ON SETS AND FUNCTIONS IN A METRIC SPACE

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In Chapter I, a metric space is defined and some examples of metric spaces are given. General theorems on open and closed sets in metric spaces are proved, thus laying the groundwork for later study. A metric space is shown to be a topological space. The first five separation axioms, which are topological properties known as T_0, T_1, T_2 (Hausdorff), T_3 (regular), and T_4 (normal), are defined. Examples of general topological spaces which satisfy one axiom but not the next one are presented. However, it is proved that a metric space is T_0, T_1, T_2, T_3, and T_4, showing that a metric space is less general than a topological space.

In Chapter II, some specific set properties of metric spaces are considered. The central property studied is compactness. The equivalence of compact, sequentially

compact, countably compact, and complete and totally bounded sets in a metric space is established. Compactness is studied in its relationship to other properties such as separability and the Lindelof Property. The Baire Category Theorem is proved, thus giving a property related to completeness in a metric space.

Chapter III concerns some of the properties of functions in metric spaces. Attention is given mainly to functions from a metric space (M,d) into the set of real numbers R. Continuity and semi-continuity are the central properties.

ON SETS AND FUNCTIONS IN A METRIC SPACE

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CHAPTER I

BASIC PROPERTIES

The purpose of this thesis is to study some of the properties of metric spaces. An effort will be made to show that many of the properties of a metric space are generalized properties of R, the set of real numbers, or Euclidean n-space, and are specific cases of the properties of a general topological space.

In Chapter I, a metric space will be defined and some examples of metric spaces will be given. General theorems on open and closed sets in metric spaces will be proved, thus laying the groundwork for later study. A metric space is shown to be a topological space. The first five separation axioms, which are topological properties known as T_0, T_1, T_2 (Hausdorff), T_3 (regular), and T_4 (normal), are defined. Examples of general topological spaces which satisfy one axiom but not the next one are presented. However, it is proved that a metric space is T_0, T_1, T_2, T_3, and T_4, showing that a metric space is less general than a topological space.

In Chapter II some specific set properties of metric spaces will be considered. The central property to be studied is compactness. The equivalence of compact, sequentially compact, countably compact, and complete and totally bounded sets in a metric space will be established. Compactness will then be studied in its relationship to other properties such as separability and the Lindelof Property. The Baire Category Theorem is proved, thus giving a property related to completeness in a metric space.

Chapter III concerns some of the properties of functions in metric spaces. Attention will be given mainly to functions from a metric space (M,d) into the set of real numbers R. Continuity and semi-continuity are the central properties.

1.1. Definition. Let M be a non-empty set. A metric on M is a real function d whose domain is the set of all ordered pairs of elements of M and which satisfies the following three conditions:

i) if $x,y \in M$, $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;

ii) if $x,y \in M$, $d(x,y) = d(y,x)$ (symmetry law);

iii) if $x,y,z \in M$, $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

If d is a metric on M, then the ordered pair (M,d) is called a metric space.

The following are examples of metric spaces:

1.2. Example. Let M be a non-empty set, and d such that for $x, y \in M$,

$$d(x,y) = \begin{cases} 0 \text{ if } x = y, \\ 1 \text{ if } x = y. \end{cases}$$

Call this metric space N_d. It is easy to verify that d satisfies properties i, ii, and iii.

1.3. Example. Consider the real number line R and the absolute value function $|x|$ defined on R. For $x, y \in R$, let $d(x,y) = |x - y|$. Then (R,d) is a metric space. For choose $x, y, z \in R$. Then $|x - y| \geq 0$ and $|x - y| = 0$ if and only if $x = y$. Thus i) holds. Since $|x - y| = |y - x|$, ii) holds. Finally,

$$|x - z| = |x - y + y - z| \leq |x - y| + |y - z|$$

and iii) holds.

1.4. Example. Consider the complex plane C. If $z_1 = a_1 + b_1 i \in C$ and $z_2 = a_2 + b_2 i \in C$, then

$$|z_1 - z_2| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}.$$

Define $d(z_1, z_2) = |z_1 - z_2|$. The proof that (C,d) is a metric space is similar to that given in Example 1.3.

1.5. Example. For any $n \in J$, let

$$x = (x_1, x_2, \ldots, x_n) \text{ and } y = (y_1, y_2, \ldots, y_n)$$

be ordered n-tuples of real numbers, that is, $x, y \in R^n$. Define $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$. For $n = 2$, this is the usual distance formula for the plane and is equivalent to Example 1.4. The only non-trivial part of the proof is the triangle inequality, and the standard Minkowski inequality can be employed to prove this.

1.6 Definition. A linear space over the reals is a mathematical system (X, R, "+", ".") which consists of a set of elements X, the set of real numbers R, a function "+" from $X \times X$ into X, and a function "." from $R \times X$ into X which satisfies the following conditions:

1) $x + y = y + x$, for $x, y \in X$.

2) $(x + y) + z = x + (y + z)$, for $x, y, z \in X$.

3) There is a vector $0^* \in X$ such that $x + 0^* = x$ for all $x \in X$.

4) $a(x + y) = ax + ay$, for $x, y \in X$, $a \in R$.

5) $(a + b)x = ax + bx$, for $x \in X$, $a, b \in R$.

6) $a(bx) = (ab)x$, for $x \in X$, $a, b \in R$.

7) $0x = 0^*$, $1x = x$, for $x \in X$, $0, 1 \in R$.

1.7. Definition. A non-negative real-valued function defined on a linear space X is called a norm if:

1) $\|x\| = 0$ if and only if $x = 0^*$, for $x \in X$,

2) $\|x + y\| \leq \|x\| + \|y\|$, for $x, y \in X$,

3) $\|ax\| = |a| \|x\|$, for $x \in X$, $a \in R$.

A normed linear space becomes a metric space if the metric is defined as the norm of the difference between two elements, that is, $d(x,y) = \|x - y\|$.

1.8. Example. The following is a normed linear space which becomes a metric space. Consider the set of all bounded continuous real functions defined on $[0,1]$. Define the norm as $\|f\| = \sup|f(x)|$. Then the metric on this space will be

$$d(f,g) = \|f(x) - g(x)\|.$$

Property i) follows almost immediately from 1) of 1.7 because $\|f\| \geq$ and $\|f\| = 0$ if and only if $f = 0$. Property ii) is clear from 3) of 1.7 because,

$$d(f,g) = \|f(x) - g(x)\| = \|-(g(x) - f(x))\|$$
$$= |-1|\,\|g(x) - f(x)\| = \|g(x) - f(x)\| = d(g,f).$$

Hence $d(f,g) = d(g,f)$. Since the sum of two bounded functions is bounded, then

$$\|f + g\| = \sup|f(x) + g(x)| \leq \sup|f(x)| + \sup|g(x)|$$
$$= \|f\| + \|g\|.$$

Therefore $\|f + g\| \leq \|f\| + \|g\|$ and hence this is a metric space.

1.9. Definition. Let (M,d) be a metric space and p a point of M. If $r > 0$, then the open sphere with center p and radius r, denoted by $S(p,r)$, is the set of all points x in M such that $d(x,p) < r$. Obviously $S(p,r)$ contains p. The closed sphere denoted by $S[p,r]$, is the set of all points $x \in M$ such that $d(x,y) \leq r$.

1.10. Definition. Let (M,d) be a metric space and E be a subset of M. A limit point or cluster point of E is a point $p \in M$ such that each open sphere $S(p,r)$ contains at least one point $q \in E$, $q \neq p$.

1.11. Definition. Let (M,d) be a metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in M. Then $\{x_n\}_{n=1}^{\infty}$ converges to a point x, called the limit of the sequence $\{x_n\}_{n=1}^{\infty}$, if for each open sphere $S(x,r)$, there exists

1.12. Remark. The above definition is equivalent to the following. Given $\epsilon > 0$, there exists an $n_0 \in J$ such that if $n \geq n_0$, then $d(x_n, x) < \epsilon$.

1.13. Definition. Let (M,d) be a metric space, and let E be a subset of M. A point x in E is called an interior point of M if it is the center of some open sphere contained in E. The interior of E is the set of all its interior points. If every point of E is an interior point of E then E is said to be an open set.

1.14. Some examples of open sets. An open sphere $S(x_0, r)$ on the real number line is just an interval $(x_0 - r, x_0 + r)$ with center x_0 and length $2r$. In the two dimensional plane an open sphere is just a circle. A sphere in R^3 is the usual three dimensional sphere. An open sphere about $x \in N_d$ with radius 1 is the single point x.

1.15. Theorem. In any metric space (M,d), each open sphere $S(x_0, r)$ is an open set.

Proof. Let $x \in S(x_0, r)$. Choose $t = r - d(x_0, x)$. Now consider the open sphere $S(x,t)$. Choose $y \in S(x,t)$. This implies $d(x,y) < t$. Then

$$\begin{aligned} d(x_0,y) &\leq d(x_0,x) + d(x,y) \\ &< r - t + t \\ &< r. \end{aligned}$$

Hence $d(x_0, y) < r$ which implies $y \in S(x_0, r)$. Thus $S(x,t) \subset S(x_0, r)$. Therefore, $S(x_0, r)$ is an open set.

1.16. *Theorem*. *In any metric space* (M,d) *both* M *and* $\emptyset$ *are open sets*.

Proof. If $x \in M$, $r < 0$, then $S(x,r) \subset M$. Hence x is an interior point of M, Therefore M is open.

The empty set $\emptyset$ is open simply because there is no $x \in \emptyset$ and therefore in a vacuous sense each $x \in \emptyset$ is an interior point of $\emptyset$. Therefore $\emptyset$ is an open set.

1.17. *Theorem*. *Let* (M,d) *be a metric space*. *A subset* G *of* M *is open if and only if it is a union of open spheres*.

Proof. Suppose G, a subset of M, is the union of a collection $\mathcal{S}$ of open spheres $s = S(x,r)$. Then $G = \mathcal{S}^*$, where $\mathcal{S}^* = \bigcup_{s \in \mathcal{S}} s$.

Suppose x is a point in G. Thus x is in at least one of the open spheres of $\mathcal{S}$ say $S(x_0,r_0)$. Since $S(x_0,r_0)$ is an open set by Theorem 1.15, for $x \in S(x_0,r_0)$ there exists an open sphere $S(x,r_1)$ such that

$$S(x,r_1) \subset S(x_0,r_0) \subset G$$

and therefore $S(x,r_1) \subset G$. Therefore G is open.

Let G be an open subset of M. If $x \in G$ then there exists an open sphere $S(x,r)$ which is contained in G. Let $\mathcal{S}$ be the collection of all such open spheres. Then $G = \mathcal{S}^*$. Therefore G is the union of a collection of open spheres.

1.18. *Theorem*. *Let* (M,d) *be a metric space*. *Then*

1) *any union of open sets in M is open; and*

2) *any finite intersection of open sets in M is open.*

Proof. 1) Let $\mathcal{G}$ be any collection of open sets in M. Let $\mathcal{G}^* = \bigcup_{G \in \mathcal{G}} G$. If $\mathcal{G}$ is empty then $\mathcal{G}^* = \emptyset$ and $\mathcal{G}^*$ is open by Theorem 1.16. Suppose $\mathcal{G}$ is not empty. Then by Theorem 1.17, every open set in $\mathcal{G}$ is the union of open spheres. It follows by Theorem 1.17 that $\mathcal{G}^*$ is open.

2) Let $\{G_i\}_{i=1}^{n}$ be a finite collection of open sets in M. Let $x \in \bigcap_{i=1}^{n} G_i$; then x is an element of each G_i. Since each G_i is an open set there is an open sphere of radius r_i such that $S(x,r_i) \subset G_i$. Thus there is the set of radii $\{r_1, r_2, \ldots, r_n\}$. Let r be the smallest of these radii. Then the open sphere $S(x,r) \subset S(x,r_i)$, for $1 \leq i \leq n$. Therefore $S(x,r) \subset G_i$, for $1 \leq i \leq n$ and hence $S(x,r) \subset \bigcap_{i=1}^{n} G_i$. Therefore $\bigcap_{i=1}^{n} G_i$ is an open set.

1.19. *Definition.* A subset F of the metric space (M,d) is called a *closed set* if it contains each of its limit points.

1.20. *Theorem.* *In any metric space* (M,d), *the empty set* $\emptyset$ *and the full space* M *are closed sets.*

Proof. Since the empty set $\emptyset$ has no elements it will have no limit points. Therefore $\emptyset$ contains all of its limit points.

The space M contains all points; therefore if it has any limit points, it will surely contain them.

1.21. *Theorem.* *Let* (M,d) *be a metric space.* *A subset* F *of* M *is closed if and only if its complement* $\tilde{F}$ *is open.*

Proof. Let F be a closed subset of M. Consider $x \in \tilde{F}$; then x is not a limit point of F because F is closed. Therefore, for some $r > 0$, $S(x,r) \subset \tilde{F}$ and thus $\tilde{F}$ is open.

Suppose $\tilde{F}$ is open. Let x be a limit point of F. If $x \in \tilde{F}$ then there is an open sphere $S(x,r)$ for some $r > 0$ such that $S(x,r) \subset \tilde{F}$. This is a contradiction. Therefore x must be an element of F which implies F contains all of its limit points and is closed.

1.22. *Theorem*. *In any metric space* (M,d), *each closed sphere is a closed set*.

Proof. Let $S[x,r]$ be a closed sphere. Let $p \in \widetilde{S[x,r]}$. Then $d(p,x) = \epsilon > r$. Since for any point x' of $S[x,r]$, $d(x,x') \leq r$, then $d(x',p) \geq \epsilon - r$ and $x' \notin S(p, \epsilon - r)$. This implies $S(p, \epsilon - r) \subset \widetilde{S[x,r]}$. Therefore $\widetilde{S[x,r]}$ is an open set which implies $S[x,r]$ is a closed set.

1.23. *Remark*. For arbitrary unions and intersections the following properties hold:

$$\widetilde{(\bigcap A_\alpha)} = \bigcup \tilde{A}_\alpha \text{ and } \widetilde{(\bigcup A_\alpha)} = \bigcap \tilde{A}_\alpha .$$

1.24. *Theorem*. *Let* (M,d) *be a metric space*. *Then*

1) *any intersection of closed sets in* M *is closed*,

2) *any finite intersection of closed sets in* M *is closed*.

Proof. 1) Let $\{F_\alpha\}$ be any collection of closed sets and $F = \bigcap F_\alpha$. Since each F is closed its complement $\tilde{F}$ is open by Theorem 1.21. Therefore $\bigcup \tilde{F}_\alpha$ is open by Theorem 1.18. By Theorem 1.21, $\widetilde{(\bigcup \tilde{F}_\alpha)}$ is a closed set. By Remark 1.23 $\widetilde{(\bigcup \tilde{F}_\alpha)} = \bigcap F_\alpha$. Therefore $\bigcap F_\alpha$ is a closed set.

2) Let $\{F_j\}_{j=1}^{n}$ be a finite collection of closed sets.

Let $F = \bigcup_{j=1}^{n} F_j$. By Theorem 1.21, $\tilde{F}_j$ is open for $1 \le j \le n$. Therefore by Theorem 1.18, $\bigcap_{j=1}^{n} \tilde{F}_j$ is open. By Remark 1.23, $(\widetilde{\bigcap_{j=1}^{n} \tilde{F}_j}) = \bigcup_{j=1}^{n} F_j$ which is closed by Theorem 1.21.

1.25. Theorem. A singleton $\{x\}$ in a metric space (M,d) is a closed set.

Proof. Pick an arbitrary point $x_0 \in M$. Consider the closed spheres $S[x_0,r]$, $r > 0$.

Let $x \in \{x_0\}$. This implies that $x = x_0$. Each of the closed spheres $S[x_0,r]$ contains x_0, so $x_0 \in \bigcap S[x_0,r]$. Hence $\{x_0\} \subset \bigcap S[x_0,r]$.

To prove $\bigcap S[x_0,r] \subset \{x_0\}$, it is sufficient to prove the contrapositive, that is, if $x \notin \{x_0\}$, then $x \notin \bigcap S[x_0,r]$. Since $x \notin \{x_0\}$, then $x \neq x_0$. Let $r_1 = d(x_0,x)$. Then $x \notin S[x_0,r_1/2]$, which implies $x \notin \bigcap S[x_0,r]$. Therefore, $\bigcap S[x_0,r] \subset \{x_0\}$. Hence $\{x_0\} = \bigcap S[x_0,r]$. Each of the $S[x_0,r]$ is a closed set by Theorem 1.22. By Theorem 1.24, $\bigcap S[x_0,r]$ is a closed set. Thus $\{x_0\}$ is a closed set.

1.26. Corollary. A finite set A of a metric space (M,d) is closed.

Proof. Let $A = \{x_1, x_2, \ldots, x_n\}$. Then $A = \bigcup_{i=1}^{n} \{x_i\}$. By Theorem 1.25, each $\{x_i\}$ is a closed set. Therefore A is the union of a finite collection of closed sets, which by Theorem 1.24, is closed.

1.27. Theorem. The surface of a sphere is a closed set.

Proof. The surface E of a sphere $S(p,r)$ is the set of points $q \in M$ such that $d(p,q) = r$. The closed sphere $S[p,r]$ is the set of points $q \in M$ such that $d(p,q) \leq r$ and $\widetilde{S(p,r)}$ is the set of points $q \in M$ such that $d(p,q) \geq r$. Therefore, $E = S[p,r] \cap \widetilde{S(p,r)}$. Each of $S[p,r]$ and $\widetilde{S(p,r)}$ are closed sets by Theorem 1.22 and 1.21, respectively. Therefore E is the intersection of two closed sets, so by Theorem 1.24, E is closed.

1.28. Definition. The derived set of E, denoted by E', is the set of all limit points of E. The closure of E, denoted by $\bar{E}$, is the set $E' \cup E$.

1.29. Theorem. The closure of $S(p,r)$ is contained in $S[p,r]$.

Proof. Obviously, $S(p,r) \subset S[p,r]$. If $\overline{S(p,r)} \not\subset S[p,r]$ then there is some limit point y of $S(p,r)$ which is not an element of $S[p,r]$. Thus $d(p,y) > r$. Therefore $d(p,y) - r = t$ for some $t > 0$. Hence $S(y,t/2)$ does not contain a point of $S(p,r)$. This is a contradiction to the fact that y is a limit point of $S(p,r)$ and therefore $\overline{S(p,r)} \subset S[p,r]$.

It is useful to note:

1.30. Remark. If E is any subset of a metric space (M,d), then $\bar{E}$ and E' are closed sets.

A metric space is actually a generalization of a concept called a topological space.

1.31. Definition. If S is a set and $\mathcal{J}$ is a collection of subsets G of S such that

1) $\emptyset$ and S are in $\mathcal{J}$,

2) $\bigcup G_\alpha$ is in $\mathcal{J}$ for any subcollection $\{G_\alpha\}$ of $\mathcal{J}$,

3) $\bigcap_{i=1}^{n} G_i \in \mathcal{J}$, if $G_i \in \mathcal{J}$,

then $\mathcal{J}$ is said to be a topology for S, and (S, $\mathcal{J}$) is said to be a topological space. The sets G of $\mathcal{J}$ are called open sets. A closed set is the complement of an open set.

1.32. Example. Let S be any set. Let the open sets be all of the subsets of S. This is called the discrete topology.

1.33. Example. If S is any set, then the indiscrete topology has as its open sets S and $\emptyset$.

1.34. Example. Let S be the set of real numbers and the open sets be the unions of collections of intervals.

1.35. Example. Let S be the set $\{a,b,c,d\}$. Let the open sets be $\{a,b,c,d\}$, $\{a,b\}$, $\{b,c,d\}$, and $\{b\}$. Note that $\{c\}$ is neither open nor closed.

1.36. Theorem. *If* (M,d) *is a metric space then the collection of open sets is a topology for* M.

Proof. By Theorem 1.16, 1) of Definition 1.31 is true. Parts 2) and 3) of Definition 1.31 follow from Theorem 1.18 1) and 1.18 2) respectively. Therefore (M,d) is a topological space.

The following properties of a topological space are called <u>separation axioms</u>.

1.37. <u>Definition</u>. A topological space $(S, \mathcal{J})$ is T_0 means that if p_1 and p_2 are two points of S, then there exists an open set containing one of p_1 and p_2, but not the other.

1.38. <u>Definition</u>. A topological space $(S, \mathcal{J})$ is T_1 means that if p_1 and p_2 are two points of S, there exists an open set G_1 such that G_1 contains p_1 and not p_2.

1.39. <u>Definition</u>. A topological space $(S, \mathcal{J})$ is T_2 means that if p_1 and p_2 are two points of S, there exist two open sets G_1 and G_2 such that G_1 contains p_1, G_2 contains p_2, and $G_1 \cap G_2 = \emptyset$. A space satisfying T_2 is called <u>Hausdorff</u>.

1.40. <u>Definition</u>. A topological space $(S, \mathcal{J})$ is T_3 means that it is T_1 and if p is a point of S and F is a closed set of S not containing p, there exist two open sets G_1 and G_2 such that G_1 contains p and G_2 contains F and $G_1 \cap G_2 = \emptyset$. A space satisfying T_3 is called <u>regular.</u>

1.41. <u>Definition</u>. A topological space $(S, \mathcal{J})$ is T_4 means that it is T_1 and if F_1 and F_2 are disjoint closed sets of S, then there exists open sets G_1 and G_2, $G_1 \supseteq F_1$ and $G_2 \supseteq F_2$, $G_1 \cap G_2 = \emptyset$. A space satisfying T_4 is called <u>normal</u>.

1.42. <u>Example</u>. The following is an example of a T_0-space which is not a T_1-space.

Let $S = R$ and the open sets consist of $\emptyset$, R, and the sets of the form (x,∞) where $x \in R$. This is a T_0-space because for $x_1 < x_2$, $((x_1 + x_2)/2, \infty)$, contains x_2 but does not contain x_1. However, any open set that contains x_1 contains x_2, so it is clearly not a T_1-space.

1.43. Example. This is an example of a T_1-space which is not a T_2-space.

Consider the real line R. Let S be the following subset of R: $\{0\} \cup [1,2]$. Let $\mathcal{J}$ consist of unions of collections of subsets of S of the form; (a,b), $[1,b)$, $(a,2]$, and $\{0\} \cup (1,b)$ for $a,b \in (1,2)$, $a < b$. For any two points x_1 and x_2 of this space, there is an open set G_1 which contains x_1 and not x_2 and a G_2 which contains x_2 and not x_1. However, G_1 and G_2 may not be disjoint. For example, let the two points be 0 and 1. An open set containing 0 contains a set of the form $\{0\} \cup (1,b)$ and an open set containing 1 contains a set of the form $[1,b')$. The intersection of these two sets contains $(1,b)$ if $b < b'$ and $(1,b')$ if $b' < b$. Therefore this T_1-space is not a T_2-space.

1.44. Example. As an example of a T_2-space which is not a T_3-space consider the upper half of the two dimensional plane and the x-axis. Let the open sets be the unions of collections of interiors of circles which are completely above the x-axis and interiors of semi-circles with centers on the x-axis plus the center which is on the x-axis. This is clearly a T_2-space.

However, consider the real line R minus the point p. Call this set F_p. This is a closed set because about any point x, x not on R, there exists an open circle containing x and no point of R. There is also a half open circle about p which contains no other point of R. By Theorem 1.15, each of these open circles is an open set. The union of all such circles is, by Theorem 1.18, an open set. Thus F_p is open and therefore F_p is closed. Any open set G_1 containing p also contains the interior of a semi-circle with center at p. In order to cover the rest of the real line the open sets must also be the interiors of semi-circles plus the points on the x-axis. Let the union of all of these be G_2. Thus if G_1 contains p and G_2 contains F_p, than $G_1 \cap G_2 \neq \emptyset$. Hence this space is a T_2-space but not a T_3-space.

1.45. Example. The following is a T_3-space but not a T_4-space.

Consider the upper half of the two dimensional plane plus the real line R. Let the open sets be unions of collections of interiors of circles entirely above the x-axis or interiors of open circles and the point of tangency if they are tangent to the real line. The fact that this space is a T_3-space is fairly obvious except in the case where F_p is the real line R minus the point p. This is closed as in Example 1.44. There exists an open circle G_1 with radius one tangent to the x-axis at p. This set does not contain any of the points of F_p. For any $x \in F_p$ there is an open circle G_x tangent to x with radius less than half the distance of x to G_1. This circle does not intersect G_1. Let $G_2 = \bigcup G_x$, $x \in F_p$. Therefore $G_1 \cap G_2 = \emptyset$, and thus this is a T_3-space.

To show that this space is not a T_4-space, consider F_1 as the set of rationals and F_2 as the set of irrationals in R. The sets F_1 and F_2 are closed. It can be shown that if an open set G_1 contains F_1 and an open set G_2 contains F_2, then $G_1 \cap G_2 \neq \emptyset$, but the proof is omitted.

1.46. <u>Theorem</u>. *A metric space* (M,d) *is a* T_0, T_1, *and* T_2-*space*.

<u>Proof</u>. Let p_1 and p_2 be two points of M. Let $r = d(p_1,p_2)$. Then $S(p_1,r/2)$ contains p_1 and $S(p_2,r/2)$ contains p_2. Suppose $S(p_1,r/2) \cap S(p_2,r/2) \neq \emptyset$. Thus there is some x such that $x \in S(p_1,r/2) \cap S(p_2,r/2)$. Hence $d(p_1,x) < r/2$ and $d(p_2,x) < r/2$. Therefore, by iii),

$$d(p_1,p_2) \leq d(p_1,x) + d(x,p_2) < r/2 + r/2 = r,$$

a contradiction. Thus $S(p_1,r/2) \cap S(p_2,r/2) = \emptyset$ and therefore M is a T_2-space. The theorem follows, since clearly a T_2-space is a T_0-space and a T_1-space.

1.47. <u>Theorem</u>. *A metric space* (M,d) *is a* T_3-*space*.

<u>Proof</u>. Let p be a point in M and F be a closed set in M

and F be a closed set in M not containing p. The set F is the complement of some open set G. Therefore $p \in G$ since $p \notin F$. Choose $t > 0$ such that $S(p,t) \subset G$. Then $\widetilde{S(p,t)} \supset F$. Let $G_1 = S(p,t/2)$ and $G_2 = \widetilde{S[p,t/2]}$, which is open since by Theorem 1.21, $S[p,t/2]$ is closed. Then $p \in G$, and $F \subset \widetilde{S(p,t)} \subset \widetilde{S[p,t/2]}$, since $S[p,t/2] \subset S(p,t)$. Now $G_1 \cap G_2 = \emptyset$, and hence (M,d) is a T_3-space.

1.48. <u>Theorem</u>. <u>A metric space</u> (M,d) <u>is a</u> T_4-<u>space</u>.

<u>Proof</u>. Let F_1 and F_2 be disjoint closed sets in M. Define

$$G_1 = \{x \mid d(x,F_1) < d(x,F_2)\} \text{ and}$$
$$G_2 = \{x \mid d(x,F_2) < d(x,F_1)\} .$$

Note that $G_1 \supset F_1$ and $G_2 \supset F_2$. Suppose $G_1 \cap G_2 = \emptyset$. This implies that there exists an $x \in G_1 \cap G_2$ such that $d(x,F_1) < d(x,F_2)$ and $d(x,F_2) < d(x,F_1)$. This implies that $d(x,F_1) < d(x,F_1)$ which is impossible. Therefore $G_1 \cap G_2 = \emptyset$.

Suppose G_1 is not open; then $\widetilde{G}_1$ is not closed. This implies that there exists some limit point p of G_1 which is not in $\widetilde{G}_1$. Thus $p \in G_1$. Also, p is not a limit point of F_2 because F_2 is closed. For otherwise it would follow that $p \in F_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$, a contradiction. Therefore, there exists a positive number k such that $d(p,F_2) = k$.

<u>Case 1</u>. Let $p \in F_1$. Then there exists an $x \in \widetilde{G}_1$ such that $d(x,p) < k/3$. Then $d(x,F_2) > k/3$. Thus $d(x,F_1) < k/3$ which implies $x \in G_1$. This is a contradiction to the fact that $x \in \widetilde{G}_1$. Therefore G_1 is open.

Case 2. Let $p \notin F_1$. Thus there is some $j > 0$ such that $d(p,F_1) = j$. Since $p \in G_1$ then $d(p,F_1) < d(p,F_2)$ which implies that $j < k$. Hence $(k - j)/2 > 0$. Since p is a limit point of G_1, there exists an $x \in \widetilde{G}_1$ such that $d(x,p) < (k - j)/2$. Thus

$$d(x,F_2) > k - (k - j)/2 = (k + j)/2 \text{ and}$$

$$d(x,F_1) < j + (k - j)/2 = (k + j)/2.$$

Therefore $x \in G_1$. This is a contradiction to $x \in \widetilde{G}_1$. Therefore G_1 is open.

A very similar proof follows through in proving that G_2 is an open set. Thus (M,d) is a T_4-space.

CHAPTER II

SET PROPERTIES

2.1. Definition. An open cover of E is a collection of open sets $\{G_\alpha\}$, $\alpha \in A$, such that $E \subset \bigcup_{\alpha \in A} G_\alpha$.

2.2. Definition. A set E of a metric space (M,d) is said to be compact if every open cover $\mathcal{G} = \{G_\alpha\}$ of E has a finite subcover $\{G_{\alpha_i}\}_{i=1}^{n}$ where $G_{\alpha_i} \in \mathcal{G}$.

2.3. Remark. A closed and bounded interval in the set of real numbers $R = R^1$ is a compact set. In general, in Euclidean n-space, R^n, a set which is closed and bounded is compact.

The following are examples of sets which are not compact.

2.4. Example. Consider the real line R and as the set, the open interval (0,1). Let an open cover be;

$\{(1/2,1), (1/3,1), (1/4,1),\ldots,(1/n,1),\ldots\}$. Thus $(0,1) \subset \bigcup_{n=1}^{\infty}(1/n,1)$. Suppose that there is a finite subcover, $\bigcup_{i=1}^{n}(1/n_i,1)$. Then the interval $(0,1/n_k]$ is not covered and so (0,1) is not compact.

2.5. Example. Consider the closed interval [0,1] minus the point 1/2. Then,

$$[0,1] - \{1/2\} \subset (\bigcup_{n=1}^{\infty}[0,1/2 - 1/(n+1)) \cup (\bigcup_{n=1}^{\infty}(1/2+1/(n+1),1]).$$

In a similar manner to Example 2.4, it is seen that

$$\left(\bigcup_{n=1}^{\infty}[0,\ 1/2 - 1/(n+1))\right) \cup \left(\bigcup_{n=1}^{\infty}(1/2 + 1/(n+1), 1]\right)$$

has no finite subcover and therefore $[0,1] - \{1/2\}$ is not compact.

2.6. Example. Consider the discrete metric N_d as given in Example 1.2. Let N_d in this case be any infinite set. Let the open cover be $\{S(x,1/2)\}$, $x \in N_d$. Suppose that for some $n \in J$, $N_d \subset \bigcup_{i=1}^{n} S(x_i,1/2)$. This implies that N_d is finite, hence a contradiction. Therefore N_d is not compact.

The following theorems are fundamental theorems relating to compactness in a metric space.

2.7. Theorem. A compact subset E of a metric space (M,d) is closed.

Proof. Let $p \in M$ and $p \notin E$. Let $q \in E$. Define $r_q = d(p,q)$. Then consider the open spheres $S(p,r_q/2)$ and $S(q,r_q/2)$. Thus $S(p,r_q/2) \cap S(q,r_q/2) = \emptyset$. Thus $\{S(q,r_q/2)\}$, $q \in E$, is an open cover of E. Hence some finite subcollection covers E. Call the centers of those spheres $\{q_1,q_2,\ldots,q_n\}$. For each q_i, $1 \leq i \leq n$, there is an open sphere $S(p,r_i/2)$, where $r_i = r_{q_i}$. Clearly, $p \in \bigcap_{i=1}^{n} S(p,r_i/2) = S_p$ and S_p is an open set by Theorem 1.15. Also, $\bigcup_{i=1}^{n} S(q_i,r_i/2) \cap S_p = \emptyset$. Therefore $S_p \subset \tilde{E}$.

Repeating this process for each $p \in \tilde{E}$ yields a collection of open sets which covers $\tilde{E}$ and whose union is a subset of $\tilde{E}$. Thus $\tilde{E}$ is an open set which, by Theorem 1.21, implies E is a closed set.

2.8. Theorem. A closed subset E of a compact metric space (M,d) is compact.

Proof. Let $\mathcal{G}$ be a collection of open sets G which covers E. Let $\mathcal{G}' = \bigcup_{G \in \mathcal{G}} G \cup \{\tilde{E}\}$. By Theorem 1.21, E is an open set. Thus $\mathcal{G}'$ is an open set and $\mathcal{G}'$ is an open cover of M Since M is compact, some finite subcover, say $\mathcal{G}''$, of $\mathcal{G}'$ covers M. Let $\mathcal{G}''' = \mathcal{G}'' - \{\tilde{E}\}$. Thus $\mathcal{G}'''$ also covers E. Since $\mathcal{G}''$ is finite $\mathcal{G}''' \subset \mathcal{G}''$ is finite. Therefore E is compact.

2.9. Definition. A collection $\{F_\alpha\}$, $\alpha \in A$. of closed sets is said to have the finite intersection property if, for every finite subset A_0 of A,

$$\bigcap_{\alpha \in A_0} F_\alpha \neq \emptyset .$$

2.10. Theorem. A metric space (M,d) is compact if and only if, for every collection $\{F_\alpha\}$, $\alpha \in A$, of closed sets with the finite intersection property,

$$\bigcap_{\alpha \in A} F_\alpha \neq \emptyset .$$

Proof. Let M be compact. Suppose $\bigcap F_\alpha$ is empty. Then, by Remark 1.23, $\bigcap_{\alpha \in A} F_\alpha = \widetilde{\bigcup_{\alpha \in A} \tilde{F}} = \emptyset$. Therefore, $\bigcup_{\alpha \in A} \tilde{F}_\alpha = M$. By Theorem 1.21, each $\tilde{F}_\alpha$ is an open set, so $\{F_\alpha\}$, $\alpha \in A$, form an infinite open cover of M, Since M is compact there exists a finite subset A_0 of A such that $\bigcup_{\alpha \in A_0} F_\alpha = M$. By Remark 1.23, $\bigcup_{\alpha \in A_0} \tilde{F}_\alpha = \bigcap_{\alpha \in A_0} F_\alpha = \tilde{M} = \emptyset$. Therefore $\bigcap_{\alpha \in A_0} F_\alpha = \emptyset$. However, this is a contradiction since the collection $\{F_\alpha\}$, $\alpha \in A$ has the finite intersection property. Hence $\bigcap_{\alpha \in A} F_\alpha$ is not empty.

To prove the other half of the theorem the contrapositive form will be used. Therefore suppose M is not compact. Then there exists an open cover $\{G_\alpha\}$, $\alpha \in A$, of M such that for all finite subsets A_0 of A, $\bigcup_{\alpha \in A_0} G \not\supset M$. Therefore, $\widetilde{\bigcup_{\alpha \in A_0} G_\alpha} = \bigcap_{\alpha \in A_0} \tilde{G}_\alpha$, is non-empty. Thus the collection $\{\tilde{G}_\alpha\}$, $\alpha \in A$, has the finite intersection property. However, $\bigcup_{\alpha \in A} G_\alpha = M$, so by Remark 1.23, $\bigcap_{\alpha \in A} \tilde{G}_\alpha = \emptyset$. Therefore, if M is not compact, then there exists a collection of closed sets with the finite intersection property which has an empty intersection. Thus the theorem follows.

2.11. Definition. A set E of a metric space (M,d) is said to be sequentially compact if every sequence in E has a subsequence which converges to a point in E.

2.12. Definition. A set E of a metric space (M,d) is said to have the Bolzano-Weierstrass Property if every infinite subset E_1 of E has a limit point in E.

2.13. Theorem. *A set E in a metric space (M,d) is sequentially compact if and only if E has the Bolzano-Weierstrass Property.*

Proof. Assume $E \subset M$ is sequentially compact. Let E_1 denote an infinite subset of E. Let

$x_1 \in E_1$,

$x_2 \in E_1 - \{x_1\}$,

$x_3 \in E_1 - \{x_1, x_2\}$,

...

$x_n \in E_1 - \{x_{n-2}, x_{n-1}\}$,

...

By the hypothesis E is sequentially compact so $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ which converges to a point $x \in E$. The values of the subsequence also are distinct, that is, $x_{n_i} \neq x_{n_j}$ if $i \neq j$. This implies that the range of $\{x_{n_i}\}_{i=1}^{\infty}$ is infinite. Choose $r > 0$. Then there exists $N \in J$ such that $\{x_{n_i}\}_{i=1}^{N} \subset S(x,r)$. Thus $S(x,r)$ contains an infinite number of points of E_1, that is, x is a limit point of E_1.

Now assume E has the Bolzano-Weierstrass Property. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in E. Suppose also the range of $\{x_n\}_{n=1}^{\infty}$ is infinite. Then it has a cluster point, say x, $x \in E$. Choose x_n, $x_n \neq x$. Choose x_{n_2}, $n_2 > n_1$, $0 < d(x,x_{n_2}) < d(x,x_{n_1})$. If x_{n_i} has been chosen, choose $x_{n_{i+1}}$ so that $0 < d(x,x_{n_{i+1}}) < d(x,x_{n_i})$. It follows that $\lim_{i\to\infty} x_{n_i} = x$.

If the range of $\{x_n\}_{n=1}^{\infty}$ is finite, then there exists an x, an element of the range, such that $x = x_n$ for an infinite number of n. Let $\{x_{n_i}\}_{i=1}^{\infty}$ be the subsequence of $\{x_n\}_{n=1}^{\infty}$ consisting of these points. Then $\lim_{i\to\infty} x_{n_i} = x$. Therefore, the theorem follows.

2.14. <u>Theorem</u>. <u>A compact set E of a metric space (M,d) is sequentially compact</u>.

<u>Proof</u>. By Theorem 2.13, it is sufficient to show that every infinite subset E_1 of E has a limit point in E.

Suppose that E_1 is an infinite subset of E which has no limit point in E. Therefore, about each point

$x \in E$ there is an open sphere $S(x, r_x)$, for some $r_x > 0$, which contains no point of E_1 except possibly x. The open spheres $\{S(x,r_x)\}$, $x \in E$, form an infinite open cover of E. Since E is compact, there exists a finite subset of $\{S(x,r_x)\}$ which forms an open cover of E. This implies that E_1 is finite, a contradiction.

2.15. Definition. A set E of a metric space (M,d) is said to be countably compact if every countable open cover $\mathcal{G} = \{G_\alpha\}_{\alpha \in A}$ has a finite sub-cover $\{G_{\alpha_i}\}_{i=1}^{n}$.

2.16. Theorem. A compact set E of a metric space (M,d) is countably compact.

Proof. Let $\mathcal{G}$ be any countable open cover of E. Since E is compact, E has a finite open subcover $\mathcal{G}'$. Therefore, E is countably compact.

2.17. Definition. If E is a subset of a metric space (M,d), then the diameter of E, denoted by D(E) or D(E,d), is the sup $d(x,y)$, $x,y \in E$.

2.18. Definition. A subset E of a metric space (M,d) is said to be bounded if its diameter D(E) is finite.

2.19. Definition. A set E of a metric space (M,d) is said to be totally bounded if given $\epsilon > 0$, there exists a finite set $\{x_1, x_2, \ldots, x_n\}$ of elements of E such that $E \subset (\bigcup_{i=1}^{n} S(x_i, \epsilon))$.

2.20. Remark. If a subset E of a metric space (M,d) is totally bounded, then E is bounded.

For the set of real numbers R^1, boundedness implies total boundedness. The same thing is true for Euclidean n-space, R^n. However, in a general metric space it is not true that boundedness implies total boundedness.

2.21. Example. Consider the discrete metric N_d where N_d is infinite. Since $D(N_d) = 1$, N_d is bounded. Consider any subset E of N_d such that $D(E) \leq 1/2$. There is at most one point in E. A finite number of such subsets covers only a finite number of points of N_d. Therefore, N_d is bounded but not totally bounded.

2.22. Example. Let M = J. Let $d_1(x,y) = |x - y|$ for $x,y \in M$ and let $d(x,y) = (d_1(x,y))/(1 + d_1(x,y))$. (M,d) is a metric space. The diameter D(M,d) is bounded because $D(M,d) \leq 1$. If x and y are any two distinct integers then $d(x,y) \geq 1/2$. Let $\epsilon = 1/4$. Then any S(x,1/4) can contain only one integer since $d(x,y) \geq 1/2$ for $x \neq y$. Therefore M cannot be covered by a finite number of spheres of radius less than 1/4 because M = J is infinite. Thus M is bounded but not totally bounded.

The following examples are those of totally bounded metric spaces which are not compact.

2.23. Example. Let E be the set (0,1) of the real line R. By Remark 2.20, boundedness in Euclidean n-space implies total boundedness. Thus (0,1) is totally bounded. However, as shown in Example 2.3, E is not compact.

2.24. Example. Let E be [0,1] minus the point 1/2. Again by Remark 2.20, E is totally bounded. However, by Example 2.5, E is not compact.

The following theorem shows that at least one type of compactness implies total boundedness.

2.25. Theorem. A set E of a sequentially compact metric space (M,d) is totally bounded.

Proof. Let $\epsilon > 0$. Pick a point $x_1 \in E$. If $S(x_1, \epsilon)$ covers E, then E is totally bounded. Suppose $S(x_1, \epsilon)$ does not cover E; then pick $x_2 \in E$, $x_2 \notin S(x_1, \epsilon)$. If $S(x_1, \epsilon) \cup S(x_2, \epsilon)$ covers E, then E is totally bounded. However, if $S(x_1, \epsilon) \cup S(x_2, \epsilon)$ does not cover E, then pick $x_3 \in E$, $x_3 \notin S(x_1 \epsilon) \cup S(x_2, \epsilon)$. Continue the process by means of mathematical induction. If the process can be continued for only a finite number of times, then E is totally bounded. Otherwise, consider $\{S(x_n, \epsilon)\}_{n=1}^{\infty}$. The centers of these spheres form an infinite sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points of E. Since E is sequentially compact, then $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ which converges to a point $p \in E$. Now consider the sphere $S(p, \epsilon/2)$. Hence $S(p, \epsilon/2)$ contains all but a finite number of terms of $\{x_{n_i}\}_{i=1}^{\infty}$. This is a contradiction to the fact that any two terms of $\{x_{n_i}\}_{i=1}^{\infty}$ are at least a distance of ϵ apart. Therefore E can be covered by a finite number of spheres with radius less than ϵ. This implies that E is totally bounded.

2.26. Theorem. *If $\mathcal{G}$ is an open cover of a set E with the Bolzano-Weierstrass Property, then there exists a number $\epsilon > 0$ such that every sphere of radius ϵ with center in E is contained in some $G \in \mathcal{G}$.*

Proof. Suppose that there does not exist a number $\epsilon > 0$ such that every sphere of radius ϵ with center in E is contained in some $G \in \mathcal{G}$. Then there exists some $x_1 \in E$ such that $S(x_1, 1)$ is not contained in any $G \in \mathcal{G}$. There will also exist an $x_2 \in E$ such that $S(x_2, 1/2)$ is not contained in any $G \in \mathcal{G}$. In general, for every $n \in J$, there will be an $x_n \in E$, $n \in J$, such that $S(x_n, 1/n)$ is not contained in any $G \in \mathcal{G}$. If the range of $\{x_i\}_{i=1}^{\infty}$ is finite, then let x be a number such that $x = x_n$ for an infinite number of $n \in J$. If the range of $\{x_i\}_{i=1}^{\infty}$ is infinite, then it has a limit point $x \in E$ because E has the Bolzano-Weierstrass Property. Whichever is the case, there is some $G \in \mathcal{G}$ such that $x \in G$. Denote the G by G_1. There exists an integer $n_0 \in J$ such that $S(x, 2/n_0)$ is contained in G_1. There exists an integer $n' > n_0$ such that $x_{n'}$ belongs to $S(x, 1/n_0)$. Hence

$$S(x_{n'}, 1/n') \subset S(x, 2/n_0) \subset G_1.$$

This is a contradiction of the fact that $S(x_n, 1/n)$ is not contained in any $G \in \mathcal{G}$. Therefore the theorem follows.

2.27. Theorem. *If a set E of a metric space (M,d) has the Bolzano-Weierstrass Property, then E is compact.*

Proof. Let $\mathcal{G}$ be an open cover of E. By Theorem 2.26, there exists an $\epsilon > 0$ such that every sphere of radius ϵ with center in E is in at least one $G \in \mathcal{G}$. Since by Theorem 2.25, E is totally bounded, then there exists a finite number of $S(x_i, \epsilon)$, $x_i \in E$, such that $E \subset \bigcup_{i=1}^{n} S(x_i, \epsilon)$. Since $\{G_i\}_{i=1}^{n}$ is a finite open cover of E, E is compact.

2.28. Definition. A subset A of E is said to be *closed in* E if every limit point of A contained in E is a point of A.

2.29. Lemma. *A closed subset A of a countably compact set E of a metric space (M,d) is countably compact.*

Proof. The proof follows exactly as the proof to Theorem 2.28, except on the restriction of M to E.

2.30. Theorem. *If a set E of a metric space (M,d) is countably compact, then E has the Bolzano-Weierstrass Property.*

Proof. Let E_1 be an infinite subset of E. Suppose E_1 has no cluster points in E. Then E_1 is closed in E and by Lemma 2.29, E_1 is countably compact. Let $\{x_i\}_{i=1}^{\infty} = E_2$ denote a countably infinite subset of E_1. Choose $n \in J$. Since x_n is not a cluster point of E_1, there exists a number $r_n > 0$ such that $S(x_n, r_n)$ contains no point of E_1 other than x_n. If x is a point of $E_1 - E_2$ then since x is not a cluster point of E_1, there exists an $r_x > 0$ such that $S(x, r_x)$ contains no point of E_1 other than x. Let $\bigcup S(x, r_x)$, $x \in E_1 - E_2$, be denoted by D. Then $\{D\} \cup \{S(x_i, r_i)\}$

is a countable cover of E_1. Since no member of E_2 belongs to D or to more than one of $S(x_i, r_i)$, this countable cover has no finite subcover. Thus E_1 is not countably compact, a contradiction.

2.31. Definition. The sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a metric space (M,d) if given $\epsilon > 0$, there exists an $N \in J$ such that for $m,n \geq N$, then $d(x_n, x_m) < \epsilon$.

2.32. Definition. A set E in a metric space (M,d) is complete if every Cauchy sequence is convergent to a point in E.

The following are examples of metric spaces which are not complete.

2.33. Example. Let M be the set of rational numbers. Some Cauchy sequences of rational numbers converge to irrational numbers which are not in M; therefore M is not complete.

However, the set $M = R$ of real numbers is complete.

2.34. Example. Consider as the entire space M, the open interval (0,1). Consider the Cauchy sequence $\{1, 1/2, 1/3, 1/4, \ldots, 1/n, \ldots\}$. Since 0 is not in M then $\{1, 1/2, 1/3, 1/4, \ldots, 1/n, \ldots\}$ is not convergent and therefore (0,1) is not complete.

2.35. Lemma. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and $\{x_{n_i}\}_{i=1}^{\infty}$ is a subsequence which converges to x, then $\{x_n\}_{n=1}^{\infty}$ also converges to x.

2.36. Theorem. *A set E of a metric space (M,d) is compact if and only if it is complete and totally bounded.*

Proof. Suppose that E is complete and totally bounded. Since compactness is equivalent to sequential compactness, it is sufficient to show that every infinite sequence $\{x_n\}_{n=1}^{\infty}$ in E has a convergent subsequence with respect to E.

Since E is totally bounded E can be covered by a finite number of spheres with radius less than one. Since $\{x_n\}_{n=1}^{\infty}$ has an infinite number of terms (not necessarily distinct), then one of these spheres, say S_1, has an infinite number of terms of $\{x_n\}_{n=1}^{\infty}$ in it. Now E can also be covered by a finite number of spheres with radius less than 1/2. One of these spheres, say S_2, is such that $S_1 \cap S_2$ contains an infinite number of terms of $\{x_n\}_{n=1}^{\infty}$. Assume that S_i, $1 \leq i \leq k$, is a sphere of radius less than $1/i$ such that $\bigcap_{i=1}^{k} S_i$ contains an infinite number of terms of $\{x_n\}_{n=1}^{\infty}$. Now E can be covered by a finite number of spheres of radius less than $1/(k + 1)$, and one of these, say S_{k+1}, is such that $\bigcap_{i=1}^{k+1} S_i$ contains an infinite number of terms of $\{x_n\}_{n=1}^{\infty}$. By finite mathematical induction we have defined a sequence $\{S_k\}$, S_k a sphere of radius less than $1/k$ such that for every k, $\bigcap_{i=1}^{\infty} S_i$ contains an infinite number of terms of $\{x_n\}_{n=1}^{\infty}$.

Choose $n_1 \in J$ such that $x_{n_1} \in S_1$. Choose $n_2 \in J$ such that $n_2 > n_1$ and $x_{n_2} \in S_1 \cap S_2$. Continuing this process, using mathematical induction, we define a sequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that for $k \in J$, $n_k < n_{k+1}$ and $x_{n_k} \in \bigcap_{i=1}^{k} S_i$, that is, $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that, for $N \in J$ and $k \geq N$, $x_{n_k} \in \bigcap_{i=1}^{N} S_i$.

Choose $\epsilon > 0$. Then there exists an $N \in J$ such that $2/N < \epsilon$. Choose $k, j \in J$, $k, j \geq N$. Then $x_{n_k}, x_{n_j} \in S_N$, a sphere of radius less than $1/N$. Hence, $d(x_{n_k}, x_{n_j}) < 2/N < \epsilon$, and $\{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence. Since E is complete $\{x_{n_k}\}$ converges to some $x \in E$. This implies that E is sequentially compact and therefore E is compact.

Now suppose E is compact. Total boundedness of E is implies by Theorem 2.25. Since E is compact, then E is sequentially compact by Theorem 2.14. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Since E is sequentially compact then $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ which converges to $x \in E$. Therefore by Lemma 2.35, $\{x_n\}_{n=1}^{\infty}$ is also convergent to $x \in E$. Therefore E is complete.

2.37. Corollary. Let E be a set of a metric space (M,d). Then the following statements are equivalent:

1) E is compact.

2) E is countably compact.

3) E is sequentially compact.

4) E has the Bolzano-Weierstrass Property.

5) E is complete and totally bounded.

Proof. The proof follows immediately from Theorems 2.13, 2.14, 2.16, 2.27, 2.30, and 2.36.

2.38. <u>Definition</u>. Let E be a set of a metric space (M,d) and E_1 a subset of E. Then E_1 is said to be <u>dense</u> (or everywhere dense) if $\bar{E}_1 = E$.

2.39. <u>Definition</u>. A set E of a metric space (M,d) is said to be <u>separable</u> if it has a subset E_1 which has a countable number of points and which is dense in M, that is, $\bar{E}_1 = E$.

2.40. <u>Theorem</u>. <u>If E is a compact set of a metric space (M,d), then E is separable</u>.

<u>Proof</u>. Let E be a compact set. By Theorem 2.25, E is totally bounded. Let $n \in J$. Then there exists a finite set $F_n = \{x_{n1}, x_{n2}, \ldots, x_{nN_n}\} \subset E$ such that if $x \in E$, there exists an i, $1 \leq i \leq N_n$, with $d(x, x_{ni}) < 1/n$. Let $F = \bigcup_{n=1}^{\infty} F_n$. This is a countable subset of E.

Consider a point $p \in E$, $p \notin F$. Let G be an open set containing p. There exists some $m \in J$ such that $S(p, 1/m) \subset G$. Choose $n > m$. Then $d(p, x_{ni}) < 1/n$ for some $x_{ni} \in F_n \subset F$. Thus, p is a limit point of F. Hence $E \subset \bar{F}$. Therefore E is separable.

2.41. <u>Definition</u>. A set E of a metric space (M,d) is said to have the <u>Lindelöf Property</u> if every open cover $\mathcal{G}$ of E has a countable open subcover $\mathcal{G}'$.

2.42. <u>Theorem</u>. <u>A set E of a metric space (M,d) has the Lindelöf Property if and only if it is separable</u>.

<u>Proof</u>. Assume E is separable. Let $\mathcal{G} = \{G_\alpha\}$ be an open cover of E. Let $D = \{x_i\}_{i=1}^{\infty}$ be a countable dense subset such that $\bar{D} = E$. Pick an $x_i \in D$. This x_i

is an element of at least one G_α, call it G_{α_i}. There exists a rational r such that the sphere $S(x_i, r) \subset G_{\alpha_i}$. If $x_i \in D$ and r is a rational number such that there exists a member of $\mathcal{Y}$ containing $S(x_i, r)$, then let one such member of $\mathcal{Y}$ belong to a collection $\mathcal{Y}'$. Hence $\mathcal{Y}'$ is countable.

Now pick a $p \in E$. If $p \in D$, then p is covered by at least one member of $\mathcal{Y}'$. Suppose $p \in E$, $p \notin D$. Since D is dense on E, p must then be a limit point of D. Consider a $G_\alpha^* \in \mathcal{Y}'$ such that $p \in G_\alpha^*$. Then there is an open sphere $S(p,r)$ with r rational such that $S(p,r) \subset G_\alpha^*$. Consider $S(p,r/2)$. Then there is an $x_i \in S(p,r/2)$, $x_i \in D$. Thus $p \in S(x_i, r/2) \subset S(p,r) \subset G_\alpha^*$. Hence $S(x_i, r/2)$ is contained in some member of $\mathcal{Y}'$. Thus E is covered by $\mathcal{Y}'$, and E has the Lindelöf Property.

Assume E has the Lindelöf Property. The sequence $\{S(x,1/n)\}$, $x \in E$ covers E. Since E has the Lindelöf Property, for each $n \in J$, there exists a countable open subcover $\{S(x_{n_i}, 1/n)\}$, that is, $E \subset \bigcup_{i=1}^{\infty} S(x_{n_i}, 1/n)$. For each $n \in J$, $\{x_{n_i}\}_{i=1}^{\infty}$ is countable. Therefore, $\bigcup_{n=1}^{\infty} \{x_{n_i}\}_{i=1}^{\infty}$ is a countable set. Call it D.

Pick a point $p \in E$, $p \notin D$. Consider $S(p, r_0)$ for some $r_0 > 0$. There exists some $n \in J$ such that $S(p,1/n) \subset S(p, r_0)$. Since E is covered by $\{S(x_{n_i}, 1/n)\}_{i=1}^{\infty}$, then $p \in S(x_{n_i}, 1/n)$ for some i. Therefore $d(p, x_{n_i}) < 1/n$, so $x_{n_i} \in S(p,1/n) \subset S(p, r_0)$. Thus p is a limit point of D. Hence $\bar{D} = E$ and E is separable.

2.43. Lemma. *Let E be a closed set of a complete metric space (M,d). Then E is complete.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in E. Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in M and converges to some $x \in M$. However, since E is closed, $x \in E$. Therefore E is complete.

2.44. Theorem. *Let E be a complete set of a metric space (M,d) and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of E such that $D(F_n)$ approaches 0. Then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.*

Proof. Suppose $\bigcap_{n=1}^{\infty} F_n$ contains two points x_1 and x_2. Since $d(x_1,x_2) = \epsilon > 0$ and $\lim_{n\to\infty} D(F_n) = 0$, there exists a number $n' \in J$ such that $D(F_{n'}) < \epsilon$. But this implies x_1 and x_2 cannot both belong to $F_{n'}$ and hence not to $\bigcap_{n=1}^{\infty} F_n$. Hence $\bigcap_{n=1}^{\infty} F_n$ does not contain two points.

Let x_i be a member of F_i. If $\epsilon > 0$ then there exists an $n' \in J$ such that $D(F_{n'}) < \epsilon$ and hence if $m,m' \geq F_{n'}$ and $d(x_m,x_{m'}) < \epsilon$. Hence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Since E is complete $\{x_i\}_{i=1}^{\infty}$ converges to a point $x \in E$. If $n \in J$, $x_i \in F_n$ for $i \geq n$, and since F_n is complete by Lemma 2.43, $\{x_i\}_{i=1}^{\infty}$ has the limit x and $x \in F_n$. Hence $x \in F$.

2.45. Definition. A subset E of a metric space (M,d) is said to be *nowhere dense* in M if it is dense in no open sphere at all, that is, if every sphere $S \subset R$ contains another sphere S' such that $S' \cap E = \emptyset$.

2.46. Theorem. (Baire Category Theorem). *If $\{A_n\}_{n=1}^{\infty}$ is a sequence of nowhere dense sets in a complete set E of a metric space (M,d), then every open sphere S in E contains a point which does not belong to any member of the sequence $\{A_n\}_{n=1}^{\infty}$.*

Proof. Since A_1 is nowhere dense in E, there exists an open sphere $S_1 \subset S$ with radius less than one such that $A_1 \cap S_1 = \emptyset$. There exists a closed sphere D_1 contained in S_1 with radius less than 1/2. Since the set A_2 is nowhere dense, the interior of D_1 contains an open sphere S_2 such that $A_2 \cap S_2 = \emptyset$. There exists a closed sphere D_2 contained in S_2 with radius less than 1/4. Since the set A_3 is nowhere dense, the interior of D_2 contains an open sphere S_3 such that $A_3 \cap S_3 = \emptyset$. Continue this process. Then a descending sequence $\{D_n\}_{n=1}^{\infty}$ of (non-empty) closed spheres is defined as well as a sequence $\{S_n\}_{n=1}^{\infty}$ of open spheres such that for $n \in J$, $D_n \subset S_n$, the radius of S_n is less than 1/n and $S_n \cap A_n = \emptyset$. By Theorem 1.22, $\{D_n\}_{n=1}^{\infty}$ is a descending sequence of closed sets such that $\lim_{n\to\infty} D(D_n) = 0$. Therefore, by Theorem 2.44, $\bigcap_{n=1}^{\infty} D_n$ contains exactly one point. Call it x. Since each of the D_n's is contained in its corresponding S_n, then x belongs to $\bigcap_{n=1}^{\infty} S_n$. For $n \in J$, $x \in S_n$ and $S_n \cap A_n = \emptyset$, and therefore $x \notin A_n$, that is, x does not belong to any member of the sequence $\{A_n\}_{n=1}^{\infty}$.

The preceding statement of the Baire Category Theorem is equivalent to the following statement:

2.46'. Theorem. *A complete set E in a metric space (M,d) cannot be represented as the union of a countable number of nowhere dense sets.*

2.47. Definition. A point x in a set E of a metric space (M,d) is called *isolated* if the set $\{x\}$ is open.

2.48. Corollary. *A complete set E in a metric space (M,d) without isolated points is uncountable.*

Proof. No single point is nowhere dense. Then the union of single points must be uncountable.

CHAPTER III

FUNCTION PROPERTIES

3.1. Definition. A function f from a metric space (M_1,d_1) into a metric space (M_2,d_2) is continuous at x_0 if given $\epsilon > 0$, there exists a $\delta > 0$ such that if $d_1(x,x_0) < \delta$, then $d_2(f(x),f(x_0)) < \epsilon$. The preceding definition can also be stated as follows:

For each open sphere $S(f(x_0),\epsilon)$ there exists an open sphere $S(x_0,\delta)$ such that $f(S(x_0,\delta)) \subset S(f(x_0),\epsilon)$.

The function f is continuous on M_1 if f is continuous at each point of M_1.

3.2. Definition. A function f from a metric space (M,d) into the set R is continuous if given $\epsilon > 0$, there exists a $\delta > 0$ such that if $d(x,x_0) < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

3.3. Theorem. Let f be a function from the metric space (M_1,d_1) into the metric space (M_2,d_2). Then f is continuous if and only if $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

Proof. Suppose f is continuous and G is an open set in M_2. Let $x \in f^{-1}(G)$. Then $f(x) \in G$. Since G is open in M_2 there exists an open shpere $S(f(x),\epsilon) \subset G$. Since f is continuous at x, there exists an open sphere $S(x,\delta)$ such that $f(S(x,\delta)) \subset S(f(x),\epsilon)$. Since $S(f(x),\epsilon) \subset G$, then $f(S(x,\delta)) \subset G$. Therefore $S(x,\delta) \subset f^{-1}(G)$. Thus $f^{-1}(G)$ is open.

Now suppose $f^{-1}(G)$ is open in M_1 whenever G is open in M_2. Pick an $x_0 \in M_1$ and let $\in > 0$. Then consider the open sphere $S(f(x_0), \in)$. Thus $f^{-1}(S(f(x_0), \in))$ is open. Therefore there exists a $\delta > 0$ such that $S(x_0, \delta) \subset f^{-1}(S(f(x_0), \in))$. Therefore, $f(s(x_0, \delta)) \subset S(f(x_o), \in)$, so f is continuous at x_0. Since x_0 was any point in M_1, then f is continuous.

3.4. Theorem. <u>If f is a continuous function from a compact metric space (M_1, d_1) into a metric space (M_2, d_2), then $f(M_1)$ is compact.</u>

Proof. Let $\mathcal{B} = \{G_\alpha\}$ be an open cover of $f(M_1)$. By Theorem 3.3, $f^{-1}(G_\alpha)$ is open for each α. Since M_1 is compact, there exists a finite subcover of M_1, say $\{f^{-1}(G_{\alpha 1}), f^{-1}(G_{\alpha 2}), \ldots, f^{-1}(G_{\alpha n})\}$. Then

$$M_1 \subset f^{-1}(G_{\alpha 1}) \cup f^{-1}(G_{\alpha 2}) \cup \ldots \cup f^{-1}(G_{\alpha n}), \text{ and}$$

$$f(M_1) \subset f(f^{-1}(G_{\alpha 1}) \cup f^{-1}(G_{\alpha 2}) \cup \ldots \cup f^{-1}(G_{\alpha n})).$$

Since the images of the unions is equal to the union of the images, then

$$f(M_1) \subset f(f^{-1}(G_{\alpha 1})) \cup f(f^{-1}(G_{\alpha 2})) \cup \ldots \cup f(f^{-1}(G_{\alpha n})).$$

This implies,

$$f(M_1) \subset G_{\alpha 1} \cup G_{\alpha 2} \cup \ldots \cup G_{\alpha n} \text{ since } f(f^{-1}(G_{\alpha i})) \subset G_{\alpha i}$$

for each i. Therefore $f(M_1)$ is compact.

3.5. Definition. Let f be a function from a metric space (M_1, d_1) into a metric space (M_2, d_2). The function f is <u>uniformly continuous</u> if for every $\epsilon > 0$, there exists a

$\delta > 0$ such that if x_1 and x_2 are two points of M_1 with $d_1(x_1,x_2) < \delta$, then $d_2(f(x_1),f(x_2)) < \epsilon$.

3.6. <u>Theorem</u>. <u>If f is a continuous function from a compact metric space</u> (M_1,d_1) <u>into a metric space</u> (M_2,d_2), <u>then it is uniformly continuous</u>.

<u>Proof</u>. Choose $\epsilon > 0$ and $p \in M_1$. Then since f is continuous at p there exists a $\delta_p > 0$ such that if $d_1(p,x) < \delta_p$ then $d_2(f(x),f(p)) < \epsilon/2$. The collection $\{S(p,\delta_p/2)\}$, for $p \in M_1$ forms an open cover of M_1. Since M_1 is compact, then for some $n \in J$ there is a finite sub-cover $\{S(p_i,\delta_{p_i}/2)\}$, $1 \le i \le n$. Let
$\delta = \min\{\delta_{p_1}/2, \delta_{p_2}/2, \ldots, \delta_{p_n}/2\}$.

Suppose $x,y \in M_1$ and $d_1(x,y) < \delta$. Then there exists a p_k, $1 \le k \le n$, such that $d_1(p_k,x) < \delta_{p_k}/2$. Since $d_1(x,y) < \delta < \delta_{p_k}/2$, then

$$d_1(p_k,y) \le d_1(p_k,x) + d_1(x,y) < \delta_{p_k}/2 + \delta_{p_k}/2 = \delta_{p_k}.$$

Since $d_1(p_k,x) < \delta_{p_k}/2 < \delta_{p_k}$ then $d_2(f(p_k),f(x)) < \epsilon/2$, and since $d_1(p_k,y) < \delta_{p_k}$ then $d_2(f(p_k),f(y)) < \epsilon/2$. Thus

$$d_2(f(x),f(y)) \le d_2(f(x),f(p_k)) + d_2(f(p_k),f(y))$$
$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore $d_2(f(x),f(y)) < \epsilon$, which implies that f is uniformly continuous.

The following functions are from a metric space (M,d) into R.

3.7. <u>Definition</u>. Let $E \subset R$, then
$u(f;E) = \text{lub } f(E)$ where $f(E) = \{f(x) | x \in E\}$,

$l(f;E) = \text{glb } f(E)$ where $f(E) = \{f(x) | x \in E\}$,

$s(f;E) = \text{lub}\{|f(x_2) - f(x_1)| \mid x_1, x_2 \in E\}$.

The $s(f;E)$ is often called the saltus. Even if f is from a metric space (M_1,d_1) into a metric space (M_2,d_2) the saltus of f on $E \subset M_1$ may be defined:

$s(f;E) = \text{lub}\{d(x_1,x_2) | x_1,x_2 \in E\}$.

However, $u(f;E)$ and $l(f;E)$ have no meaning in this case.

3.8. Remark. If f is bounded, the functions $u(f;E)$, $l(f;E)$, and $s(f;E)$ are finite.

3.9. Definition. Let

$U(f;x) = \text{glb}\{u(f;S(x,r))\ r > 0\}$,

$L(f;x) = \text{lub}\{l(f;S(x,r))\ r > 0\}$,

$S(f;x) = \text{glb}\{s(f;S(x,r))\ r > 0\}$.

3.10. Remark. If $A \subset B$, then $D(A) \leq D(B)$.

3.11. Lemma. If $A \subset B$, $s(f;A) \leq s(f;B)$.

Proof. By Remark 3.10, $D(A) \leq D(B)$. This implies

$u(f;A) - l(f;A) \leq u(f;B) - l(f;B)$.

Thus $\text{lub}\{|f(a_1) - f(a_2)| \mid a_1,a_2 \in A\} \leq \text{lub}\{|f(b_1) - f(b_2)| \mid b_1,b_2 \in B\}$.

Therefore , $s(f;A) \leq s(f;B)$.

It may be instructive to include another proof to

3.12. Theorem. If f is a continuous function from a compact set E of a metric space (M_1,d_1) into a metric space (M_2,d_2), then it is uniformly continuous.

Proof. Choose $\epsilon > 0$. Then there exists an r_x, $x \in E$, such that for $S(x,r_x) = I_x$, $s(f;I_x \cap E) < \epsilon$. Then $\mathcal{G} = \{I_x\}$, $x \in E$, is an open cover for E. Since by

Corollary 2.37, E has the Bolzano-Weierstrass Property, then by Theorem 2.26, there exists a $\delta > 0$ such that every sphere of radius δ with center in E is contained in some $I_x \in \mathcal{G}$. Choose $x_1, x_2 \in E$, such that $d_1(x_1, x_2) < \delta$. Then the open sphere $S(x_1, \delta)$ contains x_1 and x_2. Therefore, $S(x_1, \delta) \subset I_x$. Hence

$$d_2(f(x_2), f(x_1)) \leq s(f; S(x_1, \delta) \cap E)$$
$$\leq s(f; I_x \cap E) < \epsilon .$$

Thus $d_2(f(x_2), f(x_1)) < \epsilon$ and f is uniformly continuous.

3.13. Definition. A function f is said to be upper semi-continuous at $x = c$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $f(x) < f(c) + \epsilon$ whenever $d(x,c) < \delta$.

Similarly, f is said to be lower semi-continuous at $x = c$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $f(x) > f(c) - \epsilon$ whenever $d(x,c) < \delta$.

3.14. Theorem. If f is a bounded function from a metric space (M,d) into R, then U(f;x) is an upper semi-continuous function and L(f;x) is a lower semi-continuous function.

Proof. Let $x_0 \in M$. Choose $\epsilon > 0$. Then, by the definition of U(f;x), there exists some $\delta > 0$ such that $u(f; S(x_0, \delta)) < U(f;x) + \epsilon$. Now choose $x \in S(x_0, \delta)$, that is, $d(x, x_0) < \delta$. Then there exists a $\delta_1 > 0$ so that $S(x, \delta_1) \subseteq S(x_0, \delta)$ and therefore

$$U(f;x) \leq u(f; S(x, \delta_1)) \leq u(f; S(x_0, \delta)) < U(f; x_0) + \epsilon .$$

Hence, U(f;x) is upper semi-continuous at $x = x_0$. In a similar way, L(f;x) is lower semi-continuous at $x = x_0$.

3.15. Remark. If $\delta > 0$, then

$$l(f;S(x_0,\delta)) \leq L(f;x_0) \leq f(x_0) \leq U(f;x_0) \leq u(f;S(x_0,\delta)).$$

3.16. Theorem. *Let f be a function from a metric space (M,d) into R. Then f is upper semi-continuous if and only if $U(f;x) = f(x)$ and f is lower semi-continuous if and only if $L(f;x) = f(x)$.*

Proof. Suppose f is upper semi-continuous at $x_0 = x$. Choose $\epsilon > 0$. Then there exists a $\delta > 0$ such that for $x \in S(x_0,\delta)$, $f(x) < f(x_0) + \epsilon$.

$$U(f;x_0) \leq u(f;S(x_0,\delta)) \leq f(x_0) + \epsilon .$$

Therefore, $U(f;x_0) \leq f(x_0) + \epsilon$ which implies $U(f;x_0) \leq f(x_0)$. By Remark 3.15, $f(x_0) \leq U(f;x_0)$. Therefore $U(f;x_0) = f(x_0)$.

Suppose $U(f;x_0) = f(x_0)$. Given $\epsilon > 0$, there exists a $\delta > 0$ such that $u(f;S(x_0,\delta)) - f(x_0) < \epsilon$. Hence for $x \in S(x_0,\delta)$, $f(x) - f(x_0) < \epsilon$ which says that f is upper semi-continuous at x_0. Since x_0 was any element in the domain of the function, then f is upper semi-continuous.

A similar proof can be used to show that $L(f;x_0) = f(x_0)$ if and only if f is a lower semi-continuous function.

3.17. Corollary. *Let f be a function from a metric space (M,d) into the set R. Then f is continuous if and only if $U(f;x) = L(f;x) = f(x)$.*

Proof. A continuous function is both upper and lower semi-continuous.

3.18. Lemma. If $A \subset R$, then

$$\text{lub } A - \text{glb } A = \text{lub } d(x_1,x_2),\ x_1,x_2 \in A.$$

3.19. Corollary. *Let f be a function from a metric space (M,d) into R. Let $E \subset M$, then*

$$s(f;E) = u(f;E) - l(f;E).$$

3.20. Remark. Let f be a function from a metric space (M,d) into R. Then $S(f;x_0) \leq s(f;S(x_0, \delta))$ for $\delta > 0$.

3.21. Theorem. $S(f;x) = U(f;x) - L(f;x)$.

Proof. Let $x = x_0$. Given $\epsilon > 0$, there exists a $\delta > 0$ such that $s(f;S(x_0, \delta)) - S(f;x_0) < \epsilon$.

By Corollary 3.19,

$$u(f;S(x_0, \delta)) - l(f;S(x_0, \delta)) - S(f;x_0) < \epsilon .$$

$$u(f;S(x_0, \delta)) < l(f;S(x_0, \delta)) + S(f;x_0) + \epsilon .$$

By Remark 3.15,

$U(f;x_0) < L(f;x_0) + S(f;x_0) + \epsilon$. Then,

$U(f;x_0) - L(f;x_0) < S(f;x_0) + \epsilon$. Since ϵ is any positive number, $U(f;x_0) - L(f;x_0) \leq S(f;x_0)$.

Choose $\epsilon > 0$. There exists $\delta_1 > 0$ such that $u(f;S(x_0, \delta_1)) - U(f;x_0) < \epsilon/2$ and there exists $\delta_2 > 0$ such that $L(f;x_0) - l(f;S(x_0,\delta_2)) < \epsilon/2$. Let $\delta = \min(\delta_1, \delta_2)$. Then $u(f;S(x_0, \delta)) - U(f;x_0) < \epsilon/2$ and $L(f;x_0) - l(f;S(x_0, \delta)) < \epsilon/2$. Thus,

$u(f;S(x_0, \delta)) - l(f;S(x_0, \delta)) < U(f;x_0) - L(f;x_0) + \epsilon$.

By Corollary 3.19, $s(f;S(x_0, \delta)) < U(f;x_0) - L(f;x_0) + \epsilon$.

Since ϵ is any positive number, then

$s(f;S(x_0, \delta)) \leq U(f;x_0) - L(f;x_0)$. By Remark 3.20,

$S(f;x_0) \leq U(f;x_0) - L(f;x_0)$. Therefore,

$S(f;x_0) = U(f;x_0) - L(f;x_0)$.

3.22. Theorem. $S(f;x) = 0$ if and only if f is continuous at x.

Proof. Suppose f is continuous. By Theorem 3.21,

$$S(f;x) = U(f;x) - L(f;x).$$

By Corollary 3.17,

$$U(f;x) - L(f;x) = f(x) - f(x) = 0.$$

Therefore $S(f;x) = 0$.

Let $x = x_0$. Suppose $S(f;x_0) = 0$. Thus by Theorem 3.21, $U(f;x_0) - L(f;x_0) = 0$. Therefore, given $\epsilon > 0$, there exists a $\delta > 0$ such that,

$u(f;S(x_0, \delta)) - l(f;S(x_0, \delta)) < \epsilon$. This implies that for $x \in S(x_0, \delta)$, $|f(x) - f(x_0)| < \epsilon$. Therefore f is continuous at x_0, which was any value of x. Therefore f is continuous.

3.23. Corollary. $S(f;x) > 0$ if and only if f is discontinuous at x.

3.24. Theorem. Let f be a function from a metric space (M,d) into R. If $k \geq 0$, then the set $F_k = \left\{x \middle| S(f;x) \geq k\right\}$ is closed.

Proof. Let x_0 be a limit point of F_k. Consider $S(x_0,r)$ for some r. Since x_0 is a limit point of F_k, then $S(x_0,r/2)$, must contain a $y \in F_k$. Since $y \in F_k$, then $S(f;y) = U(f;y) - L(f;y) \geq k$. Therefore since $S(y,r/2) \subset S(x_0,r)$, then by Lemma 3.11,

$$k \leq s(f;S(y,r/2)) \leq s(f;S(x_0,r)).$$

Since this is true for any $r > 0$, then

$$\text{glb}\left\{s(f;S(x_0,r)\right\} \geq k$$

which implies $S(f;x_0) \geq k$, which implies $x_0 \in F_k$. Hence, F_k is closed.

3.25. Corollary. *Let f be a function from a metric space (M,d) into R. If $k > 0$ then the set $F_k = \{x | S(f;x) < k\}$ is open.*

3.26. Definition. A set is called a G-*delta*, G_δ, if it is the intersection of a countable collection of open sets.

A set is called an F-*sigma*, F_δ, if it is the union of a countable collection of closed sets.

3.27. Remark. The complement of an F_δ is a G_δ.

3.28. Theorem. *If f is a function from a metric space (M,d) into R, then the set of points D of discontinuity is an F .*

Proof. Let $x \in D$. Then by Corollary 3.23, $S(f;x) > 0$. Thus for some $n \in J$, $S(f;x) \geq 1/n$. Let

$$F_n = \{x | S(f;x) \geq 1/n\} .$$

By Theorem 3.24, F_n is closed. It is also true that $x \in F_n$. Therefore $x \in \bigcup_{n=1}^{\infty} F_n$ which says $D \subset \bigcup_{n=1}^{\infty} F_n$.

Suppose $x \in \bigcup_{n=1}^{\infty} F_n$, then $x \in F_n$ for some $n \in J$. Thus $S(f;x) \geq 1/n$ which implies that $x \in D$. Hence $\bigcup_{n=1}^{\infty} F_n \subset D$. Therefore $D = \bigcup_{n=1}^{\infty} F_n$ which is an F_δ.

3.25. Corollary. *If f is a function from a metric space (M,d) into R, then the set of points C of continuity is a G_δ.*

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