-1}(M_n)$. Let $K = \bigcup_{n=1}^{\infty} K_n$. Then $K = p^{-1}(N)$. Since for each n, K_n is open in G, K is open in G. Hence, the inverse image of open sets is open. Therefore, p is continuous.

The function p determines a mapping p_* of $\pi(G)$ into $\pi(G^i)$, given by $p_*([A]) = [p(A)]$, where $[A] \in \pi(G)$. This mapping $p_* : \pi(G) \to \pi(G^i)$ is a homomorphism, as seen in the proof of Theorem 1.17. Note that $\mathfrak{p}$ maps the subspace G^i onto itself.

To show $\mathfrak{p}$ is onto, choose any basepoint p in G^i. Let [A] be an element in $\pi(G^i,p)$. The mapping $j: G^i \to G$ is the inclusion mapping, or the identity. Then $j_*([A]) \in \pi(G,p)$, where $j_*: \pi(G^i) \to \pi(G)$, and $\mathfrak{p}_*(j_*([A])) = [A]$. Hence, the composition $\mathfrak{p}_*(j_*)$ is the identity map, and $\mathfrak{p}_*$ is onto.

Lemma 2.6.1. For any point p on the circle G^i, any p-based loop A in G is equivalent to its projection $\mathfrak{p}(A)$ on the circle G^i.

Proof: Let p be a point on the circle G^i. Let $A:[0, \|A\|] \to G$ be a p-based loop in G. The loop A can be described as $(f(t), g(t))$, where f and g are continuous functions of $t \in [0, \|A\|]$. Define a collection $\{H_s\}, 0 \le s \le 1$, of p-based loops by the formula

$$H_s(t) = \left(f(t)\left[s + \frac{1-s}{\sqrt{f^2(t) + g^2(t)}}\right], \quad g(t)\left[s + \frac{1-s}{\sqrt{f^2(t) + g^2(t)}}\right]\right),$$

for $0 \le t \le \|H_s\| = \|A\|$.

$$H_0(t) = \left(\frac{f(t)}{\sqrt{f^2(t) + g^2(t)}} , \frac{g(t)}{\sqrt{f^2(t) + g^2(t)}} \right) = \mathfrak{p}(A).$$

$H_1(t) = (f(t), g(t)), 0 \le t \le \|A\|, = A.$

For every $s \in [0,1]$, $\|H_s\| = \|A\|$. Then the stopping time $\|H_s\|$ is a constant function of s, and thus, continuous.

Let R be the closed region $0 \leq s \leq 1$, $0 \leq t \leq \| H_s \| = \| A \|$. Let (s_o, t_o) be a point in R. Consider $H(s_o, t_o) = H_{s_o}(t_o) =$

$$\left(f(t_o)\left[s_o + \frac{1-s_o}{\sqrt{f^2(t_o) + g^2(t_o)}}\right],\ g(t_o)\left[s_o + \frac{1-s_o}{\sqrt{f^2(t_o) + g^2(t_o)}}\right]\right).$$

Let N be an open disc centered at $H(s_o, t_o)$ with an arbitrary radius r.

By the continuity of A, there exists a neighborhood I' about t_o such that for all $t_1, t_2 \in I = I' \cap [0, \| A \|]$, the distance between $(f(t_1), g(t_1))$ and $(f(t_2), g(t_2))$ is less than $\frac{r}{2}$. If x and y are points, denote the distance between x and y by $d(x,y)$. Then $d[(f(t_1), g(t_1)), (f(t_2), g(t_2))] < \frac{r}{2}$. But

$H_1(t) = (f(t), g(t))$, $0 \leq t \leq \| A \|$. Hence $d(H_1(t_1), H_1(t_2)) < \frac{r}{2}$.

Now for a fixed t, the relation between $d[H_s(t), H_o(t)]$ and $d[H_1(t), H_o(t)]$ will be investigated, where $s \in [0,1]$. Pick $t \in [0, \| A \|]$, and let $s \in [0,1]$. Note that $H_o(t)$ is on the circle G^1.

$$d[H_s(t), H_o(t)] = d\left[\left(f(t)\cdot\left[s + \frac{1-s}{\sqrt{f^2(t)+g^2(t)}}\right],\right.\right.$$

$$\left.\left. g(t)\cdot\left[s + \frac{1-s}{\sqrt{f^2(t)+g^2(t)}}\right]\right),\ \left(\frac{f(t)}{\sqrt{f^2(t)+g^2(t)}},\ \frac{g(t)}{\sqrt{f^2(t)+g^2(t)}}\right)\right]$$

$$= \left\{f^2(t)\cdot\left[\left(s + \frac{1-s}{\sqrt{f^2(t)+g^2(t)}}\right) - \left(\frac{1}{\sqrt{f^2(t)+g^2(t)}}\right)\right]^2 + \right.$$

$$\left. g^2(t)\cdot\left[\left(s + \frac{1-s}{\sqrt{f^2(t)+g^2(t)}}\right) - \left(\frac{1}{\sqrt{f^2(t)+g^2(t)}}\right)\right]^2\right\}^{1/2}$$

$$= s\cdot\left|1 - \frac{1}{\sqrt{f^2(t)+g^2(t)}}\right|\cdot\sqrt{f^2(t)+g^2(t)}\ .$$

Now

$$d\left[H_1(t),H_o(t)\right] = d\left[(f(t),\ g(t)),\ \left(\frac{f(t)}{\sqrt{f^2(t)+g^2(t)}},\ \frac{g(t)}{\sqrt{f^2(t)+g^2(t)}}\right)\right]$$

$$= \left[f^2(t)\cdot\left(1 - \frac{1}{\sqrt{f^2(t)+g^2(t)}}\right)^2 + g^2(t)\cdot\left(1 - \frac{1}{\sqrt{f^2(t)+g^2(t)}}\right)\right]^{1/2}$$

$$= \left|1 - \frac{1}{\sqrt{f^2(t)+g^2(t)}}\right|\cdot\sqrt{f^2(t)+g^2(t)}\ .$$

Hence, $d\ [H_s(t),\ H_o(t)] = s\cdot\{d[H_1(t),\ H_o(t)]\}$.

Consider $H_o(t_o)$. This point is on the unit circle G^1. Denote the point $(0,0)$ by A. For any t, denote the line segment with end points A and $H_o(t)$ by $\overline{At}$. Then there exist numbers $\epsilon_1,\ \epsilon_2 > 0$ such that the angle between $\overline{At_o+\epsilon_1}$ and $\overline{At_o-\epsilon_2}$ is less than 90^o. Let $I_1 = [t_o-\epsilon_2,\ t_o+\epsilon_1]$. Let $I_2 = I \cap I_1$. Then $t_o \in I_2$, and for $t_1,\ t_2 \in I_2$, the angle between $\overline{Ot_1}$ and $\overline{Ot_2}$ is less than 90^o.

Let t_1 and t_2 be points in I_2. The objective here is to show that for any $s \in [0,1]$, $d[H_s(t_1),\ H_s(t_2)] \leq d[H_1(t_1),H_1(t_2)]$. Choose $s \in [0,1]$. Let $H_1(t_1) = P_1$, $H_s(t_1) = P_1'$, $H_o(t_1) = P_1^o$, $H_1(t_2) = P_2$, $H_s(t_2) = P_2'$. Denote the angle at A between $\overline{P_1A}$ and $\overline{P_2A}$ as angle A. Construct a line C parallel to $\overline{AP_2}$ and intersecting $AP_,$ at P_1^o. Construct lines h and k parallel to the y-axis, such that h intersects $\overline{AP_2}$ at P_2' and k intersects $\overline{AP_2}$ at P_2. Denote the intersection of h and C by $P'_{2,h}$ and the intersection of k and C by $P_{2,k}$.

Let c denote $d(P_1^o, P_{2,k})$, c' denote $d(P_1^o, P'_{2,h})$, b denote $d(P_1^o, P_1)$, b' denote $d(P_1^o, P'_1)$, a denote $d(P_1, P_2)$ and a' denote $d(P'_1, P'_{2,h})$.

Since C and $\overline{AP_2}$ are parallel, angle A is equal to the angle at P_1^o between $\overline{P_1P_1^o}$ and $\overline{P_{2,k}P_1^o}$. Since $d[H_s(t), H_o(t)] = s\cdot[d(H_1(t), H_o(t)]$, $b' = s\cdot b$ and $c' = s\cdot c$. Hence, $b' \le b$ and $c' \le c$. By the Law of Cosines, $a'^2 = b'^2 + c'^2 - 2b'c'\cos A = s^2b^2 + s^2c^2 - 2s^2bc\cos A = s^2(b^2 + c^2 - 2bc\cos A) = s^2a^2$. Since $0 \le s \le 1$, $0 \le s^2 \le 1$, so $a'^2 \le a^2$ and $a' \le a$.

Let $\alpha = d(P_1, P_2)$ and $\alpha' = d(P'_1, P'_2)$. Note that $d(A, P_1) = 1 + b$, $d(A, P'_1) = 1 + b'$, $d(A, P_2) = 1 + c$, and $d(A, P'_2) = 1 + c'$.

Then, by the Law of Cosines,

$$\alpha'^2 = (b'+1)^2 + (c'+1)^2 - 2(b'+1)(c'+1)\cos A$$

$$= b'^2 + 2b' + 1 + c'^2 + 2c' + 1 + (-2b'c' - 2b' - 2c' - 2)\cos A$$

$$= b'^2 + c'^2 - 2b'c'\cos A + 2b' + 2c' + 2 + (-2b' - 2c' - 2)\cos A$$

$$\alpha'^2 = a'^2 + 2(1-\cos A)(b' + c' + 1)$$

Following the same steps, it is clear that $\alpha^2 = a^2 + 2(1-\cos A)(b + c + 1)$. Since angle $A < 90^o$, $1-\cos A$ is positive. Hence,

$$\begin{aligned}\alpha'^2 &\le a^2 + 2(1-\cos A)(b' + c' + 1)\\ &\le a^2 + 2(1-\cos A)(b + c + 1)\\ &= \alpha^2.\end{aligned}$$

Then, since $\alpha'^2 \le \alpha^2$, $\alpha' \le \alpha$. But, $\alpha' = d[H_s(t_1), H_s(t_2)]$ and $\alpha = d[H_1(t_1), H_1(t_2)]$. Hence, $d[H_s(t_1), H_s(t_2)] \le d[H_1(t_1), H_1(t_2)]$.

Let $J' = \left\{s : |s-s_o| < \dfrac{r}{2[d[H_1(t_o), H_o(t_o)] + 1]}\right\}$, and let $J = J' \cap [0,1]$. Let $D = J \times I$. Then D is a neighborhood of (s_o, t_o) in R.

Let (s,t) be a point in D. For $H(s,t)$ to be in N, it must be true that $d[H(s,t), H(s_o,t_o)] < r$. By the triangle inequality, for the point (s,t_o) in D, $d(H_s(t), H_{s_o}(t_o)]$
$\leq d[H_s(t), H_s(t_o)] + d[H_s(t_o), H_{s_o}(t_o)]$.

Since $t_1, t_2 \in I$, $d[H_s(t), H_s(t_o)] \leq d[H_1(t_1), H_1(t_2)] < \frac{r}{2}$.

Now $d[(H_s(t_o), H_{s_o}(t_o)] = |d[H_s(t_o), H_o(t_o)] - d[H_{s_o}(t_o), H_o(t_o)]|$
$= |s \cdot \{d[H_1(t_o), H_o(t_o)]\} - s_o \cdot \{d[H_1(t_o), H_o(t_o)]\}|$
$= |s-s_o| \cdot \{d[H_1(t_o), H_o(t_o)]\} < \frac{r}{2}$, since $s \in J$.

Hence, $d[H_s(t), H_{s_o}(t_o)] < \frac{r}{2} + \frac{r}{2} = r$, so $H(s,t) \in N$, and $H(D) \subset N$. Therefore, the function H, defined by $H(s,t) = H_s(t)$, maps the region R continuously into G. Thus, $\{H_s\}$, $0 \leq s \leq 1$, is a continuous family, and $\mathfrak{p}(A) \simeq A$.

<u>Lemma 2.6.2</u>. If A and B are p-based loops in G^i such that $A \simeq B$ in G then $A \simeq B$ in G^i.

<u>Proof</u>: Let A and B be p-based loops in G^i such that $A \simeq B$ in G. Let $\{K_s\}$, $0 \leq s \leq 1$, be the continuous family of p-based loops in G which exhibit $A \simeq B$. By Lemma 1, for all $s \in [0,1]$, $K_s \simeq \mathfrak{p}(K_s)$, and $\mathfrak{p}(K_s) \in G^i$.

Consider the family $\{\mathfrak{p}(K_s)\}$, $0 \leq s \leq 1$. Assume $K_o = A$ and $K_1 = B$. Then $\mathfrak{p}(K_o) = \mathfrak{p}(A)$ and $\mathfrak{p}(K_1) = \mathfrak{p}(B)$. But A and B are contained in G^i, so $\mathfrak{p}(A) = A$ and $\mathfrak{p}(B) = B$. Hence $\mathfrak{p}(K_o) = A$ and $\mathfrak{p}(K_1) = B$.

For every $s \in [0,1]$, $\| K_s \| = \| p(K_s) \|$. Hence, since $\{K_s\}$ is a continuous family, the stopping time $\| K_s \| = \| p(K_s) \|$ is a continuous function of s.

Let R be the closed region $0 \leq s \leq 1$, $0 \leq t \leq \| p(K_s) \|$. Choose $(s,t) \in R$. $p[K(s,t)] = p(K_s(t))$. Let N be a neighborhood of $p(K_s(t))$ in G^i. Then there exists an open arc I about $p(H_s(t))$ in G^i such that $I \subset N$. Let a and b be the end-points of I. Let N_1 be the set in G such that $p(N_1) = I$. Then N_1 is a neighborhood of $K(s,t) = K_s(t)$. Since $\{K_s\}$ is a continuous family, there exists a neighborhood D about (s,t) such that $K(D) \subset N_1$. Then $p(K(D)) \subset I \subset N$. Hence, $p(K)$ maps the closed region R continuously into G^i. Therefore, $A \simeq B$ in G^i.

<u>Lemma 2.6.3</u>. If A and B are p-based loops in G, then $A \approx B$ if and only if $p(A) \simeq p(B)$ in G^i.

<u>Proof</u>: The proof of this lemma is the same as the proof of Lemma 2.4.3, changing X to G, Y to G^i, and f to p.

<u>Lemma 2.6.4</u>. The induced homomorphism p_* maps an equivalence class in G to an equivalence class in G^i if and only if they have an element in common.

<u>Proof</u>: The proof of this lemma is the same as the proof of Lemma 2.4.4, changing X to G, Y to G^i, f to p, and f_* to p_*.

Therefore, with Lemma 2.6.4 established, $p_*: \pi(G,p) \to \pi(G^i,p)$ is one-to-one. Hence, p_* is an isomorphism. Since G and G^i are both pathwise connected, the fundamental group $\pi(G)$ is isomorphic to the fundamental group $\pi(G^i)$. By Theorem 2.5, $\pi(G^i)$ is infinite cyclic. Therefore, the fundamental group $\pi(G)$ of the annulus G is infinite cyclic.

<u>(2.7) $E^2-(0,0)$</u>. The fundamental group of the space $E^2-(0,0)$ is also infinite cyclic. However, the approach used to show this fact will be different from previous proofs in that the space $E^2-(0,0)$ will be shown to be homeomorphic to the open annulus.

In polar coordinates, let $G_o = \{(\rho,\varphi): 0 < \rho < \infty,\ 0 \leq \varphi < 2\pi\}$, let $G_1 = \{(\rho,\varphi): 0 < \rho < 1,\ 0 \leq \varphi < 2\pi\}$, and let $G_2 = \{(\rho,\varphi): 1 < \rho < 2,\ 0 \leq \varphi < 2\pi\}$.

<u>Lemma 2.7.1</u>. G_o is homeomorhpic to G_1.

<u>Proof</u>: Define a mapping g as follows: for (ρ,φ) in G_1, $g(\rho,\varphi) = (\frac{\rho}{1-\rho}, \varphi)$. Then $g: G_1 \to G_o$. To show g is onto, let (ρ,φ) be a point in G_o. Then $0 < \rho < \infty$. Let (ρ',φ) be a point such that $\rho' = \frac{\rho}{1+\rho}$. Then $\rho = \frac{\rho'}{1-\rho'}$. Now $0 < \rho' < 1$, so $(\rho',\varphi) \in G_1$. Since $\rho = \frac{\rho'}{1-\rho'}$, $g(\rho',\varphi) = (\rho,\varphi)$. Therefore, g is onto.

To show g is one-to-one, let (ρ_1,φ_1) and (ρ_2,φ_2) be distinct points in G_1. Then either $\rho_1 \neq \rho_2$ or $\varphi_1 \neq \varphi_2$. Note that $g(\rho_1,\varphi_1) = \left(\frac{\rho_1}{1-\rho_1}, \varphi_1\right)$ and $g(\rho_2,\varphi_2) = \left(\frac{\rho_2}{1-\rho_2}, \varphi_2\right)$.

If $\rho_1 \neq \rho_2$, then either $\rho_1 > \rho_2$ or $\rho_1 < \rho_2$. Suppose, without loss of generality, that $\rho_1 > \rho_2$. Then $1-\rho_1 < 1-\rho_2$. Hence $\frac{\rho_1}{1-\rho_1} > \frac{\rho_2}{1-\rho_2}$. Therefore, $g(\rho_1,\varphi_1) \neq g(\rho_2,\varphi_2)$. If $\varphi_1 \neq \varphi_2$, then $g(\rho_1,\varphi_1) \neq g(\rho_2,\varphi_2)$.

Hence, g is one-to-one.

Let N be an open set in G_o. Let $P_o = (\rho_o,\varphi_o)$ be a point in $g^{-1}(N)$. Since N is open, there exist $\rho_1, \rho_2, \varphi_1$, and φ_2 such that the set $M = \{(p, \varphi): p_1 < p < p_2, \varphi_1 < \varphi < \varphi_2\} \subset N$ and $g(P_o) \in M$. Let $J = \{(p, \varphi): p_1 < \frac{p}{1-p} < p_2, 0 < p < 1, \varphi_1 < \varphi < \varphi_2\}$. Then $P_o = (p_o, \varphi_o) \in J$, since $g(P_o) \in M$. Also, J is open. Let (p', φ') be a point in J. Then $\varphi_1 < \varphi' < \varphi_2$ and $p_1 < \frac{p'}{1-p'} < p_1$. Since $g(p', \varphi') = (\frac{p'}{1-p'}, \varphi')$, $g(p', \varphi') \in M \subset N$. Hence, $J \subset g^{-1}(N)$. Therefore, $g^{-1}(N)$ is open.

Now let Q be an open set in G_1. Let $P' = (p', \varphi')$ be a point in $g(Q)$. There exist p_1, p_2, φ_1, and φ_2 such that the set $Q_1 = \{(p, \varphi): p_1 < p < p_2, \varphi_1 < \varphi < \varphi_2\} \subset Q$ and $P' \in Q_1$. Let $Q_2 = \{(p, \varphi): \frac{p_1}{1-p_1} < p < \frac{p_2}{1-p_2}, \varphi_1 < \varphi < \varphi_2\}$.

Hence, $0 < p_1 < p_2 < 1$ since $Q_1 \subset G_1$. Then $g(P') \in Q_2$, since $P' \in Q_1$ implies $\frac{p_1}{1-p'} < \frac{p'}{1-p'} < \frac{p_2}{1-p_2}$.

Let (p_o, φ_o) be a point in Q_2. Then $g^{-1}(p_o, \varphi_o) \in Q_1 \subset Q$. Hence, $Q_2 \subset f(Q)$. Since Q_2 is open, $f(Q)$ is open. Therefore, G_o is homeomorphic to G_1.

Lemma 2.7.2. G_1 is homeomorphic to G_2.

Proof: Define a mapping f such that for $(p, \varphi) \in G_1$, $f(p, \varphi) = (p+1, \varphi)$. Then $f: G_1 \to G_2$. To show p is onto, let (p', φ') be a point in G_2. Then $1 < p' < 2$ and $0 \leq \varphi' < 2\pi$. Let (p, φ) be a point such that $p = p'-1$ and $\varphi = \varphi'$. Then $0 < p < 1$ and $0 \leq \varphi < 2\pi$, so $(p, \varphi) \in G_1$. Since $p + 1 = p'$ and $\varphi = \varphi'$, $f(p, \varphi) = (p', \varphi')$. Hence, f is onto.

To show p is one-to-one, let (p_1, φ_1) and (p_2, φ_2) be distinct points in G_1. Then either $p_1 \neq p_2$ or $\varphi_1 \neq \varphi_2$. Now $f(p_1, \varphi_1) = (p_1 + 1, \varphi_1)$ and $f(p_2, \varphi_2) = (p_2+1, \varphi_2)$. If $p_1 \neq p_2$, then $p_1 + 1 \neq p_2 + 1$ and $f(p_1, \varphi_1) \neq f(p_2, \varphi_2)$.

If $\varphi_1 \neq \varphi_2$, then $f(p_1, \varphi_1) \neq f(p_2, \varphi_2)$. Hence, f is one-to-one.

Let N be an open set in G_2. Let $P = (p_o, \varphi_o)$ be a point in $f^{-1}(N)$. Then $f(P) = (p_o+1, \varphi_o) \in N$. Since N is open there exist p_1, p_2, φ_1, and φ_2 such that the set $M = \{p, \varphi) : p_1 < p < p_2, \varphi_1 < \varphi < \varphi_2\}$ $\subset D$, and $f(P) \in M$.

Let $J = \{(p-1, \varphi) : (p, \varphi) \in M\}$. Then $P \in J$. Choose $(p', \varphi') \in J$. Then $f(p', \varphi') = (p'+1, \varphi') \in M \subset N$. Hence $J \subset f^{-1}(N)$. Now $J = \{(p, \varphi) : p_1 - 1 < p < p_2 - 1, \varphi_1 < \varphi < \varphi_2\}$. Therefore, J is open, and so $f^{-1}(N)$ is open.

Let Q be an open set in G_1. Let $P' = (p', \varphi')$ be a point in $f(Q)$. Since Q is open, there exist p_1', p_2', φ_1', and φ_2' such that the set $M' = \{(p, \varphi) : p_1' < p < p_2', \varphi_1' < \varphi < \varphi_2'\} \subset Q$, and $f^{-1}(P') \in M'$. Let $J' = \{(p, \varphi) : p_1' + 1 < p < p_2' + 1, \varphi_1' < \varphi < \varphi_2'\}$. Then $(p', \varphi') \in J'$, since $f^{-1}(p', \varphi') = (p'-1, \varphi') \in M'$. Let (p, φ) be a point in J'. Then $p_1' + 1 < p < p_2' + 1$, $f^{-1}(p, \varphi)$ $= (p-1, \varphi)$, and $p_1' < p - 1 < p_2'$. Hence $f^{-1}(p, \varphi) \in M' \subset Q$. Therefore, $J' \subset f(Q)$ and so $f(Q)$ is open. Hence, G_1 is homeomorphic to G_2.

By Lemma 2.7.1 G_o is homeomorphic to G_1, and by Lemma 2.7.2 G_1 is homeomorphic to G_2. Hence, G_o is homeomorphic to G_2. Then by Theorem 1.17, $\pi(G_o)$ is isomorphic to $\pi(G_2)$. But, by (2.6), $\pi(G_2)$ is infinite cyclic. Hence, $\pi(G_o)$ is infinite cyclic.

<u>(2.8) The space E^3 minus the z-axis</u>. The fundamental group of E^3 minus the z-axis is infinite cyclic. To show this let $X = E^3 - \{(x,y,z) : x = y = 0\}$, and let $Y = \{x,y,z) : z=0\}$. Define a mapping f by the formula $f(x,y,z) = (x,y,0)$, for $(x,y,z) \in X$. Then $f : X \rightarrow Y$.

Let N be an open set in Y. $f^{-1}(N) = \{x,y,z):(x,y,0) \in N\}$. Let (x',y',z') be a point in $f^{-1}(N)$. Then $f(x',y',z') = (x',y',0) \in N$. Thus, there exists an open disc D with center at $(x',y',0)$ such that $D \subset N$. Denote the radius of D by r. Let $N_1 = \{(x,y,z): d[(x,y,z),(x',y',z')] < r\}$. Then N_1 is an open set about (x',y',z'). Let (x_1,y_1,z_1) be a point in N_1. $d[(x',y',0),(x_1,y_1,0)] \leq d[(x',y',z'),(x_1,y_1,z_1)] < r$. Hence $f(x_1,y_1,z_1) \in D \subset N$. Therefore, $f(N_1) \subset N$, so $N_1 \subset f^{-1}(N)$. Then $f^{-1}(N)$ is open. Hence, the inverse image of open sets is open, so f is a continuous mapping.

f determines a mapping f_* of $\pi(X)$ into $\pi(Y)$ given by $f_*([A]) = [f(A)]$, where $[A] \in \pi(X)$. This mapping $f_*:\pi(X) \to \pi(Y)$ is a homomorphism by the proof of Theorem 1.17.

To show f is onto, choose any basepoint p in Y. Let $[A]$ be an element in $\pi(Y,p)$. The mapping $j:Y \to X$ is the inclusion mapping. Then $j_*([A]) \in \pi(X,p)$, where $j_*: \pi(Y) \to \pi(X)$, and $f_*(j_*([A]) = [A]$. Hence, the composition $f_*(j_*)$ is the identity map, and f_* is onto.

Lemma 2.8.1. For any point $p = (x,y,0)$, any p-based loop A in X is equivalent to its projection $f(A)$ in Y.

Proof: Let $p = (x,y,0)$ be a point in X. Note that p is also a point in Y. Let $A:[0,\|A\| \to X$ be a p-based loop in X. Denote A as $(g(t), h(t), k(t))$, where g,h and k are continuous functions of $t \in [0,\|A\|]$. Define a collection $\{H_s\}$, $0 \leq s \leq 1$, of p-based loops by the formula $H_s(t) = (g(t),h(t), s\cdot k(t))$, for $0 \leq t \leq \|H_s\| = \|A\|$. $H_0(t) = (g(t), h(t),0)$, $0 \leq t \leq \|A\|$, so $H_0 = f(A)$. $H_1(t) = (g(t),h(t),k(t))$, $0 \leq t \leq \|A\|$, so $H_1 = A$. For every $s \in [0,1]$, $\|H_s\| = \|A\|$.

Then the stopping time $\|H_s\|$ is a constant function of s, so that function is continuous.

Let R be the closed region $0 \le s \le 1$, $0 \le t \le \|H_s\| = \|A\|$, and let (s,t) be a point in R. $H(s,t) = H_s(t) = (g(t),h(t),s\cdot k(t))$. Let N be the interior of a sphere centered at H(s,t) with an arbitrary radius r. By the continuity of A, there exists a neighborhood I' about t, such that for $t_1,t_2 \in I = I' \cap [0,\|A\|]$, the distance between $(g(t_1), h(t_1), k(t_1))$ and $(g(t_2),k(t_2))$ is less than $\frac{r}{2}$.

Let $J' = \left\{s' : |s-s'| < \frac{r}{2(|k(t)|+1)}\right\}$. Then J' is an interval about s. Let $J = J' \cap [0,1]$. Let $D = J \times I$. Then D is a neighborhood of (s,t). Choose a point (s',t') in D.
$d[H_{s'}(t'), H_{s'}(t)] =$

$$\sqrt{[g(t')-g(t)]^2+[h(t')-h(t)]^2+s'^2[k(t')-k(t)]^2} \le$$

$$s'\cdot\sqrt{[g(t')-g(t)]^2+[h(t')-h(t)]^2+[k(t')-k(t)]^2} \le$$

$$\sqrt{[g(t')-g(t)]^2+[h(t')-h(t)]^2+[k(t')-k(t)]^2} < \frac{r}{2}.$$

Also, $d[H_{s'}(t), H_s(t)] =$

$$\sqrt{[g(t)-g(t)]^2+[h(t)-h(t)]^2+[s'-s]^2\cdot[k(t)]^2} =$$

$$|s'-s|\cdot|k(t)| < \left(\frac{r}{2(|k(t)|+1)}\right)\left(|k(t)|\right) \le \frac{r}{2}.$$

Now
$d[H_{s'}(t'), H_s(t)] \le d[H_{s'}(t'),H_{s'}(t)] + d[H_{s'}(t),H_s(t)]$

$$< \frac{r}{2}+\frac{r}{2} = r.$$

Hence, $H_{s'}(t') \in N$, so $H(D) \subset N$. Then the function H defined by the formula $H(s,t) = H_s(t)$ maps the region R continuously into X. Therefore, $\{H_s\}$, $0 \leq s \leq 1$, is a continuous family, and $A \simeq f(A)$.

Lemma 2.8.2. If A and B are p-based loops in Y such that $A \simeq B$ in X, then $A \simeq B$ in Y.

Proof: Let A and B be p-based loops in Y such that $A \simeq B$ in X. Let $\{K_s\}$, $0 \leq s \leq 1$, be the continuous family of p-based loops in G which exhibit $A \simeq B$. By Lemma 2.8.1 for all $s \in [0,1]$, $K_s \simeq f(K_s)$, and $f(K_s) \in Y$.

Consider the family $\{f(K_s)\}$, $0 \leq s \leq 1$. Assume $K_o = A$ and $K_1 = B$. Then $f(K_o) = f(A)$ and $f(K_1) = f(B)$. But A and B are contained in Y, so $f(A) = A$ and $f(B) = B$. Hence $f(K_o) = A$ and $f(K_1) = B$. For every $s \in [0,1]$, $\| K_s \| = \| f(K_s) \|$. Hence, since $\{K_s\}$ is a continuous family, the stopping time $\| K_s \| = \| f(K_s) \|$ is a continuous function of s.

Let R be the closed region $0 \leq s \leq 1$, $0 \leq t \leq \| f(K_s) \|$. Choose a point (s,t) in R. $f[K(s,t)] = f(K_s(t))$. Let N be a neighborhood of $f(K_s(t))$ in Y. Then there exists an open disc D with center at $f(K_s(t))$ such that $D \subset N$. Denote the radius of D by r. Let N_1 be the set in X such that $f(N_1) = D$. Then N_1 is a neighborhood of $K(s,t) = K_s(t)$. Since $\{K_s\}$ is a continuous family, there exists a neighborhood D_1 about (s,t) such that $K(D_1) \subset N_1$. Then $f(K(D_1)) \subset D \subset N$. Hence, f(K) maps the closed region R continuously into Y. Therefore, $A \simeq B$ in Y.

Lemma 2.8.3. If A and B are p-based loops in X, then $A \simeq B$ if and only if $f(A) \simeq f(B)$ in Y.

Proof: The proof of this lemma is the same as the proof of Lemma 2.4.3.

Lemma 2.8.4. The induced homomorphism f_* maps an equivalence class in X to an equivalence class in Y if and only if they have an element in common.

Proof: The proof of this lemma is the same as the proof of Lemma 2.4.4.

Therefore, by Lemma 2.8.4, $f_* : \pi(X,p) \to \pi(Y,p)$ is one-to-one. Hence, f_* is an isomorphism. Then the fundamental group $\pi(X)$ is isomorphic to the fundamental group $\pi(Y)$. But Y is homeomorphic to $E^2-(0,0)$, which has an infinite cyclic fundamental group by (2.7). Therefore, by Theorem 1.17, the fundamental group $\pi(X)$ of the space E^3 minus the z-axis is infinite cyclic.

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