COMPLETENESS AXIOMS IN AN ORDERED FIELD

APPROVED:

George Cappe
Major professor

D. F. Dawson
Minor Professor

Frank F. Ann
Director of the Department of Mathematics

Robert Toulouse
Dean of the Graduate School

The purpose of this paper was to prove the equivalence of the following completeness axioms.

1. If $H$ and $K$ are subsets of $F$, $H \neq \emptyset$, $K \neq \emptyset$, $H \cap K = \emptyset$, $H \cup K = F$ and $h \leq k$ for every $h$ in $H$ and $k$ in $K$, then there is an element $v$ in $F$ so that if $h$ is in $H$ and $k$ is in $K$, then
   
   (a) $h \leq v$ and $v \geq k$, or
   
   (b) $h = v$ and $v \geq k$.

2. If $S$ is bounded above and is not empty, then it has a least upper bound.

3. Let $\{a_n\}$ be a sequence so that for each positive integer $i$, $a_i$ is in $F$. If for any $p$ in $P$ there is a positive integer $N$ so that if $n$ and $m$ are greater than $N$, then $p \leq (a_n + a_m)$, then $\{a_n\}$ has a limit.

4. If $\{a_n\}$ is a monotone non-decreasing sequence bounded above, then it has a limit.

5. If, for every positive integer $i$, $M_i$ is a closed interval and $M_{i+1} \subseteq M_i$, then there is a $v$ in $F$ such that, for each positive integer $i$, $v$ is in $M_i$.

This purpose was carried out by first defining an ordered field and developing some basic theorems relative to it, then proving that $\lim_{n \to \infty} [(u+u)^*]^n = z$ (where $u$ is the multiplicative identity, $z$ is the additive identity, and
* indicates the multiplicative inverse of an element), and finally proving the equivalence of the five axioms. In order to do this, the Archimedean property was assumed to be true in the field, although it was pointed out that this property could be proved to be true if the first, second or fourth axiom were true in the field. One example of such a proof concluded the paper.
COMPLETENESS AXIOMS IN AN ORDERED FIELD

THESIS

Presented to the Graduate Council of the North Texas State University in Partial Fulfillment of the Requirements

For the Degree of

MASTER OF ARTS

By

Sister Louis Marie Carter, B. A.

Denton, Texas

December, 1971
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. PRELIMINARY THEOREMS</td>
<td>5</td>
</tr>
<tr>
<td>III. COMPLETE ORDERED FIELDS</td>
<td>13</td>
</tr>
</tbody>
</table>
A field is a set of at least two elements having two binary operations (addition and multiplication) defined on it which satisfy certain properties. The operations will be indicated in the usual manner.

The properties required of a field $F$ are as follows:

1. For every $a$ and $b$ in $F$, $a+b = b+a$ and $ab = ba$.
2. There is an element $z$ in $F$ such that $a+z = z+a = a$ for every $a$ in $F$.
3. For every $a$ in $F$ there is an $\bar{a}$ in $F$ so that $a+\bar{a} = \bar{a}+a = z$.
4. For every $a$, $b$, $c$ in $F$, $(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$.
5. There is an element $u$ in $F$ such that $au = ua = a$ for every $a$ in $F$.
6. For every $a$ in $F$, provided $a \neq z$, there is an $a^*$ in $F$ so that $aa^* = a^*a = u$.
7. For every $a$, $b$, $c$ in $F$, $a(b+c) = (ab)+(ac)$.

A field is called an ordered field if there exists a subset $P$ in $F$ so that if $a$ is in $F$, then one and only one of the following is true: $a$ is in $P$, $\bar{a}$ is in $P$, or $a = z$; if $a$ is in $P$ and $b$ is in $P$, then $a+b$ and $ab$ are in $P$. An order relation $R$ is defined on $F$ in this manner: if $a$ is in $F$ and
b is in \( F \), then \( a \mathcal{R} b \) if and only if \( b + a \) is in \( P \). This relation \( \mathcal{R} \) is transitive (if \( a \mathcal{R} b \) and \( b \mathcal{R} c \), then \( a \mathcal{R} c \)) but is neither reflexive nor symmetric as will be shown.

Let \( a \mathcal{R} b \) and \( b \mathcal{R} c \). Then \( b + a \) and \( c + b \) are in \( P \). But by the properties of a field \( (b + a) + (c + b) = c + a \). So \( a \mathcal{R} c \) by definition of the order relation. Hence the relation \( \mathcal{R} \) is transitive.

Now let \( a \mathcal{R} b \), and suppose by way of contradiction that \( b \mathcal{R} a \). Then \( b + a \) and \( a + b \) are in \( P \); hence \( (b + a) + (a + b) \) is in \( P \). This is clearly impossible since \( (b + a) + (a + b) = z \). Thus \( \mathcal{R} \) is not symmetric.

Finally, suppose that \( a \mathcal{R} a \). Then \( a + a \) is in \( P \). But \( a + a = z \), which cannot be in \( P \). So again there is a contradiction; hence \( \mathcal{R} \) is not reflexive.

Let \( n \) be a positive integer and let \( na \) be \( a \) added to itself \( n-1 \) times. The Archimedean property states that, given an element \( a \) in \( P \) and an element \( b \) in \( P \), there is a positive integer \( n \) such that \( b \mathcal{R} na \). Let \( F \) have the Archimedean property.

The following are some definitions of terms used in the course of this paper.

A set \( S \) is **bounded above** if and only if there is an element \( M \) in \( F \) such that \( s \mathcal{R} M \) for every \( s \) in \( S \). A sequence \( a_n \) is bounded above if and only if there is an element \( M \) in \( F \) such that for every positive integer \( n \), \( a_n \mathcal{R} M \).

\( U \) is the **least upper bound** of \( S \) if and only if

(a) \( s \mathcal{R} U \) for every \( s \) in \( S \), and

(b) for any \( p \) in \( P \) there is at least one \( s \) in \( S \) such that \( U + p \mathcal{R} s \).
A sequence \( \{a_n\} \) is **monotone non-decreasing** if and only if, for every \( i \) and \( j \) such that \( i < j \), \( a_i \leq a_j \).

\( L \) is the limit of a sequence \( \{a_n\} \) if and only if for any \( p \) in \( P \) there is a positive integer \( N \) such that if \( n > N \), then \((L+p) \leq a_n \leq (L+p)\).

If \( a \) and \( b \) are elements of \( F \) so that \( a \leq b \) and \( M \) is a set such that \( x \) is in \( M \) if and only if \( x \) is in \( F \) and \( a \leq x \leq b \), then \( M \) is a **closed interval**, sometimes denoted by \([a,b]\).

The **length** of \([a,b]\) is \( b+\alpha \).

The **midpoint** of \([a,b]\) is \((b+\alpha)/(u+\alpha)*\).

For any \( a \) in \( F \) the **absolute value** of \( a \), \( |a| \), is \( a \) if \( a = z \) or if \( a \) is in \( P \), and \( |a| = \alpha \) if \( a \) is in \( P \).

\( a^1 = a \). If \( k \) is a positive integer, then \( a^{k+1} = (a^k)a \). It follows that \( a^n(a^m) = a^{n+m} \).

\( 1a = a \). If \( k \) is a positive integer, then \( (k+1)a = ka + a \). It follows that \( na + ma = (n+m)a \).

The purpose of this paper is to prove the equivalence of five completeness axioms stated as follows.

1. If \( H \) and \( K \) are subsets of \( F \), \( H \neq \emptyset , K \neq \emptyset \), \( H \cap K = \emptyset \), \( H \cup K = F \) and \( h \leq k \) for every \( h \) in \( H \) and \( k \) in \( K \), then there is an element \( v \) in \( F \) so that if \( h \) is in \( H \) and \( k \) is in \( K \), then
   
   (a) \( h \leq v \) and \( v \leq k \), or
   
   (b) \( h = v \) and \( v \leq k \).

2. If \( S \) is bounded above and is not empty, then it has a least upper bound.
(3) Let \( \{a_n\} \) be a sequence so that for each positive integer \( i \), \( a_i \) is in \( F \). If for any \( p \) in \( P \) there is a positive integer \( N \) so that if \( n \) and \( m \) are greater than \( N \), then
\[
\overline{p} \leq (a_n + a_m) \leq p, \text{ then } \{a_n\} \text{ has a limit.}
\]

(4) If \( \{a_n\} \) is a monotone non-decreasing sequence bounded above, then it has a limit.

(5) If, for every positive integer \( i \), \( M_i \) is a closed interval and \( M_{i+1} \subseteq M_i \), then there is a \( v \) in \( F \) such that, for each positive integer \( i \), \( v \) is in \( M_i \).

Since the proof of the equivalence of these statements involves the use of several theorems concerning the elements of fields, these theorems will be proved first.
CHAPTER II

PRELIMINARY THEOREMS

(A) Uniqueness of inverses

Suppose that both \( \overline{a} \) and \( b \) are additive inverses of \( a \), an element of \( F \). Then \( a+\overline{a} = z \) and \( a+b = z \). Thus

\[
\begin{align*}
    a+\overline{a} &= a+b \\
    \overline{a}+(a+\overline{a}) &= \overline{a}+(a+b) \\
    (\overline{a}+a)+\overline{a} &= (\overline{a}+a)+b \\
    z+\overline{a} &= z+b \\
    \overline{a} &= b.
\end{align*}
\]

Hence each element has a unique additive inverse.

Similarly, suppose that both \( a^* \) and \( b \) are multiplicative inverses of \( a \), a non-zero element of \( F \). Then \( aa^* = u \) and \( ab = u \). Therefore

\[
\begin{align*}
    aa^* &= ab \\
    a^*(aa^*) &= a^*(ab) \\
    a^*u &= (a^*a)b \\
    a^* &= ub \\
    a^* &= b.
\end{align*}
\]

Hence each non-zero element of \( F \) has a unique multiplicative inverse.
(B) Uniqueness of additive and multiplicative identities

Suppose \( a+b = a \) and \( a+z = a \). Then

\[
\begin{align*}
a+b &= a+z \\
\bar{a}+(a+b) &= \bar{a}+(a+z) \\
(\bar{a}+a)+b &= (\bar{a}+a)+z \\
z+b &= z+z \\
b &= z.
\end{align*}
\]

There is an \( a \) in \( F \) such \( a \neq z \). Suppose that \( ab = a \) and \( au = a \). Then \( ab = au \). There is an \( a^* \) in \( F \) such that

\[
\begin{align*}
a^*(ab) &= a^*(au) \\
(a^*a)b &= (a^*a)u \\
ub &= uu \\
b &= u.
\end{align*}
\]

Hence if two elements of \( F \) serve as additive or as multiplicative identities, they must be equal.

(C) \( az = z \).

\[
\begin{align*}
a+az &= au+az \\
&= a(u+z) \\
&= au \\
&= a.
\end{align*}
\]

But \( a+z = a \). Therefore, by uniqueness of the additive identity, \( az = z \).
(D) \( a\bar{c} = \bar{a}c = \bar{a}c \)

\[
\begin{align*}
ac + \bar{ac} &= z. \\
ac + a\bar{c} &= a(c + \bar{c}) \\
&= az \\
&= z. \\
ac + \bar{ac} &= (a + \bar{a})c \\
&= zc \\
&= z.
\end{align*}
\]

Hence, by uniqueness of the additive inverse,

\( a\bar{c} = \bar{a}c = \bar{a}c. \)

(E) \( u = u^*; \ z = \bar{z}. \)

If \( a \) is in \( F \), then \( a + z = a. \)

\[
\begin{align*}
a + \bar{z} &= (a + z) + \bar{z} \\
&= a + (z + \bar{z}) \\
&= a + z \\
&= a.
\end{align*}
\]

So by uniqueness of the additive identity \( z = \bar{z}. \)

If \( a \) is in \( F \), then \( ua = a. \)

\[
\begin{align*}
u*a &= u*(ua) \\
&= (u*u)a \\
&= ua \\
&= a.
\end{align*}
\]

So by uniqueness of the multiplicative identity,

\( u = u^*. \)
(F) \( u \neq z \).

There is an \( a \) in \( F \) such that \( a \neq z \). Suppose \( u = z \).

Then \( au = az \). But \( z = az + az \)  
\[ = au + az \]
\[ = a(u + z) \]
\[ = a(u + z) \]
\[ = a. \]

This is a contradiction; hence \( u \neq z \).

(G) \( \bar{a} = a; (a^*)^* = a \).

\[ a + \bar{a} = z. \]
\[ \bar{a} + a = \bar{a} + a \]
\[ = z. \]

Hence, by uniqueness of the additive inverse, \( \bar{a} = a \).

\[ a a^* = u. \]
\[ (a^*)^* a^* = a^* (a^*)^* \]
\[ = u. \]

Hence, by uniqueness of the multiplicative inverse,  
\( (a^*)^* = a \).

(H) \( P \) contains \( u \).

Let \( a \) be in \( P \). Since \( u \neq z \), either \( u \) is in \( P \) or \( \bar{u} \) is in \( P \). Suppose \( \bar{u} \) is in \( P \). Then \( a\bar{u} \) is in \( P \). But
\[ au = \overline{au} \]
\[ = \overline{a}. \]

This is a contradiction; hence \( u \) is in \( P \).

**\( I \)** If \( a \) is in \( P \), then \( a^* \) is in \( P \).

Suppose \( a^* \) is not in \( P \). Then \( a^* = z \) or \( \overline{a^*} \) is in \( P \).

If \( a^* = z \), then \( u = aa^* = az = z \). This is a contradiction; so \( a^* \neq z \).

Now suppose \( \overline{a^*} \) is in \( P \). Then \( aa^* \) is in \( P \). But \( aa^* = \overline{aa^*} = \overline{u} \). Again this is a contradiction; so \( \overline{a^*} \) is not in \( P \). Hence \( a^* \) must be in \( P \).

**\( J \)** \( \overline{a+b} = \overline{a+b} \).

\[ (a+b)+(\overline{a+b}) = z. \]
\[ (a+b)+(\overline{a+b}) = (a+\overline{a})+(b+\overline{b}) \]
\[ = z+z \]
\[ = z. \]

So by uniqueness of the additive inverse \( \overline{a+b} = \overline{a+b} \).

**\( K \)** If \( a \) is in \( P \) and \( b \) is not in \( P \), then \( b \) R \( a \).

If \( b \) is not in \( P \), then \( b = z \) or \( \overline{b} \) is in \( P \).

If \( b = z \), then \( a+\overline{b} = a+z = a \); so \( b \) R \( a \).

If \( \overline{b} \) is in \( P \), then \( a+\overline{b} \) is in \( P \). Hence \( b \) R \( a \).
(L) If \( aRb \) and \( c \) is in \( P \), then \( ac R bc \).

Let \( aRb \). Then \( b+a \) is in \( P \). Thus \( (b+a)c \) is also in \( P \). But 
\[
(b+a)c = bc+ac
\]
\[
= bc+ac.
\]
Hence \( ac R bc \).

(M) If \( aRb \), then \( (c+a) R (c+b) \).

Let \( aRb \). Then \( (b+a) \) is in \( P \). But 
\[
b+a = (b+a)+z
\]
\[
= (b+a)+(c+c)
\]
\[
= (b+c)+(a+c)
\]
\[
= (c+b)+(c+a).
\]
Therefore \( (c+a) R (c+b) \).

(N) If \( aRb \), then \( \bar{b} R \bar{a} \).

Let \( aRb \). Then \( (b+a) \) is in \( P \). But \( b+a = \bar{a}+b = \bar{a}+\bar{b} \).
Therefore \( \bar{b} R \bar{a} \).

(0) If \( a \) and \( b \) are in \( P \) and \( aRb \), then \( b^* R a^* \).

Let \( a \) and \( b \) be in \( P \) and \( aRb \). Then \( a^* \) and \( b^* \) are in \( P \).

\[
, \ a^* R b^*
\]
\[
\bar{u} R b^*
\]
\[
b^*u R b^*b^*
\]
\[
b^* R u^*
\]
\[
b^* R a^*.
\]
(P) If each of \( a, b, c \) and \( d \) is in \( P \), \( a \ R \ b \), and \( c \ R \ d \), then \( ac \ R bd \).

Since \( a \ R \ b \), \( c \ R \ d \), and \( b \) and \( c \) are in \( P \), then \( ac \ R bc \) and \( bc \ R bd \). Therefore \( ac \ R bd \).

(Q) If \( m \) and \( n \) are positive integers such that \( n < m \), then \( [(u+u)^*]^m \ R \ [(u+u)^*]^n \).

First it will be shown that, if \( a \) is in \( F \), then \((a^n)^* = (a^*)^n\).

\[
\begin{align*}
\quad & a^n(a^n)^* = u. \\
\quad & a^n(a^*)^n = (aa^*)^n \\
\quad & = u^n \\
\quad & = u.
\end{align*}
\]

So by uniqueness of the multiplicative inverse \((a^n)^* = (a^*)^n\).

Let \( p = m - n \). Then \( u \ R (u+u) \ R (u+u)^n \) and \( u \ R (u+u) \ R (u+u)^p \).

\[
\begin{align*}
\quad & u(u+u)^n \ R (u+u)^p(u+u)^n \\
\quad & (u+u)^n \ R (u+u)^m \\
\quad & [(u+u)^m]^* \ R [(u+u)^n]^* \\
\quad & [(u+u)^*]^m \ R [(u+u)^*]^n.*
\end{align*}
\]
\( \lim_{n \to \infty} \left[ (u+u)^* \right]^n = z. \)

\( u R (u+u)^1. \)

Assume that for some positive integer \( k \), \( ku R (u+u)^k. \)

Then \( ku(u+u) R (u+u)^k(u+u) \)
\( (ku+ku) R (u+u)^{k+1} \)

But \( u \equiv ku \), so \( ku+u R (u+u)^{k+1} \)
\( (k+1)u R (u+u)^{k+1}. \)

Hence for any positive integer \( n \), \( u \equiv nu R (u+u)^n. \)

Let \( p \) be in \( P \). By the Archimedean principle there is a positive integer \( N \) such that
\( p^* R Nu R (u+u)^N. \)

If \( n > N \), then \( (u+u)^N R (u+u)^n \). So
\[ p^* R (u+u)^n \]
\[ \left[ (u+u)^n \right]^* R (p^*)^* \]
\[ \left[ (u+u)^* \right]^n R p. \]

Hence \( \overline{p} R \left[ (u+u)^* \right]^n R p \)
and \( \overline{z+p} R \left[ (u+u)^* \right]^n R z+p. \)

Therefore \( \lim_{n \to \infty} \left[ (u+u)^* \right]^n = z. \)
CHAPTER III

COMPLETE ORDERED FIELDS

It is now possible to prove the equivalence of the five completeness axioms mentioned earlier in this paper. The procedure to be used is as follows: it will be shown that statement (1) implies statement (2), statement (2) implies statement (3), ..., statement (5) implies statement (1).

Proof I

ASSUME: If H and K are subsets of F, H ≠ Ø, K ≠ Ø, H ∩ K = Ø, H ∪ K = F and h R k for every h in H and k in K, then there is an element v in F so that if h is in H and k is in K, then (a) h R v and v R k, or

(b) h = v and v R k.

PROVE: If a non-empty set S in F is bounded above, then it has a least upper bound.

PROOF: Let S be bounded above. Let K be the set of all upper bounds of S not contained in S. Let H ∪ K = F, H ∩ K = Ø, S ⊆ H. If h is in H, then h is in S or h is not in S. If h is in S and k is in K, then h R k. Since h is in S, h is not in K; hence h ≠ k. Therefore h R k. If h
is not in $S$, then $h$ is not an upper bound of $S$. Therefore there is an element $q$ of $S$ such that $h \not R q$. Since $k$ is in $K$, it follows that $q \not R k$; therefore $h \not R k$.

$S$ is not empty; therefore $H$ is not empty.

There is a $j$ such that $j$ is an upper bound of $S$. If $s$ is in $S$, then $s \not R j \not R j+u$. So $s \not \in j+u$, which is an upper bound of $S$ and cannot be in $S$. Therefore $K$ is not empty; it at least contains $j+u$.

By the assumption there is an element $v$ in $F$ so that if $h$ is in $H$ and $k$ is in $K$, then

(a) $h \not R v$ and $v \not R k$, or

(b) $h = v$ and $v \not R k$.

Let $a$ be in $P$. From previous proofs

$z \not R (u+u)\ast R u$

$az \not R a(u+u)\ast R au$

$z \not R a(u+u)\ast R a$

$a \not R a(u+u)\ast \not R z$

$(v+a) \not R (v+a(u+u)\ast) R (v+z)$

$(v+a) \not R (v+a(u+u)\ast) R v R (v+a)$.

Thus $v+a(u+u)\ast$ is in $H$, since $K$ consists only of elements such that $v \not R k$. If $v+a(u+u)\ast$ is an upper bound of $S$, then it is in $S$. If it is not an upper bound of $S$, then there is an element $g$ in $S$ such that

$(v+a) \not R (v+a(u+u)\ast) R g \not R v R (v+a)$.

In either case there is an element $s$ of $S$ such that

$(v+a) \not R s R (v+a)$. Thus $v$ is the least upper bound of $S$. 

Proof II

ASSUME: If a non-empty set $S$ is bounded above, then it has a least upper bound.

PROVE: Let $\{a_n\}$ be a sequence. If for any $p$ in $P$ there is a positive integer $N$ such that if $n$ and $m$ are greater than $N$, then $p R (a_n + a_m) R p$, then $\{a_n\}$ has a limit.

PROOF: $P$ contains $u$. By hypothesis there is a positive integer $N$ such that if $n > N$ and $m > N$, then $u R (a_n + a_m) R u$. If $n > N$, then $u R (a_n + a_{n+1}) R u$.

Let $K = \{a_1 + |a_2| + |a_3| + \ldots + |a_N| + |a_{N+1}| + u\}$. If $j$ is a positive integer, then $j \leq N+1$ or $j > N+1$. If $j \leq N+1$, then $|a_j| R K$. Since $a_j R \{a_j\}$, then $a_j R K$.

If $j > N+1$, then by the hypothesis $a_j R (a_{N+1} + u)$, then $a_j R (|a_{N+1}| + u) R K$.

Let $L = R$. If $j \leq N+1$, then $L R a_j$. If $j > N+1$, then $(a_{N+1} + u) R a_j$. But $L R (a_{N+1} + u)$; hence $a_j R a_j$.

Define $H = \{x: x$ is in $H$ if and only if there is a positive integer $N$ such that if $n > N$, then $a R a_n\}$. $L$ is in $H$ since $L R a_j$ for every positive integer $j$.

$K$ is not in $H$ because $a_j R K$ for every positive integer $j$.

If $x$ is in $H$, then there is a positive integer $i$ such that $a_i R K$ and $x R a_i$. So $K$ is an upper bound for $H$. 

Thus by the assumption $H$ has a least upper bound; call it $q$.

Let $p$ be in $P$. There is an $x$ in $H$ such that

$$(q + p(u+u)^*) \leq q.$$ 

There is a positive integer $N_1$ such that if $n > N_1$, then $x \leq a_n$. And there is a positive integer $N_2$ such that if $n > N_2$ and $m > N_2$, then

$$p(u+u)^* \leq p(u+u)^*.$$ 

Since $H$ does not contain $(q + p(u+u)^*)$, there is a positive integer $N_3 > N_1 + N_2$ such that

$$x \leq a_{N_3} \leq q + p(u+u)^*.$$ 

If $n > N_3$, then $n > N_1$ and $n > N_2$. Thus $x \leq a_n$ and

$$p(u+u)^* \leq p(u+u)^*.$$ 

So $a_{N_3} + p(u+u)^* \leq a_n + p(u+u)^*$. But $a_{N_3} \leq q + p(u+u)^*$. Thus

$$a_n \leq q + p(u+u)^* + p(u+u)^* \leq a_n.$$ 

Also $x \leq a_{N_3}$, so $x + p(u+u)^* \leq a_n$. And $q + p(u+u)^* \leq x$.

Thus

$$q + p(u+u)^* + p(u+u)^* \leq a_n.$$ 

Hence $\{a_n\}$ has a limit.
Proof III

ASSUME: Let \( \{a_n\} \) be a sequence. If for any \( p \) in \( P \) there is a positive integer \( N \) such that for \( n \) and \( m \) greater than \( N \), \( p \ R \ (a_n + a_m) \ R p \), then \( \{a_n\} \) has a limit.

PROVE: If \( \{a_n\} \) is a monotone non-decreasing sequence bounded above, then \( \{a_n\} \) has a limit.

PROOF: Let \( \{a_n\} \) be a monotone non-decreasing sequence bounded above by \( B_1 \). \( \{a_n\} \) is bounded below by \( a_1 \). Let \( a_1 = A_1 \) and consider the closed interval \( [A_1, B_1] \). It has length \( B_1 + A_1 \) and contains \( \{a_n\} \). The midpoint of \( [A_1, B_1] \) is \( (A_1 + B_1)(u+u)^* \). If the interval \( [A_1, (A_1 + B_1)(u+u)^*] \) contains infinitely points of \( \{a_n\} \), let \( A_1 = A_2 \) and \( (A_1 + B_1)(u+u)^* = B_2 \). If \( [A_1, (A_1 + B_1)(u+u)^*] \) does not contain infinitely many points of \( \{a_n\} \), let \( A_1 = A_2 \) and \( (A_1 + B_1)(u+u)^* = B_2 \). The midpoint of \( [A_2, B_2] \) is \( (A_2 + B_2)(u+u)^* \). If the interval \( [A_2, (A_2 + B_2)(u+u)^*] \) contains infinitely many elements of \( \{a_n\} \), let \( A_2 = A_3 \) and \( (A_2 + B_2)(u+u)^* = B_3 \). If it does not contain infinitely many elements of \( \{a_n\} \), let \( (A_2 + B_2)(u+u)^* = A_3 \) and \( B_2 = B_3 \). Suppose \( B_2 = B_1 \). The length of \( [A_2, B_2] = B_2 + A_2 \). But

\[
B_2 + A_2 = B_1 + (B_1 + A_1)(u+u)^* \\
= B_1 + (B_1 + A_1)(u+u)^* \\
= B_1 + B_1(u+u)^* + A_1(u+u)^* \\
= B_1(u+u)(u+u)^* + B_1(u+u)^* + A_1(u+u)^*
\]
\[= B_1(\overline{u+u})^*(\overline{u+u+u})+A_1(\overline{u+u})^*\]

\[= B_1(\overline{u+u})^*(\overline{u+u+u})+A_1(\overline{u+u})^*\]

\[= B_1(\overline{u+u})^*(\overline{u+z})+A_1(\overline{u+u})^*\]

\[= B_1(\overline{u+u})^*u+A_1(\overline{u+u})^*\]

\[= B_1(u+u)^*+\overline{A_1}(u+u)^*\]

\[(B_1+\overline{A_1})(u+u)^*\]  

Similar reasoning holds if \(A_2 = A_1\) and \(B_2 = (A_1+B_1)(u+u)^*\).

Using the same line of reasoning it is found that the length of \([A_2,B_2]\) is \((B_1+\overline{A_1})[(u+u)*]^2\). This process may be continued indefinitely. It will be found that the interval \([A_n,B_n]\) has length \((B_1+\overline{A_1})[(u+u)*]^n\).

Either \(B_1+\overline{A_1} = z\) or \(B_1+\overline{A_1}^\prime\) is in \(P\). Since \([(u+u)*]^n\) has the limit \(z\), if \(p\) is in \(P\) and \(B_1+\overline{A_1}\) is in \(P\), then there is a positive integer \(N\) such that \([(u+u)*]^N R p(B_1+\overline{A_1})^*\).

If \(p\) is in \(P\) and \(B_1+\overline{A_1} = z\), there is a positive integer \(N\) such that \([(u+u)*]^N R p\). In either case, if \(n > N\), then \((B_1+\overline{A_1})[(u+u)*]^N R (B_1+\overline{A_1})[(u+u)*]^N R p\), and if \(i \geq j > n\), then \(a_i, a_j\) are in \([A_n,B_n]\), \(a_i+\overline{a_j}\) is in \(P\), and \(a_i+\overline{a_j} R B_n+\overline{A_n}\). But \(B_n+\overline{A_n} = (B_1+\overline{A_1})[(u+u)*]^n\), and so

\[p R (a_i+\overline{a_j}) R (B_1+\overline{A_1})[(u+u)*]^N R P.\]

Hence by the assumption \(\{a_n\}\) has a limit.
ASSUME: If \( \{a_n\} \) is a monotone non-decreasing sequence bounded above, then it has a limit.

PROVE: If for any positive integer \( i \), \( M_i \) is a closed interval and \( M_{i+1} \subseteq M_i \), then there is an element \( v \) in \( F \) such that \( v \) is in \( M_i \) for every \( i \).

PROOF: Consider the set of all closed intervals \( [a_i, b_i] = M_i ; \)
\[ a_1 \subseteq a_2 \subseteq \cdots \subseteq a_i \subseteq \cdots \subseteq b_1 \subseteq \cdots \subseteq \]
is a monotone non-decreasing sequence bounded above by \( b_i \); therefore \( \{a_i\} \) has a limit \( v \).
Suppose there is an \( a_i \) such that \( v \not\subseteq a_i \). Then \( a_i + v \) is in \( P \). Thus there is a positive integer \( M \) such that if \( m > M \), then \( (v+a_i + v) \subseteq a_m \subseteq (v+a_i + v) \) or \( (v+a_i + v) \subseteq a_m \subseteq a_i \).
Since \( \{a_i\} \) is monotone non-decreasing, this is impossible.
Therefore \( a_i \subseteq v \) for all \( i \).
Suppose \( b_i \subseteq v \) for some positive integer \( i \). Then \( v + b_i \) is in \( P \). There is a \( j \) such that
\[
\begin{align*}
    v + a_j &\subseteq v + b_i \\
    a_j &\subseteq b_i \\
    b_i &\subseteq a_j.
\end{align*}
\]
This is a contradiction; therefore \( v \not\subseteq b_i \). So \( v \) is in the closed interval \( [a_i, b_i] = M_i \) for every \( i \).
ASSUME: If for any positive integer $i$, $M_i$ is a closed interval and $M_{i+1} \subseteq M_i$, then there is an element $v$ in $F$ such that $v$ is in $M_i$ for every $i$.

PROVE: If $H$ and $K$ are subsets of $F$ such that $H \neq \emptyset$, $K \neq \emptyset$, $H \cap K = \emptyset$, $H \cup K = F$ and $h R k$ for every $h$ in $H$ and $k$ in $K$, then there is an element $v$ in $F$ such that if $h$ is in $H$ and $k$ is in $K$, then (a) $h R v$ and $v = k$, or (b) $h = v$ and $v R k$.

PROOF: Let $H$ and $K$ be defined as in the hypothesis. Let $a_1$ be in $H$, and let $b_1$ be in $K$. Consider the midpoint of the closed interval $[a_1, b_1]$; it is either in $H$ or in $K$. If it is in $H$, call it $a_2$ and let $b_1 = b_2$. If the midpoint is in $K$, call it $b_2$ and let $a_1 = a_2$. The interval $[a_2, b_2]$ is contained in $[a_1, b_1]$. Likewise consider the midpoint of $[a_2, b_2]$. If it is in $H$, call it $a_3$ and let $b_2 = b_3$; if it is in $K$, call it $b_3$ and let $a_2 = a_3$. The interval $[a_3, b_3] \subseteq [a_2, b_2]$. This procedure may be continued indefinitely. By the assumption there is an element $v$ in $F$ such that $v$ is contained in every one of the closed intervals.

Suppose there is an $h$ in $H$ such that $v R h$. If $i$ is any positive integer, then $a_{i+1} \subseteq v R h R b_i$. So $h + v$ is in $P$. Thus $(h + v)(b_1 + a_1)$ is in $P$. There is a positive integer
n such that \[ [(u+u)^*]^n R (h+v)(b_i+a_{i^n})^*. \] Hence
\[ (b_i+a_{i^n})[(u+u)^*]^n R (h+v). \]
So
\[ (b_i+a_{i+n}) R (h+v). \]
Thus
\[ (v+a_{i+n}) R (h+b_{i+n}). \]
But \( v+a_{i+n} = z \) or \( v+a_{i+n} \) is in \( P \). Therefore \( h+b_{i+n} \) is in \( P \). This is a contradiction. Hence \( h \neq v \) for every \( h \) in \( H \).

If there is an \( h \) in \( H \) such that \( h = v \), then \( v \) is in \( H \).

Hence \( v \not\in k \) for every \( k \) in \( K \).

If \( h \not\in v \) for every \( h \) in \( H \), then \( v \) is in \( K \). Suppose there is a \( k \) in \( K \) such that \( k \not\in v \). Then for any positive integer \( i \), \( a_i R k R v \not\in b_i \). So \( v+k \) is in \( P \) and \( b_i+a_i \) is in \( P \).

Thus \( (v+k)(b_i+a_i)^* \) is in \( P \). There is a positive integer \( n \) such that \[ [(u+u)^*]^n R (v+k)(b_i+a_i)^*. \] Thus
\[ (b_i+a_i)[(u+u)^*]^n R (v+k). \]
So
\[ (b_{i+n}+a_{i+n}) R (v+k). \]
Thus
\[ (k+a_{i+n}) R (v+b_{i+n}). \]
But \( k+a_{i+n} = z \) or \( k+a_{i+n} \) is in \( P \). Thus \( v+b_{i+n} \) is in \( P \).

This is a contradiction. Hence if \( h \not\in v \) for every \( h \) in \( H \), then \( v \not\in k \) for every \( k \) in \( K \).

This concludes the proof of the equivalence of the five axioms.
Although the Archimedean property was assumed to be true in $F$ and was used in proofs III and V, it can be proved to be true if either statement 1 or statement 2 or statement 4 is true in $F$. For example, if the least upper bound property (statement 2) is assumed true in $F$, the Archimedean property can be proved as follows.

Let $a$ be in $P$ and $b$ be in $P$.

If $b R a$, then $b R 1a$.

If $b = a$, then $b R 2a$.

If $a R b$, suppose that for every positive integer $n$, $na \not\in P$. Let $S = \{na\}$. $S$ is bounded above by $b$; therefore $S$ has a least upper bound. Call it $L$.

\[
\begin{align*}
  z & R a R L \, . \\
  (u+u)^* & R u. \\
  L(u+u)^* & R Lu. \\
  L(u+u)^* & R L. \\
\end{align*}
\]

Since $L$ is the least upper bound of $S$, there is a positive integer $i$ such that $L(u+u)^* R ia$.

So

\[
\begin{align*}
  L(u+u)^*(u+u) & R ia(u+u) \\
  L R ia+ia \\
  L R 2ia. \\
\end{align*}
\]

This is a contradiction since $2i$ is a positive integer; therefore there is a positive integer $n$ such that $b R na$. 