RINGS OF CONTINUOUS FUNCTIONS

APPROVED:

[Signature]
Major Professor

[Signature]
Minor Professor

[Signature]
Director of the Department of Mathematics

[Signature]
Dean of the Graduate School

The purpose of this paper is to examine properties of the ring \( C(X) \) of complex or real-valued continuous functions on an arbitrary topological space \( X \). Of particular interest are the zero-sets of these continuous functions and the relationships between these zero-sets in \( X \) and ideals in \( C(X) \).

A primary source for this study was Leonard Gillman and Meyer Jerison's *Rings of Continuous Functions*, used extensively in Chapter II in discussing algebraic properties of the ring \( C(X) \). H. L. Royden's *Real Analysis* was used as a general reference, particularly in the development which this paper presents of the Stone–Čech compactification. Frank Connor contributed an example of a certain function needed in Chapter II. Discussions with Paul Lewis were the major source of information for this paper.

The paper is organized into sections dealing with \( C(X) \) from algebraic and topological viewpoints. Chapter I presents two concepts — nets, which are used a great deal in proofs, and filters, which are used in connection with zero-sets.

Chapter II deals with algebraic properties of \( C(X) \) and \( BC(X) \), the subring of bounded continuous functions on \( X \). Topics discussed include order properties of \( C(X) \), invariants under homomorphisms, zero-sets in \( X \), the extension of continuous
functions, and connections between ideals in $C(X)$ and zero-set filters in $X$.

Chapter III is concerned with locally compact Hausdorff spaces, the one-point compactification $A(X)$ of such a space $X$, and problems involved in extending functions in $C(X)$ to functions in $C[A(X)]$. This discussion brings up in a natural way the Stone–Čech compactification of $X$. In this chapter, it is shown that in studying rings of continuous functions from an algebraic viewpoint, there is no need to consider more general spaces than completely regular spaces.

Chapter IV defines a metric on $C(X)$, where $X$ is a compact space, and discusses a natural correspondence between the space $X$ and the maximal ideal space of $C(X)$. 
RINGS OF CONTINUOUS FUNCTIONS

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Carolyn Connell, B. A.
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CHAPTER I

INTRODUCTION AND PRELIMINARIES

The purpose of this paper is to examine properties of the ring $C(X)$ of all complex or real-valued continuous functions on an arbitrary topological space $X$.

Chapter II deals with algebraic properties of $C(X)$, restricted in this chapter to the real-valued continuous functions on $X$, and $BC(X)$, the subring of bounded continuous functions on $X$. Topics discussed include order properties of $C(X)$, invariants under homomorphisms, zero-sets in $X$, the extension of continuous functions, and connections between ideals in $C(X)$ and zero-set filters in $X$.

Chapter III is concerned with topological aspects of $C(X)$, particularly with compactifications of a space $X$ and with problems in extending a continuous function on $X$ to a continuous function on a compactification of $X$. In this chapter the study is reduced to the case of completely regular spaces.

Chapter IV defines a metric on $C(K)$, where $K$ is a compact space, and discusses a natural correspondence between the space $K$ and the maximal ideal space of $C(K)$.

Several preliminary concepts are discussed here which will prove useful in later chapters. In particular, there
will be several occasions to use nets, especially in Chapters III and IV.

**DEFINITION 1.1.** A directed set $A$ is a set together with a binary relation "<" such that
(a) if $m$, $n$, and $p$ are elements of $A$ such that $m < n$ and $n < p$, then $m < p$; and
(b) if $m$ and $n$ are in $A$, then there is an element $p$ in $A$ such that $m < p$ and $n < p$.

**DEFINITION 1.2.** A net is a mapping $f$ from a directed set $A$ into a topological space $X$. Denote the image of $f$ at $a \in A$ by $x_a$, and denote the net by $\{x_a\}_{a \in A}$.

The following are some important definitions and theorems concerning nets.

**DEFINITION 1.3.** A net $\{x_a\}_{a \in A}$ is said to converge to $x$ if for each open set $O$ containing $x$, there is an $a_0 \in A$ such that if $a \in A$, then $x_a \in O$.

**DEFINITION 1.4.** A net $\{x_a\}_{a \in A}$ is said to cluster at $x$ if for every open set $O$ containing $x$ and for every $a_0 \in A$, there is an $a > a_0$ such that $x_a \in O$.

**THEOREM 1.1.** Given a space $X$ and $Y \subseteq X$, $Y$ is compact iff every net in $Y$ clusters at some point in $Y$.

**THEOREM 1.2.** A mapping $f$ from a space $X$ into a space $Y$ is continuous iff $f$ preserves converging nets.

**THEOREM 1.3.** A topological space is Hausdorff iff each net in the space converges to at most one point.
THEOREM 1.4. A point \( p \) is a limit point of a space \( X \) iff there exists a net \( \{x_a\}_{a \in A} \) in \( X \backslash \{p\} \) converging to \( p \).

Nets are a generalization of sequences, but there are some spaces in which it is necessary to work with nets since sequences are insufficient; the following is an example of such a space.

Let \( X = \left[ \mathbb{Z}^+ \times \mathbb{Z}^+ \right] \cup \{(0,0)\} \), where \( \mathbb{Z}^+ \) denotes the positive integers. For the topology on \( X \), define the discrete topology on \( \mathbb{Z}^+ \times \mathbb{Z}^+ \), and let \( U \) be a neighborhood of \((0,0)\) if \( U \) contains \((0,0)\) and if there exists an integer \( N \) such that for every \( n \geq N \), \( U \) contains all but finitely many points of \( n \times \mathbb{Z}^+ \). With open sets defined in this way, \( X \) is a topological space.

Although \((0,0)\) is a limit point of \( X \), no sequence in \( X \) converges to \((0,0)\). To see this, assume there exists such a sequence. Suppose the column \( k \times \mathbb{Z}^+ \) contains infinitely many points of the sequence, and consider the open set \( Q \) containing \((0,0)\) which contains no points of \( k \times \mathbb{Z}^+ \) and which contains all but finitely many points of every column after \( k \times \mathbb{Z}^+ \). Then \( Q \) is an open set about \((0,0)\) which omits an infinite number of points of the sequence, so the sequence could not converge to \((0,0)\). Hence the sequence has only a finite number of points on each column. Let \( T \) be a set containing \((0,0)\) which omits each point of the sequence. Then since \( T \) is an open set containing \((0,0)\) and containing no points of the sequence, the sequence does not converge.
to the point \((0,0)\), a contradiction. Therefore, no sequence in \(X\) converges to \((0,0)\).

There is no sequence in \(X\) converging to the limit point \((0,0)\), but by Theorem I.4, there is a net in \(X\) converging to \((0,0)\). To characterize limit points in spaces such as this one, therefore, it is necessary to use nets rather than sequences.

Another concept that will prove useful later is the notion of a filter. Filters will be used several times in Chapter II.

**DEFINITION I.5.** Let \(E\) be any set. A nonempty set \(\mathcal{F}\) of nonempty subsets of \(E\) is called a filter if it satisfies the following:

(a) if \(A \in \mathcal{F}\) and \(B \in \mathcal{F}\), then \(A \cap B \in \mathcal{F}\); and

(b) if \(A \in \mathcal{F}\) and \(A \subseteq B\), then \(B \in \mathcal{F}\).
CHAPTER II

ALGEBRAIC STRUCTURE

The collection $C(X)$ of all continuous real-valued functions on a space $X$ is a commutative ring with unity: the sum and product of continuous functions are continuous, $f = 0$ is the additive identity, $f = 1$ is the multiplicative identity, $-f(x)$ is the additive inverse of $f(x)$, and associativity, commutativity and distributivity all hold in $C(X)$ since these properties hold for the real number field.

The set $C(X)$ is also partially-ordered, with the ordering defined as follows: $f \geq g$ iff $f(x) \geq g(x)$ for every $x$ in $X$. In fact, $C(X)$ is a partially-ordered ring.

DEFINITION II.1. A partially-ordered ring is called a lattice-ordered ring if for each $a$ and $b$ in the ring, $a \wedge b$ and $a \vee b$ are also in the ring, where $a \wedge b = \min \{a, b\}$, and $a \vee b = \max \{a, b\}$.

If $f$ and $g$ are in $C(X)$, then $f - g$ is in $C(X)$, and $f \vee g = \frac{1}{2} (f + g + |f - g|)$ is in $C(X)$. Similarly, $f \wedge g = \frac{1}{2} (f + g - |f - g|)$ is in $C(X)$, and it follows that $C(X)$ is a lattice-ordered ring.

If the space $X$ is discrete, then every function on $X$ is continuous, and $C(X)$ contains all functions on $X$. Conversely, if every real-valued function on $X$ is continuous, then the
characteristic function of every set in \( X \) is continuous, and \( X \) is discrete.

The subset \( \text{BG}(X) \) consisting of all bounded functions in \( C(X) \) is closed with respect to addition and multiplication, and if \( f \in \text{BC}(X) \), then \(-f \in \text{BC}(X)\); therefore, \( \text{BC}(X) \) is a subring of \( C(X) \). The ring \( \text{BG}(X) \) is also a sublattice of \( C(X) \) since if \( f \) and \( g \) are in \( \text{BG}(X) \), then \( f \wedge g \) and \( f \vee g \) are also in \( \text{BG}(X) \).

Note that if \( X \) is a countably compact topological space, then \( C(X) = \text{BG}(X) \).

To describe order in \( C(X) \), it is sufficient to note that \( f \succeq 0 \) iff \( f = k^2 \) for some real number \( k \).

**THEOREM II.1.** Every homomorphism from \( C(Y) \) into \( C(X) \) preserves order.

**PROOF.** Let \( \Phi \) be a homomorphism from \( C(Y) \) into \( C(X) \). Suppose that \( f < g \) for some functions \( f \) and \( g \) in \( Y \). Then \( g - f = k^2 \) for some \( k \). Therefore \( \Phi(g) - \Phi(f) = \Phi(g-f) = \Phi(k^2) \), and \( \Phi(k^2) = [\Phi(k)]^2 \) is positive. Therefore \( \Phi(g) > \Phi(f) \), and order is preserved. \( \text{Q.E.D.} \)

**DEFINITION II.2.** A mapping \( \Phi \) from a lattice \( A \) into a lattice \( E \) is called a lattice homomorphism into \( E \) provided that \( \Phi(a \vee b) = \Phi(a) \vee \Phi(b) \) and \( \Phi(a \wedge b) = \Phi(a) \wedge \Phi(b) \).

**THEOREM II.2.** Every homomorphism \( \Phi \) from \( C(Y) \) or \( \text{BG}(Y) \) into \( C(X) \) is a lattice homomorphism.

**PROOF.** Let \( g \) and \( h \) be functions in \( C(Y) \). Then \( (|g||h|)^2 = (|g|)(|h|) = t(|g|^2) = t(g^2) = (tg)^2 \). Also, \( |g|^2 \geq 0 \) implies that \( t|g|^2 \geq 0 \) by Theorem II.1. Therefore
\[(tg)^2 = (tg)^2 \text{ implies } t|g| = |tg|. \] Since \( g \in h = \frac{1}{2}(g+h+|g-h|) \), then
\[ t(g \lor h) + t(g \land h) = t(g + h + |g - h|) \]
\[ = tg + th + |tg - th| \]
\[ = (tg \lor th) + (tg \land th). \]
Hence \( t(g \lor h) = (tg \lor th) \). Similarly, \( t(g \land h) = (tg \land th) \), and \( t \) is a lattice homomorphism. Q.E.D.

**THEOREM II.3.** Every homomorphism \( t \) from \( C(Y) \) or \( BG(Y) \) into \( G(X) \) takes bounded functions to bounded functions.

**PROOF.** Since \( t1 = t(1 \cdot 1) = (t1)(t1) \), \( t1 \) is an idempotent in \( C(X) \), so it can assume no values on \( X \) other than 0 or 1. Hence for each \( n \in N \), \( tn = t1 + \ldots + t1 \) assumes no values other than 0 or \( n \). If \( g \in BC(Y) \), then there exists some \( n \in N \) such that \( |g| \leq n \); thus \( |tg| = t|g| \leq tn \leq n \). Q.E.D.

A consequence of Theorem II.3 is that an isomorphism from \( C(Y) \) onto \( C(X) \) carries \( BC(Y) \) onto \( BC(X) \). This is also a corollary of the next theorem.

**THEOREM II.4.** Let \( t \) be a homomorphism from \( C(Y) \) into \( C(X) \) whose image contains \( BG(X) \). Then \( t \) carries \( BC(Y) \) onto \( BC(X) \).

**PROOF.** Since the image of \( t \) contains \( BC(X) \), there exists some \( f \in C(Y) \) such that \( tf = 1 \). Then \( t1 = (tf)(t1) = t(f \cdot 1) \), which equals 1, so \( tn = n \) for each \( n \in N \).

Given \( k \in BC(X) \), choose \( h \in C(Y) \) for which \( th = k \). There exists an \( n \in N \) satisfying \( |k| \leq n \). Define \( g = (-n \lor h) \land n \).
Then $g$ is in $BC(Y)$ and

$$tg = t[(-n \lor h) \land n]$$
$$= t(-n \lor h) \land tn$$
$$=[t(-n) \lor th] \land tn$$
$$= (-n \lor k) \land n$$
$$= k. \quad \text{Q.E.D.}$$

The next several results will be concerned with zero-sets of continuous functions.

**DEFINITION II.3.** The set $Z(f) = \{x \in X: f(x) = 0\}$ is called the zero-set of $f$, where $f \in C(X)$.

Any set that is a zero-set of some function in $C(X)$ is called a zero-set in $X$. Clearly, $Z(f) = Z(|f|) = Z(f^n)$ for all $n \in \mathbb{N}$; $Z(0) = X$, and $Z(1) = \emptyset$. In addition,

$Z(f) \cup Z(g) = Z(fg)$, and $Z(f^2 + g^2) = Z(|f| + |g|) = Z(f) \cap Z(g)$.

For any $f \in C(X)$, $|f|A1 \in BC(X)$, and $Z(f) = Z(|f|A1)$. Therefore, $C(X)$ and $BC(X)$ yield the same zero-sets.

For every $f$ in $C(X)$, $Z(f)$ is closed since it is the inverse image of a closed set. Also, $Z(f)$ is a $G_δ$ since

$$Z(f) = \bigcap_{n \in \mathbb{N}} \{x \in X: |f(x)| < 1/n\}.$$ 

Hence every zero-set is a closed $G_δ$. It is somewhat surprising that in a normal space the converse is also true.

**THEOREM II.5.** In a normal space, every closed $G_δ$ is a zero-set.

**PROOF.** Let $F$ be a closed $G_δ$ in a normal space $X$. Then there exists \(\{O_i\}_{i=1}^{\infty}\), with $O_i$ open for each $i$, such that

$$F = \bigcap_{i=1}^{\infty} O_i.$$

For each $i$, $F \subseteq O_i$ implies that $F$ and $\sim_{O_i}$ are
disjoint closed sets. By the normality of $X$, there exists a continuous function $f_i$ such that $0 \leq f_i \leq 1$ and $f_i(\overset{\sim}{O}_i) = 0$. For each $i$, define $f'_i = 1/2 \cdot f_i$, and let $f = \sum_{i=1}^{\infty} f'_i$. For $x \in F$, $f(x) = 1$. For $x \notin F$, there exists some $k$ such that $x \notin O_k$; therefore $x \in \overset{\sim}{O}_k$, and $f'_k(x) = 0$. Therefore, $f(x) = f'_k(x) + \sum_{i \neq k} f'_i(x) = \frac{\infty}{i \neq k} f_i(x) \leq 1 - 1/2^k < 1$. Hence, $F = \{x: f(x) \geq 1\} = Z(g)$, where $g = (f-1)\wedge 0$, and $F$ is a zero-set.

The following theorem and observation make the abundance of zero-sets in a metric space clear.

**THEOREM II.6.** Every closed subset of a metric space is a $G_\delta$.

**PROOF.** Let $F$ be a closed set in a metric space $M$. For every $n \in \mathbb{N}$, define $G_n = \bigcup_{x \in F} N(x, 1/n)$. Each $G_n$ is open, and $F \subseteq \bigcap_n G_n$ since $F \subseteq G_n$ for each $n$. Let $y \in \bigcap_n G_n$. Then for any given $n \in \mathbb{N}$, $y \in G_n$, so it follows that $y \in N(x, 1/n)$ for some $x \in F$. Therefore, $y$ is a limit point of $F$, so $y \in F$. Hence $F = \bigcap_n G_n$, and $F$ is a $G_\delta$. Q.E.D.

Therefore, since every metric space is normal, every closed set in a metric space is a zero-set.

**DEFINITION II.4.** A set $G \subset X$ is called a cozero-set if $G$ is the complement of a zero-set. Since $\{x: f(x) \geq 0\} = Z(f \wedge 0)$ and $\{x: f(x) < 0 = Z(f \vee 0)$, then $\text{pos } f = \{x: f(x) > 0\}$ and $\text{neg } f = \{x: f(x) < 0\} = \text{pos } (-f)$ are cozero-sets. Conversely, every cozero-set is of this form: $X \setminus Z(f) = (\text{pos } f) \cup (\text{neg } f) = \text{pos } |f|$. 
For a function $f \in C(X)$, $f^{-1}$ exists iff $f$ vanishes nowhere on $X$. Therefore $f$ is a unit iff $Z(f) = \emptyset$. For a function $f \in BC(X)$, if $f$ is a unit, then $Z(f) = \emptyset$. The converse is not necessarily true, however, since for $f \in BC(X)$, $f^{-1}$ may be unbounded. A function $f \in BC(X)$ is a unit in $BC(X)$ iff $f$ is bounded away from zero: if $|f| \geq r$ for some $r > 0$, then $f^{-1}$ is bounded by $1/r$; conversely, if $f$ is a unit in $BC(X)$, then $f^{-1}$ is bounded by some $n > 0$, and $|f| \geq 1/n$, so $f$ is bounded away from zero.

An example of a bounded continuous function whose inverse is unbounded is $f(n) = 1/n$ in $C(N)$. The function $f$ is bounded by 1, and $f$ is continuous since the space $N$ is discrete, but $f^{-1}(n) = n$ is clearly unbounded.

As noted before, $Z(X)$, the set of all zero-sets in $X$, is closed under the formation of finite unions and finite intersections.

**THEOREM II.7.** The set $Z(X)$ is closed under countable intersections.

**PROOF.** Let $\{f_n\}$ be a sequence in $C(X)$, and define $g_n = |f_n|/\sqrt{2^n}$. For $x \in X$, define $g(x) = \sum_{n \in \mathbb{N}} g_n(x)$. Since $|g_n| \leq 2^{-n}$, the series converges uniformly, so $g$ is a continuous function. Also, $Z(g) = \bigcap_{n \in \mathbb{N}} Z(g_n) = \bigcap_{n \in \mathbb{N}} Z(f_n)$. Hence $Z(X)$ is closed under countable intersections. Q.E.D.

The set $Z(X)$ need not be closed under infinite unions, however, since an infinite union of closed sets is not necessarily closed: In fact, even a closed, countable union of
zero-sets need not be a zero-set; for example, see \([1, \text{p. 97}]\). In addition, \(Z(X)\) need not be closed under arbitrary intersections. For example, let \(S\) be an uncountable space in which all points are isolated except for a distinguished point \(s\), a neighborhood of \(s\) being any set containing \(s\) whose complement is countable. Then \(s\) is not a zero-set since \(s\) is not a closed \(G_\delta\). But \(s\) is the intersection of the zero-sets of all characteristic functions \(\chi_x\) such that \(x \in S \setminus \{s\}\).

**DEFINITION II.5.** Two sets \(A\) and \(B\) of \(X\) are said to be completely separated in \(X\) if there exists a function \(f \in BC(X)\) such that \(f(A) = \{0\}\), \(f(B) = \{1\}\), and \(0 \leq f \leq 1\).

To show that two sets \(A\) and \(B\) are completely separated, it is sufficient to find a function \(g\) in \(C(X)\) satisfying \(g(x) \geq 1\) for \(x \in B\) and \(g(x) \leq 0\) for all \(x \in A\), for then the function \((0 \lor g) \land 1\) has the required properties. Also, the numbers 0 and 1 may be replaced by any real numbers \(r\) and \(s\), with \(r < s\).

Two sets contained in completely separated sets are completely separated.

**THEOREM II.8.** Two sets are completely separated iff their closures are.

**PROOF.** If two sets \(A\) and \(B\) are completely separated, then there exists an \(f \in BC(X)\) such that \(f(A) = 0\), \(f(B) = 1\), and \(0 \leq f \leq 1\). Let \(x\) be a limit point of \(A\). Then there exists a net \(\{x_a\}_a\) in \(A\) converging to \(x\). For each \(x_a \in \{x_a\}_a\), \(f(x_a) = 0\), but \(f(x_a)\) converges to \(f(x)\) since \(f\) is continuous. Therefore \(f(\overline{A}) = 0\), and similarly, \(f(\overline{B}) = 1\).

Q.E.D.
DEFINITION II.6. If a zero-set \( Z \) is a neighborhood of a set \( A \), \( Z \) is called a zero-set neighborhood of \( A \).

THEOREM II.9. Two sets are completely separated iff they are contained in disjoint zero-sets. Moreover, completely separated sets have disjoint zero-set neighborhoods.

PROOF. Suppose that \( Z(f) \cap Z(g) = \emptyset \). Then \( |f| + |g| \) has no zeros, so \( h(x) \) may be defined as follows: \( h(x) = \frac{|f(x)|}{|f(x)| + |g(x)|} \).

Then \( h \in C(X) \), \( h[Z(f)] = 0 \), and \( h[Z(g)] = 1 \); hence if two sets are contained in disjoint zero-sets, then they are completely separated.

Conversely, if \( A \) and \( A' \) are completely separated, there exists an \( f \in C(X) \) such that \( f(A) = 0 \), and \( f(A') = 1 \). Then the disjoint sets \( P = \{ x : f(x) \leq 1/3 \} \) and \( P' = \{ x : f(x) \geq 2/3 \} \) are zero-set neighborhoods of \( A \) and \( A' \), respectively. Q.E.D.

The following few results are concerned with extending continuous functions.

DEFINITION II.7. A subspace \( S \) of \( X \) is said to be \( C \)-embedded in \( X \) if every function in \( C(X) \) can be extended to a function in \( C(X) \). Similarly, \( S \) is \( BC \)-embedded in \( X \) if every function in \( BC(S) \) can be extended to a function in \( BC(X) \).

If a function \( f \) in \( BC(S) \) has an extension \( g \) in \( C(X) \), then \( f \) also has a bounded extension: if \( n \) is a bound for \( |f| \), then \((-n \vee g) \wedge n \) belongs to \( BC(X) \) and agrees with \( f \) on \( S \).

Therefore, a set \( S \) is \( BC \)-embedded in \( X \) iff every function in \( BC(S) \) can be extended to a function in \( C(X) \).
If SCXCY, and X is C-embedded in Y, then S is C-embedded in Y iff S is C-embedded in X. The transitivity also holds for BC-embedded sets.

An example of a subspace that is not BC-embedded is \( \mathbb{R} \setminus \{0\} \) in the space \( \mathbb{R} \) of real numbers: the function which is 1 for all positive \( r \) and -1 for negative \( r \) has no continuous extension. On the other hand, the set \( \mathbb{N} \) is not only BC-embedded in \( \mathbb{R} \), but even C-embedded.

THEOREM II.10. Urysohn's Extension Theorem.

A subspace \( S \) of \( X \) is BC-embedded in \( X \) iff any two completely separated sets in \( S \) are completely separated in \( X \).

PROOF. Suppose that \( S \) is BC-embedded in \( X \) and that \( A \) and \( B \) are completely separated sets in \( S \). Then there exists a function \( f \) in \( BC(S) \) that is 0 on \( A \) and 1 on \( B \). Since \( f \) has an extension to a function \( g \) in \( BC(X) \), where \( g \) is 0 on \( A \) and 1 on \( B \), then \( A \) and \( B \) are completely separated in \( X \).

Conversely, suppose that any two completely separated sets in \( S \) are completely separated in \( X \). Let \( f_1 \) be a function in \( BC(S) \). Then there exists some \( m \in \mathbb{N} \) so that \( |f_1| \leq m \). Define \( r_n = \frac{m}{2^n} \). Then \( |f_1| \leq m = 3r_1 \). Inductively, given a function \( f_n \) in \( BC(S) \) with \( |f_n| \leq 3r_n \), define the sets \( A_n = \{ s \in S : f_n(s) \leq -r_n \} \) and \( B_n = \{ s \in S : f_n(s) \geq r_n \} \). Then \( A_n \) and \( B_n \) are disjoint zero-sets; thus, they are completely separated in \( S \). Therefore, by assumption, \( A_n \) and \( B_n \) are completely separated in \( X \), and there exists a function \( g_n \) in \( BC(X) \) that equals \(-r_n \) on \( A_n \) and \( r_n \) on \( B_n \), with \( |g_n| \leq r_n \). Now
define \( f_{n+1} = (f_n - g_n)|_S \). Since \( f_n \) and \( g_n \) differ on \( S \) by no more than \( 2r_n \), then \( |f_{n+1}| \leq 2r_n \); therefore, \( |f_{n+1}| \leq 3r_{n+1} \).

So given \( f_n \) in \( \text{BC}(S) \) with \( |f_n| \leq 3r_n \), an \( f_{n+1} \) may be found in \( \text{BC}(S) \) with \( |f_{n+1}| \leq 3r_{n+1} \). This completes the induction step.

Now let \( g(x) = \sum_n g_n(x) \) for \( x \in X \). The series \( \left\{ \sum_n g_n(x) \right\} \) converges uniformly since the series \( \left\{ \sum_n \frac{2}{3^n} \right\} \) converges. Therefore, \( g \) is continuous on \( X \).

Observe that

\[
(g_1 + \cdots + g_n)_S = (f_1 - f_2) + \cdots + (f_n - f_{n+1}) = f_1 - f_{n+1}.
\]

The sequence \( \{f_{n+1}(s)\} \) approaches 0 at every point \( s \) of \( S \) since \( |f_n| \leq 3r_n \), and \( \{r_n\} \) converges to 0. Therefore,

\[
g(s) = \lim_n \sum_{i=1}^n g_i(s) = \lim_n (f_1 - f_{n+1})(s) = f_1(s). \text{ Thus, } g \text{ equals } f_1 \text{ on } S. \quad \text{Q.E.D.}
\]

If \( X \) is a metric space, every closed set is a zero-set; thus, any two disjoint closed sets are disjoint zero-sets and are, therefore, completely separated in \( X \). If \( S \) is closed, then closed sets in \( S \) are also closed in \( X \). Furthermore, two completely separated sets in \( S \) have completely separated closures in \( S \). Thus, their closures are completely separated in \( X \), and it follows that the sets themselves are completely separated in \( X \). Then by Urysohn's Extension Theorem, any closed set in a metric space is \( \text{BC-embedded} \).

This result is Tietze's Extension Theorem for metric spaces.

**THEOREM II.11.** A \( \text{BC-embedded} \) subset of \( X \) is \( \text{C-embedded} \) iff it is completely separated from every zero-set disjoint from it.
PROOF. Let $S$ be BC-embedded in $X$, and suppose that $S$ is also C-embedded in $X$. Let $Z(h)$ be a zero-set in $X$, disjoint from $S$. Define $f(s) = 1/h(s)$ for $s \in S$. The function $h$ is continuous, and $h$ is zero at no point in $S$ since $S$ and $Z(h)$ are disjoint; therefore, $f$ is continuous on $S$. Thus, $f$ has an extension $g$ in $C(X)$. Then $gh$ is in $C(X)$ and equals 1 on $S$ and 0 on $Z(h)$. It is then clear that $S$ and $Z(h)$ are completely separated.

Conversely, let $f$ be a function in $C(S)$, and suppose that $S$ is completely separated from every zero-set disjoint from it. Then $\arctan f$ is in $BC(S)$ and has an extension to a function $g$ in $C(X)$. The set $Z = \{ x \in X : |g(x)| < \sqrt{2} \}$ is a zero-set of $X$, and $Z$ is disjoint from $S$ since on $S$, $g$ equals $\arctan f$, and $|\arctan f| < \sqrt{2}$. By hypothesis, $Z$ and $S$ are completely separated. Hence there exists a function $h \in C(X)$ satisfying $h(S) = 1$, $h(Z) = 0$, and $|h| \leq 1$. Then $gh$ agrees with $\arctan f$ on $S$, and $|gh(x)| < \sqrt{2}$ for every $x \in X$. Hence $\tan gh$ is a real-valued continuous extension of $f$ to all of $X$. Q.E.D.

As was noted following Theorem II.10, a closed set $F$ in a metric space is BC-embedded. If $Z$ is a zero-set disjoint from $F$, then $Z$ and $F$ are closed, disjoint sets in a metric space. Therefore, they are completely separated, and, by Theorem II.11, it follows that a closed set in a metric space is C-embedded.
In the next several results, a natural connection will be demonstrated between ideals in \( C(X) \) and a certain class of filters on \( X \).

**DEFINITION II.8.** A proper subset \( I \) of \( C(X) \) is an ideal in \( C(X) \) provided that \( I \) is a subring such that \( gf \in I \) for any \( f \in I \) and \( g \in C(X) \).

The intersection of any nonempty family of ideals is an ideal. Also, by the maximal principle, every ideal is contained in a maximal ideal. Every maximal ideal \( M \) is prime; that is, if \( fg \in M \), then \( f \in M \) or \( g \in M \).

The smallest ideal (possibly the improper ideal \( C(X) \) ) containing a given collection of ideals \( I, \ldots, I \), and elements \( f, \ldots \), is denoted by \( (I, \ldots, f, \ldots) \). It consists of all elements of \( C(X) \) expressible as finite sums \( i_1 + \cdots + s f + \cdots \), where \( i \in I \) and where \( s, \ldots \) are arbitrary functions in \( C(X) \).

Corresponding remarks are also true for \( B_0(X) \). If \( I \) is an ideal in \( C(X) \), then \( I \cap B_0(X) \) is an ideal in \( B_0(X) \).

**DEFINITION II.9.** A nonempty subfamily \( F \) of \( Z(X) \) is called a \( z \)-filter on \( X \) provided that

\( a) \) \( \emptyset \notin F \);

\( b) \) if \( Z_1, Z_2 \in F \), then \( Z_1 \cap Z_2 \in F \); and

\( c) \) if \( Z \in F \) and \( Z' \in Z(X) \) such that \( Z' \supseteq Z \), then \( Z' \in F \).

By (c), \( X \) belongs to every \( z \)-filter. Because of (c), (b) may be replaced by (b') if \( Z_1, Z_2 \in F \), then \( Z_1 \cap Z_2 \) contains a member of \( F \).
Every family $B$ of zero-sets that has the finite intersection property is contained in a z-filter: one z-filter containing $B$ is the family $F$ of all zero-sets containing finite intersections of members of $B$. Since none of the finite intersections are empty, $\emptyset \notin F$. For any $Z_1$ and $Z_2$ in $F$, each contains a finite intersection of members of $B$, and $Z_1 \cap Z_2$ must contain such a finite intersection. Thus, $Z_1 \cap Z_2 \in F$. Also, if $Z \in Z(X)$ and $Z' \subseteq Z$ for some $Z'$ in $F$, then $Z$ contains a finite intersection of members of $B$, so $Z \in F$. Therefore, $F$ is a z-filter containing $B$. In fact, $F$ is clearly the smallest z-filter containing $B$. The family $B$ is said to generate the z-filter $F$, and when $B$ is closed under finite intersection, it is called a base for $F$.

The definition of z-filters is clearly analogous to that of filters. Filters and z-filters are exactly the same in discrete spaces since every set in a discrete space is a zero-set.

In any space $X$, the intersection with $Z(X)$ of any filter is a z-filter. Conversely, if $F'$ is the smallest filter containing a given z-filter $F$, then $F' \cap Z(X) = F$ since (a) $F' \supset F$ and $Z(X) \supset F$, so $[F' \cap Z(X)] \supset F$; and (b) every zero-set in $F'$ is in $F$, so $[F' \cap Z(X)] \subseteq F$.

**Theorem 11.12.** (a) If $I$ is an ideal in $C(X)$, then $Z(I) = \{Z(f): f \in I\}$ is a z-filter on $X$.

(b) If $F$ is a z-filter on $X$, then $Z^+(f) = \{f: Z(f) \in F\}$ is an ideal in $C(X)$. 
PROOF. (a) Suppose that \( I \) is an ideal in \( C(X) \). Since \( I \) contains no unit, \( \phi \not\in Z(I) \). Let \( Z_1, Z_2 \in Z(I) \). Then there exist functions \( f_1, f_2 \in I \) such that \( Z(f_1) = Z_1 \) and \( Z(f_2) = Z_2 \). Then \( f_1^2 + f_2^2 \in I \), and \( Z(f_1^2 + f_2^2) = Z_1 \cap Z_2 \in Z(I) \). Next, let \( Z \) be in \( Z(I) \) and \( Z' \) be in \( Z(X) \). Then there exists an \( f \in I \) and a \( g \in C(X) \) such that \( Z = Z(f) \) and \( Z' = Z(g) \). But \( fg \in I \); therefore, if \( Z' \supset Z \), then \( Z' = Z \cup Z' = Z(fg) \in Z(I) \). It follows that \( Z(I) \) is a \( z \)-filter on \( X \).

(b) Let \( J = Z^+(F) \), where \( F \) is a \( z \)-filter. The set \( J \) contains no unit since if \( J \) contains a unit \( f \), then \( Z(f) = \phi \), and \( \phi \not\in F \). Let \( f, g \in J \), and let \( h \in C(X) \). Then \( Z(f - g) \) contains \( Z(f) \cap Z(g) \in F \), and \( Z(f - g) \in F \). Also, \( Z(hf) \supset Z(f) \in F \), and this implies that \( Z(hf) \in F \). Hence \( f - g \in J \) and \( hf \in J \), and it follows that \( J \) is an ideal in \( C(X) \). Q.E.D.

If \( F \subset Z(X) \), then \( Z[Z^+(F)] = F \). Therefore, every \( z \)-filter is of the form \( Z(J) \) for some ideal \( J \) in \( C(X) \). Also, \( Z^+[Z(I)] \supset I \); the following is an example showing that this inclusion may be proper.

Let \( I \) be the principle ideal in \( C(R) \) generated by the identity mapping. Then \( I \) consists of all functions \( f \) in \( C(R) \) such that \( f(x) = x \cdot g(x) \) for some \( g \) in \( C(R) \). Every function in \( I \) vanishes at \( x = 0 \); therefore, every zero-set in \( Z(I) \) contains the point \( 0 \). Also, since \( Z(f) = \{0\} \) where \( f(x) = x \), then \( Z(I) \) contains \( \{0\} \). Thus, \( Z(I) \) must consist of all zero-sets which contain \( 0 \) since \( Z(I) \) is a \( z \)-filter.

The ideal \( M = Z^+[Z(I)] \) consists of all functions in \( C(R) \) which vanish at \( 0 \). So \( M \supset I \), but \( M \neq I \). For example, the
function $i^{1/3}$ is in $M$, where $i$ is the identity mapping. But if $i^{1/3} \in I$, then $i^{1/3} = gi$ for some $g \in C(R)$. In fact, $g(x) = x^{-2/3}$ for $x \neq 0$; therefore, $g$ is not continuous at the point 0. Thus, $i^{1/3} \in M \setminus I$, and $I \neq Z^*[Z(I)]$.

Theorem II.12 (a) is not true in general if $C(X)$ is replaced by $BC(X)$. If $J$ is an ideal in $BC(X)$, properties (b) and (c) in the definition of $z$-filters do hold for $Z(J)$, but (a) need not hold. For example, the set $J$ of all sequences that converge to zero is an ideal in $BC(N)$. Since $j = \{1/n\}^\infty_{n=1} \in J$, and $Z(j) = \phi$, $\phi \in Z(J)$, and $Z(J)$ is not a $z$-filter.

DEFINITION II.10. A $z$-filter on $X$ which is not contained in any other $z$-filter on $X$ is called a $z$-ultrafilter on $X$.

THEOREM II.13. (a) If $M$ is a maximal ideal in $C(X)$, then $Z(M)$ is a $z$-ultrafilter on $X$.

(b) If $A$ is a $z$-ultrafilter on $X$, then $Z^*(A)$ is a maximal ideal in $C(X)$.

PROOF. (a) Suppose $M$ is a maximal ideal in $C(X)$. By Theorem II.12 (a), $Z(M)$ is a $z$-filter. Let $P$ be a $z$-ultrafilter that contains $Z(M)$. Then $Z^*(P)$ is an ideal such that $M \subseteq Z^*(P)$. From the maximality of $M$, it follows that $Z(M) = P$, and $Z(M)$ is a $z$-ultrafilter.

(b) Suppose that $A$ is a $z$-ultrafilter on $X$. By Theorem II.12 (b), $Z^*(A)$ is an ideal in $C(X)$. Let $I$ be a maximal ideal in $C(X)$ that contains $Z^*(A)$. Then $Z(I)$ contains
$Z[Z^+(A)] = A$. But since $A$ is a $z$-ultrafilter, then $Z(I) = A$, and $Z^+(A) = Z^+[Z(I)].$ Therefore, $Z^+(A) = I,$ and $Z^+(A)$ is maximal. \[\text{Q.E.D.}\]

Since the containment $I \subseteq Z^+[Z(I)]$ may be proper, it does not necessarily follow that an ideal is maximal from the fact that its $z$-filter is maximal.

**Theorem II.14.** (a) Let $M$ be a maximal ideal in $C(X);$ if $Z(f)$ meets every member of $Z(M),$ then $f \in M.$

(b) Let $A$ be a $z$-ultrafilter on $X;$ if a zero-set $Z$ meets every member of $A,$ then $Z \in A.$

**Proof.** (a) Let $M$ be a maximal ideal in $C(X)$ such that $Z(f)$ meets every member of $Z(M).$ Then $Z(M) \cup \{Z(f)\}$ generates a $z$-filter, but this $z$-filter must equal $Z(M)$ since $Z(M)$ is an ultrafilter. Therefore, $Z(f) \in Z(M),$ and since the maximality of $M$ implies that $Z^+[Z(M)] = M,$ it follows that $f \in M.$

(b) By Theorem II.13, statements (a) and (b) are equivalent. \[\text{Q.E.D.}\]

**Definition II.11.** An ideal $I$ in $C(X)$ is called a $z$-ideal if $Z(f) \in Z(I)$ implies $f \in I,$ or equivalently, if $I = Z^+[Z(I)].$

If $F$ is any $z$-filter on $X,$ then $Z^+(F)$ is a $z$-ideal since $F = Z[Z^+(F)].$ Hence if $J$ is any ideal in $C(X),$ then $Z(J)$ is a $z$-filter, and $I = Z^+[Z(J)]$ is a $z$-ideal; in fact, $I$ is the smallest $z$-ideal containing $J.$

It follows from Theorem II.13 that for a maximal ideal $I,$ $Z^+[Z(I)] = I;$ hence every maximal ideal is a $z$-ideal.

Also, the intersection of an arbitrary family of $z$-ideals is
a z-ideal since if \( \{I_a\}_{a \in A} \) is a family of z-ideals, then 
\[
Z^+[Z(\bigcap_a I_a)] = \bigcap_a Z^+[Z(I_a)] = \bigcap_a I_a.
\]

The discussion following Theorem 11.12 shows that the principle ideal (I) in \( C(R) \) is not a z-ideal. If \( S \) is a non-empty set in any space \( X \), then the family \( J \) of all functions in \( C(X) \) that vanish everywhere on \( S \) is a z-ideal: if \( Z(f) \) is in \( Z(J) \), then \( Z(f) \) must contain \( S \), and \( f \in J \).

In \( C(N) \), every ideal \( I \) is a z-ideal. To show this, suppose that \( Z(f) = Z(g) \), where \( g \in I \), and define \( h \) as follows: 
\[
h(n) = 0 \text{ for } n \in Z(g), \quad \text{and} \quad h(n) = f(n)/g(n) \text{ for } n \notin Z(g).
\]
Since \( N \) is discrete, \( h \) is continuous, and \( f = hg \). Therefore, \( f \) is in \( I \), and \( I \) is a z-ideal.

**LEMMA.** [1, p. 7] The intersection of all the prime ideals containing a given ideal \( I \) is precisely the set of all elements for which some power belongs to \( I \).

**THEOREM 11.15.** Every z-ideal in \( C(X) \) is an intersection of prime ideals.

**PROOF.** For every \( n \in N \), \( Z(f^n) = Z(f) \). Therefore, if \( I \) is a z-ideal, and if \( f^n \in I \), then \( Z(f) = Z(f^n) \in Z(I) \), from which it follows that \( f \in I \). Since for every \( f^n \in I \), \( f \) is also in \( I \), then by the preceding lemma, \( I \) is the intersection of all prime ideals containing it. Q.E.D.

Since every maximal ideal is a z-ideal, and since the intersection of any family of z-ideals is a z-ideal, it follows that every intersection of maximal ideals is a z-ideal. It is not true, however, that every z-ideal is the intersection
of maximal ideals; for example, let $K$ be the set of all functions $f$ in $C(R)$ for which $Z(f)$ is a neighborhood of 0. The set $K$ is a z-ideal, and it is contained properly in the maximal ideal $M$ of all functions that vanish at 0. If $I$ is any ideal containing $K$, then $Z(K) \subseteq Z(I)$, and every member of $Z(I)$ must meet every neighborhood of 0; therefore, every member of $Z(I)$ contains 0. It follows that $I \subseteq M$, and $M$ must be the only maximal ideal containing $K$. Therefore, $K$ is not an intersection of maximal ideals.

In addition, the fact that $K$ is an intersection of prime ideals, each of which is contained in $M$, proves the existence of non-maximal, prime ideals in $C(R)$.

To show that the converse of Theorem 11.15 is false, it is enough to find one prime ideal that is not a z-ideal. For an example of such a prime ideal, see [1, p. 31].

The next result clarifies further the relation between prime ideals and z-ideals.

**Theorem 11.16.** For any z-ideal $I$ in $C(X)$, the following are equivalent:

(a) $I$ is prime.

(b) $I$ contains a prime ideal.

(c) For all $g, h \in C(X)$, if $gh = 0$, then $g \in I$ or $h \in I$.

(d) For every $f \in C(X)$, there is a zero-set in $Z(I)$ on which $f$ does not change sign.

**Proof.** Clearly, (a) implies (b). On the other hand, if $I$ contains a prime ideal $P$, and if $gh = 0$, then $gh \in P$. 

Hence either \( g \in PC \) or \( h \in PC \). Therefore, (b) implies (c).

Suppose next that (c) is true. For every \( f \in C(X) \), \( (f \land 0) \) and \( (f \lor 0) \) are in \( C(X) \). Since \( (f \land 0)(f \lor 0) = 0 \), then either \( (f \land 0) \) or \( (f \lor 0) \) must be in \( I \). Since \( Z(f \land 0) = \{ x \in X : f(x) \geq 0 \} \) and \( Z(f \lor 0) = \{ x \in X : f(x) \leq 0 \} \), then there is a zero-set in \( Z(I) \) on which \( f \) does not change sign. Thus, (c) implies (d).

To complete the cycle of implications, suppose that (d) is true. Given that \( gh \) is in \( I \) for some functions \( g \) and \( h \) in \( C(X) \), consider \( |g| - |h| \), a function in \( C(X) \). By the hypothesis, there is a zero-set \( Z \) in \( Z(I) \) on which \( |g| - |h| \) does not change sign; assume without loss of generality that \( |g| - |h| \) is non-negative on \( Z \). Then if \( g(x) = 0 \) for \( x \in Z \), it follows that \( h(x) = 0 \). Hence \( Z(h) \cap Z(h) = Z \cap Z(gh) \in Z(I) \). But since \( Z(h) \) contains a zero-set in \( Z(I) \), then \( Z(h) \) is itself in \( Z(I) \). Since \( I \) is a \( z \)-ideal, it follows that \( h \) is in \( I \). Thus, \( I \) is prime.

Q.E.D.

In any commutative ring, if \( I \) and \( I' \) are ideals, neither containing the other, then \( I \cap I' \) is not prime: if \( a \in I \setminus I' \) and \( a' \in I' \setminus I \), then neither \( a \) nor \( a' \) is in \( I \cap I' \), but \( aa' \) is in \( I \cap I' \). An example in \( C(R) \) is the intersection \( J \) of the maximal ideals \( I \), consisting of all functions in \( C(R) \) that vanish at \( 0 \), and \( I' \), consisting of all functions in \( C(R) \) that vanish at \( 1 \). The ideal \( J \) is clearly not prime: for example, take \( a = i \) and \( a' = i - 1 \), and although neither function is in \( J \), \( aa' \) is in \( J \). In fact, by Theorem II.16, \( J \) cannot even contain a prime ideal.
The following theorem generalizes this result to arbitrary maximal ideals in $C(X)$, for any space $X$.

**THEOREM 11.17.** Every prime ideal in $C(X)$ is contained in a unique maximal ideal.

**PROOF.** By the maximal principle, every ideal is contained in at least one maximal ideal. If $M$ and $M'$ are distinct maximal ideals, then $M$ and $M'$ are both $z$-ideals, and $M \cap M'$ is a $z$-ideal. As shown previously, $M \cap M'$ is not prime; hence by Theorem II.16, $M \cap M'$ does not contain a prime ideal. Q.E.D.

**DEFINITION 11.12.** A $z$-filter $F$ is called a prime $z$-filter if whenever the union of two zero-sets belongs to $F$, then at least one of them belongs to $F$.

**THEOREM 11.18.** (a) If $P$ is a prime ideal in $C(X)$, then $Z(P)$ is a prime $z$-filter.

(b) If $F$ is a prime $z$-filter, then $Z^{-1}(F)$ is a prime $z$-ideal.

**PROOF.** (a) Suppose that $P$ is a prime ideal in $C(X)$. Let $Q = Z^{-1}[Z(P)]$; then $Z(Q) = Z(P)$, and $Q$ is a $z$-ideal containing a prime ideal. By Theorem II.16, $Q$ is prime. Suppose that $Z(f) \cup Z(g) \in Z(P)$ for some $f, g \in C(X)$. Then $Z(fg) \in Z(Q)$, and $fg$ belongs to the $z$-ideal $Q$. Since $Q$ is prime, either $f$ or $g$ is in $Q$. Suppose without loss of generality that $f \in Q$. Then $Z(f) \in Z(Q) = Z(P)$.

(b) Suppose that $F$ is a prime $z$-filter, and suppose that $fg \in P$ where $P = Z^{-1}(F)$. Then $Z(fg) = Z(f) \cup Z(g) \in Z(P) = F$. By hypothesis, either $Z(f)$ or $Z(g)$ belongs to $F = Z(P)$. 
Suppose that $Z(f) \subseteq Z(P)$. Then since $P$ is a z-ideal, $f$ is in $P$. Therefore, $P = Z^+(F)$ is a prime z-filter. Q.E.D.

From Theorems II.17 and II.18, it follows that a prime z-filter is contained in a unique z-ultrafilter. In addition, since every maximal ideal in $C(X)$ is prime, every z-ultrafilter is a prime z-filter.

In a discrete space, there is no difference between prime and maximal; that is, every prime filter $U$ is an ultrafilter. For if $A \notin U$, then $X \setminus A \in U$ since $A \cup (X \setminus A) = X \in U$, and $U$ is prime. Hence $A$ cannot be adjoined to $U$ since $A \cap (X \setminus A) = \emptyset$. 
CHAPTER III

ALEXANDROFF AND STONE-Cech COMPACTIFICATIONS

Up to this point, no conditions have been specified for the topological space $X$ on which the ring of all continuous functions has been defined. In Chapter IV, it will be useful to consider continuous functions defined on a locally compact Hausdorff space. Locally compact spaces will also be discussed in connection with one-point compactifications.

DEFINITION III.1. A topological space $X$ is said to be locally compact iff for each $x$ in $X$, there is an open set $0$ containing $x$ such that $\overline{0}$ is compact.

A locally compact space is clearly not necessarily compact — the real line $\mathbb{R}$, the Euclidean plane $\mathbb{R}^2$, and any discrete space containing infinitely many points are some examples of locally compact, noncompact spaces.

In considering a noncompact space $X$, it is often useful to construct a compact space $Y$ in which $X$ may be embedded as a dense subset; that is, $Y$ is a compact space for which there is a homeomorphism $\varphi$ of $X$ into $Y$ such that $\varphi(X)$ is dense in $Y$. Such a space $Y$ is called a compactification of $X$.

If the space $X$ is a locally compact Hausdorff space, then it is possible to compactify $X$ merely by adding to $X$ a point $w$ not in $X$. The point $w$ is called the point at
infinity, and \( X \cup \{w\} = A(X) \) is called the one-point, or Alexandroff, compactification of \( X \). A set \( 0 \) in the space \( A(X) \) is defined to be open either if \( 0 \) is open in the relative topology of \( X \), or if \( 0 \) is the complement of a compact set in \( X \), or if \( 0 = A(X) \).

**Theorem III.1.** If \( X \) is a locally compact space, then the open sets defined for \( A(X) \) form a topology for \( A(X) \).

**Proof.** By the definition of open sets in \( A(X) \), \( \emptyset \) and \( A(X) \) are both open. Assume that \( A \) and \( B \) are open sets in \( A(X) \). If \( B = \emptyset \), then \( B \cap A = \emptyset \), which is open. If \( B = A(X) \), then \( B \cap A = A \), which is open. If \( A \) and \( B \) are both open in \( X \), then clearly \( A \cap B \) is open in \( X \) and in \( A(X) \). If \( A \) is open in \( X \) and if \( B \) is the complement of a compact set \( K \) in \( X \), then \( K = 0 \cup \{w\} \), where \( 0 \) is open in \( X \); hence \( A \cap B = A \cap (0 \cup \{w\}) = A \cap 0 \), which is open in \( X \) and in \( A(X) \). If \( A = \hat{K} \) and \( B = \hat{M} \) for \( K \) and \( M \), compact sets in \( X \), then \( A \cap B = \hat{K} \cap \hat{M} = \hat{K} \cup \hat{M} \), which is open in \( A(X) \) since \( K \cup M \) is compact in \( X \). Therefore, for every pair \( A, B \) of open sets in \( A(X) \), \( A \cap B \) is open in \( A(X) \).

Next, let \( \{A_\alpha\}_{\alpha \in \mathcal{U}} \) be a collection of open sets in \( A(X) \). If \( A_\alpha \subseteq X \) for each \( \alpha \in \mathcal{U} \), then \( \bigcup A_\alpha \subseteq X \), and \( \bigcup A_\alpha \) is open in \( A(X) \). If there exists \( \alpha_0 \in \mathcal{U} \) such that \( A_{\alpha_0} = \hat{K} \) for some compact set \( K \) in \( X \), then \( K \setminus \{\bigcup A_\alpha\} \) is a closed subset of a compact set; hence \( \bigcup A_\alpha = K \setminus \{\bigcup A_\alpha\} \), and \( \bigcup A_\alpha \) is open in \( A(X) \). Q.E.D.

**Theorem III.2.** If \( X \) is a locally compact \( T_2 \) space, then \( A(X) \) is compact and \( T_2 \).

**Proof.** Let \( U \) be any open covering of \( A(X) \). Then there is an open set \( 0 \) in \( U \) containing \( w \), and \( A(X) \setminus 0 \) is a compact
set $K$ in $X$. For each $x$ in $K$, there is a $V_x \in U$ such that $x \in V_x \setminus \{w\} = V_x'$. Then $\{V_x' : x \in K\}$ forms an open cover of $K$, and there is a finite subset $\{V_{x_1}', V_{x_2}', \ldots, V_{x_m}'\}$ that also covers $K$. Then $\{V_{x_1}, V_{x_2}, \ldots, V_{x_m}\}$ also covers $K$, and $\{0, V_{x_1}, V_{x_2}, \ldots, V_{x_m}\}$ is a finite subset of $U$ which covers $A(X)$. Hence $A(X)$ is compact.

If $x$ and $y$ are points in $X$ such that $x \neq y$, then there exist disjoint open sets $O_1$ and $O_2$ in $X$ such that $x \in O_1$ and $y \in O_2$. If $x \in X$ and $y = w$, then since $X$ is locally compact, there exists an open set $O$ in $X$ such that $x \in O \subset \overline{O}$, where $\overline{O}$ is compact. Hence $\overline{O}$ is an open set containing $y = w$, and $O$ is an open set disjoint from $\overline{O}$ which contains $x$. Therefore, since any two distinct points in $A(X)$ can be separated by disjoint open sets, $A(X)$ is Hausdorff. Q.E.D.

The space $X$ and the space $A(X) \setminus \{w\}$ are identical and are, therefore, homeomorphic. In addition, the one-point compactification of any locally compact $T_2$ space $X$ is unique in that any two such compactifications are homeomorphic. To see this, let $A(X) = X \cup \{w\}$, and let $A'(X) = X \cup \{p\}$, where $w$ and $p$ are distinct points not in $X$, and define the mapping $\psi : A(X) \to A'(X)$ such that $\psi(x) = x$ for each $x \in X$ and $\psi(w) = p$. Defined in this way, $\psi$ is one-to-one, onto, and bi-continuous.

The following are some examples of locally compact $T_2$ spaces and their compactifications:

(1) The one-point compactification of $(a,b]$ is $[a,b]$. 
(2) The interval \([a, b]\) is also a compactification of \((a, b)\).

To form the one-point compactification of \((a, b)\), add the single point \(w\) instead, obtaining a homeomorphic copy of the unit circle in \(\mathbb{R}^2\).

(3) The one-point compactification of \(\mathbb{R}^2\) is any closed sphere \(S^2\).

One important use of the one-point compactification of a locally compact Hausdorff space is in proving that a locally compact Hausdorff space is completely regular. This theorem and another theorem concerning locally compact spaces follow.

**THEOREM III.3.** A locally compact \(T_2\) space is completely regular.

**PROOF.** If \(X\) is a locally compact \(T_2\) space, then \(A(X)\) is compact and \(T_2\); therefore, \(A(X)\) is normal. Let \(F\) be a closed set in \(X\) and let \(x \in X \setminus F\). The point \(#\{x\}\) and the set \(F \cup \{w\}\) are disjoint closed sets in \(A(X)\). By Urysohn's Lemma, there is a continuous real-valued function \(f\) on \(A(X)\) such that \(0 \leq f \leq 1\), \(f(F \cup \{w\}) = 0\), and \(f(x) = 1\). Hence \(f\big|_X\) is a continuous function on \(X\) which is \(0\) on \(F\) and \(1\) on \(x\). Q.E.D.

**THEOREM III.4.** Any open subset of a locally compact \(T_2\) space is locally compact.

**PROOF.** Let \(S\) be an open subset of \(X\), a locally compact \(T_2\) space. Pick \(t \in S\). Then there exists an open set \(\bar{0}\) such that \(t \in \bar{0}\) and \(\bar{0}\) is compact. The point \(t\) is contained in the open set \(0 \cap S\). Since \(X\) is regular by Theorem III.3, there exists an open set \(G\) such that \(t \in G \subset \bar{G} \subset (0 \cap S)\).
contained in the compact set $\bar{0}$, $G$ is compact. Hence $S$ is locally compact. Q.E.D.

The problem of extending continuous and bounded continuous functions on a subspace to the whole space was discussed in Chapter II. It would also be interesting to know that given a locally compact $T_2$ space $X$, a compact space $K$ containing $X$ could be found in which $X$ is $C$-embedded or $BC$-embedded.

The compact space $A(X)$ does not have the property that every continuous function on $X$ may be extended continuously to $A(X)$; in fact, this extension property does not hold even for bounded continuous functions. For example, consider the function $f(x) = \sin \frac{1}{x}$, defined on $(0,1]$. The sequences $\{x_n\}$ and $\{y_n\}$, with $x_n = \frac{1}{n!}$ and $y_n = \frac{2}{(1+4n)!}$ for $n = 1, 2, \ldots$, both converge to 0, but $\{f(x_n)\}$ converges to 0 and $\{f(y_n)\}$ converges to 1. Hence $f$ does not have a limit at 0, and it would be impossible to extend $f$ continuously to $[0,1]$, the one-point compactification of $(0,1]$.

However, since every locally compact $T_2$ space $X$ is completely regular, it is true that every bounded continuous function on $X$ may be extended continuously to $\beta(X)$, the Stone-Čech compactification of $X$.

The following theorem establishes the existence and several properties of the Stone-Čech compactification.

THEOREM III.5. Let $X$ be a completely regular topological space and $F$ the family of continuous real-valued
functions $f$ on $X$ with $|f| \leq 1$.

(a) If $I = [-1,1]$, then $X$ is homeomorphic with a set $E \subseteq I^F$.

(b) Each bounded continuous real-valued function on $X$ extends to a continuous function on $\beta(X)$.

(c) Defining $\beta(X) = \overline{E}$, $\beta(X)$ is a compactification of $X$, and $\beta(X)$ is unique in that if $Z$ is any other space with the same properties, then $Z$ and $\beta(X)$ are homeomorphic.

**PROOF.** (a) Define the mapping $\varphi: X \to I^F$ such that $\varphi$ maps a point $x \in X$ into its $F$-tuple $\{f(x)\}_{f \in F}$. Let $x, y \in X$ such that $x \neq y$. Since $X$ is completely regular, there exists a function $f_\alpha$ in $F$ such that $f_\alpha(x) = 0$ and $f_\alpha(y) = 1$. Hence the $f_\alpha$-th co-ordinates of $\varphi(x)$ and $\varphi(y)$ differ, and $\varphi(x) \neq \varphi(y)$. Therefore, the mapping $\varphi$ is one-to-one, mapping $X$ onto a subspace $E$ of $I^F$.

Let $\{x_a\}$ be a net in $X$ converging to $x \in X$. Then $\varphi(x_a) = \{f(x_a)\}_{f \in F}$, and $\varphi(x) = \{f(x)\}_{f \in F}$. Since $\{f(x_a)\}_{f \in F}$ converges to $f(x)$ for each $f$ in $F$, then $\{f(x_a)\}_{f \in F}$ converges to $\{f(x)\}_{f \in F}$. Therefore, $\varphi(x_a)$ converges to $\varphi(x)$, and $\varphi$ is continuous.

Conversely, if $\{f(x_a)\}_{f \in F}$ is a net in $E$ converging to $\{f(x)\}_{f \in F}$ in $E$, then since the net must converge co-ordinate-wise, it follows that $f(x_a) \to f(x)$ for each $f \in F$. Since $X$ is completely regular, $x_a \to x$, and $\varphi^{-1}$ is continuous.

Since $\varphi$ is a one-to-one bi-continuous mapping from $X$ onto $E$, $X$ and $E$ are homeomorphic.
(b) If \( f \) is a bounded, continuous, real-valued function on \( X \), then there exists a positive integer \( N \) such that \( |f(x)| < N \) for all \( x \in X \). Then \( g = (1/N)f \) is a continuous function bounded by 1; hence \( g \in \mathcal{F} \). Since for any \( x \in X \), \( \mathcal{H}_g [\mathcal{O}(x)] = g(x) \), \( \mathcal{H}_g \) is a function on \( C \), the image of \( X \), which equals the function \( g \) on \( X \). Since extension mappings are continuous, \( \mathcal{H}_g \) is continuous on \( \mathcal{F} \) and in particular, on \( \beta(X) \). Therefore, \( \mathcal{H}_g \) is a continuous extension of \( g \) to all of \( \beta(X) \), and \( N \mathcal{H}_g \) is a continuous extension of \( f \) to \( \beta(X) \).

(c) Each \( I = [-1, 1] \) is compact; therefore by the Tychonoff Product Theorem, \( I^\mathcal{F} \) is compact in the product topology. Since \( X \) is homeomorphic with \( E \), where \( \beta(X) = E \), \( X \) is embedded as a dense subset of \( \beta(X) \). Also, \( \beta(X) = E \) is a closed subset of the compact space \( I^\mathcal{F} \) and is, therefore, compact. Hence, \( \beta(X) \) is a compactification of \( X \).

Suppose that \( Z \) is another compact space with the same properties as \( \beta(X) \). Then there is a homeomorphism \( \sigma : X \to C \), where \( C = Z \). Since \( \Phi : X \to E \) is a homeomorphism where \( E = \beta(X) \), then \( \gamma = \sigma \Phi^{-1} \) is a homeomorphism between \( E \) and \( C \).

Suppose that \( p \) is a point in \( \beta(X) \setminus X \). Then there exists a net \( \{x_a\} \) in \( X \) that converges to \( p \). Since \( Z \) is compact, \( \{x_a\} \) must cluster in \( Z \); in addition, \( \{x_a\} \) must cluster in \( Z \setminus X \) since if \( \{x_a\} \) clusters in \( X \), then \( \{x_a\} \) could not converge to a point \( p \) not in \( X \).
Suppose that \( \{x_a^i\} \) clusters at two distinct points \( r \) and \( q \) in \( Z \setminus X \). Then there exist subnets \( \{x_a'^i\} \) and \( \{x_a''^i\} \) such that \( \{x_a'^i\} \to r \) and \( \{x_a''^i\} \to q \). Since \( Z \) is completely regular, there exists a function \( f \) in \( C(Z) \) such that \( f(r) \) and \( f(q) \) are not equal. Then \( f(x_a'^i) \to f(r) \), and \( f(x_a''^i) \to f(q) \).

Since \( f \bigg|_X \) is in \( C(X) \), by part (b) there exists an extension \( \hat{f} \) of \( f \bigg|_X \) in \( C[\beta(X)] \). Then \( \hat{f}(x_a'^i) = f(x_a'^i) \to f(p) \), and \( \hat{f}(x_a''^i) = f(x_a''^i) \to f(p) \), a contradiction. Hence, \( \{x_a^i\} \) clusters at only one point \( q \) in \( Z \setminus X \).

Suppose that \( \{x_a^i\} \) does not converge to \( q \). Then there exists a subnet \( \{x_a'^i\} \) of \( \{x_a^i\} \) that does not cluster at \( q \). Since each net in \( Z \) must cluster, there exists an \( r \neq q \) such that \( \{x_a'^i\} \not\to r \), a contradiction. Therefore, \( x_a \to q \).

Define the mapping \( \psi : \beta(X) \to Z \) such that \( \psi(x) = x \) for \( x \in X \), and \( \psi(p) = q \) for the \( p \in \beta(X) \setminus X \) and \( q \in Z \setminus X \) mentioned above. The mapping \( \psi \) is an onto mapping since for any \( q \) in \( Z \setminus X \), there exists a net \( \{x_a^i\} \) converging to \( q \), and \( \{x_a^i\} \) converges to some \( p \) in \( \beta(X) \); hence \( \psi(p) = q \). The mapping is one-to-one since if \( p_1 \) and \( p_2 \) are distinct points in \( \beta(X) \setminus X \), then there exist nets \( \{x_a^i\} \) and \( \{y_b^i\} \) such that \( \{x_a^i\} \) converges to \( p_1 \) and \( \{y_b^i\} \) converges to \( p_2 \). Then there exist distinct points \( q_1 \) and \( q_2 \) in \( Z \) such that \( \psi(p_1) = q_1 \) and \( \psi(p_2) = q_2 \); hence \( \psi(p_1) = \psi(p_2) \).

To see that \( \psi \) is continuous, note first that if \( f : Z \to \mathbb{R} \) is in \( C(Z) \), then there exists an \( \hat{f} \) in \( C[\beta(X)] \) such that \( \hat{f} \) is the extension of \( f \bigg|_X \) to \( \beta(X) \). If \( u \) is in \( Z \), there
exists some point \( t \) in \( \hat{\beta}(X) \) such that \( \psi(t) = u \). Then \( \hat{f}(t) = f[\psi(t)] = f(u) \).

Suppose that \( \{u_a\} \) is a net in \( Z \) converging to \( u \in Z \).

Since \( Z \) is completely regular, \( \{u_a\} \) converges to \( u \) iff \( \{f(u_a)\} \) converges to \( f(u) \) for each \( f \in C(Z) \). Let \( t_a = \psi^{-1}(u_a) \) and \( t = \psi^{-1}(u) \). Then \( f[\psi(t_a)] \) converges to \( f[\psi(t)] \) for each \( f \) in \( C(Z) \). But this means that \( \hat{f}(t_a) \) converges to \( \hat{f}(t) \) for each \( \hat{f} \) in \( C[\hat{\beta}(X)] \); this is true iff \( \{t_a\} \) converges to \( t \) since \( \hat{\beta}(X) \) is completely regular. Thus, \( \psi \) and \( \psi^{-1} \) are continuous, and \( \hat{\beta}(X) \) and \( Z \) are homeomorphic. Q.E.D.

Remark: It is stated in [2, p. 170] that \( X \) is always embedded as a dense open subset of \( \hat{\beta}(X) \). This is incorrect, for since a compact space is also locally compact, this would imply by Theorem III.4 that every completely regular space is locally compact. The correct statement of the theorem should be the following: The space \( X \) is embedded as a dense open subset of \( \hat{\beta}(X) \) iff \( X \) is locally compact.

From Theorem III.5, it is clear that if the space \( X \) is completely regular, then every function in \( BC(X) \) may be extended continuously to a continuous function on \( \hat{\beta}(X) \).

What is especially nice, however, is that if \( X \) is any topological space, then there is a completely regular space \( Y \) such that \( C(X) \) and \( C(Y) \) are isomorphic. Hence in considering rings of continuous functions, algebraically there is no need to consider more general spaces than completely
regular spaces. The following discussion leads to this important result.

DEFINITION III.2. A collection $B$ of closed sets in a space $X$ is a base for the closed sets in $X$ if every closed set in $X$ is an intersection of members of $B$. Equivalently, $B$ is a base if whenever $F$ is closed and $x \in X \setminus F$, there is a member of $B$ that contains $F$ but not $x$.

THEOREM III.6. A $T_2$ space $X$ is completely regular iff the family $Z(X)$ of all zero-sets is a base for the closed sets.

PROOF. Suppose that $X$ is completely regular. If $F$ is a closed set in $X$ and $x \in X \setminus F$, then there is an $f \in C(X)$ such that $F(F) = 0$ and $f(x) = 1$. Then $F \subseteq Z(f)$ and $x \notin Z(f)$. Hence $Z(X)$ is a base for the closed sets.

Conversely, suppose that $Z(X)$ is a base for the closed sets. Let $F$ be a closed subset of $X$ and $x \in X \setminus F$. Then there is a $g \in C(X)$ such that $F \subseteq Z(g)$ and $x \notin Z(g)$. Therefore, $g(x) = r$ for some $r \in \mathbb{R}$, $r \neq 0$. The function $f = gr^{-1}$ is in $C(X)$; $f(x) = (gr^{-1})(x) = rr^{-1} = 1$, and $f(F) = 0$. Q.E.D.

In addition, the following are true:

(a) Every closed set $F$ in a completely regular space is an intersection of zero-set-neighborhoods of $F$.
(b) Every neighborhood of a point in a completely regular space contains a zero-set-neighborhood of the point.

The topology on a space $X$ always determines the continuous functions from $X$ to $\mathbb{R}$; Theorem III.6 implies that
if $X$ is completely regular, then the converse is true: its topology is determined by the continuous real-valued functions defined on it.

**DEFINITION III.3.** If $X$ is an abstract set, and if $C'$ is a subfamily of the set of all real-valued functions on $X$, then the weak topology induced by $C'$ on $X$ is defined to be the smallest topology on $X$ such that all functions in $C'$ are continuous.

Hence, in a completely regular space $X$, the weak topology determined by all continuous real-valued functions is equivalent to the regular topology on $X$; conversely, if these two topologies are equivalent, then $X$ is completely regular. The following theorem is a restatement of this fact.

**THEOREM III.7.** If $X$ is a $T_2$ space whose topology is determined by some subfamily $C'$ of the set of all real-valued functions on $X$, then $X$ is completely regular.

**PROOF.** Since every function in $C'$ is continuous, then the weak topology induced by $C'$ is contained in the weak topology induced by $C(X)$. But the weak topology induced by $C(X)$ is always contained in the given topology of $X$, which is the topology induced by $C'$. Therefore, the two coincide, and $X$ must be completely regular. Q.E.D.

To obtain the weak topology, it is not necessary to consider preimages of all open sets in $R$ since preimages of subbasic open sets in $R$ constitute a subbase for the weak topology. A subbasic system of neighborhoods for a point
x in X is given by all sets of the form \( \{ y \in X : |f(x) - f(y)| < \varepsilon \} \), where \( f \in C' \) and \( \varepsilon > 0 \).

Similarly, a family of closed sets is a subbase for the closed sets if the finite unions of its members constitute a base. Since the closed rays form a subbase for the closed sets in \( \mathbb{R} \), their preimages form a subbase for the closed sets in \( X \): \( \{ x \in X : f(x) \geq r \} \) and \( \{ x \in X : f(x) \leq r \} \), for \( f \in C' \) and \( r \in \mathbb{R} \).

Since the continuous functions determine the topology of a completely regular space, they determine the continuous mappings into the space, as the next theorem shows.

**THEOREM III.8.** Let \( C' \) be a subfamily of \( C(Y) \) that determines the topology of \( Y \). A mapping \( \gamma \) from a space \( S \) into \( Y \) is continuous iff the composite function \( g \circ \gamma \) is in \( C(S) \) for every \( g \in C' \).

**PROOF.** Suppose that \( \gamma : S \to Y \) is continuous. Then for any \( g \in C' \), \( g \circ \gamma \) is the composition of two continuous functions and is, therefore, in \( C(S) \).

Next, suppose that for every \( g \in C' \), \( g \circ \gamma \) is in \( C(S) \). The sets of the form \( g^{-1}(F) \), where \( F \) is a closed set in \( \mathbb{R} \), have been shown to form a subbasis of the closed sets in \( S \). The set \( \gamma^{-1}[(g^{-1}(F))] = (g \circ \gamma)^{-1}(F) \) is closed in \( S \) since \( g \circ \gamma \) is continuous. Therefore, \( \gamma \) is continuous. Q.E.D.

**THEOREM III.9.** For every topological space \( X \), there exists a completely regular space \( Y \) and a continuous mapping \( \gamma \) of \( X \) onto \( Y \) such that the mapping \( g \to g \circ \gamma \) is an isomorphism of \( C(Y) \) onto \( C(X) \).
PROOF. Define $x \equiv x'$ in $X$ to mean that $f(x) = f(x')$ for all $f$ in $C(X)$. Let $Y$ be the set of all equivalence classes defined by this relation. Define the mapping $\tau$ from $X$ onto $Y$ as follows: $\tau x$ is the equivalence class that contains $x$. With each $f$ in $C(X)$, associate a function $g$, where $g$ is a real-valued function on $Y$, as follows: $g(y)$ is the common value of $f(x)$ at every point $x$ in $y$. Thus $f = g \circ \tau$. Let $C'$ denote the family of all such functions $g$; that is, $g \in C'$ iff $g \circ \tau \in C(X)$. Let the topology on $Y$ be the weak topology induced by $C'$. By definition, every function in $C'$ is continuous. Hence by Theorem III.8, $\tau$ is continuous.

If $y$ and $y'$ are distinct points of $Y$, then for some $g$ in $C'$, $g(y) \neq g(y')$. Therefore, there exist open sets $O_1$ and $O_2$ such that $g(y) \in O_1$ and $g(y') \in O_2$ and $O_1 \cap O_2 = \emptyset$. It follows that $g^{-1}(O_1) \cap g^{-1}(O_2) = \emptyset$, where $y \in g^{-1}(O_1)$ and $y' \in g^{-1}(O_2)$, and $Y$ is Hausdorff. Then by Theorem III.7, $Y$ is completely regular.

Finally, consider any function $h$ in $C(Y)$. Since $\tau$ is continuous, $h \circ \tau$ is continuous on $X$. Since $h \circ \tau \in C(X)$, then by the definition of $C'$, $h$ is in $C'$. Therefore $C(Y) \subseteq C'$, so $C(Y) = C'$. Hence the mapping $g \mapsto g \circ \tau$ is surjective. That $g \mapsto g \circ \tau$ is a homomorphism and injective is clear. Q.E.D.
CHAPTER IV

METRIC STRUCTURE

If the space K is compact, it is possible to define a metric on the set C(K) of all continuous complex-valued functions on K such that C(K) is a complete metric space. To see this, note that C(K) is a linear space since if f and g are in C(K), then af + bg is in C(K) for any complex numbers a and b. In addition, since K is compact, all continuous functions on K are bounded, and the uniform norm may be defined on C(K): \[ ||f|| = \max_{x \in K} |f(x)| \]. The space is also complete in this norm since any Cauchy sequence of functions in C(K) converges to a function in C(K). Hence C(K) is a Banach space -- a complete normed linear space. The space C(K) becomes a metric space by defining \[ d(f, g) = ||f - g|| \].

Similarly, if X is an arbitrary topological space, the space BC(X) of all continuous bounded complex-valued functions on X is a complete metric space, with the same metric defined as above.

If H is a locally compact space, two subspaces of C(H) will be defined in the following way.

DEFINITION IV.1. The space C_0(H) consists of all functions f in C(H) with the property that for a given \( \epsilon > 0 \),
there exists a compact set $K$ such that if $t$ is a point not in $K$, then $|f(t)| < \xi$.

**DEFINITION IV.2.** The space $C_c(H)$ consists of all functions in $C(H)$ with compact support; that is, all functions $f$ for which there exists a compact set $K$ such that $f(x) = 0$ for $x \notin K$.

If $f$ is a continuous function on $A(H)$, the one-point compactification of $H$, such that $f(w) = 0$, then for a given $\xi > 0$, there is an open set $U$ containing $w$ such that $f(U) < \xi$. But since $U$ is an open set containing $w$, $U = \tilde{K}$ for some compact set $K$ in $H$. It follows that $f(x) < \xi$ for $x \notin K$, and $f|_H$ is in $C_0(H)$. Similarly, for any function $g$ in $C_0(H)$, if $g$ is extended continuously to $g'$ on $A(X)$, then $g'$ must be zero at $w$ since $g$ is arbitrarily small in neighborhoods around $w$. It is clear, therefore, that $C_0(H)$ is isometric and isomorphic to the set of all functions in $C[A(H)]$ which vanish at $w$.

**THEOREM IV.1.** The space $C_0(H)$ is complete with respect to the uniform norm topology.

**PROOF.** Let $\{f_n\}$ be a Cauchy sequence in $C_0(H)$. Then $\{f_n\}$ may be interpreted as a Cauchy sequence in $C_w[A(H)]$, the set of all functions in $C[A(H)]$ that vanish at $w$. Since $C[A(H)]$ is complete, there exists an $f \in C[A(H)]$ such that $f_n \to f$. But then $f_n \to f$ pointwise, and since $f_n(w) = 0$ for each $n$, it follows that $f(w) = 0$. Therefore $f|_H$ is in $C_0(H)$, and $C_0(H)$ is complete. Q.E.D.
If the space K is compact, then $C_0(K)$ has an identity since the identity function $f_\equiv 1$ on K may be considered to be small outside the compact set K. If a space X is not compact, however, $C_0(X)$ cannot contain an identity f since f can be less than one nowhere on X.

**DEFINITION IV.3.** A net $\{e_a\}_{a \in A}$ in $BC(H)$ is said to be an approximate identity for $BC(H)$ if for any f in $BC(H)$,

$$\lim_{a} e_a f = f$$

in the uniform norm.

**THEOREM IV.2.** If H is a locally compact, noncompact space, then $C_0(H)$ does not contain an identity, but $C_0(H)$ does have an approximate identity.

**PROOF.** From a previous discussion, it is clear that $C_0(H)$ does not contain an identity.

Let $M$ be the set of all compact subsets of H. The set M is clearly not empty since H is locally compact, and for every $x \in H$ there is a compact set in H containing x. Define the relation "$\subseteq$" on elements $K_1$ and $K_2$ in M such that $K_1 \subseteq K_2$ iff $K_1 \subset K_2$. Thus M is a directed set. For each set K in M, there exists a function $e_K$ in $C_0(H)$ such that $e_K(K) = 1$, and $e_K$ is zero outside of some larger compact set.

Let $f$ be a function in $C_0(H)$, and let $\epsilon > 0$ be given. Then there exists a compact set $K_\epsilon$ such that if $x \notin K_\epsilon$, then $|f(x)| < \epsilon$. For $x \in K_\epsilon$, $e_{K_\epsilon}(x) = 1$; hence $(e_{K_\epsilon} \cdot f)(x) = f(x)$, and $|f(x) - (e_{K_\epsilon} \cdot f)(x)| = 0$. For $x \notin K_\epsilon$, $e_{K_\epsilon}(x) < 1$ and $|f(x)| < \epsilon$; hence $|e_{K_\epsilon} \cdot f| < \epsilon$, and $|f(x) - (e_{K_\epsilon} \cdot f)(x)| < \epsilon$. Thus, $\lim_{K \in M} e_K f = f$. Q.E.D.
Note: Theorem IV.2 also holds for $C_c(H)$ since the approximate identity $\{e_k\}$ is contained in $C_c(H)$.

**THEOREM IV.3.** The completion of the space $C_c(H)$ is $C_0(H)$.

**PROOF.** Clearly, $C_c(H) \subseteq C_0(H)$. Let $f$ be any function in $C_0(H)$. Since the approximate identity $\{e_k\}$ is contained in $C_c(H)$, then for each $K$ in $M$, $e_kf$ is contained in $C_c(H)$. The fact that $\{e_kf\}_{K \in M}$ is a net in $C_c(H)$ that converges to $f$ implies that $f$ is a limit point of $C_c(H)$. Therefore, $C_c(H) = C_0(H)$. Q.E.D.

The next few theorems are concerned with an identification between a compact set $K$ and the set of all maximal ideals in $C(K)$.

**THEOREM IV.4.** (a) If $K$ is a compact space and $I$ is an ideal in $C(K)$, then there exists some point $x$ in $K$ such that $f(x) = 0$ for each $f$ in $I$.

(b) If $K$ is a compact space and $M$ is a maximal ideal in $C(K)$, then there exists exactly one point $x$ in $K$ such that $M = \{f \in C(K) : f(x) = 0\}$.

**PROOF.** (a) Suppose that for each $x$ in $K$, there exists an $f_x$ in $I$ such that $f_x(x) = 0$. Then there exists an open set $O_x$ in $K$ such that $x \in O_x$ and $f(O_x) \neq 0$. The set $\{O_x\}_{x \in K}$ is an open covering of $K$. Hence there exists a finite sub-cover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_r}\}$. If $f$ is in $I$, then $\overline{f}$ (f conjugate) is in $C(K)$; it follows that $f \cdot \overline{f} = |f|^2$ is in $I$. Therefore the function $\hat{g} = |f_{x_1}|^2 + |f_{x_2}|^2 + \ldots + |f_{x_r}|^2$ is in $I$, and
\[ Z(g) = \emptyset \]. Therefore, \( g \) is a unit, and \( I \) is not an ideal, a contradiction.

(b) Suppose that \( M \) is a maximal ideal in \( C(K) \). Then by (a), there exists some \( x \in K \) such that \( f(x) = 0 \) for all \( f \in M \). Let \( J = \{ f \in C(K) : f(x) = 0 \} \). The set \( J \) is clearly an ideal containing \( M \); hence the maximality of \( M \) implies that \( J = M \).

Suppose next that there are two distinct points \( x \) and \( y \) at which every function in \( M \) is zero. Since \( K \) is compact, \( K \) is normal, and by Urysohn's Lemma, there exists a continuous function \( g \) such that \( g(x) = 0 \) and \( g(y) = 1 \). It follows that the ideal \( J \) contains \( M \) properly, which is a contradiction. Q.E.D.

THEOREM IV.5. If \( K \) is a compact space and \( M \) is a maximal ideal in \( C(K) \), then \( M \) is closed with respect to the uniform norm topology.

PROOF. Since \( M \) is maximal, there exists some \( \bar{x} \) in \( K \) such that \( M = \{ f \in C(K) : f(\bar{x}) = 0 \} \). Let \( \{ f_n \} \) be a Cauchy sequence in \( M \). Since \( C(K) \) is complete, \( f_n \to f \) for some \( f \) in \( C(K) \). But uniform convergence implies pointwise convergence. In particular, \( f_n(\bar{x}) \to f(\bar{x}) \), and since \( f_n(\bar{x}) = 0 \) for each \( n \), it follows that \( f(\bar{x}) = 0 \). Therefore \( f \) is in \( M \), and \( M \) is closed. Q.E.D.

An interesting example of a maximal ideal is the set \( C_w[A(H)] \) of all functions in \( C[A(H)] \) vanishing at \( w \). Since this set is closed by the preceding theorem, and since this
set is isometric and isomorphic to the set $C_0(H)$, then $C_0(H)$ must be closed as well. This provides another proof of the fact that $C_0(H)$ is complete.

It is clear that there is a natural one-to-one correspondence between the compact space $K$ and the set of all maximal ideals in $K$. There is also a correspondence between the maximal ideals and complex-valued homomorphisms on $C(K)$.

**THEOREM IV.6.** If $K$ is a compact space and $M$ is a maximal ideal in $C(K)$, then $M$ corresponds naturally to a continuous complex-valued homomorphism on $C(K)$.

**PROOF.** By Theorem IV.4, $M = \{ f \in C(K): f(x) = 0 \}$ for some $x$ in $K$. Consider the complex-valued function $\hat{x}$ on $C(K)$, where $\hat{x}(f) = f(x)$ for each $f$ in $C(K)$. Since

$$\hat{x}(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha \hat{x}(f) + \beta \hat{x}(g),$$

and $\hat{x}(fg)(x) = f(x)g(x) = \hat{x}(f)\hat{x}(g)$, then $\hat{x}$ is a homomorphism. Also, $\hat{x}$ is continuous since if $f_n \to f$, then $f_n(x) \to f(x)$ for each $x$ in $K$, and $\hat{x}(f_n) \to \hat{x}(f)$. In addition, if $f \in M$, then $f(x) = 0$, and it follows that $\hat{x}(f) = 0$. Therefore $f \in \hat{x}^{-1}(0)$. Similarly, if $g \in \hat{x}^{-1}(0)$, then $\hat{x}(g) = 0$; thus $g(x) = 0$ and $g \in M$. Thus, $\hat{x}^{-1}(0) = M$. Q.E.D.

**THEOREM IV.7.** Let $h$ be a non-zero complex homomorphism on $C(H)$.

(a) Then $h$ must be continuous with respect to the uniform norm topology.

(b) The set $h^{-1}(0)$ is a maximal ideal.

(c) If $h$ and $g$ are complex homomorphisms on $C(H)$ and if $h^{-1}(0) = g^{-1}(0)$, then $h = g$. 
PROOF. (a) Since \( h \) being bounded implies that \( h \) is continuous, it suffices to show that \( h \) is bounded. The norm of \( h \) is defined as follows: 
\[
\| h \| = \sup_{\| f \| \leq 1} \{ h(f) \}.
\]
Since \( \| f \| \leq 1 \) implies that the range of \( f \) is bounded by 1, to show that \( h \) is bounded, it suffices to show that \( h(f) \) is contained in \( \mathbb{R}_f \), the range of \( f \).

Suppose \( h(f) \) is not contained in \( \mathbb{R}_f \). If \( 1 \) is the function that is identically one, then \( h(f) \cdot 1 = h(f) \). Since \( h(f) \notin \mathbb{R}_f \), then \( g = h(f) \cdot 1 - f \) is never zero, and it follows that \( g^{-1} \) exists. Then

\[
\begin{align*}
    h(g) &= h\left[ h(f) \cdot 1 - f \right] \\
        &= h\left[ h(f) \cdot 1 \right] - h(f) \\
        &= h(f) \cdot h(1) - h(f) \\
        &= h(f) - h(f) \\
        &= 0.
\end{align*}
\]

Thus \( 0 = h(g) = h(g) \cdot h(g^{-1}) = h(g \cdot g^{-1}) = h(1) \), a contradiction.

(b) It is clear that \( h^{-1}(0) \) is an ideal. If \( L : E \rightarrow \mathbb{F} \) is any linear functional on an arbitrary Banach space \( E \), consider \( \text{Ker}(L) \oplus \langle z \rangle \), where \( z \notin \text{Ker}(L) \), and \( \langle z \rangle = \{ \alpha z : \alpha \in \mathbb{C} \} \).

Suppose that \( w \) is in \( E \). If \( w \) is in \( \text{Ker}(L) \), then \( w \) is in \( \text{Ker}(L) \oplus \langle z \rangle \). If \( w \notin \text{Ker}(L) \), then \( L(w) \neq 0 \). Also, \( L(z) \neq 0 \), and

\[
L(w) = \left[ L(w) / L(z) \right] L(z) = L\left[ \frac{L(w)}{L(z)} \cdot z \right].
\]

Therefore \( w - \frac{L(w)}{L(z)} \cdot z \) is in \( \text{Ker}(L) \), and \( w = \left[ w - \frac{L(w)}{L(z)} \cdot z \right] + \frac{L(w)}{L(z)} \cdot z \) is in \( \text{Ker}(L) \oplus \langle z \rangle \). Therefore, \( E = \text{Ker}(L) \oplus \langle z \rangle \), and \( \text{Ker}(L) \) is a maximal proper subspace. Since \( h^{-1}(0) \) is an ideal and is maximal, then \( h^{-1}(0) \) is a maximal ideal. Q.E.D.
(c) If \( h^{-1}(0) = g^{-1}(0) \), then there exists a maximal ideal \( M_x \) such that \( M_x = \{ f \in \mathcal{C}(X) : f(x) = 0 \} \), and such that \( M_x = \left( h^{-1}(0) = g^{-1}(0) \right) \). Since \( M_x \) is maximal, if \( f \notin M_x \), then 
\[ M_x \oplus \langle f \rangle = \mathcal{C}(X). \]
Then there exists some \( k \in M_x \) such that \( k + \alpha f = 1 \), where \( 1 \) is the identically 1 function. Then
\[ 1 = h(1) = h(k + \alpha f) = h(k) + h(\alpha f) = h(\alpha f) = \alpha h(f); \]
also, \( 1 = g(1) = g(k + \alpha f) = \alpha g(f) \). Therefore \( g(f) = h(f) \), and \( g = h \).

Q.E.D.

From these results, it follows that there is a natural one-to-one correspondence between the maximal ideals in \( \mathcal{C}(K) \) and the complex-valued homomorphisms on \( \mathcal{C}(K) \). It also follows that the only type of homomorphisms on \( \mathcal{C}(K) \) are the point-evaluation homomorphisms (\( \hat{x} \), for each \( x \) in \( K \)).

In fact, it is possible to define a topology on the set \( S = \{ \hat{x} : x \in K \} \) such that \( S \) and \( K \) will be homeomorphic. This topology is the Gelfand topology, defined as follows:
\[ \hat{x}_a \to \hat{x} \iff \hat{x}_a(f) \to \hat{x}(f) \text{ for each } f \text{ in } \mathcal{C}(K); \text{ that is, } \]
\[ \hat{x}_a \to \hat{x} \iff f(\hat{x}_a) \to f(x) \text{ for each } f \text{ in } \mathcal{C}(K). \]

**THEOREM IV.8.** If \( K \) is a compact \( T_2 \) space, then the space \( S = \{ \hat{x} : x \in K \} \) (with the Gelfand topology) is homeomorphic with the space \( K \).

**PROOF.** To show that \( S \) is \( T_2 \), suppose that \( \hat{x}_a \to \hat{x} \) and \( \hat{x}_a \to \hat{y} \) where \( \hat{x} \neq \hat{y} \). Then \( f(\hat{x}_a) \to f(x) \) and \( f(\hat{x}_a) \to f(y) \) for each \( f \) in \( \mathcal{C}(K) \); hence \( f(x) = f(y) \) for each \( f \). By Urysohn's Lemma, if \( f(x) = f(y) \) for every \( f \) in \( \mathcal{C}(K) \), then \( x = y \), a contradiction. Therefore a net \( \{ \hat{x}_a \} \) in \( S \) converges to only one point, and by Theorem I.3, \( S \) is \( T_2 \).
To show S is compact, let \( \{ x_a \} \) be a net in S. Since K is compact, by Theorem I.1 it follows that \( x_a \to x \) for some \( x \in K \). Then for each \( f \in C(K) \), \( f(x_a) \to f(x) \). Therefore \( \hat{x}_a \to \hat{x} \), and S is compact.

Define the mapping \( \Lambda : K \to S \) such that \( \Lambda(x) = \hat{x} \). Then if \( x_a \to x \) in K, it follows that \( f(x_a) \to f(x) \) for each \( f \) in \( C(K) \). Therefore \( \hat{x}_a \to \hat{x} \), and \( \Lambda \) is a continuous mapping.

Since K and S are compact \( T_2 \) spaces such that \( \Lambda : K \to S \) is continuous (and clearly \( \Lambda \) is a bijection), then \( \Lambda \) is a homeomorphism. Q.E.D.
BIBLIOGRAPHY
