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**NONLINEAR ASPECTS OF
SHOCK RESPONSE IN
ISOLATED ACCELEROMETERS**

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Thomas L. Paez
Experimental Mechanics Department
Sandia National Laboratories
Albuquerque, New Mexico

Norman Hunter
Vibration Testing Section
Los Alamos National Laboratory
Los Alamos, New Mexico

Biographies

Thomas Paez is a senior member of the technical staff in the Experimental Mechanics Department at Sandia National Laboratories. He earned his BS and MS degrees in Civil Engineering at the University of New Mexico, and his PhD at Purdue University. He has worked in both education and industry and has nineteen years of experience in random vibration, structural mechanics, and mechanical system testing.

Norman Hunter is a staff member in the Vibration Testing Section at Los Alamos National Laboratory. He earned his BS degree in Electrical Engineering at Mississippi State University and his MS at the University of New Mexico. He has been involved in testing at Los Alamos National Laboratory and Sandia National Laboratories, and has twenty-eight years of experience in mechanical system testing, test analysis and control, and chaotic vibrations.

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Abstract

Numerous investigations have studied the potential for chaotic vibrations of nonlinear systems. It has been shown for many simple nonlinear systems, that when they are excited severely enough, or with the appropriate parametric combinations, that they will execute chaotic vibrations. The present investigation considers the potential for the occurrence of chaos in a practical nonlinear system - the isolated accelerometer. A simple, first order model is proposed for the isolated accelerometer, and it is shown that chaos can occur in the isolated accelerometer. A preliminary investigation into the bearing that this chaos potential has on the measurement of shock response is summarized.

Introduction

Recognition of the possibility of chaotic vibrations of deterministic systems is usually traced to Henri Poincare [1], a French mathematician and philosopher who lived and practiced at the turn of the century. He recognized that the classical equations of physical systems driven by deterministic excitations can yield unpredictable solutions, and that small changes in the initial conditions can lead to large changes in the final solution of some nonlinear problems. These are some of the characteristics of the

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phenomenon that we now call chaos. The fundamental ideas and some advanced topics in chaotic vibrations are discussed in a number of texts, for example, References 2 and 3.

Chaotic vibrations occur in nonlinear systems with the appropriate parameters, sometimes in response to initial conditions only, and sometimes in response to periodic, deterministic excitations with the appropriate parameters. In typical physical systems, one manifestation of chaos is an aperiodic response to periodic inputs. For example, it will be shown later that some sinusoidal inputs with the appropriate frequencies and amplitudes, excite aperiodic responses in a single-degree-of-freedom (SDF) system with linear viscous damping and stiffness that is governed by a tangent function.

Another manifestation of chaos in nonlinear systems is known as sensitive dependence on initial conditions. This means that when a response is chaotic, small changes in the initial conditions of the system can lead to large differences in the responses. This characteristic of chaotic response is related to quantities known as the Lyapunov exponents of a dynamic system, and has associated with it a graphic explanation of why chaos occurs. (See Reference 4.) The explanation is outlined here.

Every physical system has equations governing its motion, and these can be written in a general form as

$$\dot{\{z\}} = g(\{z\}, t) \quad (1)$$

where $\{z\}$ is the vector of state variables, and $g(\dots)$ is a vector function (in general, nonlinear) of system parameters, the state variables and other functions of time (namely, the input). The dot denotes differentiation with respect to time. Equation 1 presents a state space representation, described, for example, in Reference 7. This representation tracks the response through the space of response measures, like displacement, velocity, etc. When the system parameters are selected and the input is established Eq. 1 can be solved to determine the sys-

tem response. The response that evolves from a specific set of initial conditions establishes a trajectory in the state space of the system.

When the function $g(\dots)$ is nonlinear, there is associated with Eq. 1 a variational equation that establishes a local (in time) linear approximation to the equation of motion. It is denoted

$$\dot{\{z\}} = [a]\{z\} \quad (2)$$

where $[a]$ is the constant coefficient matrix whose i th row and j th column element is defined as

$$a_{ij} = \frac{\partial g_i}{\partial z_j} \quad (2a)$$

where g_i is the i th row in the vector function on the right hand side in Eq. 1. The matrix $[a]$ is known as the system Jacobian. Equation 2 indicates that over a short period of time the response is approximately governed by a linear, first order, ordinary differential equation. The response trajectory over the period of time governed by Eq. 2 is the single response related to the initial conditions at the beginning of the short time period. The collection of possible responses that might evolve if the initial conditions at the beginning of the short time period were changed by a small amount is governed by the characteristics of the matrix $[a]$. In particular, the evolution of the response relates directly to the eigenvalues of the matrix $[a]$.

To consider this issue graphically, refer to the diagrams in Figures 1a, 1b, and 1c. Consider the potential response trajectories that the system under consideration can execute over the short period of time when Eq. 2 governs system motion. At the initial time the response is described by a point in the state space (shown by the dot in the circle in each of the figures). The response evolves over time to another value (shown by the dot in the ellipse in each of the figures). Other initial conditions can be considered, for example, the set of points in an

Figure 1a. Unstable state space trajectories.

Figure 1b. Stable state space trajectories.

Figure 1c. Chaotic state space trajectories.

infinitesimal spheroid surrounding the actual initial condition (shown by a circle in each figure). These other initial conditions would evolve into other response trajectories, and would end up near the actual final point in the trajectory. Specifically, they would evolve into responses in an infinitesimal ellipsoid surrounding the actual final point (shown by an ellipse in each of the figures). The trajectories starting in a spheroid evolve into responses falling in an ellipsoid because of the linear approximation made in Eq. 2. The relation between the ellipsoid and the spheroid establishes the character of system response. This issue needs to be considered because in real systems finite perturbations to any given trajectory always occur.

There are three situations that can occur. First, if the volume of the ellipsoid is greater than the volume of the spheroid then the response is locally (in time) unstable. In this situation, at least one of the eigenvalues of $[a]$ is positive, and the sum of all the eigenvalues is also positive. If this behavior predominates over a long period of time, then the response of the system will diverge, i.e., system motion will be unstable. The Lyapunov exponents of the dynamic system quantify the average rate of change of the major axes of the ellipsoid, and thereby characterize its stability or instability. The sum of the Lyapunov exponents of this system is positive, and reflects the general expansion of trajectories in the state space.

Second, if the volume of the ellipsoid is smaller than the volume of the spheroid and the largest of the major axes of the ellipsoid is smaller than the diameter of the spheroid, then the response is locally stable. In this situation, the eigenvalues of the matrix $[a]$ are all negative. If this behavior predominates over a long period of time, then the response of the system will converge, i.e., system motion will be stable. The Lyapunov exponents of this system are all negative, and reflect the general contraction of trajectories in the state space.

Third, if the volume of the ellipsoid is smaller

than the volume of the spheroid and the largest major axis of the ellipsoid is greater than the diameter of the spheroid, then the response is locally chaotic. In this situation at least some of the eigenvalues of $[a]$ are positive and some are negative. The sum of the eigenvalues is negative. If this behavior predominates over a long period of time, then the response will be chaotic. At least one of the Lyapunov exponents of the system is positive, and one or more are negative. These reflect the long term expansion of trajectories in some directions in state space and the long term contraction of trajectories in other directions in state space.

The potential for chaotic vibrations is a matter of concern in the application of any nonlinear mechanical system. In relation to the isolated accelerometer this potential for chaos means that similar inputs may lead to drastically different accelerometer responses. If chaotic motion occurs during the use of an isolated accelerometer then the acceleration readings do not accurately characterize the measured environment.

This is an important consideration because the use of isolated accelerometers is practically essential in some applications. Specifically, isolation is needed when it is necessary to measure environments that have signal content over a wide spectrum of frequencies which includes the resonant frequency of the accelerometer. When an accelerometer is excited at its resonant frequency it is driven into a nonlinear response range, and it gives an incorrect estimate of the acceleration waveform at all input frequencies.

Equation of Motion for the Isolated System

In modelling an isolated accelerometer, it is assumed that the isolated accelerometer is constructed by enclosing a standard, commercial accelerometer in an external case. The accelerometer is separated from the external case by an elastic, viscously damped material. The

accelerometer case and the external case are assumed to be very stiff relative to the isolating material. It is assumed that the accelerometer is symmetrically isolated from the external case.

The mathematical model to be used to simulate the behavior of the isolated accelerometer is a simple, first order approximation of an actual isolated accelerometer. Specifically, the isolated accelerometer is modeled as a rigid mass on a nonlinear spring, as shown in Fig. 2. In the figure, the mass simulates the accelerometer, and the nonlinear spring simulates the material used to isolate the accelerometer from its external case. The equation governing motion of this system is

$$\ddot{z} = -c\dot{y} - R(y) \quad (3)$$

where m is the mass of the accelerometer, c is the viscous damping of the isolation material, $R(y)$ is the spring restoring force function, z is the absolute displacement of the mass, x is the absolute displacement of the base, $y = z - x$ is the relative displacement of the mass, and dots denote differentiation with respect to time. Figure 3 shows the restoring force function. It is nearly linear for small displacements, and when the displacement nears plus or minus y_m , the restoring force approaches infinity (theoretically). The restoring force function is taken to be a tangent function in the following. In the model, y_m is the thickness of the isolation material.

The equation of motion can be normalized by dividing through by m , and expressing z in terms of y and x . The result is

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 \frac{2y_m}{\pi} \tan\left(\frac{\pi y}{2y_m}\right) = -\ddot{x} \quad (4)$$

where ω_n is the natural frequency of the system when it responds to a low level excitation and ζ is the viscous damping factor.

Figure 2. Schematic of isolated accelerometer.

Figure 3. Restoring force curve.

In our investigation of chaotic vibrations we are interested in considering the response of the system to a sinusoidal input. Therefore, we define the base acceleration as follows.

$$-\ddot{x} = A \cos(\omega_p t) \quad (5)$$

where A is the amplitude of the input acceleration and ω_p is its circular frequency. Substitute Eq. 5 into Eq. 4. The resulting expression can be further normalized with respect to time by setting

$$\tau = \omega_p t \quad (6)$$

The reason we have chosen to use ω_p in the temporal normalization instead of ω_n is that we are interested in studying the behavior of the long-term response, and if it is periodic, its period will be an integer multiple of the period of the input. When Eq. 6 is used in Eq. 5 and the resulting expression is divided by ω_p^2 the result is

$$y'' + \frac{2\zeta}{\alpha} y' + \frac{2y_m}{\pi\alpha^2} \tan\left(\frac{\pi y}{2 y_m}\right) = \frac{A}{\omega_p^2} \cos(\tau) \quad (7)$$

where $\alpha = \omega_p / \omega_n$ is the ratio of input frequency to system small displacement natural frequency, and a prime denotes differentiation with respect to τ .

We can normalize the equation with respect to response amplitude by dividing through by the absolute value of the maximum potential displacement, y_m . The resulting equation is

$$Y'' + \frac{2\zeta}{\alpha} Y' + \frac{2}{\pi\alpha^2} \tan\left(\frac{\pi Y}{2}\right) = B \cos(\tau) \quad (8)$$

where Y is the ratio of actual displacement to maximum potential displacement and

$$B = \frac{A}{\omega_p^2 y_m} \quad (8a)$$

is the normalized input amplitude. Equation 8 is the equation we investigate in our search for chaos in the motion of an isolated accelerometer.

The Shooting Technique for Chaos Detection

There are several approaches to establish whether or not chaos occurs in a system. Among the most direct is the simple observation of the response (or some measure of the response) to specific initial conditions and a sinusoidal input over a long period of time. When the response is aperiodic, then the response is in a chaotic regime. However, it is not always clear when chaos occurs when we directly observe the response because in some situations many cycles of periodic response occur before the system enters a chaotic regime, and in other situations it takes many cycles for the initial transient to die out before the periodic response is realized.

To avoid this problem we can directly seek periodic solutions to an equation of motion to establish their existence. If we can show that they do not exist then there is chaos. The shooting technique is a method for seeking periodic solutions to an equation of motion. In the application of this technique we start the system response with arbitrary initial conditions and compute the solution over one period of the sinusoidal input. We compare the final state of the system in state space to the initial conditions and modify the initial conditions until the initial conditions and final state in the state space match as closely as possible. (We use a minimization technique to accomplish this.) If the initial conditions and final state match identically then the solution is called "one periodic." It is called one periodic because the system executes one period of response during one period of the input. (Note that an identical match for the exact equation is impossible because of computation roundoff and the fact that we must use an approximate method to compute the solution of the nonlinear differential equation. Therefore, judgement must be used in deciding whether a match exists.)

If a periodic solution is not found in the manner

described above then we repeat the procedure where we start the system response with arbitrary initial conditions and compute the solution over two periods of the sinusoidal input. If a periodic solution is found then it is called "two periodic" for obvious reasons. While a periodic response is not found we repeat the procedure seeking at each step the n periodic response, where n increases to some reasonable limit. If a periodic response is established then this characterizes the system in some sense. If none is found then the response is assumed to have achieved or to be approaching a chaotic state. That is, the input and system parameters in the vicinity of the ones being tested yield a chaotic solution.

A combination of the shooting technique and the direct observation of system responses was used to assess the potential for chaos in the mathematical model for the isolated accelerometer. The existence of chaos was studied for a collection of parameters covering the grid shown in Fig. 4. The damping value chosen for the investigation is ten percent. At each of ten α values and each of ten B values shown on the grid, periodic solutions were sought. Whether or not one was found was noted, and the area on the graph that is shaded denotes the area where no periodic solutions were found. It is assumed that system responses are chaotic when the input and system have parameters in this region. Figure 5 shows an example of a periodic solution, and Fig. 6 shows an example of a chaotic solution.

Shock Response of the Isolated Accelerometer

In order to consider in detail the shock response of an isolated accelerometer, we consider the system of Eq 8. Simulation of these equations is accomplished using a fourth order Runge-Kutta algorithm with variable stepsize.

We illustrate the complex nature of the time history response of an isolated accelerometer by considering the following drive.

Figure 4. Chaotic region of the normalized, isolated accelerometer.

$$\ddot{x} = 0.2 \sin\left(\frac{1}{4} \frac{2\pi t}{T}\right) + g \sin\left(4 \frac{2\pi t}{T}\right), 0 \leq t \leq 2T$$

$$= 0, \quad \text{elsewhere}$$

(9)

Here $T=2\pi/\omega_n$ is the normalized natural period of the accelerometer, so we are exciting the system with inputs of 1/4 the natural frequency and four times the natural frequency, simultaneously. We desire to measure only the lower frequency at 1/4 the natural frequency while "isolating", or filtering out, the higher frequency. In the context of a chaotic response to this input the critical parameter is the gain factor g which multiplies the higher frequency term. Three cases are considered: $g=0.2$ where the response is clearly nonchaotic, $g=3.2$, where the response is clearly chaotic, and $g=0.8$, where the response is not chaotic, but the acceleration response does not accurately reflect the nature of the lower frequency term in Eq. 9.

In the first case, we excite the isolated

Figure 5. Periodic response for $\alpha=1.0$, $B=0.2$.

Figure 6. Chaotic response for $\alpha=3.5$, $B=0.6$.

accelerometer with an unwanted waveform at four times its natural frequency and a gain, g , of 0.2. This excitation maintains the accelerometer inside its predominantly linear response range and results in the response shown in Fig. 7a. We expect to obtain only the half sine extending from $T=0$ to $T=2.0$. This is indeed most of the response, though some contamination occurs from excitation of the natural frequency at $T>2.0$ and from the input

at four times the natural frequency at $T \leq 2.0$. In a typical analysis case, the measured response would be filtered at a frequency below the accelerometer's natural frequency. The effects of this filtering (at 0.5 times the natural frequency) are illustrated in Fig. 7b. We judge this approximation of the half sine input to be a reasonable one.

Figure 7a. Input acceleration and response of isolated accelerometer for $g=0.2$.

Figure 7b. Input acceleration and filtered response of isolated accelerometer for $g=0.2$.

Figures 8a and 8b illustrate the results when the gain is increased to 3.2, a factor 16 times higher than that used in the previous case. The unfiltered response clearly indicates very

nonlinear behavior in the measured acceleration response. In fact, we have excited the accelerometer to levels where the displacements are close to the maximum allowed displacements (near the barrier, which in a practical case, may be about 0.01 inch from the accelerometer in the equilibrium position) and the stiffness is quite high. Large stiffness values result in the high acceleration peaks evident in Figure 8a. This strongly nonlinear response means that low pass filtering cannot recover the low frequency input waveform. The low pass filtered response is compared to the desired response in Figure 8b. Unlike the previous case, where the gain is 0.2, at these excitation levels, both the shape and magnitude of the acceleration response differ significantly from the desired response.

Figure 8a. Input acceleration and response of isolated accelerometer for $g=3.2$.

input levels intermediate to those shown above show progressively greater degrees of distortion between the measured waveform and the desired waveform. The character of the dynamic behavior of the isolated accelerometer system is investigated by analyzing the behavior of the input and response time series. (See references 5 and 6.)

The effect of the isolated accelerometer system on an infinitesimal error in the state is shown in Figs. 9a and 9b for the excitation levels

Figure 8b. Input acceleration and filtered response of isolated accelerometer for $g=3.2$.

corresponding to gains of 0.2 and 3.2. As expected, a gain of 0.2 corresponds to a nonchaotic case, where the error decays at a rate approximately consistent with a system damping coefficient of 0.10. A gain of 3.2 corresponds to a case where the estimated Lyapunov exponent is slightly positive, indicating that the isolated accelerometer is behaving in a chaotic manner.

Figure 9a. Error propagation in response of isolated accelerometer for $g=0.2$.

Figure 9b. Error propagation in response of isolated accelerometer for $g=3.2$.

Conclusion

The present study established a first order model for the isolated accelerometer, then normalized the model in several respects. The normalized model was used to establish parameter regimes where chaos occurs in the system. It was shown that when the input frequency ratio, α , is greater than 1.75 (approximately) and the normalized input amplitude, B , is greater than a value of 0.45 to 0.55 then chaos can occur. This result indicates that a sinusoidal input whose frequency is greater than about twice the small displacement natural frequency of the isolated accelerometer may cause chaotic vibrations when the actual input amplitude is greater than about 0.5 times $\omega_p^2 y_m$. (See Eq. 8a.) This is a very severe input for typical isolated accelerometer design parameters, and it is anticipated that sustained periodic inputs of this magnitude are not likely to occur very frequently.

Very severe input "spikes" can, however, occur during shock excitation, and the second part of this study showed what can happen when an isolated accelerometer is subjected to shock inputs with two components. The first component was relatively low frequency (one that

should be measurable by the isolated accelerometer), and the second component is relatively high frequency (one that might cause the isolated accelerometer to enter a chaotic regime). When the high frequency component is relatively severe, then the measurement of the low frequency component is distorted. In fact, it is shown that a criterion for chaotic motion is satisfied when a severe motion is input to the system, i.e., that small "errors" in the response tend to increase with time.

Additional portions of this study (not summarized here) show that even when the system response is not chaotic, the inferred acceleration may be substantially different from the actual low frequency component of acceleration.

The most important conclusion is that there may be some potential for chaotic motion in the isolated accelerometer when it is subjected to extreme environments. The topic merits further investigation.

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