ON LANE'S INTEGRAL

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The problem and purpose of this paper is to develop Lane's Integral in two-space, and then to expand these concepts into three-space and n-space. Lane's Integral can be used by both mathematicians and statisticians as one of the tools in the calculation of certain probabilities and expectations. The method of presentation is straightforward with the basic concepts of integration theory and Stieltjes Integral assumed.

The paper is divided into five main chapters and includes a preface and a bibliography. In this paper only real functions which are defined over the set of all real numbers are considered. The first chapter provides the basic groundwork for the remaining chapters, and, therefore, the properties relative to Lane's Integral in two-space are presented with the necessary definitions and theorems. The second chapter is devoted to the development and expansion of the integral into three-space with the necessary analogous definitions and theorems from Chapter I.

In the third chapter natural analogues of the theorems and material in Chapter I and Chapter II are presented for n-space. Since the material extends readily to n-space, just a few sample definitions were presented. Chapter IV contains the material necessary to expand the domain of integration from a finite interval to an infinite interval. Therefore,

in the fourth chapter the definitions and theorems presented in the preceding chapters for the definite integral were expanded to fit the concept of the improper integral.

The paper concludes with Chapter V which gives some of the advantages of the Lane Integral over the more commonly used Stieltjes Integral. It is, therefore, hoped that eventually the more encompassing Lane Integral will come more into vogue.

ON LANE'S INTEGRAL

THESIS

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PREFACE

The concepts of integration theory have been a tool of both the mathematician and the statistician for many years. However, in statistical theory more general concepts and integrals are needed in the calculation of certain probabilities and expectations. Today with the aid of integrals such as Lane's Integral these areas are more exacting.

The purpose of this paper is to develop and expand Lane's Integral into three-space, and then to present some fundamental theorems. The first chapter of this paper will provide and investigate some properties of Lane's Integral in two-space. The second chapter is devoted to the development of the integral in three-space for quasi-continuous real functions and functions of bounded variation.

In the third chapter natural analogues of theorems in Chapter I and Chapter II are presented for n-space dimensions. The fourth chapter presents the material necessary to expand the domain of integration from a finite interval to an infinite interval. Therefore, in Chapter IV the definitions and theorems presented in the preceding chapters for the definite integral will be expanded to fit the concept of the improper integral. The final chapter gives some of the advantages of Lane's Integral over the more popular Stieltjes Integral.

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CHAPTER I

THE INTEGRAL $\int_a^b f(x)dg(x)$

In this paper only real functions which are defined over the set of all real numbers are considered. A basic knowledge and understanding of the Stieltjes integral is assumed and consequently no rigorous derivation of it will be presented. In order to simplify some of the statements which are to be made, certain preliminary definitions and remarks are necessary.

Definition 1.1. The statement that f is a function implies that if t is a real number then there is just one number c such that $f(t) = c$.

Definition 1.2. The statement that G is the graph of the function y with respect to the function x in the interval $[a,b]$ means that G is the set of ordered triples such that p,q,r is in G if and only if it is true that $a \leq p \leq b$ and $q = x(p)$ and $r = y(p)$. The statement that P is a point of G means that there is a number t such that $a \leq t \leq b$ and $x(t)$ is the abscissa of P and $y(t)$ is the ordinate of P.

Definition 1.3. The statement that D is a subdivision of the interval $[a,b]$ means that D is a finite set of (one or more) intervals $[t_0,t_1]$, $[t_1,t_2]$, $\cdots$, $[t_{n-1},t_n]$ such that $t_0 = a$ and $t_n = b$.

Definition 1.4. The statement that E is a refinement of D means that E is a subdivision of $[a,b]$ such that, if $[p,q]$ is an interval in D then there is a subset of E which is a subdivision of $[p,q]$.

Definition 1.5. The statement that the function f is quasi-continuous in the interval $[a,b]$ means that if $\epsilon > 0$ then there is a subdivision D of $[a,b]$ such that, if $[p,q]$ is one of the intervals in D and s and t are in the segment (p,q), then $| f(s) - f(t) | < \epsilon$.

Notation. If x is a function, y is a function, and D is a subdivision of the interval $[a,b]$, then $S_D(x,y)$ denotes the number

$$\sum_{p=0}^{n-1} 1/2\ [y(t_p) + y(t_{p+1})]\ [x(t_{p+1}) - x(t_p)].$$

Theorem 1.1. For numerical work it is often convenient to make use of the fact that

$$S_D(x,y) = y(t_0) \cdot 1/2\ [x(t_1) - x(t_0)]$$

$$+ \sum_{p=1}^{n-1} y(t_p) \cdot 1/2\ [x(t_{p+1}) - x(t_{p-1})]$$

$$+ y(t_n) \cdot 1/2\ [x(t_n) - x(t_{n-1})].$$

Proof:

$$S_D(x,y) = \sum_{p=0}^{n-1} 1/2\ [y(t_p) + y(t_{p+1})][x(t_{p+1}) - x(t_p)]$$

$$= \sum_{p=0}^{n-1} 1/2 \cdot y(t_p) \cdot [x(t_{p+1}) - x(t_p)]$$

$$+ \sum_{p=0}^{n-1} 1/2\ y(t_{p+1}) \cdot [x(t_{p+1}) - x(t_p)].$$

Then

$$S_D(x,y) = y(t_0) 1/2\ [x(t_1) - x(t_0)]$$

$$+ \{ \sum_{p=1}^{n-1} 1/2 \cdot y(t_p) \cdot [x(t_{p+1}) - x(t_p)]$$

$$+ \sum_{p=0}^{n-2} 1/2 \cdot y(t_{p+1}) \cdot [x(t_{p+1}) - x(t_p)] \}$$

$$+ y(t_n)\ 1/2\ [x(t_n) - x(t_{n-1})]$$

$$= y(t_0) 1/2 [x(t_1) - x(t_0)] + \{ \sum_{p=1}^{n-1} 1/2\ y(t_p)[x(t_{p+1}) - x(t_p)]$$

$$+ \sum_{p=1}^{n-1} 1/2\ y(t_p)[x(t_p) - x(t_{p-1})] \}$$

$$+ y(t_n)\ 1/2\ [x(t_n) - x(t_{n-1})]$$

$$= y(t_0)\ 1/2\ [x(t_1) - x(t_0)]$$

$$+ \{ \sum_{p=1}^{n-1} 1/2\ y\ (t_p)\ [x(t_{p+1}) - x(t_{p-1})] \}$$

$$+ y(t_n)\ 1/2\ [x(t_n) - x(t_{n-1})].$$

This completes the proof of Theorem 1.1.

Definition 1.6. The statement that the function y is integrable with respect to the function x in the interval $[a,b]$ means that if $\epsilon > 0$ then there is a subdivision D of $[a,b]$ such that $| S_D(x,y) - S_E(x,y) | < \epsilon$ if E is a refinement of D.

Theorem 1.2. For the function y to be integrable with respect to the function x in the interval $[a,b]$ it is necessary and sufficient that there be one and only one number c such that if $\epsilon > 0$, then there is a subdivision D of $[a,b]$ such that $|S_E(x,y) - c | < \epsilon$ if E is a refinement of D.

Proof: Assume that there exists a number L such that for every $\epsilon > 0$ there exists a subdivision D of $[a,b]$ such that $| S_E(x,y) - L| < 1/2\ \epsilon$, where E is any refinement of D. Now D is a refinement of D. Therefore, $| S_D(x,y) - L | < 1/2\ \epsilon$. Then by using the Triangle Inequality,

$$| S_D(x,y) - S_E(x,y) | \leq | S_D(x,y) - L | + | S_E(x,y) - L |$$

$$< 1/2\ \epsilon + 1/2\ \epsilon = \epsilon.$$

Then by Definition 1.6 the function y is integrable with respect to the function x in $[a,b]$. This completes the proof of the sufficiency.

Now suppose y is integrable with respect to x on $[a,b]$. Then if $\epsilon > 0$ there exists a subdivision D of $[a,b]$ such that if E is any refinement of D, then $| S_D(x,y) - S_E(x,y) | < \epsilon$.

Now we need to produce a candidate for the value of the integral. Let $\{D_j\}$ be a sequence of subdivisions of $[a,b]$ where each D_j is associated with $\epsilon = 1/2^j$ for $j = 1, 2, \ldots, n$ and D_j is the subdivision referred to in Definition 1.6. Let $\{D_j'\}$ be a sequence of refinements such that D_j' is the common refinement of D_j and D_{j-1} and D_{j-2} and $\cdots$ and D_2 and D_1. It can be seen that each D_{j+1}' is a refinement of D_j'. Now consider the sequence of sums $\{S_{D_j'}(x,y)\}$. This sequence is a Cauchy sequence of real numbers. Therefore the sequence converges to some real number; call it L.

Hence, given any $\epsilon > 0$ there exists a D_j' such that $1/2^j < \epsilon/4$, then $| S_{D_j'}(x,y) - S_{D_{j+1}'}(x,y) | < 1/2^j < \epsilon/4$. Moreover, there exist an integer m such that $| S_{D_m'}(x,y) - L | < \epsilon/4$. Pick the higher of j and m and call it p. Then $| S_{D_p'}(x,y) - S_{D_{p+1}'}(x,y) | < \epsilon/4$ and $| S_{D_p'}(x,y) - L | < \epsilon/4$. Now, by Definition 1.6 it is true that if E is any refinement of D_p, then $| S_{D_p}(x,y) - S_E(x,y)| < 1/4\ \epsilon$ and that $| S_{D_p}(x,y) - S_{D_p}'(x,y)| < \epsilon/4$. Now adding inequalities:

$$| S_E(x,y) - L | \leq | S_{D_p}(x,y) - S_{D_p'}(x,y)| + | S_{D_p'}(x,y) - L|$$

$$+ | S_E(x,y) - S_{D_p}(x,y) | < 1/4\ \epsilon + 1/4\ \epsilon + 1/4\ \epsilon < \epsilon.$$

Therefore, there exists an L such that for any $\epsilon > 0$ there exists a subdivision D of $[a,b]$ such that, if E is any refinement of D, then $| S_E(x,y) - L | < \epsilon$.

Now, if there exists a number c, then it is unique. Assume L_1 and L_2 are values of the integral on [a,b] of the function y with respect to the function x. Now $\frac{|L_2 - L_1|}{2} > 0$, and there exists a subdivision D of [a,b] such that if E is a refinement of D then $|S_E(x,y) - L_1| < \frac{|L_2 - L_1|}{2}$.

Also, there exists a subdivision P of [a,b] such that if Q is a refinement of P then $|L_2 - S_Q(x,y)| < \frac{|L_2 - L_1|}{2}$. Let H be a subdivision of [a,b], which is the common refinement of E and Q; then $|S_H(x,y) - L_1| < \frac{|L_2 - L_1|}{2}$ and $|L_2 - S_H(x,y)| < \frac{|L_2 - L_1|}{2}$. Now adding inequalities:

$$|S_H(x,y) - L_1| + |L_2 - S_H(x,y)| < |L_2 - L_1|.$$

But this contradicts the Triangle Inequality, so the original assumption must be false. Thus the value of the integral is unique. This completes the proof.

<u>Definition 1.7.</u> If $\{D_i\}$ $i = 1,2,\ldots, n$ is a sequence of subdivisions of [a,b] where the lengths of the maximum intervals of the D_i's tend toward zero, then let

$$S(D_i;f,g) = \sum_{k=1}^{n} f(\xi_k)[g(t_k) - g(t_{k-1})]$$

where $t_{k-1} \leq \xi_k \leq t_k$. Under certain conditions, the sequence $S(D_i; f,g)$ has a limit called the Stieltjes integral.

Theorem 1.3. If the Stieltjes integral exists in the interval $[a,b]$, then Lane's integral also exists in $[a,b]$, and they are equal.

Proof Outline: Assume that Stieltjes integral exists and let its value be represented by I. Then both the left-Stieltjes and right-Stieltjes integrals exist, and both equal I. If $S(D_L;f,g)$ and $S(D_R;f,g)$ denote the approximating sums over any subdivision D of $[a,b]$, then $\frac{S(D_L;f,g) + S(D_R;f,g)}{2}$, their mean, approximates $\frac{I + I}{2}$ and is $S_D(g,f)$. Thus given the existence of Stieltjes integral we are also assured the existence of Lane's integral.

For the purposes of this paper Lane's integral and the Stieltjes mean integral will be the same.

Definition 1.8. Suppose that x is a function, y is a function, and $[a,b]$ is an interval. The statement that c is the integral from a to b of y with respect to x, which may be written as

$$c = \int_a^b y(t)dx(t),$$

means that c is a number and that the statement from Theorem 1.2 is true.

Theorem 1.4. The statement that $c = \int_a^b y(t)dx(t)$ means that $-c = \int_b^a y(t)dx(t)$. Moreover, $\int_a^a y(t)dx(t) = 0$.

Proof: For $\epsilon > 0$ there exists a subdivision D of $[a,b]$ such that

$$\left| \int_a^b y(t)dx(t) - \sum_{p=0}^{n-1} 1/2\, [y(t_{p+1}) + y(t_p)][x(t_{p+1}) - x(t_p)] \right| < \epsilon,$$

but

$$\left| \int_b^a y(t)dx(t) - \sum_{p=0}^{n-1} 1/2\, [y(t_{n-1-p}) + y(t_{n-p})][x(t_{n-1-p}) - x(t_{n-p})] \right| < \epsilon.$$

It is obvious from the two representations of the sums that the x differences are reversed, i.e., one is the negative of the other. Therefore,

$$\int_a^b y(t)dx(t) = - \int_b^a y(t)dx(t).$$

It is also obvious that the integral of a function with respect to another function taken over a point is zero, because each approximating sum $S_E(x,y)$ is zero.

Definition 1.9. The statement that the function f is of bounded variation in the interval $[a,b]$ means that there is a number B such that if $D:\{[t_0,t_1],[t_1,t_2], \ldots,[t_{n-1},t_n]\}$ is a subdivision of $[a,b]$; then

$$\sum_{p=0}^{n-1} | f(t_{p+1}) - f(t_p) | \leq B.$$

The least such number B is denoted by $V_a^b(f)$, and is said to be the variation of f in $[a,b]$.

Lemma 1.1. Suppose x is a function of bounded variation in the interval $[g,h]$. If $\epsilon > 0$, then there exists an interval $[g,k]$ which is a subset of $[g,h]$ such that, if $[g,w]$ is a subset of $[g,k]$, then $V_g^k(x) - V_g^w(x) < \epsilon$.

Proof: Suppose $\epsilon > 0$. Let $s_1 = h$ and then for $n > 1$, let $s_n = \frac{g + s_{n-1}}{2}$. Then $V_g^{s_n}(x)$ exists and $V_g^{s_{n+1}}(x) \leq V_g^{s_n}(x)$. Let L denote the greatest lower bound of the numbers $V_g^{s_n}(x)$. There is a positive integer p such that $V_g^{s_{p+d}}(x) - L < 1/2\ \epsilon$ if d is a non-negative integer. If $[g,w]$ is a subset of $[g,s_p]$, then there is a positive integer d such that $s_{p+d} < w < s_p$ and

$$V_g^{s_{p+d}}(x) \leq V_g^{w}(x) \leq V_g^{s_p}(x)$$

and also

$$V_g^{s_{p+d}}(x) - L < 1/2\ \epsilon.$$

Hence

$$V_g^{w}(x) - L < 1/2\ \epsilon.$$

Thus

$$V_g^{s_p}(x) - V_g^{w}(x) < \epsilon,$$

and $[g,s_p]$ qualifies as $[g,k]$.

Theorem 1.5. If y is a quasi-continuous function in the interval $[a,b]$ and x is a function of bounded variation in $[a,b]$, then y is integrable with respect to x in the interval $[a,b]$.

Proof: If $V_a^b(x) = 0$, then the proof is trivial.

Suppose that $V_a^b(x) > 0$. There is a positive number M such that $|y(t)| \leq M$, because quasi-continuity implies boundedness.

For $\epsilon > 0$, there exists a subdivision D of $[a,b]$ such that if p and q are points in the interior of an interval of

D, then $|\ y(p) - y(q)\ | < \epsilon$, because y is quasi-continuous. Let n denote the number of intervals in D.

Let D' be a refinement of D obtained as follows. In the interior of each interval $[g,h]$ of D select two numbers k and ℓ so that (in conformance with Lemma 1.1) it will be true that if $g < w < k$ and $\ell < u < h$, then

$$V_g^k(x) - V_g^w(x) < \epsilon \text{ and}$$

$$V_\ell^h(x) - V_u^h(x) < \epsilon.$$

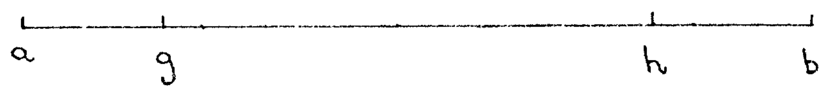
See Figure 1, page 11.

Suppose E is any refinement of D'. For each interval $[g,h]$ of D there are two intervals $[g,w]$ and $[u,h]$ of E, where $g < w \leq k$ and $\ell \leq u < h$. Let e_1 denote the two-member set $\{[g,w],[u,h]\}$. Let e_2 denote the set of all the other intervals of E which lie in $[g,k]$ or $[\ell,h]$ (i.e., all members of E which are subsets of $[w,k]$ or $[\ell,u]$).

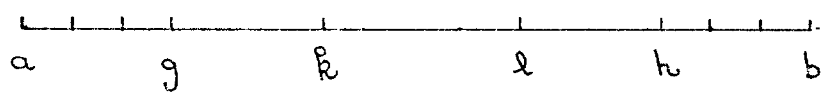
Let e_3 denote the set of all intervals of E which are subsets of $[k,\ell]$. Let E_1 denote the union of all such sets e_1, and let E_2 denote the union of all sets e_2, and let E_3 denote the union of all such sets e_3 (for all such intervals $[g,h]$ of D). See Figure 1, page 11.

Now the terms of $S_{D'}(x,y)$ which are associated with any interval $[g,h]$ of D are

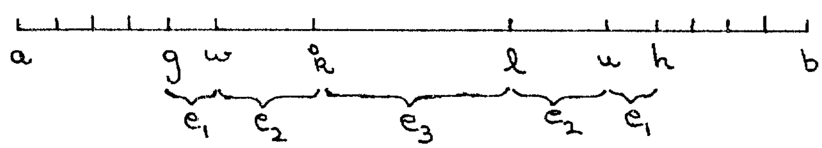
$$\frac{y(g) + y(k)}{2}[x(k) - x(g)] \quad + \quad \frac{y(k) + y(\ell)}{2}[x(\ell) - x(k)]$$
$$+ \frac{y(\ell) + y(h)}{2}[x(h) - x(\ell)].$$



The subdivision D



The subdivision D′



The subdivision E

Figure 1 - Subdivisions D, D′ and E for Theorem 1.5

Since $g < w \leq k$ and $\ell \leq u < h$, we can add and subtract an $x(w)$ in the first set of square brackets and we can add and subtract an $x(u)$ in the third set of square brackets, and then expand into the five terms

$$\frac{y(g) + y(k)}{2}[x(w) - x(g)] + \frac{y(g) + y(k)}{2}[x(k) - x(w)]$$

$$+ \frac{y(k) + y(\ell)}{2}[x(\ell) - x(k)] + \frac{y(\ell) + y(h)}{2}[x(u) - x(\ell)]$$

$$+ y(\ell) + y(h)[x(h) - x(u)].$$

Let $S_{D_1'}(x,y)$ denote the set of all such first terms and all fifth terms (for all intervals $[g,h]$ of D). Let $S_{D_2'}(x,y)$ denote the set of all such second and fourth terms. Let $S_{D_3'}(x,y)$ denote the set of all such third terms. Notice that

$$S_{D'}(x,y) = S_{D_1'}(x,y) + S_{D_2'}(x,y) + S_{D_3'}(x,y).$$

Now

$$|S_{D'}(x,y) - S_E(x,y)| =$$

$$|\{S_{D_1'}(x,y) + S_{D_2'}(x,y) + S_{D_3'}(x,y)\} - \{S_{E_1}(x,y) + S_{E_2}(x,y) + S_{E_3}(x,y)\}|$$

$$\leq |S_{D_1'}(x,y) - S_{E_1}(x,y)| + |S_{D_2'}(x,y) - S_{E_2}(x,y)|$$

$$+ |S_{D_3'}(x,y) - S_{E_3}(x,y)|.$$

Let this last expression be denoted by $|A| + |B| + |C|$.

Now, for each interval $[g,h]$ of D, the terms

$$\left\{\frac{y(g)+y(k)}{2}[x(w)-x(g)]+\frac{y(\ell)+y(h)}{2}[x(h)-x(u)]\right\}$$

$$-\left\{\frac{y(g)+y(w)}{2}[x(w)-x(g)+\frac{y(u)+y(h)}{2}[x(h)-x(u)]\right\}$$

appear in $|A|$. These net out to

$$\frac{y(k)-y(w)}{2}[x(w)-x(g)]+\frac{y(\ell)-y(u)}{2}[x(h)-x(u)].$$

But this is less than

$$\frac{\epsilon}{2}\cdot[x(w)-x(g)]+\frac{\epsilon}{2}\cdot[x(h)-x(u)],$$

because w, k, ℓ and u are all interior points of $[g,h]$, an interval of the subdivision D which was chosen as a response to ϵ in accordance with the quasi-continuous property of y. Factoring out $\frac{\epsilon}{2}$ leaves an expression which is less than the variation of the function x over $[g,w]$ and $[u,h]$. Hence

$$|A| < \frac{\epsilon}{2}\cdot(\text{variation of } x \text{ over } E_1)$$

Now in $|B|$ above, if $\frac{y(\ )+y(\ )}{2}$ is replaced by M, the result is an expression which is greater than $|B|$. Factoring out each M leaves an expression such that

$$|B| \leq M.\ (\text{variation of } x \text{ over } E_1 - \text{variation of } x \text{ over } D_2')$$

$$\leq M.\ (\text{variation of } x \text{ over } E_1)$$

But using Lemma 1.1 and the fact that there are n intervals in D, then

$$|B| < M\cdot 2n\epsilon.$$

Now for $|C|$. Each interval $[k, \ell]$ is associated with only one term in $S_{D_3'}(x,y)$ $\left\{\text{namely } \frac{y(k) + y(\ell)}{2}[x(\ell) - x(k)]\right\}$ but it may be associated with more than one term in $S_{E_3}(x,y)$, because $[k,\ell]$ may have some endpoints of E in its interior, and these, of course, would not be endpoints of D$'$. If so, let these extra endpoints of E in $[k, \ell]$'s interior be denoted by $t_1, t_2, \cdots, t_i$. Merely add and subtract $x(t)$'s in $\frac{y(k) + y(\ell)}{2}[x(\ell) - x(k)]$ to get

$$\frac{y(k) + y(\ell)}{2}\left\{[x(t_1) - x(k)] + [x(t_2) - x(t_1)] + \cdots + [x(\ell) - x(t_i)]\right\}.$$

After distributing the $\frac{y(k) + y(\ell)}{2}$ across the series, and performing this for each interval $[k, \ell]$ of D_3', we find that $S_{D_3'}(x,y)$ has as many terms as $S_{E_3}(x,y)$. Now in $|S_{D_3'}(x,y) - S_{E_3}(x,y)|$ we regroup terms according to alike x differences. An example term would be

$$\left\{\frac{y(k) + y(\ell)}{2} - \frac{y(t_p) + y(t_{p+1})}{2}\right\}[x(t_{p+1}) - x(t_p)].$$

Now each abscissa is an interior point of an interval in D, a subdivision chosen as a response to ϵ in accordance with the quasi-continuous property of y. Hence the expression in the braces above is less than ϵ. After factoring out ϵ's we have

$$|C| < \epsilon.\ (\text{variation of } x \text{ over } E_3)$$

Then

$$|A| + |B| + |C| < \frac{\epsilon}{2}\ .\ (\text{variation of } x \text{ over } E_1)$$

$$+ M\cdot 2n\epsilon + \epsilon.\ (\text{variation of } x \text{ over } E_3)$$

$$< \epsilon.\ (\text{variation of } x \text{ over } E_1 \text{ and } E_3 + 2Mn).$$

Thus

$$|S_{D'}(x,y) - S_E(x,y)| < \epsilon\cdot(V_a^b(x) + 2Mn).$$

So to meet the challenge of any positive number ϵ_1, we consider ϵ as being $\dfrac{\epsilon_1}{V_a^b(x) + 2Mn}$. This completes the proof of Theorem 1.5.

<u>Theorem 1.6.</u> If y is integrable with respect to x in the interval [a,b] and k is a number, then

$$\int_a^b y(t)d[x(t)+k] = \int_a^b y(t)dx(t)$$

<u>Proof</u>: Since y is integrable with respect to x in [a,b] then by Theorem 1.2 there exists a number $I = \int_a^b y(t)dx(t)$.

Since the integral exists then for $\epsilon > 0$ there exists a subdivision D of [a,b] such that if E is a refinement of D then $|S_E(x,y) - I| < \epsilon$. But this means $|S_E(x+k,y) - I| < \epsilon$, because $S_E(x,y) = S_E(x+k,y)$. So from Theorem 1.2 we see that $\int_a^b y(t)d[x(t)+k]$ exists and is equal to I. Therefore

$$\int_a^b y(t)d[x(t)+k] = \int_a^b y(t)dx(t).$$

Theorem 1.7. If $\int_a^b y(t)dx(t)$ exists and k is a number, then

$$\int_a^b [ky(t)]dx(t) = k \int_a^b y(t)dx(t)$$

Proof: If $k = 0$, the argument is trivial. So suppose $k \neq 0$. Let $\epsilon > 0$ be arbitrarily chosen. Then $\epsilon/|k| > 0$. Since y is integrable with respect to x in [a,b] then there exists a subdivision D of [a,b], such that if E is a refinement of D, then $|S_E(x,y) - I| < \epsilon/|k|$ where $I = \int_a^b y(t)dx(t)$. Then

$$|k|\cdot|S_E(x,y) - I| < \epsilon$$

$$|k \cdot S_E(x,y) - kI| < \epsilon$$

$$|S_E(x,ky) - kI| < \epsilon .$$

Therefore $\int_a^b [ky(t)]dx(t)$ exists and is equal to kI.

Thus it follows that $\int_a^b [ky(t)]dx(t) = k \int_a^b y(t)dx(t)$.

Theorem 1.8. Suppose that each of y and y_1 is integrable with respect to x in the interval [a,b]. Then $\int_a^b [y(t) + y_1(t)]\, dx(t)$ exists and

$$\int_a^b [y(t) + y_1(t)]dx(t) = \int_a^b y(t)dx(t) + \int_a^b y_1(t)dx(t).$$

Proof: Let I denote $\int_a^b y(t)dx(t)$ and let $I_1 = \int_a^b y_1(t)dx(t)$. Since y is integrable with respect to x in [a,b], then for $\epsilon > 0$ there exists a subdivision D of [a,b], such that, if E is a refinement of D, then $|S_E(x,y) - I| < 1/2\ \epsilon$. Also since y_1 is integrable with respect to x in [a,b], then there exists a subdivision D_1 of [a,b], such that, if E_1 is a refinement of D_1, then $|S_{E_1}(x,y) - I_1| < 1/2\ \epsilon$. Let D_2 denote the subdivision of [a,b] which is constructed from both D and D_1. Hence if E_2 is any refinement of D_2 it is also a refinement of D and D_1. So that $|S_{E_2}(x,y) - I| < 1/2\ \epsilon$ and $|S_{E_2}(x,y) - I_1| < 1/2\ \epsilon$. By the triangle inequality, we have

$$|S_{E_2}(x,y) + S_{E_2}(x,y_1) - I - I_1| \leq |S_E(x,y) - I| + |S_E(x,y_1) - I_1|$$

$$|S_{E_2}(x,y + y_1) - (I + I_1)| < 1/2\ \epsilon + 1/2\ \epsilon = \epsilon.$$

Therefore, $\int_a^b [y(t) + y_1(t)]dx(t)$ exists according to Theorem 1.2, and by Definition 1.8

$$\int_a^b [y(t)+y_1(t)]dx(t) = \int_a^b y(t)dx(t) + \int_a^b y_1(t)dx(t).$$

Theorem 1.9. Suppose that y is integrable with respect to x in the interval [a,b]. If $a < c < b$, then y is integrable with respect to x in [a,c] and in [c,b]; moreover,

$$\int_a^b y(t)dx(t) = \int_a^c y(t)dx(t) + \int_c^b y(t)dx(t).$$

Proof: Since y is integrable over [a,b], then given $\epsilon > 0$, there is a subdivision D of [a,b], such that if E is a refinement of D, then by Definition 1.6,

$$|S_P(x,y) - S_Q(x,y)| < \epsilon.$$

It is easy to show that if D is modified to include c as one of its end points, then the resulting subdivision---call it D'---also has this property for any of its refinements E'.

Let D_1' and D_2' be the two subsets of D' which lie in [a,c] and [c,b] respectively. Now if E_1' is any refinement of D_1', then $E_1' \cup D_2'$ is a refinement of D' and therefore

$$|S_{D'}(x,y) - S_{E_1' \cup E_2'}(x,y)| < \epsilon/2$$

$$|\{S_{D_1'}(s,y) + S_{D_2'}(x,y)\} - \{S_{E_1'}(x,y) - S_{D_2'}(x,y)\}| < \epsilon/2$$

$$|S_{D_1'}(x,y) - S_{E_1'}(x,y)| < \epsilon/2.$$

This means that y is integrable with respect to x over [a,c], and therefore $\int_a^c y(t)dx(t)$ exists. A similar argument establishes the integrability of y over [c,b].

Now $\int_a^c y(t)dx(t) + \int_c^b y(t)dx(t) = \int_a^b y(t)dx(t)$ is shown as follows. Suppose $\epsilon > 0$. Then $\epsilon/3 > 0$ and there exist subdivisions F, F_1 and F_2 of [a,b], [a,c], and [c,b] respectively such that if G, G_1, and G_2 are refinements of F, F_1 and F_2 respectively, then

$$\left|\int_a^b y(t)dx(t) - S_G(x,y)\right| < \epsilon/3$$

and

$$\left|S_{G_1}(x,y) - \int_a^c y(t)dx(t)\right| < \epsilon/3$$

and $$\left|S_{G_2}(x,y) - \int_c^b y(t)dx(t)\right| < \epsilon/3.$$

Now, let H denote the refinements of F constructed from all the endpoints of F, F_1, and F_2. Let H_1 and H_2 denote the portions of H which lie in $[a,c]$ and $[c,b]$, respectively. Then H, H_1, and H_2 are refinements of the subdivisions F, F_1, and F_2 respectively, and the three inequalities above are true for H, H_1, and H_2 respectively. If these three inequalities are added and the triangle property of inequalities is applied, then the $S_{H_1}(x,y)$ and $S_{H_2}(x,y)$ terms would net out against the $S_H(x,y)$ term and would leave only

$$\left|\int_a^b y(t)dx(t) - \int_a^c y(t)dx(t) - \int_b^c y(t)dx(t)\right| < \epsilon/3 + \epsilon/3 + \epsilon/3.$$

So, for any $\epsilon > 0$ it is true that $\int_a^b y(t)dx(t)$ differs from $\left\{\int_a^c y(t)dx(t) + \int_c^b y(t)dx(t)\right\}$ by less than ϵ. Hence these two quantities cannot differ at all, because if so, ϵ could be chosen such that it is less than their difference and thus reach a contradiction.

Theorem 1.10. If $\int_a^b y(t)dx(t)$ exists, then $\int_a^b x(t)dy(t)$ exists and

$$\int_a^b y(t)dx(t) = y(b)x(b) - y(a)x(a) - \int_a^b x(t)dy(t).$$

Proof: Since y is integrable with respect to x in [a,b], then for $\epsilon > 0$ there exists a subdivision D of [a,b] such that if E is a refinement of D, then

$$|S_D(x,y) - S_E(x,y)| < \epsilon.$$

It can easily be shown algebraically that

$$S_D(x,y) + S_D(y,x) = y(b)x(b) - y(a)x(a).$$

Then

$$\begin{aligned} |S_D(x,y) - S_E(x,y)| &= |-S_D(y,x)+y(b)x(b)-y(a)x(a)-[-S_E(y,x) \\ &\qquad +y(b)x(b)-y(a)x(a)]| \\ &= |S_D(y,x) - S_E(y,x)| < \epsilon. \end{aligned}$$

Therefore, by Definition 1.6, x is integrable with respect to y over the interval [a,b], and $\int_a^b x(t)dy(t)$ exists. For $\epsilon > 0$ there exists a subdivision D of [a,b] such that, if E is a refinement of D, then $|S_E(x,y) - \int_a^b y(t)dx(t)| < \epsilon$. Since it has been shown that $S_E(x,y) = y(b)x(b) - y(a)x(a) - S_E(y,x)$, substituting and rearranging terms leaves

$$\left| \left\{ y(b)x(b)-y(a)x(a) - \int_a^b y(t)dx(t) \right\} - S_E(y,x) \right| < \epsilon.$$

Now since $S_E(y,x)$ is associated with the unique number $\int_a^b x(t)dy(t)$ then, according to Theorem 1.2, the expression in the braces must be another name for $\int_a^b x(t)dy(t)$. Therefore,

$$\int_a^b x(t)dy(t) = y(b)\,x(b) - y(a)\,x(a) - \int_a^b y(t)dx(t).$$

Theorem 1.11. Suppose that y is integrable with respect to x in the interval [a,b]. If x is of bounded variation in [a,b] and $|y(t)| \leq M$ for each number t in $[a,b]$, then $\left|\int_a^b y(t)dx(t)\right| \leq MV_a^b(x)$.

Proof: Since x is of bounded variation in $[a,b]$, then $\sum_{p=0}^{n-1}|x(t_{p+1}) - x(t_p)| \leq V_a^b(x)$ for any subdivision D of $[a,b]$. Moreover $|y(t)| \leq M$ for each t in $[a,b]$, so that $1/2|y(t_1) + y(t_2)| \leq M$. Hence $|S_D(x,y)| \leq M \cdot V_a^b(x)$ for any subdivision D of $[a,b]$. Now $\int_a^b y(t)dx(t)$ cannot exceed $M \cdot V_a^b(x)$ because, if so, then for the case where $\epsilon = \dfrac{\int_a^b y(t)dx(t) - M \cdot V_a^b(x)}{4}$ there would be a subdivision D of $[a,b]$ such that $|S_D(x,y)|$ would differ from $\int_a^b y(t)dx(t)$ by less than ϵ and would therefore exceed $M \cdot V_a^b(x)$. This completes the proof.

Theorem 1.12. If x is a function and $[a,b]$ is an interval, then

$$\int_a^b 1dx(t) = x(b) - x(a).$$

Proof: Let D be a subdivision of $[a,b]$ consisting of just one member: the interval $[a,b]$. Then if $\epsilon > 0$ and E is any refinement of D, it is true that

$$|S_E(x,1) - [x(b)-x(a)]| < \epsilon$$

because

$$S_E(x,1) = \sum_{p=0}^{m-1}\frac{1+1}{2}[x(t_{p+1}) - x(t_p)]$$

$$= \sum_{p=0}^{m-1}[x(t_{p+1}) - x(t_p)]$$

$$= x(b) - x(a).$$

Theorem 1.13. If x is a function and [a,b] is an interval, then

$$\int_a^b x(t)dx(t) = 1/2[x(b)]^2 - 1/2[x(a)]^2.$$

Proof: Let $\epsilon > 0$ and let D be any subdivision of [a,b] and E be any refinement of D. Then

$$S_E(x,x) = \sum_{p=0}^{n-1} 1/2[x(t_{p+1}) + x(t_p)][x(t_{p+1}) - x(t_p)]$$

$$= \sum_{p=0}^{n-1} 1/2\{[x(t_{p+1})]^2 - [x(t_p)]^2\}$$

$$= 1/2[x(b)]^2 - 1/2[x(a)]^2.$$

But this implies that $|S_E(x,x) - \{1/2[x(b)]^2 - 1/2[x(a)]^2\}| < \epsilon$. Therefore, by Theorem 1.2, x is integrable with respect to x in [a,b], and

$$\int_a^b x(t)dx(t) = 1/2[x(b)]^2 - 1/2[x(a)]^2.$$

Definition 1.10. The statement that the function f has a derivative f′ in the interval [a,b] means that if $a \leq t \leq b$ then there is a number f′(t) such that the following statement is true:

If $\epsilon > 0$, then there is a segment (p,q) containing t such that, if s is in (p,q) and in [a,b], then

$$\left|\frac{f(s) - f(t)}{s - t} - f'(t)\right| < \epsilon.$$

Definition 1.11. The statement that the function f has a continuous derivative, f', in the interval $[a,b]$, means that f has a derivative, f', in $[a,b]$, and that if $\epsilon > 0$, then there is a subdivision D of $[a,b]$ which has the following property. If $[p,q]$ is an interval in D and s and t are numbers in $[p,q]$, then $|f'(s) - f'(t)| < \epsilon$.

Remark 1.11. Clearly, if E is any refinement of D (in Definition 1.11), then E also has that property.

Theorem 1.14. If the function f has a continuous derivative, f', in the interval $[a,b]$, and $\epsilon > 0$, then there is a subdivision D of $[a,b]$, such that, if s and t are in one of the intervals of D, then

$$\left|\frac{f(s) - f(t)}{s - t} - f'(t)\right| < \epsilon.$$

Proof: Since f has a continuous derivative f' in $[a,b]$, then for $\epsilon > 0$ there exists a subdivision D of $[a,b]$ as per Definition 1.11. Since f has a derivative in $[a,b]$, then for each t in $[a,b]$ there exists a segment (p_t, q_t) over which the inequality in Definition 1.10 is true. Let G represent the collection of all such segments (p_t, q_t). Then according to the Heine-Borel Theorem some finite subset G' of G covers $[a,b]$. The end points of G' form a subdivision of $[a,b]$; call it Δ. Now let E be the refinement of both Δ and D which is constructed by taking the end points of both Δ and D. Therefore, for any interval $[p,q]$ of E, the inequality in Definition 1.10 applies. This completes the proof.

<u>Theorem 1.15.</u> Suppose that x has a continuous derivative, x', in the interval $[a,b]$ and that there is a number M such that $|y(t)| < M$ if $a \leq t \leq b$. If

$$\int_a^b y(t)dx(t) = c_1 \text{ or } \int_a^b x'(t)y(t)dt = c_2,$$

then

$$\int_a^b y(t)dx(t) = \int_a^b x'(t)y(t)dt.$$

<u>Proof</u>: The function x has a continuous derivative; therefore, according to Theorem 1.14, if $\epsilon > 0$, then $\frac{\epsilon}{2M(b-a)} > 0$. Then there exists a subdivision D of $[a,b]$ such that for each interval $[t_p, t_{p+1}]$ in D

$$\left|\frac{x(t_{p+1}) - x(t_p)}{t_{p+1} - t_p} - x'(t_p)\right| < \epsilon_1, \text{ where } \epsilon_1 = \frac{\epsilon}{2M(b-a)}.$$

Now consider $S_D(x,y)$ and $S_D(t,x'y)$.

$$|S_D(x,y) - S_D(t,x'y)| = \left|\sum_{p=0}^{n-1} 1/2[y(t_{p+1})+y(t_p)][x(t_{p+1}-x(t_p)]\right.$$

$$\left.- \sum_{p=0}^{n-1} 1/2[x'(t_{p+1})y(t_{p+1})+x'(t_p)y(t_p)][t_{p+1}-t_p]\right|$$

$$= \sum_{p=0}^{n-1} 1/2[y(t_{p+1}) +y(t_p)][x(t_{p+1})-x(t_p)]\frac{[t_{p+1}-t_p]}{t_{p+1}-t_p}$$

$$\left.- \sum_{p=0}^{n-1} 1/2[x'(t_{p+1})y(t_{p+1})+x'(t_p)y(t_p)][t_{p+1}-t_p]\right|$$

$$= \left| \sum_{p=0}^{n-1} 1/2\ y(t_{p+1}) \left\{ \frac{[x(t_{p+1}) - x(t_p)]}{t_{p+1} - t_p} [t_{p+1} - t_p] - x'(t_{p+1})[t_{p+1} - t_p] \right\} \right.$$

$$\left. + 1/2\ y(t_p) \left\{ \frac{[x(t_{p+1}) - x(t_p)]}{t_{p+1} - t_p} [t_{p+1} - t_p] - x'(t_p)[t_{p+1} - t_p] \right\} \right|$$

$$\leq \left| \sum_{p=0}^{n-1} M\ 1/2 \left\{ \frac{x(t_{p+1}) - x(t_p)}{t_{p+1} - t_p} [t_{p+1} - t_p] - x'(t_{p+1})[t_{p+1} - t_p] \right. \right.$$

$$\left. \left. + \frac{x(t_{p+1}) - x(t_p)}{t_{p+1} - t_p} [t_{p+1} - t_p] - x'(t_p)[t_{p+1} - t_p] \right\} \right|$$

$$\leq M \sum_{p=0}^{n-1} 1/2 \left\{ \left| \frac{x(t_{p+1}) - x(t_p)}{t_{p+1} - t_p} [t_{p+1} - t_p] - x'(t_{p+1})[t_{p+1} - t_p] \right| \right.$$

$$\left. + \left| \frac{x(t_{p+1}) - x(t_p)}{t_{p+1} - t_p} [t_{p+1} - t_p] - x'(t_p)[t_{p+1} - t_p] \right| \right\}$$

$$\leq M \sum_{p=0}^{n-1} 1/2 \left\{ \left(\left| \frac{x(t_{p+1}) - x(t_p)}{t_{p+1} - t_p} - x'(t_{p+1}) \right| \right. \right.$$

$$\left. \left. + \left| \frac{x(t_{p+1}) - x(t_p)}{t_{p+1} - t_p} - x'(t_p) \right| \right) [t_{p+1} - t_p] \right\}$$

$$< M \sum_{p=0}^{n-1} 1/2\{(\epsilon_1 + \epsilon_1)[t_{p+1} - t_p]\}$$

$$< M\epsilon_1 \cdot \sum_{p=0}^{n-1} [t_{p+1} - t_p]$$

$< M\epsilon_1(b-a)$, which is $\frac{\epsilon}{2}$.

Hence $|S_D(x,y) - S_D(t,xy)| < \frac{\epsilon}{2}$. This result would also be true for any refinement E of D.

Now suppose $\int_a^b x'(t)y(t)dt = c_2$, and for $\epsilon > 0$ there exists a subdivision D_1 of $[a,b]$, such that if E_1 is a refinement of D_1 then $|S_{E_1}(t,x'y) - c_2| < 1/2\ \epsilon$. Take the subdivisions D and D_1 and form a new subdivision J, such that the endpoints of J are all those from D and D_1. Let K be any refinement of J. Then $|S_K(x,y) - S_K(t,x'y)| < 1/2\ \epsilon$ and $|S_K(t,x'y) - c_2| < 1/2\ \epsilon$. After adding inequalities and applying the triangle inequality, the two $S_K(t,x'y)$ terms net out and leave

$$|S_K(x,y) - C_2| < 1/2\ \epsilon + 1/2\ \epsilon = \epsilon.$$

Thus $\int_a^b y(t)dx(t)$ exists and is equal to c_2, according to Theorem 1.2 and Definition 1.8.

If it is the case that $\int_a^b y(t)dx(t) = c_1$, then $|S_K(t,x'y) - c_1| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon$ is obtained in a similar manner to show that $\int_a^b x'(t)y(t)dt$ exists and is equal to c_1.

Theorem 1.16. Suppose that $[a,b]$ is an interval, $[c,d]$ is an interval, u,v,x,y is a function sequence, and that if D is a subdivision of $[a,b]$, and E is a subdivision of $[c,d]$, then there is a refinement F of D and a refinement G of E such that $S_F(x,y) = S_G(u,v)$. If y is integrable with respect to x in $[a,b]$, and v is integrable with respect to u in $[c,d]$, then

$$\int_a^b y(t)dx(t) = \int_c^d v(t)du(t).$$

Proof: Suppose that $\int_a^b y(t)dx(t) \neq \int_c^d v(t)du(t)$. Then $|I - I_1|$ equals some positive number ϵ, where $I = \int_a^b y(t)dx(t)$ and $I_1 = \int_c^d v(t)du(t)$.

Since y is integrable with respect to x in $[a,b]$, then for $1/4\ \epsilon$ there exists a subdivision D of $[a,b]$, such that if F is a refinement of D, then $|S_F(x,y) - I| < 1/4\ \epsilon$. Likewise there exists a subdivision E of $[c,d]$, such that if G is a refinement of E, then $|S_G(u,v) - I_1| < 1/4\ \epsilon$. Let the refinements F of D and G of E be chosen so that $S_F(x,y) = S_G(u,v)$. Therefore, substituting we obtain $|S_F(x,y) - I| < 1/4\ \epsilon$ and $|S_F(x,y) - I_1| < 1/4\ \epsilon$. Adding these inequalities and applying the triangle property of inequalities yields

$$|I-I_1| \leq |S_F(x,y) - I| + |S_F(x,y) - I_1| < 1/4\ \epsilon + 1/4\ \epsilon = 1/2\ \epsilon.$$

But this is a contradiction since it was assumed that $|I-I_1| = \epsilon$. Therefore, the $\int_a^b y(t)dx(t) \neq \int_c^d v(t)du(t)$ supposition must be false.

Corollary 1.16a. Suppose that [a,b] is an interval, [c,d] is an interval, and that y is integrable with respect to x in [a,b]. If f is a continuous non-decreasing function such that f(c) = a and f(d) = b, then

$$\int_a^b y(t)dx(t) = \int_c^d y[f(t)]dx[f(t)].$$

Proof: Since y is integrable with respect to x in [a,b], for $\epsilon > 0$ there exists a subdivision D of [a,b] such that, if E is a refinement of D, then

$$\left|\int_a^b y(t)dx(t) - S_E(x,y)\right| < \epsilon.$$

Let D' be the subdivision of [c,d] which f maps onto D, and take any refinement E' of D'. Then there corresponds a refinement F of D such that $f(z_i) = t_i$, where t_i belongs to [a,b], and is an endpoint of the subdivision F. Then

$$S_{E'}\{x[f(t)],y[f(t)]\} = S_F(x,y).$$

Therefore,

$$\left|\int_a^b y(t)dx(t) - S_{E'}\{x[f(t)],y[f(t)]\}\right| < \epsilon.$$

Thus y[f(t)] is integrable with respect to x[f(t)]. Then by Theorem 1.16

$$\int_a^b y(t)dx(t) = \int_c^d y[f(t)]dx[f(t)].$$

<u>Corollary 1.16b</u>. Suppose that [a,b] is an interval, [c,d] is an interval, and that y is integrable with respect to x in [a,b]. If g is a continuous non-increasing function such that $g(c) = b$ and $g(d) = a$, then

$$\int_a^b y(t)dx(t) = -\int_c^d y[g(t)]dx[g(t)].$$

The proof is like that of Corollary 1.16a.

CHAPTER II

LANE'S INTEGRAL IN THREE SPACE: $\int_{[a,b;c,d]} f(x,y)dg(x,y)$

<u>Definition 2.1.</u> The statement that f is a real function implies that if t and r are real numbers, then there is just one number c such that, $f(t,r) = c$.

<u>Definition 2.2.</u> A rectangular interval denoted by $[a,b;c,d]$ is a point set such that $a < b$ and $c < d$ and a point, (x,y), belongs to $[a,b;c,d]$ if and only if $a \leq x \leq b$ and $c \leq y \leq d$. Throughout the chapter, R will be used to represent the rectangular interval, i.e., $R = [a,b;c,d]$.

<u>Definition 2.3.</u> The statement that G is the graph of the function y with respect to the functions x and z in the rectangular interval $[a,b;c,d]$, means that P is a point of G if and only if there exist numbers t and r such that, $a \leq t \leq b$, $c \leq r \leq d$ and $x(t,r)$, $z(t,r)$ represent the first two coordinates of P, and $y(t,r)$ is the ordinate of P.

<u>Definition 2.4.</u> The statement that D is a subdivision of the rectangular interval $[a,b;c,d]$ means that, D is a finite set of, (one or more), non-overlapping rectangular intervals $[t_i,t_{i+1};z_j,z_{j+1}]$ covering $[a,b;c,d]$ such that $t_0 = a$, $z_0 = c$, $t_n = b$, and $z_m = d$ where i ranges from 0 to n-1 and j ranges from 0 to m-1.

Definition 2.5. The statement that E is a refinement of D means that E is a subdivision of $[a,b;c,d]$ such that if $[p,q;r,s]$ is a rectangular interval in D, then there is a subset of E which is a subdivision of $[p,q;r,s]$.

Definition 2.6. The statement that the function f is quasi-continuous in the rectangular interval $[a,b;c,d]$ means that if $\epsilon > 0$, then there is a subdivision D of $[a,b;c,d]$ such that if $[p,q;r,w]$ is one of the rectangular intervals in D and the ordered pairs (s,t) and (g,h) belong to the segment $(p,q;r,w)$ or to one of the line segments $(p,q;r,r)$, $(p,q;w,w)$, $(p,p;r,w)$, $(q,q;r,w)$, then

$$|f(s,t) - f(g,h)| < \epsilon.$$

Notation. If x is a function, y is a function, and D is a subdivision of the rectangular interval $[a,b;c,d]$, where $[p,q;r,w]$ represents one of the intervals in D, then $S_D(x,y)$ denotes the number

$$\sum_{\text{All } [p,q;r,w] \in D} 1/4[y(p,r)+y(q,r)+y(p,w)+y(q,w)][x(p,r)-x(q,r)+x(q,w)-x(p,w)].$$

Notice that the expression in the second set of brackets is the x-second difference $\{[x(q,w)-x(p,w)]-[x(q,r)-x(p,r)]\}$.

Theorem 2.1. For numerical work it is often convenient to make use of the fact that

$$S_D(x,y) = 1/4y(r_0,t_0)\cdot[x(r_1,t_1)-x(r_1,t_0)-x(r_0,t_1)+x(r_0,t_0)]$$

$$+\sum_{i=1}^{m-1} 1/4y(r_i,t_0)[x(r_{i+1},t_1)-x(r_{i+1},t_0)-x(r_{i-1},t_1)+x(r_{i-1},t_0)]$$

$$+ 1/4y(r_m,t_0)[x(r_m,t_1)-x(r_m,t_0)-x(r_{m-1},t_1)+x(r_{m-1},t_0)]$$

$$+\sum_{j=1}^{n-1} 1/4y(r_0,t_j)[x(r_1,t_{j+1})-x(r_1,t_{j-1})-x(r_0,t_{j+1})+x(r_0,t_{j-1})]$$

$$+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} 1/4y(r_i,t_j)[x(r_{i+1}t_{j+1})-x(r_{i+1}t_{j-1})-x(r_{i-1}t_{j+1})+x(r_{i-1}t_{j-1})]$$

$$+\sum_{j=1}^{n-1} 1/4y(r_mt_j)[x(r_mt_{j+1})-x(r_mt_{j-1})-x(r_{m-1}t_{j+1})+x(r_{m-1}t_{j-1})]$$

$$+ 1/4y(r_0,t_n)[x(r_1,t_n)-x(r_1,t_{n-1})-x(r_0,t_n)+x(r_0,t_{n-1})]$$

$$+\sum_{i=1}^{m-1} 1/4y(r_it_n)[x(r_{i+1}t_n)-x(r_{i+1}t_{n-1})-x(r_{i-1}t_n)+x(r_{i-1}t_{n-1})]$$

$$+ 1/4y(r_m,t_n)[x(r_m,t_n)-x(r_m,t_{n-1})-x(r_{m-1},t_n)+x(r_{m-1},t_{n-1})].$$

Proof: The proof is omitted because it is merely algebraic manipulations.

Definition 2.7. The statement that the function y is integrable with respect to the function x in the rectangular interval $[a,b;c,d]$ means that if $\epsilon > 0$, then there is a subdivision D of $[a,b;c,d]$ such that

$$|S_D(x,y) - S_E(x,y)| < \epsilon \text{ if E is a refinement of D.}$$

Theorem 2.2. For the function y to be integrable with respect to the function x in the rectangular interval $[a,b;c,d]$, it is necessary and sufficient that there be one and only one number I such that if $\epsilon > 0$, then there is a subdivision D of [a,b;c,d_ such that $|S_E(x,y) - I| < \epsilon$ if E is a refinement of D.

Proof: The proof is omitted because it is a duplicate of the proof of Theorem 1.2.

Definition 2.8. If $\{D_i\}$, $i=1,2,\cdots,n$, is a sequence of subdivisions of $[a,b;c,d]$, where the areas of the maximum intervals of the D_i's tend toward zero, and $[p,q;r,w]$ represents one of the intervals in D, then let

$$S(D_i;f,g) = \sum_{\text{All } [p,q;r,w] \in D_i} f(\xi,\phi)[g(p,r)-g(q,r)+g(q,w)-g(p,w)]$$

where $p \leq \xi \leq q$ and $r \leq \phi \leq w$. Under certain conditions, the sequence $S(D_i;f,g)$ has a limit called the Stieltjes integral.

Theorem 2.3. If the Stieltjes integral exists in the interval $[a,b;c,d]$, then Lane's integral also exists in $[a,b;c,d]$, and they are equal.

Proof: The proof is analogous to the proof of Theorem 1.3 presented in Chapter I.

Remark 2.3. In cases where the Stieltjes integral is evaluated as a double integral, the same is true for Lane's integral.

Definition 2.9. Suppose that x is a function, y is a function, and [a,b;c,d] is a rectangular interval. The statement that I is the integral from a to b and c to d of y with respect to x, which may be written as

$$I = \int_{[a,b;c,d]} y(t,r)dx(t,r), \text{ or } \int_R y(t,r)dx(t,r),$$

means that I is a number and that the statement from Theorem 2.2 is true.

Theorem 2.4. The statement that $I = \int_{[a,b;c,d]} y(t,r)dx(t,r)$ means that $I = -\int_{[b,a;d,c]} y(t,r)dx(t,r)$. Moreover,

$$\int_{[a,a;c,c]} y(t,r)dx(t,r) = 0.$$

Proof: For the case of $\int_{[a,b;c,d]} y(t,r)dx(t,r)$ each approximating sum $S_D(x,y)$ contains x-second-differences of the form $\{[x(t_{i+1}r_{j+1})-x(t_i,r_{j+1})]-[x(t_{i+1},r_j)-x(t_i,r_j)]\}$, but for the case of $\int_{[b,a;d,c]} y(t,r)dx(t,r)$ each approximating sum $S_D(x,y)$ contains x-second-differences of the form

$$\{[x(t_i,r_{j+1}) - x(t_{i+1},r_{j+1})]-[x(t_i,r_j) -x(t_{i+1},r_j)]\}.$$

Their algebraic signs are opposite, thus guaranteeing that $\int_{[a,b;c,d]} y(t,r)dx(t,r)$ and $\int_{[b,a;d,c]} y(t,r)dx(t,r)$ are of opposite sign.

The equation $\int_{[a,a;c,c]}y(t,r)dx(t,r) = 0$ is true because since the rectangle $[a,a;c,c]$ has zero area, each x-second-difference in $S_D(x,y)$ is zero.

Definition 2.10. The statement that the function f is of bounded variation in the interval $[a,b;c,d]$ means that there is a number B such that if D is a subdivision of $[a,b;c,d]$ then

$$\sum_{i=1}^{n} \sum_{j=1}^{m} |f(t_{i+1},r_{j+1}) -f(t_{i+1},r_j)+f(t_i,r_j)-f(t_i,r_{j+1})| \leq B.$$

The least such number B is denoted by $V_{a,c}^{b,d}(f)$, and the above sum will be denoted by $S_D|f,1|$. This is arrived at by considering the variation in the x-direction and then the variation in the y-direction (i.e., by summing the second differences).

Lemma 2.1. Suppose x is a function of bounded variation in the interval $[g,h;u,v]$. If $\epsilon > 0$, then there exists an interval $[g,k;u,r]$, which is a subset of $[g,h;u,v]$ such that if $[g,w;u.t]$ is a subset of $[g,k;u,r]$, then

$$V_{g,u}^{k,r}(x) - V_{g,u}^{w,t}(x) < \epsilon.$$

Proof: Suppose $\epsilon > 0$. Let $s_1 = h$ and $z_1 = v$ and then for $n > 1$, let $s_n = \frac{g + s_{n-1}}{2}$ and $z_n = \frac{u + z_{n-1}}{2}$. Then $V_{g,u}^{s_n, z_n}(x)$ exists and

$$V_{g,u}^{s_{n+1}, z_{n+1}}(x) \leq V_{g,u}^{s_n, z_n}(x).$$

Let L denote the greatest lower bound of the numbers $V_{g,u}^{s_n, z_n}(x)$.

There are positive integers p and q such that

$V_{g,u}^{s_{p+d}, z_{q+d}}(x) - L < 1/2\ \epsilon$ if d is a non-negative integer.

If $[g,w;u,t]$ is a subset of $[g,s_p;u,z_q]$, then there is a positive integer d such that $s_{p+d} < w < s_p$ and $z_{q+d} < t < z_q$

and

$$V_{g,u}^{s_{p+d}, z_{q+d}}(x) \leq V_{g,u}^{w,t}(x) \leq V_{g,u}^{s_n, z_n}(x)$$

and also

$$V_{g,u}^{s_{p+d}, z_{q+d}}(x) - L < 1/2\ \epsilon.$$

Hence

$$V_{g,u}^{w,t}(x) - L < 1/2\ \epsilon,$$

Thus

$$V_{g,u}^{s_p, z_q}(x) - V_{g,u}^{w,t}(x) < \epsilon,$$

and

$[g,s_p;u,z_q]$ qualifies as $[g,k;u,r]$.

Theorem 2.5. If y is a quasi-continuous function in the rectangular interval $[a,b;c,d]$ and x is a function of bounded variation in $[a,b;c,d]$, then y is integrable with respect to x in the interval $[a,b;c,d]$.

<u>Proof:</u> If $V_{a,c}^{b,d}(x) = 0$, then the proof is trivial. Suppose that $V_{a,c}^{b,d}(x) > 0$. There is a positive number M such that $|y(t,r)| \leq M$, if t, r belong to $[a,b;c,d]$.

For $\epsilon > 0$, there exists a subdivision D of $[a,b;c,d]$ such that if the ordered pairs (s,t) and (g,h) meet the requirements of Definition 2.6, then $|y(s,t) - y(g,h)| < \epsilon$, because y is quasi-continuous. Let n denote the number of intervals in D.

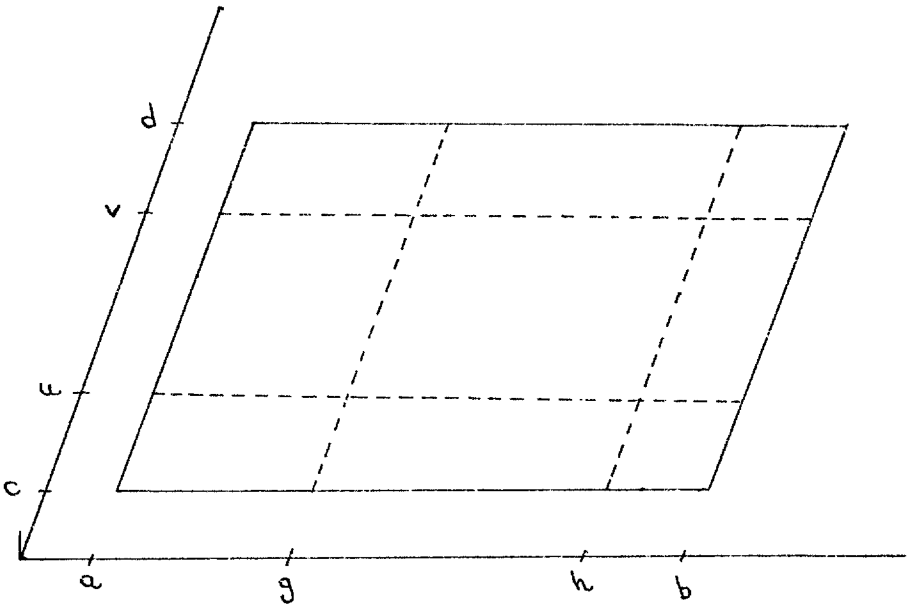
Let D′ be a refinement of D obtained as follows. In the interior of each interval $[g,h;u,v]$ of D select four numbers k and ℓ, and r and s so that $g < k < \ell < h$ and $u < r < s < v$ and (in conformance with Lemma 2.1) it will be true that if $g < w \leq k$ and $u < t \leq r$ and $\ell \leq p < h$ and $s \leq z < v$ then

$$V_{g,u}^{k,r}(x) - V_{g,u}^{w,t}(x) < \epsilon \text{ and}$$

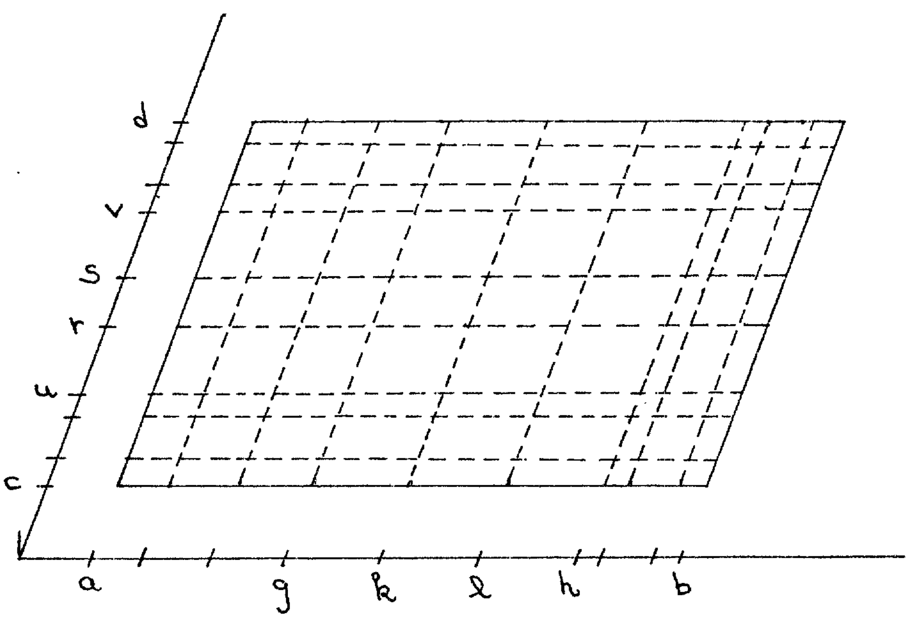
$$V_{\ell,u}^{h,r}(x) - V_{p,u}^{h,t}(x) < \epsilon \text{ and } V_{g,s}^{k,v}(x) - V_{g,z}^{w,v}(x) < \epsilon \text{ and}$$

$V_{\ell,s}^{h,v}(x) - V_{p,z}^{h,v}(x) < \epsilon$. See Figure 2, Page 38.

Suppose E is any refinement of D′. For each interval $[g,h;u,v]$ of D there are four intervals



The subdivision D



The subdivision D′

Figure 2 - Subdivisions D and D′ for Theorem 2.5

$[g,w;u,t]$, $[g,w;z,v]$, $[p,h;u,t]$ and $[p,h;z,v]$ of E, where $g < w \leq k, \ell \leq p < h$, $u < t \leq r$ and $s \leq z < v$. Let e_1 denote the four-member set

$$\{[g,w;u,t], [g,w;z,v], [p,h;u,t], [p,h;z,v]\}.$$

Let e_2 denote the set of all the other intervals of E which lie in $[g,k;u,r]$, $[\ell,h;u,r]$, $[g,k;s,v]$ or $[\ell,h;s,v]$. Let e_3 denote the set of all intervals of E which lie in $[k,\ell;u,r]$, $[k,\ell;s,v]$, $[g,k;r,s]$, or $[\ell,h;r,s]$. Let e_4 denote the set of all intervals of E which are subsets of $[k,\ell;r,s]$. Let E_1 denote the union of all such sets e_1, and let E_2 denote the union of all such sets e_2, and let E_3 denote the union of all such sets e_3 and let E_4 denote the union of all such sets e_4(for all such intervals $[g,h;u,v]$ of D). See Figure 3, Page 40.

Now the terms of $S_{D'}(x,y)$ which are associated with any interval $[g,h;u,v]$ of D are

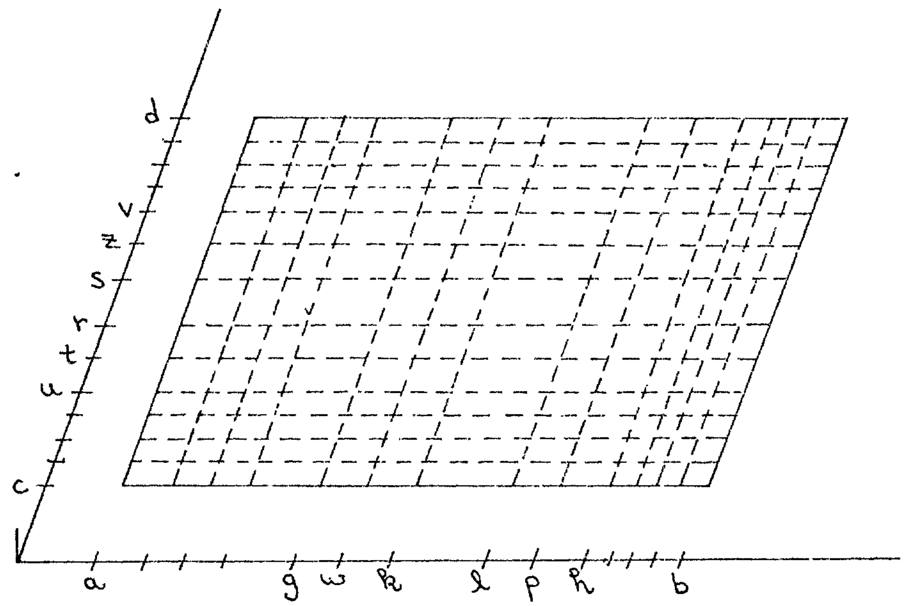
$$\frac{y(g,u)+g(k,u)+y(g,r)+y(k,r)}{4}[x(k,r)-x(g,r)+x(g,u)-x(k,r)]$$

$$+\frac{y(\ell,u)+y(h,u)+y(\ell,r)+y(h,r)}{4}[x(h,r)-x(\ell,r)+x(\ell,u)-x(h,u)]$$

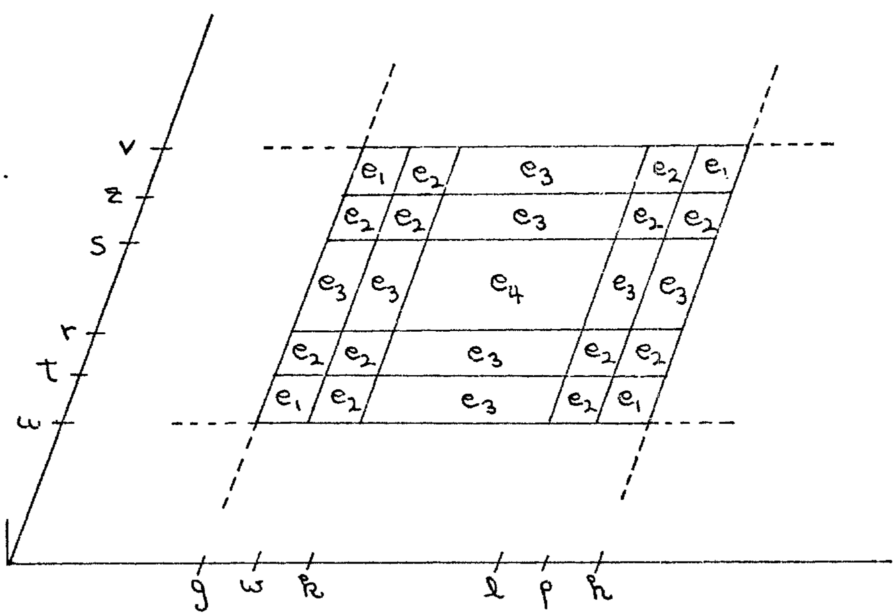
$$+\frac{y(k,u)+y(\ell,u)+y(k,r)+y(\ell,r)}{4}[x(\ell,r)-x(k,r)+x(k,u)-x(\ell,u)]$$

$$+\frac{y(g,r)+y(k,r)+y(g,s)+y(k,s)}{4}[x(k,s)-x(g,s)+x(g,r)-x(k,r)]$$

$$+\frac{y(p,r)+y(h,r)+y(\ell,s)+y(h,s)}{4}[x(h,s)-x(\ell,s)+x(\ell,r)-x(h,r)]$$



The Subdivision E



The Subinterval [g,h;u,v]

Figure 3 - Subdivision E and Subinterval [g,h;u,v] for Theorem 2.5

$$+ \frac{y(k,r)+y(\ell,r)+y(k,s)+y(\ell,s)}{4}[x(\ell,s)-x(h,s)+x(k,r)-x(\ell,r)]$$

$$+ \frac{y(g,s)+y(k,s)+y(g,v)+y(k,v)}{4}[x(k,v)-x(g,v)+x(g,s)-x(k,s)]$$

$$+ \frac{y(\ell,s)+y(h,s)+y(\ell,v)+y(h,v)}{4}[x(h,v)-x(\ell,v)+x(\ell,s)-x(h,s)]$$

$$+ \frac{y(k,s)+y(\ell,s)+y(k,v)+y(\ell,v)}{4}[x(\ell,v) - x(k,v)+x(k,s)-x(\ell,s)].$$

Since $g < w \leq k$ and $u < t \leq r$ and $\ell \leq p < h$ and $s \leq z < v$, add and subtract an $x(w,t)$, $x(w,u)$ and $x(g,t)$ in the first set of square brackets and add and subtract an $x(p,u)$, $x(p,t)$ and $x(h,t)$ in the second set of square brackets. Also add and subtract an $x(g,z)$, $x(w,z)$ and $x(w,v)$ in the seventh set of square brackets and add and subtract an $x(p,z)$, $x(h,z)$ and $x(p,v)$ in the eighth set of square brackets. Then expand this into the twenty-one terms,

$$\frac{y(g,u)+Y(k,u)+y(g,r)+y(k,r)}{4}[x(w,t)-x(g,t)+x(g,u)-x(w,u)]$$

$$+ \frac{y(g,u)+y(k,u)+y(g,r)+y(k,r)}{4}[x(k,t)-x(w,t)+x(w,u)-x(k,u)]$$

$$+ \frac{y(g,u)+y(k,u)+y(g,r)+y(k,r)}{4}[x(k,r)-x(w,r)+x(w,t)-x(k,t)]$$

$$+ \frac{y(g,u)+y(k,u)+y(g,r)+y(k,r)}{4}[x(w,r)-x(g,r)+x(g,t)-x(w,t)]$$

$$+ \frac{y(\ell,u)+y(h,u)+y(\ell,r)+y(h,r)}{4}[x(h,t)-x(p,t)+x(p,u)-x(h,u)]$$

$$+\frac{y(\ell,u)+y(h,u)+y(\ell,r)+y(h,r)}{4}[x(p,t)-x(\ell,t)+x(\ell,u)-x(p,u)]$$

$$+\frac{y(\ell,u)+y(h,u)+y(\ell,r)+y(h,r)}{4}[x(p,r)-x(\ell,r)+x(\ell,t)-x(p,t)]$$

$$+\frac{y(\ell,u)+y(h,u)+y(\ell,r)+y(h,r)}{4}[x(h,r)-x(p,r)+x(p,t)-x(h,t)]$$

$$+\frac{y(k,u)+y(\ell,u)+y(k,r)+y(\ell,r)}{4}[x(\ell,r)-x(k,r)+x(k,u)-x(\ell,u)]$$

$$+\frac{y(g,r)+y(k,r)+y(g,s)+y(k,s)}{4}[x(k,s)-x(g,s)+x(g,r)-x(k,r)]$$

$$+\frac{y(k,r)+y(\ell,r)+y(k,s)+y(\ell,s)}{4}[x(\ell,s)-x(k,s)+x(k,r)-x(\ell,r)]$$

$$+\frac{y(\ell,r)+y(h,r)+y(\ell,s)+y(h,s)}{4}[x(h,s)-x(\ell,s)+x(\ell,r)-x(h,r)]$$

$$+\frac{y(g,s)+y(k,s)+y(g,v)+y(k,v)}{4}[x(w,v)-x(g,v)+x(g,z)-x(w,z)]$$

$$+\frac{y(g,s)+y(k,s)+y(g,v)+y(k,v)}{4}[x(k,v)-x(w,v)+x(w,z)-x(k,z)]$$

$$+\frac{y(g,s)+y(k,s)+y(g,v)+y(k,v)}{4}[x(w,z)-x(g,z)+x(g,s)-x(w,s)]$$

$$+\frac{y(g,s)+y(k,s)+y(g,v)+y(k,v)}{4}[x(k,z)-x(w,z)+x(w,s)-x(k,s)]$$

$$+\frac{y(\ell,s)+y(h,s)+y(\ell,v)+y(h,v)}{4}[x(h,v)-x(h,p)+x(p,z)-x(h,z)]$$

$$+\frac{y(\ell,s)+y(h,s)+y(\ell,v)+y(h,v)}{4}[x(p,v)-x(\ell,v)+x(\ell,z)-x(p,z)]$$

$$+\frac{y(\ell,s)+y(h,s)+y(\ell,v)+y(h,v)}{4}[x(p,z)-x(\ell,z)+x(\ell,s)-x(p,s)]$$

$$+\ \frac{y(\ell,s)+y(h,s)+y(\ell,v)+y(h,v)}{4}[x(h,z)-x(p,z)+x(p,s)-x(h,s)]$$

$$+\ \frac{y(k,s)+y(\ell,s)+y(k,v)+y(\ell,v)}{4}[x(\ell,v)-x(k,v)+x(k,s)-x(\ell,s)]$$

Let $S_{D_1'}(x,y)$ denote the set of all such first, fifth, thirteenth and seventeenth terms (for all intervals $[g,h;u,v]$ of D). Let $S_{D_2'}$ denote the set of all such second, third, fourth, sixth, seventh, eighth, fourteenth, fifteenth, sixteenth, eighteenth, nineteenth and twentieth terms. Let $S_{D_3'}(x,y)$ denote the set of all such ninth, tenth, twelfth and twenty-first terms. Let $S_{D_4'}(x,y)$ denote the set of all such eleventh terms. Notice that

$$S_{D'}(x,y) = S_{D_1'}(x,y) + S_{D_2'}(x,y) + S_{D_3'}(x,y) + S_{D_4'}(x,y).$$

$$|S_{D'}(x,y) - S_E(x,y)| =$$

$$|\{S_{D_1'}(x,y)+S_{D_2'}(x,y)+S_{D_3'}(x,y)+S_{D_4'}(x,y)\}-\{S_{E_1}(x,y)+S_{E_2}(x,y)+S_{E_3}(x,y)+S_{E_4}(x,y)\}|$$

$$\leq |S_{D_1'}(x,y)-S_{E_1}(x,y)|+|S_{D_2'}(x,y)-S_{E_2}(x,y)|+|S_{D_3'}(x,y)-S_{E_3}(x,y)|$$

$$+\ |S_{D_4'}(x,y) - S_{E_4}(x,y)|.$$

Let this last expression be denoted by $|A| + |B| + |C| + |F|$.

Now, for each interval $[g,h;u,v]$ of D, the terms

$$[\{\frac{y(g,u)+y(k,u)+y(g,r)+y(k,r)}{4}[x(w,t)-x(g,t)+x(g,u)-x(w,u)]$$

$$+\ \frac{y(\ell,u) + y(h,u)+y(\ell,r)+y(h,r)}{4}[x(h,t)-x(p,t)+x(p,u)-x(h,u)]$$

$$+ \frac{y(g,s)+y(k,s)+y(g,v)+y(k,v)}{4}[x(w,v)-x(g,v)+x(g,z)-x(w,z)]$$

$$+ \frac{y(\ell,s)+y(h,s)+y(\ell,v)+y(h,v)}{4}[x(h,v)-x(h,p)+p,z)-x(h,z)]\}$$

$$- \{\frac{y(g,u)+y(w,u)+y(w,t)+y(g,t)}{4}[x(w,t)-x(g,t)+x(g,u)-x(w,u)]$$

$$+ \frac{y(h,t)+y(p,t)+y(p,u)+y(h,u)}{4}[x(h,t)-x(p,t)+x(p,u)-x(h,u)]$$

$$+ \frac{y(w,v)+y(g,v)+y(g,z)+y(w,z)}{4}[x(w,v)-x(g,v)+x(g,z)-x(w,z)]$$

$$+ \frac{y(h,v)+y(h,p)+y(p,z)-y(h,z)}{4}[x(h,v)-x(h,p)+x(p,z)-x(h,z)]\}]$$

appear in $|A|$. These net out and regroup to the twelve terms

$$\frac{y(k,u)-y(w,u)}{4}[x(w,t)-x(g,t)+x(g,u)-x(w,u)]$$

$$+ \frac{y(g,r)-y(g,t)}{4}[x(w,t)-x(g,t)+x(g,u)-x(w,u)]$$

$$+ \frac{y(k,r)-y(w,t)}{4}[x(w,t)-x(g,t)+x(g,u)-x(w,u)]$$

$$+ \frac{y(\ell,u)-y(p,u)}{4}[x(h,t)-x(p,t)+x(p,u)-x(h,u)]$$

$$+ \frac{y(\ell,r)-y(p,t)}{4}[x(h,t)-x(p,t)+x(p,u)-x(h,u)]$$

$$+ \frac{y(h,r)-y(h,t)}{4}[x(h,t)-x(p,t)+x(p,u)-x(h,u)]$$

$$\frac{y(g,s)-y(g,z)}{4}[x(w,v)-x(g,v)+x(g,z)-x(w,z)]$$

$$+\ \frac{y(k,s)-y(g,z)}{4}[x(w,v)-x(g,v)+x(g,z)-x(w,z)]$$

$$+\ \frac{y(k,v)-y(w,z)}{4}[x(w,v)-x(g,v)+x(g,z)-x(w,z)]$$

$$+\ \frac{y(\ell,s)-y(h,p)}{4}[x(h,v)-x(h,p)+x(p,z)-x(h,z)]$$

$$+\ \frac{y(h,s)-y(p,z)}{4}[x(h,v)-x(h,p)+x(p,z)-x(h,z)]$$

$$+\ \frac{y(\ell,v)-y(h,z)}{4}[x(h,v)-x(h,p)+x(p,z)-x(h,z)]$$

But this is less than or equal to

$$\frac{\epsilon}{2}[x(w,t)-x(g,t)+x(g,u)-x(w,u)]+\frac{\epsilon}{2}[x(w,t)-x(g,t)+x(g,u)-x(w,u)]$$

$$+\ \frac{\epsilon}{2}[x(w,t)-x(g,t)+x(g,u)-x(w,u)]+\frac{\epsilon}{2}[x(w,v)-x(g,v)+x(g,z)-x(w,z)]$$

$$+\ \frac{\epsilon}{2}[x(w,v)-x(g,v)+x(g,z)-x(w,z)]+\frac{\epsilon}{2}[x(w,v)-x(g,v)+x(g,z)-x(w,z)]$$

$$+\ \frac{\epsilon}{2}[x(h,v)-x(h,p)+x(p,z)-x(h,z)]+\frac{\epsilon}{2}[x(h,v)-x(h,p)+x(p,z)-x(h,z)]$$

$$+\ \frac{\epsilon}{2}[x(h,v)-x(h,p)+x(p,z)-x(h,z)]+\frac{\epsilon}{2}[x(h,t)-x(p,t)+x(p,u)-x(h,u)]$$

$$+\ \frac{\epsilon}{2}[x(h,t)-x(p,t)+x(p,u)-x(h,u)]+\frac{\epsilon}{2}[x(h,t)-x(p,t)+x(p,u)-x(h,u)],$$

because the points associated with the function y are all interior points of $[g,h;u,v]$, an interval of the subdivision D which was chosen as a response to ϵ in accordance with the quasi-continuous property for y. Factoring out $\frac{\epsilon}{2}$ leaves an expression which is less than the variation of the function x over $[g,w;z,v]$, $[g,w;s,z]$, $[w,k;z,v]$, $[w,k;s,z]$, $[\ell,p;z,v]$, $[\ell,p;s,z]$, $[p,h;z,v]$, $[p,h;s,v]$, $[g,w;u,t]$, $[g,w;t,r]$, $[w,k;u,t]$, $[w,k;t,r]$, $[\ell,p;u,t]$, $[\ell,p;t,r]$, $[p,h;u,t]$ and $[p,h;t,r]$.

Hence

$$|A| < \frac{\epsilon}{2}\cdot \text{(variation of x over } E_1).$$

Now in $|B|$ above, if each $\frac{y(\ ) + y(\ ) + y(\ ) + y(\ )}{4}$ is replaced by M, the result is an expression which is greater than or equal to $|B|$. Factoring out each M gives $|B| \leq M \cdot$ (variation of x over E_2-variation of x over D_2'),

$$|B| \leq M \cdot \text{(variation of x over } E_2)$$
$$|B| < M \cdot 4n\epsilon.$$

This result is obtained by using Lemma 2.1 and the fact that there are n intervals in D.

Now in $|C|$. Each interval $[k,\ell,s,z]$, $[k,\ell;r,s]$, $[g,k;r,s]$ and $[\ell,h;r,s]$ is associated with only one term in $S_{D_3'}(x,y)$ but it may be associated with more than one term in $S_{E_3}(x,y)$, because each interval of D' may have some endpoints of E in its interior, and these, of course, would not be endpoints of D'. Now consider one interval, $[k,\ell;u,r]$ of D_3. If $[k,\ell;u,r]$ has some endpoints of E in its interior, let these extra end points of E in $[k,\ell;u,r]$'s interior be denoted by

(t_1,r_1), $(t_2,r_2)\cdots,(t_i,r_i)$. Merely add and subtract $x(t,r)$'s inside the square brackets of

$\underline{\dfrac{y(k,u)+y(\ell,u)+y(k,r)+y(\ell,r)}{4}}[x(\ell,r)-x(k,r)+x(k,u)-x(\ell,u)]$ the appropriate number of times and then distribute $\underline{\dfrac{y(k,\ell)+y(\ell,u)+y(k,r)+y(\ell,r)}{4}}$ across the series. If this is performed for each interval of D_3', then $S_{D_3}'(x,y)$ has as many terms as $S_{E_3}(x,y)$. Now in $|S_{D_3'}(x,y)-S_{E_3}(x,y)|$ regroup terms according to alike x-second differences. Now each abscissa is an interior point of an interval in D, a subdivision chosen as a response to ϵ in accordance with the quasi-continuous property of y. Hence each grouping of y ordinates which is associated with like x-second differences is less than ϵ. After factoring out ϵ's

$$|C| < \epsilon \cdot (\text{variation of } x \text{ over } E_3).$$

Now in $|F|$, a similar argument for each interval $[k,\ell;r,s]$ in D_4' as presented in the above argument for $|C|$ will hold true. Therefore,

$$|F| < \epsilon \cdot (\text{variation over } E_4)$$

Then $|A| + |B| + |C| + |F| < \frac{\epsilon}{2} \cdot (\text{variation of } x \text{ over } E_1) + 4Mn\epsilon$

$$+ \epsilon \cdot (\text{variation of } x \text{ over } E_3)$$

$$+ \epsilon \cdot (\text{variation of } x \text{ over } E_4)$$

$$< \epsilon \cdot (\text{variation of } x \text{ over } E_1, E_2 \text{ and } E_4 + 4Mn)$$

$$|S_{D'}(x,y)-S_E(x,y)| < \epsilon \cdot \{V_{a,c}^{b,d}(x) + 4Mn\}.$$

So to meet the challenge of any positive number ϵ_1 we consider ϵ as being $\dfrac{\epsilon_1}{V_{a,c}^{b,d}(x) + 4Mn}$. This completes the proof of Theorem 2.5.

<u>Theorem 2.6.</u> If y is integrable with respect to x in [a,b;c,d], and k is a number, then

$$\int_{[a,b;c,d]} y(t,r)d[x(t,r)+k] = \int_{[a,b;c,d]} y(t,r)dx(t,r).$$

<u>Proof</u>: The proof is analogous to the proof of Theorem 1.6 already presented.

<u>Theorem 2.7.</u> If y is integrable with respect to x in [a,b;c,d], and k is a number, then

$$\int_{[a,b;c,d]} [ky(t,r)dx(t,r) = k\int_{[a,b;c,d]} y(t,r)dx(t,r).$$

<u>Proof</u>: The proof is analogous to the proof of Theorem 1.7.

<u>Theorem 2.8.</u> Suppose that each of y and y_1 is integrable with respect to x in the rectangular interval [a,b;c,d]. Then $\int_{[a,b;c,d]} [y(t,r)+y_1(t,r)]dx(t,r)$ exists and

$$\int_{[a,b;c,d]} [y(t,r)+y_1(t,r)]dx(t,r) = \int_{[a,b;c,d]} y(t,r)dx(t,r) + \int_{[a,b;c,d]} y_1(t,r)dx(t,r).$$

<u>Proof</u>: The proof is omitted, because it is a duplicate of the proof of Theorem 1.8.

Theorem 2.9. Suppose that y is integrable with respect to x in $[a,b;c,d]$, and if S_1 and S_2 are rectangular intervals such that $S_1 \cup S_2 = [a,b;c,d]$, and $S_1 \cap S_2 = \phi$, then y is integrable with respect to x in S_1 and in S_2; moreover

$$\int_{[a,b;c,d]} y(t,r)dx(t,r) = \int_{S_1} y(t,r)dx(t,r) + \int_{S_2} y(t,r)dx(t,r).$$

Proof: The proof is omitted because it parallels the proof of Theorem 1.9.

Theorem 2.10. If $\int_{[a,b;c,d]} y(t,r)dx(t,r)$ exists, then $\int_{[a,b;c,d]} x(t,r)dy(t,r)$ exists and

$$\begin{aligned}\int_{[a,b;c,d]} x(t,r)dy(t,r) = &\int_{[a,b;c,d]} y(t,r)dx(t,r)\\ &+ y(b,d)x(b,d) - y(b,c)x(b,c)\\ &+ y(a,c)x(a,c) - y(a,d)x(a,d)\\ &- \int_c^d y(b,r)dx(b,r)\\ &+ \int_c^d y(a,r)dx(a,r)\\ &- \int_a^b y(t,d)dx(t,d)\\ &+ \int_a^b y(t,c)dx(t,c).\end{aligned}$$

Proof: For $\epsilon > 0$, then $\frac{\epsilon}{6} > 0$ and there exists a subdivision D_1 of $[a,b;c,d]$ such that if E_1 is a refinement of D_1 then $|S_{E_1}(x,y) - \int_R y(t,r)dx(t,r)| < \frac{\epsilon}{6}$. Similarly there exists subdivisions D_2 and D_3 of $[c,d]$ and D_4 and D_5 of $[a,b]$ and their respective refinements E_2, E_3, E_4 and E_5 such that

$$| S_{E_2}(x(b,r),y(b,r)) - \int_c^d y(b,r)dx(b,r)| < \frac{\epsilon}{6},$$

$$|S_{E_3}(x(a,r),y(a,r)) - \int_c^d y(a,r)dx(a,r)| < \frac{\epsilon}{6},$$

$$|S_{E_4}(x(t,d),y(t,d)) - \int_a^b y(t,d)dx(t,d)| < \frac{\epsilon}{6} \text{ and}$$

$$|S_{E_5}(x(t,c),y(t,c)) - \int_a^b y(t,c)dx(t,c)| < \frac{\epsilon}{6}.$$

Let D be the composite subdivision formed by taking all the endpoints of all the subdivisions D_1, D_2, D_3, D_4, and D_5. Then if E is a refinement of D, then the above five inequalities are each true for E and D. Let $y(t_i, r_j) = y_{ij}$. Now consider $S_E(x,y) - S_E(y,x)$, which by definition is

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} 1/4[(y_{i+1\,j+1}+y_{ij})+(y_{i+1j}+y_{ij+1})][(x_{i+1j+1}+x_{ij})-(x_{i+1j}+x_{ij+1})]$$

$$- \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} 1/4[(x_{i+1j+1}+x_{ij})+(x_{i+1j}+x_{ij+1})][(y_{i+1j+1}+y_{ij})-(y_{i+1j}+y_{ij+1})].$$

After multiplying out, simplifying and regrouping, the above becomes

$$1/2\sum_{j=0}^{n-1}\{-\sum_{i=0}^{m-1} y_{ij}x_{ij+1}+\sum_{i=0}^{m-1} y_{i+1j}x_{i+1j+1}\}+1/2\sum_{j=0}^{n-1}\{\sum_{i=0}^{m-1} y_{ij+1}x_{ij}-\sum_{i=0}^{m-1} y_{i+1j+1}x_{i+1j}$$

$$+\ 1/2\sum_{i=0}^{m-1}\{\sum_{j=0}^{n-1} y_{i+1j}x_{ij}-\sum_{j=0}^{n-1} y_{i+1j+1}x_{ij+1}\}+1/2\sum_{i=0}^{m-1}\{-\sum_{j=0}^{n-1} y_{ij}x_{i+1j}+\sum_{j=0}^{n-1} y_{ij+1}x_{i+1j+1}\}.$$

Perform the following algebra for each set of braces: in the first sigma sign segregate out the lowest-subscripted term (so that only the $\sum_{i \text{ or } j=1}^{m-1 \text{ or } n-1}$ terms remain together) and in the second sigma sign segregate out the highest-subscripted term (so that only the $\sum_{i \text{ or } j=0}^{m-2 \text{ or } n-2}$ terms remain together); then change subscripts and limits of summation so that each is being summed over the range from i(or j) = 1 to i (or j) = m-1(or n-1). Now all summations inside the braces cancel and leave

$$1/2\sum_{j=0}^{n-1}\{-y_{oj}x_{oj+1}+y_{mj}x_{mj+1}\}+1/2\sum_{j=0}^{n-1}\{y_{oj+1}x_{oj}-y_{mj+1}x_{mj}\}$$

$$+\ 1/2\sum_{i=0}^{m-1}\{y_{i+1o}x_{io}-y_{i+1n}x_{in}\}+1/2\sum_{i=0}^{m-1}\{-y_{io}x_{i+1o}+y_{in}x_{i+1n}\}.$$

Applying the segregation technique once again, gives

$$[1/2(-y_{oo}x_{o1}-\sum_{j=1}^{n-1} y_{oj}x_{oj+1})+1/2(\sum_{j=o}^{n-2} y_{oj+1}x_{oj}+y_{on}x_{om-1})]$$

$$+\ [1/2(y_{mo}x_{m1}+\sum_{j=1}^{n-1} y_{mj}\,x_{mj+1})-1/2(\sum_{j=o}^{n-2} y_{mj+1}x_{mj}+y_{mn}x_{mn-1})]$$

m-1

$$+[1/2(y_{on}x_{1n}+\sum_{i=1}^{m-1} y_{in}\ x_{i+1n})-1/2(\sum_{i=o}^{m-2} y_{i+1n}x_n+y_{mn}x_{m-1n})].$$

In each set of brackets do the following: change subscripts so that both sigma's are summing over the range 1 to m-1(or n-1). This yields:

$$-[\frac{y_{oo}}{2}(x_{o1}-x_{oo})+\sum_{j=1}^{n-1} \frac{y_{oj}}{2}(x_{oj+1}-x_{oj-1}) + \frac{y_{on}}{2}(x_{on} - x_{on-1})]$$

$$+[\frac{y_{mo}}{2}(x_{m1}- x_{mo}) + \sum_{j=1}^{n-1} \frac{y_{mj}}{2}(x_{mj+1}-x_{mj-1}) + \frac{y_{mn}}{2}(x_{mn} - x_{mn-1})]$$

$$-[\frac{y_{oo}}{2}(x_{1o}-x_{oo}) + \sum_{i=1}^{m-1} \frac{y_{io}}{2}(x_{i+1o} - x_{i-1o}) + \frac{y_{mo}}{2}(x_{mo} - x_{m-1o})]$$

$$+[\frac{y_{on}}{2}(x_{1n} - x_{on}) + \sum_{i=1}^{m-1} \frac{y_{in}}{2}(x_{i+1n} - x_{i-1n}) + \frac{y_{mn}}{2}(x_{mn} - x_{m-1n})]$$

$$-\frac{y_{oo}}{2} x_{oo} + \frac{y_{on}}{2} x_{on} + \frac{y_{mo}}{2} x_{mo} - \frac{y_{mn}}{2} x_{mn} - \frac{y_{oo}}{2} x_{oo}$$

$$+ \frac{y_{mo}}{2} x_{mo} + \frac{y_{on}}{2} x_{on} - \frac{y_{mn}}{2} x_{mn}.$$

And by Theorem 1.1, the above equals

$$- [S_E(x(t_o,r),y(t_o,r)] + [S_E(x(t_m,r)] - [S_E(x(t,r_o),y(t,r_o))]$$

$$+ [S_E(x(t,r_m),y(t,r_m)] - [(y_{mn}x_{mn}-y_{mo}x_{mo})-(y_{on}x_{on}-y_{oo}x_{oo})].$$

But $y_{oo} = y(a,c)$, and $x_{mn} = x(b,d)$ and so forth, thus

$$S_E(x,y) - S_E(y,x) = -y(b,d)\ x\ (b,d) + y(b,c)x(b,c)-y(a,c)x(a,c)$$

$$+ y(a,d)x(a,d) - [S_E(x(a,r),y(a,r)] + [S_E(x(b,r),y(b,r))]$$

$$- [S_E(x(t,c).y(t,c))] + [S_E(x(t,d),y(t,d))].$$ Eventually this equation will be used to accomplish a substitution for $S_E(y,x)$ in a later expression.

Adding the original five inequalities together with

$$|[y(b,d)x(b,d) + y(b,c)x(b,c)-y(a,c)x(a,c)+y(a,d)x(a,d)]$$

$$-[y(b,d)x(b,d)+y(b,c)x(b,c)-y(a,c)x(a,c)+y(a,d)x(a,d)] < \frac{\epsilon}{6}$$

and then making use of the triangle inequality to remove absolute value bars, and then rearranging terms yields

$$|\{\int_R y(t,r)dx(t,r)-\int_c^d y(b,r)dx(b,r) + \int_a^b y(t,d)dx(t,d)$$

$$+ \int_a^b y(t,c)dx(t,c) + y(b,d)x(b,d)[y(b,c)x(b,c)+y(a,c)x(a,c)-y(a,d)x(a,d)$$

$$-\{y(b,d)x(b,d)-y(b,c)x(b,c)+y(a,c)x(a,c)-y(a,d)x(a,d)$$

$$-[S_E(x(a,r),y(a,r)] + [S_E(x(b,r),y(b,r))] - [S_E(x(t,c),y(t,c))]$$

$$+ [S_E(x(t,d),y(t,d))] - S_E(x,y)\}| < \epsilon.$$

But the expression in the second set of braces can be replaced by $-S_E(y,x)$. Hence there exists a number I (the contents of the first set of braces) such that if $\epsilon > 0$ then there exists a subdivision D of [a,b;c,d] such that for any refinement E of D, $|I-S_E(y,x)| < \epsilon$. Thus, according to Theorem 2.2, x is integrable with respect to y on [a,b;c,d] and $\int_{[a,b;c,d]} x(t,r)dy(t,r)$ is the number I.

Theorem 2.11. Suppose that y is integrable with respect to x in the interval [a,b;c,d]. If x is of bounded variation in [a,b;c,d], and $|y(t,r)| \leq M$ for each number t,r in [a,b;c,d], then

$$\left|\int_{[a,b;c,d]} y(t,r)dx(t,r)\right| \leq M \cdot V_{a,c}^{b,d}(x).$$

Proof: The proof is omitted since it parallels the proof of Theorem 1.11.

Theorem 2.12. If x is a function and [a,b;c,d] is a rectangular interval, then

$$\int_{[a,b;c,d]} 1dx(t,r) = x(a,c) - x(a,d) + x(b,d) - x(b,c).$$

Proof: The proof is analogous to that for Theorem 1.12.

Definition 2.11. The statement that the function f has a second alternate partial derivative, f'', in the interval [a,b;c,d] means that if $a < t < b$ and $c < r < d$, then there is a number $f''(t,r)$ such that the following statement is true:

If $\epsilon > 0$, then there is a segment $(p,q;g,h)$ containing (t,r) such that, if (e,k) is another ordered number pair in $(p,q;g,h)$ and in $[a,b;c,d]$ then

$$\left| \frac{\frac{f(e,k)-f(t.k)}{[e-t]} - \frac{f(e,r)-f(t,r)}{[e-t]}}{[k-r]} - f''(e,k) \right| < \epsilon.$$

Remark 2.11. The above notation simplifies to

$$\left| \frac{f(e,k)-f(t,k)+f(t,r)-f(e,r)}{[e-t]\cdot[k-r]} - f''(e,k) \right| < \epsilon.$$

Definition 2.12. The statement that the function f has a continuous second alternate partial derivative, f'', in the interval $[a,b;c,d]$ means that, f has a second alternate partial derivative, f'', in $[a,b;c,d]$, and if $\epsilon > 0$, then there is a subdivision D of $[a,b;c,d]$ which has the following property. If $[p,q;u,v]$ is an interval in D and (t,r) and (t_1,r_1) are points in $[p,q;u,v]$, then $|f''(t,r)-f''(t_1,r_1)| < \epsilon$.

Remark 2.12. Clearly, if E is any refinement of D (in Definition 2.12), then E also has that property.

Theorem 2.13. If the function f has a continuous second alternate partial derivative, f'', in the interval $[a,b;c,d]$, and if $\epsilon > 0$, then there is a subdivision D of $[a,b;c,d]$, such that if (t_1,r_1) and (t_2,r_2) are in one of the intervals of D then

$$\left| \frac{f(t_1,r_1)+f(t_2,r_2)-f(t_1,r_2)-f(t_2,r_1)}{[t_1 - t_2][r_1 - r_2]} - f''(t_1,r_1) \right| < \epsilon.$$

Proof: The proof is omitted since it is analogous to the proof of Theorem 1.14.

Theorem 2.14. Suppose x has a continuous second alternate partial derivative, x'', in the interval $[a,b;c,d]$, and that there is a number M, such that $|y(t,r)| < M$ if $a \leq t \leq b$ and $c \leq r \leq d$. If

$$\int_R y(t,r)dx(t,r) = c_1 \text{ or } \int_R y(t,r)x''(t,r)d\,[\alpha\cdot\beta](t,r) = c_2$$

where α is the x-identity plane and β is the y-identity plane, then

$$\int_R y(t,r)dx(t,r) = \int_R y(t,r)x''(t,r)d[\alpha\cdot\beta](t,r).$$

Proof: Since x has a continuous second alternate partial derivative, then by Theorem 2.13 for $\epsilon > 0$, there exists a subdivision D of $[a,b;c,d]$, such that if $[g,h;u,v]$ is an interval of D, then

$$\left|\frac{x(g,u)-x(h,u)+x(h,v)-x(g,v)}{[h-g][v-u]} - x''(g,u)\right| < \epsilon_1$$

where $\epsilon_1 = \dfrac{\epsilon}{2M(b-a)(d-c)}$.

Now

$S_D(x,y) =$

$$\sum_{[g,h;u,v]\epsilon D} 1/4[y(g,u)+y(h,u)+y(h,v)+y(g,v)][x(g,u)-x(h,u)+x(h,v)-x(g,v)]$$

and

$S_D(tr, x''y) =$

$$\sum_{[g,h;u,v]\epsilon D} 1/4[x''(g,u)y(g,u)+x''(h,u)y(h,u)+x''(h,v)y(h,v)+x''(g,v)y(g,v)].$$

$$\cdot\{[h - g][v - u]\}$$

Consider

$$|S_D(x,y) - S_D(tr, x''y)|.$$

It can be shown with techniques similar to those used in the proof of Theorem 1.15 that

$$|S_D(x,y) - S_D(tr, x''y)| < M\cdot\epsilon_1\cdot(b-a)(d-c) = 1/2\epsilon.$$

The remaining portion of the proof is omitted because it parallels the proof of Theorem 1.15.

Remark 2.14. Theorem 2.14 enables one to show how Theorem 2.10 is suggested when one integrates across the formula for the second alternate partial of the product of two functions. That is, if the functions x and y meet certain conditions, then

$$\partial_1[x\cdot y] = \partial_1 x\cdot y + x\cdot\partial_1 y$$

$$\partial_{12}^2[x\cdot y] = \partial_2[\partial_1 x\cdot y + x\cdot\partial_1 y]$$

$$= \partial_{12}^2 x\cdot y + \partial_1 x\cdot\partial_2 y + \partial_2 x\cdot\partial_1 y + x\cdot\partial_{12}^2 y$$

$$\int_R \partial_{12}^2[x\cdot y] = \int_R \partial_{12}^2 x\cdot y + \int_R \partial_1 x\cdot\partial_2 y + \partial_2 x\cdot\partial_1 y + \int_R x\cdot\partial_{12}^2 y$$

$$x(t,r)\cdot y(t,r)\Big|_R = \int_R y\,dx + \int_R \partial_1 x\cdot\partial_2 y + \partial_2 x\cdot\partial_1 y + \int_R x\,dy$$

$$= \int_R y\,dx + \int_R \{(\partial_1 x\cdot\partial_2 y + x\cdot\partial^2_{21} y) - x\cdot\partial^2_{21} y$$

$$+ (\partial_2 x\cdot\partial_1 y + x\cdot\partial^2_{12} y) - x\cdot\partial^2_{21} y\} + \int_R x\,dy$$

$$= \int_R y\,dx + \int_c^d\int_a^b \partial_1(x\cdot\partial_2 y) + \partial_2(x\cdot\partial_1 y) - 2x\cdot\partial^2_{21} y$$

$$+ \int_R x\,dy = \int_R y\,dx + [\int_c^d (x\cdot\partial_2 y\Big|_a^b) + \int_a^b (x\cdot\partial_1 y\Big|_c^d)]$$

$$-2\int_R x\,dy + \int_R x\,dy = \int_R y\,dx$$

$$+ \int_c^d \{x(b,r)\cdot\partial_2 y(b,r) - x(a,r)\cdot\partial_2 y(a,r)\}$$

$$+ \int_a^b \{x(t,d)\cdot\partial_1 y(t,d) - x(t,c)\cdot\partial_1 y(t,c)\} - \int_R x\,dy$$

Therefore

$$x(t,r)\cdot y(t,r)\Big|_R = \int_R y\,dx - \int_R x\,dy + \int_c^d x(b,r)dy(b,r)$$

$$- \int_c^d x(a,r)dy(a,r) + \int_a^b x(t,d)dy(t,d) - \int_a^b x(t,c)dy(t,c).$$

<u>Theorem 2.15.</u> Suppose that $[a,b;c,d]$ is an interval, $[e,f;g,h]$ is an interval, u,v,x,y is a function sequence, and that if D is a subdivision of $[a,b;c,d]$, and E is a subdivision of $[e,f;g,h]$, then there is a refinement F of D and a refinement G of E such that $S_F(x,y) = S_G(u,v)$. If y is integrable with respect to x in $[a,b;c,d]$, and v is integrable with respect to u in $[e,f;g,h]$, then

$$\int_a^b \int_c^d y(t,r)dx(t,r) = \int_e^f \int_g^h v(t,r)du(t,r).$$

<u>Proof</u>: The proof is analogous to the proof of Theorem 1.16.

<u>Definition 2.13.</u> A function $f(t,r)$ is non-decreasing on a set A, if and only if whenever (u,v) and $u_1,v_1)$ are points in A and $u < u_1$ and $v < v_1$, then $f(u,v) \leq f(u_1,v_1)$.

<u>Definition 2.14.</u> A function $f(t,r)$ is non-increasing on a set A, if and only if whenever (u,v) and (u_1,v_1) are points in A and $u < u_1$ and $v < v_1$, then $f(u,v) \leq f(u_1,v_1)$.

<u>Corollary 2.15a.</u> Suppose that $[a,b;c,d]$ is an interval, $[e,f;k,l]$ is an interval, and that F is integrable with respect to the continuous function P in $[a,b;c,d]$. Suppose the equation $t = g(u,v)$ and $r = h(u,v)$; where g and h are continuous non-decreasing functions; defines a one-to-one transformation of the region $[e,f;k,l]$ into the region $[a,b;c,d]$. Then

$$\int_{[a,b;c,d]} F(t,r)dP(t,r) = \int_{[e,f;k,l]} F[g(u,v),h(u,v)]dP[g(u,v),h(u,v)].$$

<u>Proof:</u> Since F is integrable with respect to P in $[a,b;c,d]$ then for $\epsilon > 0$, there exists a subdivision D_1 of $[a,b;c,d]$ such that if E_1 is a refinement of D_1, then

$$\left| \int_{[a,b;c,d]} F(t,r)dP(t,r) - S_{E_1}(P,F) \right| < \epsilon.$$

But since $t = g(u,v)$ and $r = h(u,v)$ forms a one-to-one transformation mapping $[a,b;c,d]$ onto $[e,f,k,\ ]$, there corresponds a subdivision of $[e,f;k,l]$, call it D_2, such that $S_{D_1}(P,F) = S_{D_2}\{P[g(u,v),\ h(u,v)],F[g(u,v),h(u,v)]\}$. And since this relationship is also valid for any refinement E_1 of D_1 and E_2 of D_2, then

$$\left| \int_{[a,b;c,d]} F(t,r)dP(t,r) - S_{E_2}\{P[g(u,v),h(u,v)],F[g(u,v),h(u,v)]\}\right| < \epsilon.$$

This implies that $F[g(u,v),h(u,v)]$ is integrable with respect to $P[g(u,v),h(u,v)]$. Then by Theorem 2.14

$$\int_{[a,b;c,d]} F(t,r)dP(t,r) = \int_{[e,f;k,l]} F[g(u,v),h(u,v)]dP[g(u,v),h(u,v)].$$

<u>Corollary 2.15b</u>. This corollary is identical to Corollary 2.15a except the word non-decreasing has been replaced by non-increasing, and a minus sign has been imposed on one side of the equation.

<u>Remark 2.15.</u> The two corollaries presented above enable mathematicians to perform integration by substitution.

CHAPTER III

LANE'S INTEGRAL IN N-SPACE: $\int_{[a_1,b_1;\cdots;a_{n-1},b_{n-1}]} f(t,r,\cdots)dg(t,r,\cdots)$

The definitions and theorems presented in Chapter II extend readily to an n-space function x and an n-space function y and to an n-space interval $[a_1,b_1;\cdots;a_{n-1},b_{n-1}]$. The proofs of the theorems will be analogous to those presented in Chapter II. A few sample definitions and theorems will indicate the necessary procedure for the extension.

<u>Definition 3.1.</u> The statement that f is a real n-space function implies that if $t_1,t_2,\cdots,t_{n-1}$ are real numbers then there is just one number c such that $f(t_1,t_2,\cdots,t_{n-1}) = c$.

<u>Definition 3.2.</u> An n-space interval denoted by $[a_1,b_1;a_2,b_2;\ \cdots a_{n-1},b_{n-1}]$ is a point set such that each $a_i < b_i$. A number sequence $(t_1,t_2,\cdots,t_{n-1})$ belongs to $[a_1,b_1;a_2,b_2;\cdots;a_{n-1},b_{n-1}]$ if and only if $a_1 < t_1 < b_1$, $a_2 < t_2 < b_2$, and so forth up to $a_{n-1} < t_{n-1} < b_{n-1}$.

<u>Definition 3.3.</u> The statement that D is a subdivision of the n-space interval $[a_1,b_1;\cdots;a_{n-1},b_{n-1}]$ means that D is a finite set of (one or more) nonoverlapping n-space intervals covering $[a_1,b_1;\cdots;a_{n-1},b_{n-1}]$.

Definition 3.4. The statement that the function f is quasi-continuous in the n-space interval

$$[a_1, b_1; a_2, b_2; \cdots; a_{n-1}, b_{n-1}]$$

means that if $\epsilon > 0$, then there exists a subdivision D of $[a_1, b_1; a_2, b_2; \cdots; a_{n-1}, b_{n-1}]$ such that if P is one of the n-space intervals in D and the points $(t_1, t_2, \cdots, t_{n-1})$ and $(r_1, r_2, \cdots, r_{n-1})$ belong to the interior of P or to one of its faces, then

$$|f(t_1, t_2, \cdots, t_{n-1}) - f(r_1, r_2, \cdots, r_{n-1})| < \epsilon.$$

Notation. If x is a function, y is a function and D is a subdivision of the n-space interval $[a_1, b_1; a_2, b_2; \cdots; a_{n-1}, b_{n-1}]$, and if the p_i's below represent all the apexes of the interval P in D, then $S_D(x,y)$ denotes the number

$$\sum_D (1/2^{n-1})[y(p_1)+y(p_2)+\cdots+y(p_2n-1)]\cdot[\Delta^{n-1}x(p_i)]$$

where $\Delta^{n-1}x(p_i)$ represents the n-1 finite differencing of the apexes of P.

Definition 3.5. Suppose that x is a function, y is a function and $[a_1, b_1; a_2, b_2; \cdots; a_{n-1}, b_{n-1}]$ is an n-space interval. The statement that c is the integral over the n-space interval of y respect to x, may be written as

$$c = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} y(t, r, \cdots) dx(t, r \cdots).$$

Definition 3.6. The statement that the n-space function f is of bounded variation over the n-space interval $[a_1, b_1; a_2, b_2; \cdots; a_{n-1}, b_{n-1}]$ means that there exists a number B such that if D is a subdivision of D, then

$$\sum_{D} |[\Delta^{n-1} x(p_i)]| \leq B.$$

CHAPTER IV

IMPROPER INTEGRALS

The Lane's Integral defined previously had meaning only if the interval over which integration was performed was finite. The following chapter will extend the definition so that under certain circumstances the Lane's Integral will be meaningful when the interval of integration is infinite. The proofs for the theorems presented will be omitted because of the similarities between them and ones already presented.

Two Space

The definition presented in Chapter I for a subdivision and refinement will be used here without alteration.

Definition 4.1. The statement that the function f is quasi-continuous in the interval $[a,\infty)$ means that if $\epsilon > 0$, then it is true that no matter what interval $[a,b]$ is picked, there exists a subdivision D of $[a,b]$ such that if $[p,q]$ is one of the intervals of D, and s and t are in the segment (p,q), then $|f(s) - f(t)| < \epsilon$.

Definition 4.2. The statement that the function y is integrable with respect to the function x in the interval $[a,\infty)$ means that if $\epsilon > 0$, then there exists an interval [a,b], such that if $b < B$, then $\left| \int_a^b y(t)dx(t) - \int_a^B y(t)dx(t) \right| < \epsilon$. This implies that y is integrable with respect to x on [a,b] and [a,B].

<u>Theorem 4.1.</u> For the function y to be integrable with respect to the function x in the interval $[a,\infty)$ it is necessary and sufficient that there be a number c such that if $\epsilon > 0$, then there exists an interval [a,b] such that if $b < B$, then

$$\left| \int_a^B y(t)dx(t) - c \right| < \epsilon.$$

<u>Definition 4.3.</u> Suppose that x is a function and y is a function. The statement that c is the integral from a to ∞ of y with respect to x, which may be written as

$$c = \int_a^\infty y(t)dx(t),$$

means that c is a number and that the statement from Theorem 4.1 is true.

<u>Definition 4.4.</u> The statement that the function f is of bounded variation in the infinite interval $[a,\infty)$ means there exists a number T such that if [a,b] is a subinterval of $[a,\infty)$, then $V_a^b(f) \leq T$. The least such number T is called the total variation over $[a,\infty)$ and will be represented by $V_a^\infty(f)$.

<u>Theorem 4.2.</u> If the function y is quasi-continuous and bounded in the interval $[a,\infty)$ and x is of bounded variation in $[a,\infty)$, then y is integrable with respect to x in the interval $[a,\infty)$.

<u>Theorem 4.3.</u> If y is integrable with respect to x in the interval $[a,\infty)$, and k is a number, then

$$\int_a^\infty y(t)d[x(t)+k] = \int_a^\infty y(t)dx(t).$$

<u>Theorem 4.4.</u> If y is integrable with respect to x in the interval $[a,\infty)$, and k is a number, then

$$\int_a^\infty [ky(t)]dx(t) = k\int_a^\infty y(t)dx(t).$$

<u>Theorem 4.5.</u> Suppose that each of y and y_1 is integrable with respect to x in the interval $[a,\infty)$. Then

$\int_a^\infty [y(t)+y_1(t)]dx(t)$ exists and

$$\int_a^\infty [y(t)+y_1(t)]dx(t) = \int_a^\infty y(t)dx(t) + \int_a^\infty y_1(t)dx(t).$$

<u>Theorem 4.6.</u> Suppose that y is integrable with respect to x in the interval $[a,\infty)$, and if $a < c < \infty$ then y is integrable with respect to x in $[a,c]$, and in the interval $[c,\infty)$; moreover

$$\int_a^\infty y(t)dx(t) = \int_a^c y(t)dx(t) + \int_c^\infty y(t)dx(t).$$

Theorem 4.7. If y is integrable with respect to x in $[a,\infty)$, and x and y are functions such that $\lim_{t\to\infty} y(t)\cdot x(t)$ exists, then

$$\int_a^\infty x(t)dy(t) \text{ exists and}$$

$$\int_a^\infty y(t)dx(t) = y(\infty)x(\infty) - y(a)x(a) - \int_a^\infty x(t)dy(t).$$

Theorem 4.8. Suppose that y is integrable with respect to x in the interval $[a,\infty)$. If x is of bounded variation in $[a,\infty)$ and $|y(t)| \leq M$ for each t in $[a,\infty)$, then

$$\left|\int_a^\infty y(t)dx(t)\right| \leq M\cdot V_a^\infty(x).$$

Theorem 4.9. If x is a function such that the $\lim_{t\to\infty} x(t)$ exists and $[a,\infty)$ is an infinite interval, then

$$\int_a^\infty 1dx(t) = x(\infty) - x(a).$$

Definition 4.5. The statement that the function f has a derivative, f', in the interval $[a,\infty)$, means that if $a \leq t \leq \infty$ then there is a number $f'(t)$, such that the following statement is true:

If $\epsilon > 0$, then there is a segment (p,q) containing t such that if s is another number in (p,q) and in $[a,\infty)$, then

$$\left|\frac{f(s) - f(t)}{s - t} - f'(t)\right| < \epsilon.$$

Definition 4.6. The statement that the function f has a continuous derivative, f', in the interval $[a,\infty)$ means that f has a derivative, f', in $[a,\infty)$, and that if $\epsilon > 0$ and $[a,b]$ is an interval, then there is a subdivision D of $[a,b]$ such that if $[p,q]$ is an interval in D and s and t are numbers in $[p,q]$ then $|f'(s) - f'(t)| < \epsilon$.

Theorem 4.10. If the function f has a continuous derivative, f', in the interval $[a,\infty)$, and $\epsilon > 0$ and $[a,b]$ is an interval, then there is a subdivision D of $[a,b]$ such that if s and t are in one of the intervals of D, then

$$\left|\frac{f(s) - f(t)}{s - t} - f'(t)\right| < \epsilon.$$

Theorem 4.11. Suppose that x has a continuous derivative x' in the interval $[a,\infty)$, and that there is a number M such that $|y(t)| < M$ if $a < t < \infty$. If

$$\int_a^\infty y(t)dx(t) = c_1 \text{ or } \int_a^\infty x'(t)y(t)dt = c_2, \text{ then}$$

$$\int_a^\infty y(t)dx(t) = \int_a^\infty x'(t)y(t)dt.$$

Theorem 4.12. Suppose that $[a,\infty)$ is an infinite interval, $[c,\infty)$ is an infinite interval, u,v,x,y is a function sequence, and that if D is a subdivision of $[a,b]$, and E is a subdivision of $[c,d]$, then there is a refinement F of D, and a refinement G of E, such that $S_F(x,y) = S_G(u,v)$. If y is integrable with respect to x in $[a,\infty)$ and v is integrable with respect to u in $[c,\infty)$, then

$$\int_a^\infty y(t)dx(t) = \int_c^\infty v(t)du(t).$$

<u>Corollary 4.12a.</u> Suppose that $[a,\infty)$ is an infinite interval, $[c,\infty)$ is an infinite interval, and y is integrable with respect to x in $[a,\infty)$. If f is a continuous non-decreasing function such that $f(c) = a$ and $\lim_{t\to\infty} f(t) = \infty$, then

$$\int_a^\infty y(t)dx(t) = \int_c^\infty y[f(t)]dx[f(t)].$$

<u>Corollary 4.12b.</u> Suppose that $[a,\infty)$ is an interval, $[c,\infty)$ is an infinite interval, and y is integrable with respect to x in $[a,\infty)$. If g is a continuous non-increasing function such that $g(c) = \infty$ and $\lim_{t\to\infty} g(t) = a$, or if $g(c) = \infty$ and $g(d) = a$, where d is the point at which the function g crosses the x-axis, then

$$\int_a^\infty y(t)dx(t) = -\int_a^\infty y[g(t)dx[g(t)].$$

Three Space

The definitions and theorems which were presented in first section of this chapter will extend readily to three-space functions and a rectangular infinite interval $[a,\infty;c,\infty)$. Therefore, the actual formal presentation of the material and proofs will be omitted and simply a definition of what is meant by an Improper Integral in three-space will be given.

Definition 4.7. Suppose x is a three-space function, y is a three-space function and a and c are numbers. Then by $\int_{[a,\infty;c,\infty)} y(t,r)dx(t,r)$ is meant a number I such that if $\epsilon > 0$, then there exists a rectangular interval $[a,b;c,d]$, such that if $b < B$ and $d < D$, then

$$\left| \int_{[a,B;c,D]} y(t,r)dx(t,r) - I \right| < \epsilon.$$

N-Space

As before, the definition, theorems and proofs extend readily to n-space functions and n-space infinite intervals $[a_1,\infty;a_2,\infty;\cdots;a_{n-1},\infty)$ and thus any formal presentation is omitted.

Definition 4.8. Suppose x is an n-space function, y is an n-space function and $a_1,a_2,\cdots,a_{n-1}$ is a number sequence. Then by $\int_{[a_1,\infty;a_2,\infty;\cdots;a_{n-1},\infty]} y(t_1,t_2,\cdots,t_{n-1})dx(t_1,t_2,\cdots,t_{n-1})$ is meant a number I such that if $\epsilon > 0$, then there exists an n-space interval $[a_1,b_1;a_2,b_2;\cdots;a_{n-1},b_{n-1}]$ such that if $b_1 < B_1$ and $b_2 < B_2$ and $\cdots$ and $b_{n-1} < B_{n-1}$, then

$$\left| \int_{[a_1,B_1;a_2,B_2;\cdots;a_{n-1},B_{n-1}]} y(t_1,t_2,\cdots,t_{m-1})dx(t_1,t_2,\cdots,t_{n-1}) - I \right| < \epsilon.$$

CHAPTER V

SOME ADVANTAGES OF THE LANE INTEGRAL

Lane's Integral and Stieltjes' Integral are very similar in many respects, but in some instances the Lane Integral is superior. It was shown on Page 7 that if the Stieltjes Integral exists, then the Lane Integral exists and they are equal. However, in certain instances when Stieltjes Integral does not exist, the Lane Integral will exist and yield the desired result for the value of the integral. Here is an example. Suppose that two probability graphs are presented, each containing n discrete points, and that one graph is plotted with respect to the other; then the result again would be a graph of n discrete points. It might then be desired to fit a polynomial P to these points such that the area under P equals the area of trapezoids formed by the n discrete points and the x-axis. The Lane Integral would give this area but the Stieltjes Integral would not exist. Furthermore, the existence theorem for Lane's Integral is more expansive.

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