CONVEX SETS IN THE PLANE

THESIS

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By

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CHAPTER I

INTRODUCTION

The purpose of this paper is to investigate some of the properties of convex sets in the plane through synthetic geometry. The definitions, axioms, and theorems of geometry assumed throughout the paper are from Synthetic Geometry, by R. F. Jolly and Mathematics 406, Foundations of Geometry, a course in the mathematics curriculum of North Texas State University. The assumption of these axioms and theorems assures the existence and uniqueness of parallel and perpendicular lines. Thus the results correspond to the intuitive ideas of convex sets in the plane.

In Chapter I the concepts which are assumed are stated. Also some special convex sets of the plane are presented with some of their more elementary properties.

In Chapter II several properties of convex sets are established for convex sets restricted to the plane. The theorems include properties of interiors and closures, properties of the linear manifold, properties of the convex hull, separation, properties of a support line, and kernels of starlike sets. The concluding theorems, Carathéodory's Theorem and Helly's Theorem, are two of the more famous theorems.

When an axiom or theorem is applied in a proof, it will be stated in brackets at the end of the statement. If the
axiom or theorem is from *Synthetic Geometry*, the axiom or theorem number will be preceded by A to indicate the axiom or theorem is stated in the Appendix.

Definitions

The definitions from *Synthetic Geometry* which are stated are Definition 1.1 through Definition 1.17. These are presented to clarify notation.

Definition 1.1. Point P is an endpoint of an interval I if for some Q, different from P, I is interval PQ.

Definition 1.2. If X, Y, and Z are distinct points then Y is between X and Z if Y is an element of interval XZ. The statement "Y is between X and Z" will be abbreviated XYZ.

Definition 1.3. The notation "ABCD" means that ABC, ACD, ABD, and BCD.

Definition 1.4. If A and B are two distinct points, then line \( \overrightarrow{AB} \) is the union of all intervals containing A and B.

Definition 1.5. If A and B are distinct points, segment \( \overline{AB} \) is the set of all points between A and B. Points A and B are the endpoints of the segment.

Definition 1.6. If A and B are distinct points, then ray \( \overrightarrow{AB} \) is the union of all intervals which contain B and have A as an endpoint.

Definition 1.17. If A and B are distinct points, then open ray \( \overrightarrow{AB} \) is ray \( \overrightarrow{AB} \) without endpoint A.
Definition 1.8. If A, B, and C are non-collinear points, then triangle $\triangle ABC$ is the union of the intervals AB, AC, and BC.

Definition 1.9. The plane $\Pi ABC$ is the union of all lines that contain two points of the triangle $\triangle ABC$.

Definition 1.10. A point set $M$ is convex if $M$ is a single point or whenever $P$ and $R$ are points in $M$, interval $PR$ is a subset of $M$.

Definition 1.11. If A, B, and C are non-collinear points, then angle $\angle ABC$ is the union of ray $\overrightarrow{BA}$ and ray $\overrightarrow{BC}$.

Definition 1.12. If $0$ is a point and $BC$ is an interval, then circle $C$ with center $0$ is the set of points $X$ such that interval $OX \equiv interval BC$.

Definition 1.13. If $X$ and $Y$ are points in circle $C$ such that $XOY$, interval $XY$ is a diameter of circle $C$.

Definition 1.14. A circular disk with center $0$ is the union of all radii of circle $C$ with center $0$.

Definition 1.15. The interior $I$ of circle $C$ is the circular disk $D \sim C$, that is, the point set consisting of center $0$ and all points $W$ such that $OWX$ where $X$ is in circle $C$.

Definition 1.16. The exterior of circle $C$ is the set of points $W$ such that $OXW$ where $X$ is in circle $C$.

Definition 1.17. A collection $C$ of sets is monotonic if for any two members of $C$, one is a subset of the other.

Definition 1.18. An open disk is the interior of some circle $C$. 
Definition 1.19. A point $P$ is an interior point of a set $S$ if there is an open disk with center $P$ contained in set $S$.

Definition 1.20. The interior of set $S$, $S^o$, is the set of all interior points of set $S$.

Definition 1.21. A set $S$ is open if $S$ is $S^o$.

Definition 1.22. The complement of set $S$, $S^c$, is the set of all points not in $S$.

Definition 1.23. Point $P$ is a frontier point of set $S$ if every open disk with center $P$ intersects $S$ and $S^c$.

Definition 1.24. The frontier of set $S$, $\text{Fr}(S)$, is the set of frontier points of $S$.

Definition 1.25. The closure of set $S$, $\overline{S}$, is the set of points $P$ such that every open disk with center $P$ intersects set $S$, that is, $S$ union $\text{Fr}(S)$.

Definition 1.26. A set $S$ is closed if $S$ is $\overline{S}$.

Definition 1.27. The linear manifold of convex set $S$, $L(S)$, is the union of all lines $\overrightarrow{XY}$ where $X$ and $Y$ are points in set $S$.

Definition 1.28. The relative interior of a convex set $S$ is the interior relative to the linear manifold $L(S)$; that is, if $P$ is a relative interior point of $S$, then there is an open disk $I$ with center $P$ such that $I \cap L(S)$ is contained in $S$.

Definition 1.29. The relative frontier of a convex set $S$ is the frontier relative to the linear manifold $L(S)$,
that is, point $P$ is in the relative frontier of set $S$ if for every open disk $I$ with center $P$, $I \cap L(S) \cap S$ and $I \cap L(S) \cap S^C$ are non-empty.

Definition 1.30. The convex hull of set $S$, $H(S)$, is the intersection of all convex sets containing $S$.

Definition 1.31. A set $S$ is bounded if there is an open disk $I$ such that $S$ is contained in $I$.

Definition 1.32. A line $L$ cuts a convex set $M$ if there are points of $M$ on both sides of line $L$.

Definition 1.33. Set $M$ is a maximal element of a collection $C$ of sets means

(i) $M$ is in $C$,

(ii) if $A$ is in $C$ and $M$ is contained in $A$, then $A$ is $M$.

Theorems

The following theorems will be assumed.

Theorem 1.1. If line $L_1$ is parallel to line $L_2$, then line $L_2$ is parallel to line $L_1$.

Theorem 1.2. If line $L_1$ and line $L_2$ are parallel and line $L_3$ is perpendicular to line $L_1$, then line $L_3$ is perpendicular to line $L_2$.

Theorem 1.3. If line $\overrightarrow{AB}$ is parallel to line $\overrightarrow{CD}$, line $\overrightarrow{AC}$ is perpendicular to line $\overrightarrow{AC}$ at $A$ and is perpendicular to line $\overrightarrow{CD}$ at $C$, and line $\overrightarrow{BD}$ is perpendicular to line $\overrightarrow{AB}$ at $B$ and is perpendicular to line $\overrightarrow{CD}$ at $C$, then interval $AC$ is congruent to interval $BD$. 
Theorem 1.4. If interval $AB \neq interval \ EF$, then
interval $AB < interval \ EF$ or interval $AB > interval \ EF$.

Theorem 1.5. If interval $AB \leq interval \ CD$ and
interval $CD < interval \ EF$, then interval $AB < interval \ EF$.

The following principle will be accepted as an axiom
(Zorn's Lemma). Let $C$ be a non-empty collection of subsets
of a fixed subset of the plane. If the union of each
monotonic subcollection in $C$ is an element of $C$, then $C$ contains
at least one maximal element.

The Principle of Mathematical Induction and the Well
Ordering Principle for positive integers will be assumed.

Convex Subsets of the Plane

Theorem 1.7. The plane is a convex set.

Since every point is in the plane, if a point is
between two points, it is certainly a point in the plane.
This result though simple is of sufficient importance that
is stated above as a theorem.

Theorem 1.8. A line is a convex set.

Proof. Let $P$ and $R$ be two points of a line $L$.
Line $L$ is the only line containing both $P$ and $R$ [A, Theorem 12].
Line $L$ is the union of all intervals containing both $P$
and $R$. Hence interval $PR$ is in line $L$. A line is a convex set.

By definition a single point is a convex set. A single
point is a convex subset of a line.
Theorem 1.9. An interval is a convex subset of a line.

Proof. Interval AB is a subset of line $\overline{AB}$. Let $P$ and $R$ be two points of interval AB and let $Q$ be a point such that $PQR$. Then $P$ is $A$ ($R$ is $A$) and $ARB$ ($APB$), $A$ is $P$ ($A$ is $R$) and $B$ is $R$ ($B$ is $P$), or $ARB$ and $APB$. If $A$ is $P$ ($A$ is $R$) and $ARB$ ($APB$), then $PRB$ ($RPB$). Hence $PRB$ ($RPB$) and $PQR$ imply $PQB$ [A, Theorem 4]. Point $Q$ is in interval PB which is interval AB. If $A$ is $P$ ($A$ is $R$) and $B$ is $R$ ($B$ is $P$), then interval AB is interval PR. If $APB$ and $ARP$, then $APRB$ or $ARPB$ [A, Theorem 8]. Assume $APRB$. Then $APR$ and $PQR$ imply $AQR$ [A, Theorem 4]. Since $ARB$, $AQB$ [A, Theorem 4]. Point $Q$ is in interval AB. For any two points $P$ and $R$ in interval AB, interval PR is in interval AB. Therefore interval AB is a convex subset of a line.

Theorem 1.10. A segment is a convex subset of a line.

Proof. Segment $\overline{AB}$ is the set of all points between $A$ and $B$. The points in segment $\overline{AB}$ are points in interval AB. Segment $\overline{AB}$ is a subset of line $\overline{AB}$. If $P$ and $R$ are points of segment $\overline{AB}$, then $APRB$ or $ARPB$. In the proof of Theorem 1.9. is was shown that point $Q$ such that $PQR$ is between $A$ and $B$. Hence interval PR is in segment $\overline{AB}$. Therefore segment $\overline{AB}$ is a convex subset of a line.

Corollary 1.1. A segment $\overline{AB}$ union endpoint $A$ is a convex subset of a line.

The proof follows from Theorem 1.9 and Theorem 1.10.
Theorem 1.11. A ray is a convex subset of a line.

Proof. Ray $\overrightarrow{AB}$ is the union of all intervals containing $B$ and having $A$ as an endpoint. Since line $\overrightarrow{AB}$ is the union of all intervals containing both $A$ and $B$, ray $\overrightarrow{AB}$ is a subset of line $\overrightarrow{AB}$. Let $P$ and $R$ be two points of ray $\overrightarrow{AB}$. If $P$ and $R$ are in interval $AB$, then interval $PR$ is in interval $AB$ [Theorem 1.9]. Consider $ABPR$ or $ARBP$. Either interval $AR$ contains $B$ or interval $AP$ contains $B$ and both have $A$ as an endpoint. Interval $AP$ or interval $AR$ is in ray $\overrightarrow{AB}$. Both are convex and contain $P$ and $R$; hence interval $PR$ is an interval in ray $\overrightarrow{AB}$. Ray $\overrightarrow{AB}$ is a convex subset of a line.

Corollary 1.2. An open ray $\overrightarrow{AB}$ is a convex subset of a line.

The proof follows immediately from Theorem 1.11.

Theorem 1.12. The only convex subsets of a line are the line, a single point, a ray (open ray), an interval, and a segment (segment union an endpoint).

Proof. Suppose there is a convex subset $K$ in a line $L$ which is not one of the above. Set $K$ is not a single point and it is not line $L$; hence there is a point $A$ of line $L$ not in $K$ and points $P$ and $R$ of line $L$ which are in $K$. Either $APR$ or $PRA$ since PAR would imply that $A$ is in $K$.

Assume $APR$. Let set $S_1 = \{X : X$ is in $K$ or $X$ is in ray $\overrightarrow{PR}\}$ and set $S_2 = \{X : X$ is in $L \cap S_1^c\}$. Points $P$ and $R$ are in set $S_1$ and point $A$ in set $S_2$. Line $L$ is in $S_1 \cup S_2$. 

and $S_1 \cap S_2$ is empty. Suppose $U\ V\ W$ where $U$ and $W$ are in set $S_1$ and $V$ is in set $S_2$. Ray $\overrightarrow{PR}$ is convex and if $U$ and $W$ are in ray $\overrightarrow{PR}$, $V$ is in ray $\overrightarrow{PR}$, a contradiction. Then $UVPR$ or $WVPR$.

Assume $WVPR$. Since $W$ is in $S_1$ and is not in ray $\overrightarrow{PR}$, $W$ is in set $K$. Interval $WP$ is in $K$ and $V$ is a point of $K$, a contradiction. Then no point of set $S_2$ is between two points of set $S_1$. Suppose $U\ V\ W$ where $U$ and $W$ are in set $S_2$ and $V$ is in set $S_1$. If $V$ is in ray $\overrightarrow{PR}$ either $U$ or $W$ is in the ray, a contradiction. Then $VUPR$ or $VWPR$. Assume $VWPR$.

Since $V$ is in set $S_1$ and not in ray $\overrightarrow{PR}$, $V$ is in set $K$. Interval $VP$ is in $K$ and $W$ is a point of set $K$, a contradiction. Therefore there is no point of set $S_1$ between two points of set $S_2$. By Axiom K, there is a point $M$ which is not between two points of set $S_1$ or two points of set $S_2$ [A, Axiom K].

If $PRM$ or $PMR$, then $M$ is between two points of set $S_1$.

If $MAP$, then $M$ is between two points of set $S_2$ since all points $B$ such that $BAP$ are not in set $K$. Then $M$ is $A$, $AMP$ and there are no points of set $K$ between $A$ and $M$, or $M$ is $P$. Set $K$ is contained in ray $\overrightarrow{MR}$ or open ray $\overrightarrow{MR}$.

Since $K$ is neither one of these rays, there is a point $C$ such that $MRC$ and $C$ is not in $K$. Further there are no points $D$ of $K$ such that $RCD$. Let set $S_1$ be the set \( \{ X : X \text{ is in } K \text{ or in ray } \overrightarrow{RM} \} \) and set $S_2$ be the set \( \{ X : X \text{ is in } L \cap S_1^c \} \). Points $P$ and $R$ are in set $S_1$ and point $C$ is in set $S_2$. Line $L$ is $S_1 \cup S_2$ and $S_1 \cap S_2$ is empty. The proof that no point of set $S_2$ is between two
points of set $S_2$ is similar to the previous proof. By Axiom K there is a point $N$ such that $N$ is not between two points of set $S_1$ or set $S_2$. If NMR or MNR, then $N$ is between two points of set $S_1$. If RCN, then $N$ is between two points of set $S_2$. Then $N$ is C or RNC. Set $K$ is in interval $MN$, segment $MN$ union an endpoint, or segment $MN$. There are no points of segment $MN$ which are not in set $K$ as that would contradict the uniqueness of $M$ and $N$. Then set $K$ is a segment, a segment union an endpoint, or an interval. Therefore the only convex subsets of a line are the line, a single point, a ray (open ray), an interval, a segment union an endpoint, and a segment.

Theorem 1.13. A half-plane is convex.

Proof: Let $H$ be an open half-plane determined by a line $L$, that is, there is a point $A$ which is not in line $L$ and $H$ is the $A$-side of line $L$. If $P$ and $R$ are points of set $H$, then segment $PR$ is in $H$ [A, Theorem 31]. Since $P$ and $R$ are in $H$, interval $PR$ is in $H$. The open half-plane is a convex set.

Let $H$ be a closed half-plane, that is, $H$ is line $L$ union the $A$-side of line $L$ where $A$ is a point not on line $L'$. Consider points $P$ and $R$ in set $H$. If $P$ and $R$ are both in line $L$ or on the $A$-side of line $L$, interval $PR$ is in $H$. Suppose $P$ is in the line and $R$ is on the $A$-side of the line. If $Q$ is a point such that $PQR$, $Q$ is in ray $PR$ and hence on the $R$-side of line $L$. Point $Q$ is in set $H$. Therefore interval $PR$ is in set $H$ and the set is convex.

The proof of Theorem 1.14 will be omitted due to its length.

Theorem 1.15. If $S$ is a subset of the plane such that for every $X$ and $Y$, points in set $S$, line $XY$ is in set $S$, then $S$ is a line or the plane.

Proof. If set $S$ is a set of collinear points, $S$ is a line since there is only one line containing two distinct points. Suppose set $S$ is a set of non-collinear points and let $A$, $B$, and $C$ be three non-collinear points in the set. For every two points $X$ and $Y$ of triangle $\triangle ABC$, line $XY$ is in set $S$, that is, set $S$ contains plane $\Pi ABC$. Since $S$ is a subset of the plane, set $S$ is the plane.

Theorem 1.16. Closed convex subsets of a line are the line, a ray, and interval and a single point.

Proof. Every point $P$ of a line $L$ is a frontier point of line $L$ since every open disk with center $P$ intersects $L$ and $L^c$. If a point is not in line $L$, then it is possible to find an open disk with the point as its center which does not intersect line $L$. Line $L$ contains all of its frontier points; hence $L \cap \text{Fr}(L)$ is line $L$. A line is a closed convex set. Similarly a ray $AB$ and an interval $AB$ contain all their respective frontier points and are closed sets. A single point is its only frontier point and hence closed.
Both the endpoints of a segment are frontier points of the segment which are not in the set. A segment is not a closed set. Likewise a segment union one endpoint is not closed and an open ray is not closed.

Therefore the closed convex subsets of a line are the line, a ray, an interval, and a single point.

Theorem 1.17. Bounded convex subsets of a line are a single point, a segment, a segment union an endpoint, and an interval.

Proof. A single point is contained in any open disk which has the point as its center; hence it is bounded.

If a line $L$ is bounded, then there is an open disk $I$ containing line $L$. Let $O$ be the center of $I$ and interval $OX$ be a radius of the open disk. Point $P$ is in line $L$ and hence interval $OP < interval OX$. Either $O$ is in line $L$ or $O$ is not in the line. Suppose $O$ is in line $L$. Since interval $OP < interval OX$, there is a point $R$ such that $OPR$ and interval $OR < interval OX$. Point $R$ is in line $L$ but $R$ is not in $I$. Then center $O$ is not in line $L$. There is only one line $L'$ containing $O$ and perpendicular to line $L$ [A, Theorem 77]. Let $Q$ be the point of intersection. Angle $\triangle PQO$ is a right angle and interval $QP < interval OP$ [A, Theorem 85]. Since interval $OP < interval OX$, interval $QP < interval QR$ [Theorem 1.5]. There is a point $R$ such that $QPR$ and interval $QR \equiv interval OX$. Angle $\triangle PQO$ is angle $\triangle RQO$. 
Hence angle \( \Delta RQO \) is a right angle and interval \( QR < \) interval \( OR \) [A, Theorem 85]. Then interval \( OX < \) interval \( OR \) [Theorem 1.5]. Hence point \( R \) is not in open disk \( I \). However \( QPR \) implies \( R \) is in line \( L \). Therefore line \( L \) is not bounded. In similar manner it may be shown that neither a ray nor an open ray is bounded.

A segment, a segment union an endpoint, and an interval may each be bounded by an open disk which has a point of the set as its center and a radius which is greater than the segment, the segment union an endpoint, or the interval. Therefore these are the bounded convex subsets of a line.

Theorem 1.181 Let \( M \) be a convex set which is not a single point.

(i) The linear manifold \( L(M) \) is a line if \( M \) is a set of collinear points.

(ii) The linear manifold \( L(M) \) is the plane if \( M \) is a set of non-collinear points.

Proof. (i) Let line \( L \) denote the line containing \( M \). Let \( A \) and \( B \) be distinct points of set \( M \). There is only one line containing both \( A \) and \( B \) [A, Theorem 12]. Since \( A \) and \( B \) are in \( M \) and \( M \) is contained in line \( L \), the union of all lines \( AB \) for \( A \) and \( B \) in \( M \) is line \( L \). Therefore \( L(M) \) is a line.

(ii) All points are coplanar [A, Theorem 30]. If \( P \) is in the linear manifold \( L(M) \), \( P \) is in the plane. The linear manifold \( L(M) \) is contained in the plane. The plane
may be denoted by any three non-collinear points. Then let $A$, $B$, and $C$ be three non-collinear points in set $M$ and denote the plane as plane $\pi_{ABC}$. Line $L$ in plane $\pi_{ABC}$ contains two points $D$ and $E$ of triangle $\triangle ABC$. Triangle $\triangle ABD$ union its interior is contained in set $M$. Line $L$ contains two points of $M$ and hence is in the linear manifold. Then plane $\pi_{ABC}$ is contained in $L(M)$. Therefore the plane is the linear manifold of a non-collinear convex set.
In this chapter some of the well known properties of convex sets are specialized to the plane and established by the methods of synthetic geometry, culminating with the proofs of the famous Carathéodory's Theorem and Helly's Theorem.

Theorem 2.1. If \( \alpha \) is an index set, if \( M_i \) for each \( i \in \alpha \) is convex, and if the intersection \( \bigcap_{i \in \alpha} M_i \) is non-empty, then the intersection is convex.

Proof. If points \( P \) and \( Q \) are in the intersection \( \bigcap_{i \in \alpha} M_i \), then \( P \) is a point of each \( M_i \) and \( Q \) is a point of each \( M_i \) for \( i \in \alpha \). Since each \( M_i \) is convex, interval \( PQ \) is a subset of each \( M_i \). Therefore interval \( PQ \) is in the intersection \( \bigcap_{i \in \alpha} M_i \), and \( \bigcap_{i \in \alpha} M_i \) is convex.

Corollary 2.1. If \( (M_i) \) is a sequence of convex sets, then \( \lim_{k=1}^{\infty} \bigcap_{i=1}^{\infty} M_i = \bigcap_{i=k}^{\infty} M_i \) is convex or empty.

Proof. If \( \bigcap_{i=k}^{\infty} M_i \) is non-empty and each \( M_i \) is convex, then \( \bigcap_{i=k}^{\infty} M_i \) is convex [Theorem 2.1]. The collection of convex sets, \( \bigcap_{i=k}^{\infty} M_i \), \( \bigcap_{i=k+1}^{\infty} M_i \), \ldots is monotonic since \( \bigcap_{i=k}^{\infty} M_i \subseteq \bigcap_{i=k+1}^{\infty} M_i \subseteq \bigcap_{i=k+2}^{\infty} M_i \subseteq \ldots \). If points \( A \) and \( B \) are in
\[ \bigcup_{i=1}^{\infty} M_i, \text{ then } A \text{ is a point in some } \bigcap_{i=k}^{a} M_i, \text{ and } B \text{ is a point of some } \bigcup_{i=k}^{b} M_i. \] Since the collection of intersections is monotonic, either \( i=k, M_i \) or \( i=k, M_i \).

Assume \( \bigcap_{i=k}^{a} M_i \subset \bigcup_{i=k}^{b} M_i. \) Then \( A \) is in \( \bigcup_{i=k}^{b} M_i \) and since the intersection is convex, interval \( AB \) is in \( \bigcap_{i=k}^{b} M_i. \)

This implies that interval \( AB \) is in \( \bigcup_{i=k}^{b} M_i \) and the union of the intersection is convex. If the intersections are empty, then the union is empty.

The union of a collection of arbitrary convex sets is not necessarily convex. For example, the union of two disjoint circular disks is not a convex set.

Theorem 2.2. Let \( M \) be a convex set, \( A \) and \( B \) be two distinct points, and \( J_X \) denote an open disk with center \( XX \) and a radius congruent to interval \( AB \). Then the set \( N = \{X : J_X \cap M \neq \emptyset\} \) is convex.

Proof. Let \( P \) and \( R \) be points of set \( N \). Let \( Q \) be a point such that \( PQR \). Then it must be shown that an open disk \( J \) with center \( Q \) and radius congruent to interval \( AB \) intersects set \( M \).

Let \( I, J, \) and \( K \) be open disks with centers \( P, Q, \) and \( R \), respectively, and having radii congruent to interval \( AB \). Denote the radius of \( I \) as interval \( PX \), the radius of \( J \) as
interval $QY$, and the radius of $K$ as interval $RZ$. Radii $PX$, $QY$, and $RZ$ are each congruent to interval $AB$ and hence congruent to each other [A, Axiom XII].

Line $\overrightarrow{PR}$ is the only line containing both $P$ and $R$ [A, Theorem 12]. Since $PQR$, $Q$ is in line $\overrightarrow{PR}$. There is only one line $L_1$ containing $P$ and perpendicular to line $\overrightarrow{PR}$; there is only one line $L_2$ containing $Q$ and perpendicular to line $\overrightarrow{PR}$; and there is only one line $L_3$ containing $R$ and perpendicular to line $\overrightarrow{PR}$ [A, Theorem 63]. Since lines $L_1$, $L_2$, and $L_3$ are perpendicular to line $\overrightarrow{PR}$, they are parallel [A, Theorem 68].

Let $C$ be a point of line $L_1$ distinct from $P$. There is a point $D$ such that $CPD$ [A, Axiom VI]. There is only one point $P_1$ on ray $\overrightarrow{PC}$ such that interval $PP_1$ is congruent to interval $PX$, and there is only one point $P_2$ on ray $\overrightarrow{PD}$ such that interval $PP_2$ is congruent to interval $PX$ [A, Theorem 64]. In a similar manner it may be shown that there are points $Q_1$ and $Q_2$ in line $L_2$ such that intervals $QQ_1$ and $QQ_2$ are congruent to interval $QY$ and points $R_1$ and $R_2$ in line $L_3$ such that intervals $RR_1$ and $RR_2$ are congruent to interval $RZ$. To simplify notation, assume $P_1$, $Q_1$, and $R_1$ are on the same side of line $\overrightarrow{PR}$. Intervals $PP_1$, $PP_2$, $QQ_1$, $QQ_2$, $RR_1$, and $RR_2$ are radii of open disks with congruent radii; hence they are congruent to each other.

Lines $L_1$, $L_2$, and $L_3$ are perpendicular to line $\overrightarrow{PR}$ and hence do not intersect [A, Theorem 68]. A point in line $L_1$ is on the non-$R$-side of line $L_2$ and the $Q$-side of line $L_3$. 

A point in line $L_2$ is on the R-side of line $L_1$ and the P-side of line $L_3$. A point in line $L_3$ is on the Q-side of line $L_1$ and the non-P-side of line $L_2$.

There is only one line $L_4$ containing $P_1$ and perpendicular to line $L_1$ \[\text{[A, Theorem 63].}\] Line $L_4$ is perpendicular to line $L_1$ and hence to lines $L_2$ and $L_3$. Line $L_4$ and line $PR$ are perpendicular to line $L_1$ $(L_2, L_3)$ and hence parallel \[\text{[A, Theorem 68].}\] Line $L_1$ is perpendicular to line $PR$ at point $P$ and to line $L_4$ at point $P_1$. Line $L_2$ is perpendicular to line $L_4$ at some point $E$ and to line $PR$ at point $Q$. Interval $PP_1$ is congruent to interval $EQ$ \[\text{[Theorem 1.3].}\] Since $E$ is in line $L_4$, $E$ is on the $P_1 (Q_1, R_1)$-side of line $PR$. Then $E$ is on ray $QQ_1$. However there is only one point on ray $QQ_1$ such that the interval from that point to $Q$ is congruent to interval $QY$, that is, point $Q$. Since interval $QQ_1$ is congruent to interval $PP_1$, interval $QQ_1$ is congruent to interval $QE$ and $E$ is $Q_1$ \[\text{[A, Axiom XII].}\] Similarly $R_1$ is on line $L_4$.

There is only one line $L_5$ containing $P_2$ and perpendicular to line $L_1$ \[\text{[A, Theorem 63].}\] Line $L_5$ is also perpendicular to lines $L_2$ and $L_3$, is parallel to line $PR$, and contains $Q_2$ and $R_2$. These may be shown as the corresponding properties were shown for line $L_4$. Lines $L_4$ and $L_5$ are perpendicular to line $L_1$ and hence are parallel \[\text{[A, Theorem 68].}\]

If $F$ is a point of $I (J, K)$, then $F$ is on the P (Q, R)-side of line $L_4$ and $L_5$. Otherwise $F$ would be on the line or the non-P-side of the line. If $F$ is $P_1$ or another point
of line $L_4$, then interval $PF \geq$ interval $PP_1$ according to whether $F$ is $P_1$ or distinct from $P_1$ and angle $\Delta PP_1F$ is a right angle [A, Theorem 85]. Point $F$ would be in the exterior of the open disk.

Let $U$ be a point of $I \cap M$ and $V$ be a point of $K \cap M$. Points $U$ and $V$ are in set $M$ and hence interval $UV$ is contained in the set $M$. If $U$ and $V$ are both in line $L_2$, interval $UV$ is in the line and in hence in open disk $J$. Then $J \cap M$ is non-empty. Consider the case when $U$ and $V$ are on the same side of line $L_2$. Assume the points are on the $P$-side of line $L_2$. Since $V$ is on the $P$-side of line $L_2$ and $R$ is on the non-$P$-side of the line, there is a point $H$ such that $VHR$ [A, Theorem 32]. If $H$ is $Q$, then $RHV$ is $RQV$.

Interval $QV <$ interval $RV$. Since $V$ is in $K$, interval $RV <$ interval $RZ$, a radius of $K$. Interval $RZ \equiv$ interval $QY$; hence interval $RV <$ interval $QY$ [Theorem 1.6]. Interval $QV < $ interval $RV$ and interval $RV \equiv$ interval $QY$ imply that interval $QV < $ interval $QY$ [Theorem 1.5]. Hence point $V$ is in $J$. Point $V$ is in $K \cap M$ and hence in set $M$. Since $V$ is in $J$, $J \cap M$ is non-empty and point $Q$ is in set $N$. If $H$ is distinct from $Q$, then angle $\Delta RQH$ is a right angle since line $L_2$ is perpendicular to line $\overrightarrow{FR}$ at $Q$. Point $H$ is in the interior of angle $\Delta VQR$ since $VHR$. Then angle $\Delta VQR >$ angle $\Delta HQR$, a right angle.

Interval $VQ < $ interval $VR$ [A, Theorem 85]. Interval $VR < $ interval $RZ$ and interval $RZ \equiv$ interval $QY$. Hence interval $VR < $ interval $QY$ [Theorem 1.6]. Interval $VQ < $ interval $VR$ and
interval VR < interval QY; hence interval QV < interval QY  
[Theorem 1.5]. Point V is in J. Point V is in K ∩ M and  
and hence in set M. Then J ∩ M is non-empty and point Q is in  
set N. Therefore if U and V are on the same side of line L₂,  
there is a point H between V and R in line L₂ and J which  
implies that V is in open disk J. Thus point Q is in set N,  
and interval PR is contained in set N.

Suppose U and V are not on the same side of line L₂.  
There is a point 0 in line L₂ such that UOV [A, Theorem 32].  
If 0 is Q, then there is a point of J between U and V and  
and hence in set M. Then J ∩ M is non-empty. If 0 is not Q,  
then 0 is on one side of line FR. Assume 0 is on the P₁-side  
of the line. If 0 is not in J, interval Q₀ ≤ interval QQ.  
If interval Q₀ ≡ interval QQ₁, then 0 is Q₁ and a point in  
line L₄. That implies that 0 is a point of line L₄ between  
U and V; however U and V are on the P-side of line L₄ and  
there is no point of the line between them [A, Theorem 31].  
Then point 0 is not Q. If interval Q₀ < interval QQ₁, then  
QQ₁₀. Point 0 is on the non-P-side of line L₄ and therefore  
there is a point O' of line L₄ such that O'O'U [A, Theorem 32].  
However UOV and O'O'U imply UO'V, a contradiction. Therefore  
0 is on the P-side of line L₄ and in open disk J. Interval PR  
is in set N as open disk J with center Q intersects set M.  
For any two points P and R in set N, interval PR is  
contained in the set and the set is convex.
Corollary 2.2. If $M$ is convex, then the closure $\overline{M}$ is convex.

Proof. The closure $\overline{M}$ is the set of points $P$ such that every open disk with center $P$ intersects $M$. The closure $\overline{M}$ is $\bigcap N_{AB}$ for all $A$ and $B$ such that $A$ is not $B$ and $N_{AB} = \{ P : \text{an open disk } J \text{ with center } P \text{ having radius congruent to interval } AB \text{ intersects } M \}$. Then $N_{AB}$ is convex [Theorem 2.2]. The intersection of convex sets is convex or empty [Theorem 2.1]. The points of set $M$ are in the intersection and hence it is not empty. Therefore the set $\overline{M}$ is convex.

Lemma 2.1. A triangle union its interior is a convex set.

Proof. The interior of triangle $\triangle ABC$ is convex [A, Theorem 35]. If $P$ is in triangle $\triangle ABC$, every open disk with center $P$ intersects the interior of the triangle. Point $P$ is a frontier point of the interior of the triangle. If $P$ is a point in the exterior of the triangle, then it is possible to find an open disk with $P$ as its center which does not intersect the interior of the triangle and hence a point in the exterior is not in the set nor a frontier point of the set. The closure of the interior of the triangle is the triangle union its interior. The closure of a convex set is convex [Corollary 2.2]. Therefore the interior of a triangle union the triangle is convex.
Theorem 2.3. Let $M$ be a convex set with a non-empty interior $M^\circ$, and let $X$ and $Y$ be two points of $M$, of which $Y$ is in $M^\circ$. Then every point of interval $XY$, except possibly $X$, is an interior point of $M$.

Proof. Line $\overrightarrow{XY}$ is the only line containing both $X$ and $Y$ [A, Theorem 12]. Let line $L$ denote the line containing $Y$ and perpendicular to line $\overrightarrow{XY}$ [A, Theorem 63]. Since $Y$ is in the interior $M^\circ$, there is an open disk $I$ with center $Y$ contained in $M$. Denote the radius of $I$ by interval $YP$.

Let $Z$ be a point of line $L$ distinct from $Y$ and $W$ be a point such that $ZYW$ [A, Axiom VI]. On ray $\overrightarrow{YZ}$ there is only one point $A$ such that interval $YA \equiv$ interval $YP$, and on ray $\overrightarrow{YW}$ there is only one point $B$ such that interval $YB \equiv$ interval $YP$ [A, Theorem 44]. Since $A$ and $B$ are on opposite rays, $AYB$. Both intervals are congruent to a radius of $I$ and have center $Y$ as an endpoint; both intervals are radii of $I$.

There is a point $C$ such that $ACY$ and a point $D$ such that $BDY$ [A, Theorem 16]. Points $C$ and $D$ are in $I$ and in line $L$.

Since $AYB$, $ACY$, and $BDY$, $CYD$ [A, Theorem 5]. Points $X$, $C$, and $D$ are non-collinear points in set $M$. Triangle $\triangle XCD$ union its interior is convex and contained in convex set $M$ [Lemma 2.1]. Since $CYD$, $Y$ is in the triangle; and a point $E$ such that $XEY$ is in the interior of the triangle [A, Theorem 39]. There is an open disk $J$ with center $E$ contained in the triangle union its interior, and hence $J$ is contained in set $M$. Point $E$ is in the interior $M^\circ$. Therefore every point of interval $XY$, except possibly $X$, is an interior point of $M$. 
Corollary 2.3. If $M$ is convex then $M^o$ is either convex or empty.

Proof. If the interior $M^o$ is non-empty, then there are two points $P$ and $R$ in $M^o$. Since point $P$ is in $M^o$, point $P$ is in $M$ and interval $PR$, except possibly $P$, is in $M^o$ [Theorem 2.3]. Point $P$ is in $M^o$ and interval $PR$ is contained in $M^o$. Therefore the interior $M^o$ is convex.

Corollary 2.4. If a convex set $M$ is a set of non-collinear points, then the interior $M^o$ is non-empty, open, and convex.

Proof. Let $A$, $B$, and $C$ be three non-collinear points of set $M$. Triangle $\triangle ABC$ union the interior of triangle $\triangle ABC$ is a convex subset of set $M$ [Lemma 2.1]. A point $X$ in the interior of the triangle is in $M$. There is an open disk $J$ with center $X$ which is contained in the triangle union its interior. Then $J$ is contained in set $M$ and point $X$ is in $M^o$. The interior $M^o$ is non-empty. From Corollary 2.3, the interior is convex. Let $X$ be a point of $M^o$ and let $J$ be the open disk with center $X$ which is contained in $M$. Let $Y$ be a point of $J$ distinct from $X$. Since $Y$ is in $I$, there is a radius $XZ$ of $I$ such that $XYZ$. There is a point $W$ such that $YWZ$ [A, Theorem 16]. Then $XYZ$ and $YWZ$ imply $XYW$ and $XWZ$ [A, Theorem 4, §]. Since $XWZ$, $W$ is a point of $I$ and hence a point of $M$. Then every point of interval $WX$, except possibly $W$, is in $M^o$ [Theorem 2.3]. Point $Y$ is in $M^o$ since $XYW$. Therefore for each point $X$ in $M^o$ there is an open disk $J$ with center $X$ which is contained in $M^o$ and $M^o$ is open.
Lemma 2.2. If \( X \) is a point in the interior of a triangle \( \triangle ABC \), then there is a point \( Z \) of segment \( \overline{AB} \) such that \( CXZ \).

The proof of Lemma 2.2 is omitted, owing to its length.

Lemma 2.3. An open disk is convex.

Proof. An open disk \( I \) is the non-empty interior of a circular disk \( D \). A circular disk is convex [Theorem 1.6]. Therefore open disk \( I \) is convex [Corollary 2.3].

Lemma 2.4. The non-empty intersection of a line \( L \) and an open disk \( K \) is a segment, and if line \( L \) contains the center of \( K \), the segment is a diameter of \( K \).

Proof. Line \( L \) and open disk \( K \) are convex [Theorem 1.8, Lemma 2.3]. The intersection of \( L \) and \( K \) is convex [Theorem 2.1]. The intersection is then a bounded convex subset of line \( L \). Then the intersection may be an interval \( AB \), a segment \( \overline{AB} \) union endpoint \( B \), or a segment \( \overline{AB} \) [Theorem 1.17]. If the intersection is an interval \( AB \) or a segment \( \overline{AB} \) union endpoint, then point \( B \) is in \( K \). There is a point \( C \) such that \( ABC \) [A, Axiom VI]. Since \( B \) is in \( K \) there is an open disk \( J \) with center \( B \) such that \( J \) is contained in \( K \). Denote a radius of \( J \) as interval \( BX \). On ray \( \overrightarrow{BC} \) there is only one point \( D \) such that interval \( BD \equiv \) interval \( BX \) [A, Theorem 44]. Segment \( \overline{BD} \) is in \( J \) and hence in \( K \). Point \( B \) union segment \( \overline{BD} \) is in \( K \cap L \) which is interval \( AB \) or segment \( \overline{AB} \) union \( B \).

Since \( D \) is on ray \( \overrightarrow{BC} \) and \( ABC \), \( ABD \). This implies \( D \) is not in interval \( AB \) or segment \( \overline{AB} \) union \( B \), a contradiction.
Therefore the intersection of line $L$ and open disk $K$ is segment $\overline{AB}$.

If line $L$ contains center $0$ of $K$, point $0$ is in segment $\overline{AB}$ and hence $AOB$. Suppose interval $OA$ is not a radius of $K$. Then there is a point $X$ of the circle $C$, which has $K$ as its interior, such that $OXA$ or $OAX$. If $OXA$, then all points of $K$ in interval $OA$ are in segment $OX$ union $O$. Interval $XA$ is not in $K$ and segment $\overline{OA}$ is not in $K \cap L$, a contradiction. If $OAX$, then there is a point $Y$ of $K$ such that $OAYX$. Then $Y$ is a point of $K$ which is not in segment $\overline{AB}$, a contradiction. Then interval $AO$ is a radius of $K$ and likewise interval $OB$ is a radius of $K$. Therefore segment $\overline{AB}$ is a diameter of $K$.

**Theorem 2.4.** Let $M$ be a convex set with a non-empty interior $M^\circ$, $X$ and $Y$ be points such that $X$ is in the closure $\overline{M}$ and $Y$ is in the interior $M^\circ$. Then every point of interval $XY$, except possibly $X$, is an interior point of $M$.

**Proof.** Point $X$ is in $\overline{M}$ and $Y$ is in $M^\circ$ where $M$ is a convex set with a non-empty interior. There is only one line $\overline{XY}$ containing both $X$ and $Y$ [A, Theorem 12]. Since point $Y$ is in $M^\circ$, there is an open disk $J$ with center $Y$ which is contained in $M$. Let interval $YR$ denote a radius of $J$. On ray $\overrightarrow{YX}$ there is only one point $T$ such that interval $YT = \overline{YR}$ [A, Theorem 44]. Since $T$ is in ray $\overrightarrow{YX}$ and $T$ is not $Y$, $YXT$, $T$ is $X$, or $YTX$. If $YXT$, then $X$ is in $J$ and in $M^\circ$. Interval $XY$ is in $M^\circ$ [Corollary 2.3]. If $X$ is $T$, then interval $XY$ is a radius of $J$ and every point of segment $\overline{XY}$ is in $J$ and in $M^\circ$. 
Either case implies that interval XY, except possibly X, is in the interior $M^\circ$.

Consider point T on ray $\overrightarrow{YX}$ such that $YTX$. There is only one line $L$ containing $Y$ and perpendicular to line $\overrightarrow{XY}$ [A, Theorem 77]. Line $L$ contains $Y$ and hence intersects open disk $J$. The intersection is a segment $\overline{AB}$ which is a diameter of $J$ [Lemma 2.4]. Then $AYB$. There are points $C$ and $D$ such that $ACY$ and $BDY$ [A, Theorem 16]. Points $C$ and $D$ are in $J$ and hence in $M^\circ$. Then $C$ and $D$ are in $M$ and the closure $\overline{M}$. The closure $\overline{M}$ is convex since $M$ is convex [Corollary 2.2]. Points $X$, $C$, and $D$ are in $\overline{M}$ and are non-collinear since $C$ and $D$ are in line $L$. Triangle $\triangle XCD$ union the interior of triangle $\triangle XCD$ is a convex subset of $M$ [Lemma 2.1]. Point $Y$ is on the triangle and hence point $T$ such that $YTX$ is in the interior of the triangle [A, Theorem 39]. Every point of segment $\overline{TY}$ is in $J$ and hence in $M^\circ$. Then it must be shown that point $T$ and segment $\overline{XT}$ are in $M^\circ$.

Segment $\overline{XY}$ is segment $\overline{XT}$ union point $T$ union segment $\overline{TY}$. Let $E$ be a point such that $XET$. There is only one line $\overrightarrow{CE}$ containing both $C$ and $E$ [A, Theorem 12]. Line $\overrightarrow{CE}$ is distinct from line $L$ since $E$ is not on line $L$. On ray $\overrightarrow{CE}$ there is a point $F$ in segment $\overrightarrow{XD}$ such that $DEF$, that is, $E$ is an interior point of triangle $\triangle XCD$ and ray $\overrightarrow{CE}$ intersects side $XD$ of the triangle [Lemma 2.2]. Points $X$, $E$, and $F$ are non-collinear. Triangle $\triangle XEF$ union its interior is contained in $\overline{M}$. Let $G$ be a point of the interior of triangle $\triangle XEF$. Point $G$ is in $\overline{M}$. 


There is an open disk $I$ with center $G$ which is contained in the interior of the triangle, and since $G$ is in $M$, $I$ intersects $M$. The intersection is contained in the interior of triangle $XEF$. Let $G'$ be in $I \cap M$. Point $G'$ is in the interior of triangle $\triangle XEF$ and is thus on the $X$-side of line $EF$ which is line $CE$. Ray $XY$ union ray $XD$ is on angle $\triangle YXD$.

Since $XEY$ and $XFD$, $E$ and $F$ are in angle $\triangle YXD$. Point $G'$ is in the interior of triangle $\triangle XEF$ and hence $G'$ is on the $E$-side of line $XF$ which is line $XD$, and the $F$-side of line $XE$ which is line $XY$. Point $G'$ is in the interior of angle $\triangle YXD$.

A point $H$ such that $DHY$ is on the $Y$-side of line $XD$ and the $D$-side of line $XY$. Point $H$ is in the interior of angle $\triangle YXD$. Interval $G'H$ is in the interior of angle $\triangle YXD$ [A, Theorem 35]. Point $H$ is on the $Y$-side of line $CE$. If there is a point $H'$ of line $CE$ such that $HH'Y$ then $CYD$, $YHD$, and $YH'H$ would imply $CYH'$ and two points of line $CE$ would be in line $L$. Hence the lines are the same, a contradiction. Therefore $H$ is on the $Y$-side of line $CE$. Since $G'$ is in the interior of triangle $\triangle XEF$, $G'$ is on the $X$-side of line $CE$. Point $H$ is on the $Y$-side of line $CE$ which is the non-$X$-side. Thus there is a point $K$ of line $CE$ such that $G'KH$ [A, Theorem 32]. Interval $G'H$ is in the interior of angle $\triangle YXD$ implies that $FKE$. Since $G'$ is in $M$ and $H$ is in $M$, point $K$ is in $M$. Then $FKE$ and $FEC$ imply $KEC$ [A, Theorem 5]. Point $K$ is in $M$ and $C$ is in $M^o$; interval $KC$, except possibly $K$, is in $M^o$. Point $E$ is in $M^o$. 
Since \(XET\) and \(XTY\), ETY [A, Theorem 5]. Interval \(EY\) is in \(M^o\) and point \(T\) is in \(M^o\). Therefore interval \(XY\), except possibly \(X\), is an interior point of \(M\).

Theorem 2.5. If \(M\) is convex and \(M^o\) is non-empty, then the closure of \(M^o\) is identical with \(\overline{M}\) and the interior of \(\overline{M}\) is identical with \(M^o\).

Proof. Let \(P\) be a point of \(\overline{M}\) and \(R\) be a point of \(M^o\). Every point of interval \(PR\), except possibly \(P\), is in \(M^o\) [Theorem 2.4]. Let \(J\) be an open disk with center \(P\) and let interval \(PX\) denote a radius of \(J\). On ray \(PR\) there is only one point \(Y\) such that interval \(PY\equiv\) interval \(PX\) [A, Theorem 44]. Interval \(PY\) is a radius of \(J\). If interval \(PY<\) interval \(PR\), then segment \(\overrightarrow{PY}\) is a subset of interval \(PR\) and hence in \(M^o\). If interval \(PY>\) interval \(PR\), then segment \(\overrightarrow{PR}\) is a subset of segment \(\overrightarrow{PY}\) and segment \(\overrightarrow{PY}\) intersects \(M^o\). Then any open disk with center \(P\) of \(\overline{M}\) intersects \(M^o\), and a point of \(\overline{M}\) is a point of the closure of \(M^o\). Therefore \(\overline{M}\) is contained in the closure of \(M^o\). Let \(P\) be a point of the closure of \(M^o\). Every open disk with center \(P\) intersects \(M^o\) and hence intersects \(M\). Point \(P\) is in \(\overline{M}\), and the closure of \(M^o\) is contained in \(\overline{M}\). Therefore \(\overline{M}\) is identical with the closure of \(M^o\).

Let \(P\) be a point in \(M^o\). There is an open disk \(I\) with center \(P\) which is a subset of \(M\) and hence contained in \(\overline{M}\). Point \(P\) is in the interior of \(\overline{M}\), and \(M^o\) is contained in the interior of \(\overline{M}\). Let \(P\) be in the interior of \(\overline{M}\). There is an
open disk $K$ with center $P$ which is contained in $\overline{M}$. Suppose there is a point $R$ in $K$ which is in $\overline{M}$ but not in $M$. Let $X$ be a point of $M^\circ$. Consider point $Y$ such that $XRY$. If point $Y$ is in $K$, then $Y$ is in $\overline{M}$ and every point of interval $YX$, except possibly $Y$, is in $M^\circ$ [Theorem 2.4]. Then $R$ is in $M^\circ$ and hence in $M$, a contradiction to the assumption. Therefore any point $Y$ such that $XRY$ cannot be in $K$. This implies that every open disk with center $R$ intersects the complement $K^c$ and $K$. There is no open disk with center $R$ which is contained in $K$, a contradiction to $K$ being open. Therefore if $R$ is a point of $K$, $R$ is a point of $M$ and $K$ is a subset of $M$. Then point $P$ of the interior of $\overline{M}$ is in $M^\circ$ and the interior of $\overline{M}$ is contained in $M^\circ$. Therefore the interior of $\overline{M}$ is identical with $M^\circ$.

Before proceeding to the theorems concerned with relative interiors and relative frontiers, the relative interiors and relative frontiers of collinear and non-collinear sets should be established. Note that a single point is its linear manifold and hence both the relative frontier and relative interior of a point are the point.

**Theorem 2.6.** The relative interior of a line is the line and the relative frontier of a line is an empty set.

**Proof.** The linear manifold of line $L$ is line $L$ [Theorem 1.18]. Let $P$ be any point of line $L$ and $I$ be any open disk with center $P$. The intersection of $I$ with line $L$ is a segment $\overline{XY}$ which is a diameter of $I$ [Lemma 2.4].
Hence \( I \cap L(L) \) is contained in \( L \) and every point of line \( L \) is a relative interior point of the line. There does not exist a point of line \( L \) such that an open disk with the point as its center intersects the line and the intersection intersects both the line and the complement of the line. Therefore the relative frontier of a line is an empty set.

**Theorem 2.7.** The relative interior of a ray (open ray) is the open ray, and the relative frontier is the endpoint.

**Proof.** The linear manifold of a ray \( \overrightarrow{AB} \) (an open ray \( \overrightarrow{AB} \)) is the line \( \overrightarrow{AB} \) [Theorem 1.18]. Let \( J \) be any open disk with center \( A \). Then \( J \cap L(\overrightarrow{AB}) \) is a segment \( XY \) which is a diameter of \( J \) [Lemma 2.4]. Since \( XAY \), either \( X \) is on ray \( \overrightarrow{AB} \) or \( Y \) is on ray \( \overrightarrow{AB} \). Assume \( Y \) is on ray \( \overrightarrow{AB} \). Then \( J \cap L(\overrightarrow{AB}) \) intersects ray \( \overrightarrow{AB} \) and \( J \cap L(\overrightarrow{AB}) \) intersects the complement of ray \( \overrightarrow{AB} \).

Point \( A \) is a relative frontier point of ray \( \overrightarrow{AB} \) (open ray \( \overrightarrow{AB} \)). Let \( P \) be a point of ray \( \overrightarrow{AB} \) distinct from \( A \), and let \( I \) be an open disk with center \( P \) which has a radius \( PA \). The intersection of \( J \) and line \( \overrightarrow{AB} \) is a segment \( WZ \) which is a diameter of \( J \) [Lemma 2.4]. Either \( W \) is \( A \) or \( Z \) is \( A \) since segment \( WZ \) is on line \( \overrightarrow{AB} \) and both intervals \( PW \) and \( PZ \) are congruent to interval \( PA \) [A, Theorem 44]. Assume \( W \) is point \( A \). Then \( APZ \). Since \( P \) is on ray \( \overrightarrow{AB} \) and distinct from \( A \), \( APB \), \( P \) is \( B \), or \( ABP \). Then \( APZ \) and \( APB \) imply \( ABZ \) or \( AZB \) [A, Theorem 9]. Point \( Z \) is on ray \( \overrightarrow{AB} \). If \( APZ \) and \( P \) is \( B \), then \( ABZ \) and \( Z \) is on ray \( \overrightarrow{AB} \).

If \( APZ \) and \( ABP \), then \( ABZ \) and \( Z \) is on the ray [A, Theorem 4]. Segment \( WZ \) is a subset of ray \( \overrightarrow{AB} \) and \( I \cap L(\overrightarrow{AB}) \) is contained
in ray $\overrightarrow{AB}$. Then $A$ is a point of the relative frontier and
the only point of the relative frontier. The ray without
the endpoint, that is, the open ray is the relative interior.

Theorem 2.8. The relative interior of a segment
(segment union an endpoint, interval) is the segment $\overline{AB}$, and
the relative frontier is the set of points consisting of
$A$ and $B$.

Proof. The linear manifold of a segment $\overline{AB}$ is the line
containing both $A$ and $B$ [Theorem 1.18]. Let $J$ be any open
disk with center $A$. The $J \cap L(AB)$ is a segment $\overline{XY}$ which is
a diameter of $J$ [Lemma 2.4]. Either $XAB$ or $YAB$ since $XAY$.
Assume $XAB$. Segment $\overline{XY}$ intersects the complement of segment $\overline{AB}$,
then $AYB$, $Y$ is $B$, or $ABY$. All three imply that segment $\overline{XY}$
intersects segment $\overline{AB}$. Point $A$ is in the relative frontier
of segment $\overline{AB}$. In like manner it may be shown that $B$ is
in the relative frontier of segment $\overline{AB}$. Let $P$ be a point
of segment $\overline{AB}$. Then $APB$ implies interval $AP < interval PB$
or interval $AP > interval PB$. Assume interval $AP < interval PB$.
Let $I$ be an open disk with center $P$ and a radius $PX$ which is
less than interval $AP$. Then $I$ intersects $L(\overline{AB})$ and the
intersection is a segment which is less than interval $\overline{AB}$ and
hence contained in segment $\overline{AB}$. Then $I \cap L(\overline{AB})$ is contained
in segment $\overline{AB}$ and $P$ is a relative interior point of segment $\overline{AB}$.
Segment $\overline{AB}$ is the relative interior of segment $\overline{AB}$ and points
$A$ and $B$ are the only points in the relative frontier. If the
set includes one endpoint or both, the relative interior and
relative frontier are the same as those for segment $\overline{AB}$. 
Theorem 2.9. The relative interior of a non-collinear convex set is the interior of the set, and the relative frontier of a convex set is the frontier of the set.

Proof. The linear manifold of a non-collinear set $M$ which is convex is the plane [Theorem 1.18]. The intersection of an open disk with the plane is the open disk. The interior of a non-collinear set which is convex is a convex set [Corollary 2.4]. Hence a point in $M^\circ$ is in the relative interior of $M$. Any point in $M^\circ$ is in the relative interior of $M$. Any point in the relative interior of $M$ has an open disk with the points as its center which is contained in $M$. Hence the relative interior of $M$ is $M^\circ$. Every open disk which has a frontier point of $M$ as its center intersects the linear manifold and the intersection intersects the set $M$ and the complement of $M$. Likewise a relative frontier point is one for which every open disk with the points as its center intersects the set $M$ and the complement of $M$. The relative frontier is the frontier of the convex set $M$.

Theorem 2.10. The relative interior of a convex set $M$ is non-empty.

Proof. If $M$ is a single point, the relative interior of $M$ is $M$.

If $M$ is a set of collinear points, then the linear manifold $L(M)$ is some line $L$ [Theorem 1.18]. Let $P$, $R$, and $Q$ be three points of $M$ such that $PRQ$, $PQR$, or $RPQ$. 
Assume PRQ. Interval PQ is a subset of the linear manifold L(M). Interval PR \leq interval RQ or interval PR > interval RQ. [Theorem 1.4]. Assume interval PR \leq interval RQ. Let H be the midpoint of interval PR [A, Theorem 52,79]. Hence PHR and interval RH < interval PR. Then since interval PR \leq interval RQ, interval RH < interval RQ [Theorem 1.6]. Let J be an open disk with R as its center and radius RH. Then J \cap L(M) is segment \overline{HN} which is a diameter of J [Lemma 2.4]. Interval RH \equiv interval RN and interval RH < interval RQ; hence interval RN < interval RQ [Theorem 1.6]. Therefore segment \overline{HN} is a subset of interval PQ and segment \overline{HN} is a subset of M and L(M). Point R is in the relative interior of M, and the relative interior is non-empty.

The linear manifold of a non-collinear convex set M is the plane [Theorem 1.18]. Let P, Q, and R be three non-collinear points of M. Triangle \triangle PQR union the interior of triangle \triangle PQR is a convex subset of M [Lemma 2.1]. Let A be in the interior of the triangle. There is an open disk I with point A as its center contained in the triangle union its interior and hence contained in M. Likewise the open disk I is contained in the plane and point A is a relative interior point of set M. The relative interior of set M is non-empty.
Theorem 2.11. A line \( L \) cuts convex set \( M \) if and only if the following conditions hold:

(i) \( L \) does not contain \( L(M) \),

(ii) \( L \) intersects the relative interior of \( M \).

Proof. If line \( L \) cuts \( M \), then there are points \( P \) and \( R \) of \( M \) such that \( P \) is on the non-\( R \)-side of line \( L \). Then \( P \) and \( R \) are two points of \( N \) and hence of \( L(M) \) which are not on line \( L \). Therefore \( L \) does not contain \( L(M) \).

If line \( L \) cuts \( M \) and \( M \) is a set of collinear points, then there are points \( P \) and \( R \) of \( M \) such that \( R \) is on the non-\( R \)-side of line \( L \). There is a point \( Q \) such that \( PQR \) and \( Q \) is on line \( L \) [\( A \), Theorem 32]. Interval \( PR \) is in \( M \) and point \( Q \) is in \( M \); further interval \( PR \) is contained in \( L(M) \). The linear manifold \( L(M) \) is a line [Theorem 1.18]. Assume interval \( PQ \) < interval \( QR \) since one of the three relations must hold [Theorem 1.4]. Let \( J \) be an open disk with center \( Q \) and radius \( QX \) which is less that interval \( PQ \). The linear manifold is the only line containing both \( P \) and \( R \) [\( A \), Theorem 12]. Let segment \( \overline{AB} \) represent the intersection of open disk \( J \) and the line \( L(M) \). Since \( Q \) is the center of \( J \) and in \( L(M) \), segment \( \overline{AB} \) is a diameter of \( J \) [Lemma 2.4]. Since segment \( \overline{AB} \) is a diameter of \( J \), segment \( \overline{QA} \) and segment \( \overline{QB} \) are both less that interval \( QX \) and hence less than interval \( PQ \) and interval \( QR \). That is, segment \( \overline{AB} \) is contained in \( M \). Then \( J \cap L(M) \) is contained in \( M \) and \( Q \) is a relative interior point of \( M \) which is line \( L \). Line \( L \) does intersect the relative interior of \( M \).
If line $L$ cuts $M$ and $M$ is a set of non-collinear points, then there are points $P$ and $R$ of $M$ such that $P$ is on the non-$R$-side of line $L$. There is a point $Q$ such that $PQR$ and $Q$ is on line $L$ \cite{A, Theorem 32}. Since $M$ is convex, $Q$ is a point of $M$. Let $S$ be a point of $M$ such that $P$, $R$, and $S$ are non-collinear and $S$ is either on line $L$ or on one side of line $L$. Assume $S$ is on line $L$ or on the non-$R$-side of the line. Triangle $\triangle PRS$ union its interior is a subset of convex set $M$. There is a point $X$ such that $SXQ$ which is in the interior of the triangle \cite{A, Theorem 39}. Since $S$ and $Q$ are on line $L$, $X$ is on line $L$. Point $X$ is in the interior of the triangle and hence there is an open disk $I$ which is contained in the triangle union its interior and in set $M$. Then $I \cap L(M)$ is contained in $M$ and point $Q$ is a relative interior point of $M$ on line $L$. If $S$ is on the non-$R$-side of line $L$ there is a point $Y$ of line $L$ such that $SYR$ \cite{A, Theorem 32}. Point $Y$ is on the triangle $\triangle PRS$, and hence there is a point $X$ in the interior of the triangle such that $YXQ$ \cite{A, Theorem 39}. As it has been shown point $X$ in the interior of the triangle is in the relative interior of $M$. Since $YXQ$ implies $X$ is on line $L$, line $L$ intersects the relative interior of set $M$.

If line $L$ does not contain the linear manifold of convex set $M$ and does intersect the relative interior of $M$, there is a point $P$ of $L(M)$ which is not on line $L$ and a point $Q$ of the relative interior of $M$ such that $Q$ is on the line.
Furthermore there is an open disk $J$ with center $Q$ such that $J \cap L(M)$ is contained in $M$. Point $Q$ is in $M$ and the linear manifold. Line $\overrightarrow{PQ}$ is the only line containing both $P$ and $Q$ and is contained in the linear manifold [A, Theorem 12]. Since line $\overrightarrow{PQ}$ is in $L(M)$, $J \cap$ line $\overrightarrow{PQ}$ is contained in $M$. Let segment $\overline{XY}$ represent $J \cap$ line $\overrightarrow{PQ}$ and note that segment $\overline{XY}$ is a diameter of $J$ [Lemma 2.4]. Then $XQY'$. Since segment $\overline{XY}$ is a subset of line $\overrightarrow{PR}$, it is not a subset of line $L$ and $Q$ is the point of intersection of the two lines. Let $X'$ be a point such that $XX'Q$ and $Y'$ be a point such that $YY'Q$. Points $X'$ and $Y'$ are in $J \cap$ line $\overrightarrow{PQ}$ and hence in $M$. Since $Q$ is a point of the line between $X'$ and $Y'$, they are on opposite sides of the line. Therefore line $L$ cuts $M$.

Corollary 2.5. If a line $L$ cuts $M$ it also cuts the relative interior of $M$.

Proof. If line $L$ cuts $M$, line $L$ intersects the relative interior of $M$ [Theorem 2.11]. Let $Q$ be in the intersection of line $L$ and the relative interior of $M$. Point $Q$ is a relative interior point of $M$ and there is an open disk $J$ with center $Q$ such that $J \cap L(M)$ is contained in $M$. Since $L$ cuts $M$, line $L$ does not contain $L(M)$ [Theorem 2.11]. Let $P$ be a point of $L(M)$ which is not on line $L$. There is only one line $\overrightarrow{PQ}$ containing both $P$ and $Q$ [A, Theorem 12]. Line $\overrightarrow{PQ}$ is in the linear manifold $L(M)$ as $P$ and $Q$ are in $L(M)$. Let line $\overrightarrow{PQ} \cap J$ be denoted by segment $\overline{XY}$ [Lemma 2.4]. Segment $\overline{XY}$ is $J \cap$ line $PQ$ and is contained in $J \cap L(M)$ which is contained in set $M$. Let $W$ and $V$ be two points such that
XWQ and QVY. Points W and V are on radii QX and QY, respectively, which are subsets of diameter \( \overline{XY} \) which is contained in \( J \) and \( M \). Points W and V are in \( J \) and \( M \). Then there is an open disk \( I \) with center W and an open disk \( K \) with center V which are contained in \( J \). If \( M \) is a set of collinear points and \( L(M) \) is a line, then \( I \cap L(M) \) and \( K \cap L(M) \) are segments contained in segment \( \overline{XY} \) [Lemma 2.4]. If \( M \) is a set of non-collinear points and \( L(M) \) is the plane, then disks \( I \) and \( K \) are in \( J \) and hence in \( M \). Therefore \( I \cap L(M) \) and \( K \cap L(M) \) are both contained in \( M \). Points W and V are in the relative interior of \( M \). Further XWQ and QVX with XQY imply WQV [A, Theorem 5]. Therefore there is a point of the relative interior of convex set \( M \) on both sides of line \( L \), and line \( L \) cuts the relative interior of \( M \).

Theorem 2.12. If the line \( L \) cuts the convex set \( M \) then

(i) the intersection of \( L \) with the relative frontier is no more than two points in the relative frontier,

(ii) the intersection of \( L \) with the frontier is no more than two points in the frontier.

Proof. Let \( M \) be a set of collinear points. Then the linear manifold \( L(M) \) is a line [Theorem 1.18]. If \( M \) is a line, then there are no relative frontier points and the intersection of \( L \) with the relative frontier is empty [Theorem 2.6]. If \( M \) is a ray \( \overrightarrow{AB} \) (open ray), then the only relative frontier point is point A [Theorem 2.7]. Since line \( L \) cuts ray \( \overrightarrow{AB} \), A is not on line \( L \) and the intersection
of line $L$ and the relative frontier is empty. Similarly if $M$ is a segment, a segment union an endpoint, or an interval, the intersection of line $L$ with the relative frontier is empty.

Let $M$ be a set of non-collinear points, then $L(M)$ is the plane and every point of the relative frontier is a frontier point [Theorem 1.18, Theorem 2.9]. Suppose line $L$ contains points $X, Y$, and $Z$ which are relative frontier points of $M$. Assume $XYZ$. Since $L$ cuts $M$ there is a point of the relative interior on line $L$ [Theorem 2.11]. Let $W$ denote the point and note that $W$ is in $M^o$ [Theorem 2.9]. One of the following is true: $WXYZ$, $XWYZ$, $XYWZ$, or $XYZW$. If $WXYZ$, then every point of interval $ZW$, except possibly $Z$, is in $M^o$ [Theorem 2.5]. Points $X$ and $Y$ are in $M^o$, a contradiction. Likewise the other three possibilities lead to contradictions. Then there can be at most two points of the relative frontier on line $L$ which cuts set $M$.

(ii) If $M$ is a set of collinear points, then every point is a frontier point since every open disk with one of the points as its center intersects $M$ and $M^c$. If line $L$ cuts $M$, line $L$ intersects the frontier in one point. If there were two points then $L$ would be the linear manifold $L(M)$ and could not cut $M$.

If $M$ is a set of non-collinear points, then the frontier is the relative frontier and it has been shown that the intersection can be no more than two points.
Theorem 2.13. If $M$ is an open convex set which is not the plane and $A$ is a linear manifold, a point or a line, which does not intersect $M$, there is a line $K$ such that $A$ is contained in $K$ and $K$ does not intersect $M$.

Proof. If $A$ is a line, then the proof is complete since $A$ is contained in $A$ and by the hypothesis $A$ does not intersect $M$.

Suppose $A$ is a point which is not in $M$. Let $L$ be a line containing $A$, and let $B$ and $C$ be two points of $L$ such that $BAC$. If ray $\overrightarrow{AB}$ intersects $M$ and ray $\overrightarrow{AC}$ intersects $M$, then there will be two points $C$ and $E$ of ray $\overrightarrow{AB}$ and ray $\overrightarrow{AC}$, respectively, in set $M$ such that $DAE$. Point $A$ would then be in $M$. Hence any line containing $A$ and intersecting $M$ contains only one ray determined by $A$ and intersecting $M$.

Since $M$ is open, point $A$ may be a frontier point of set $M$. The point $A$ is in $\bar{M}$. Let line $L$ be a line containing $A$; if line $L$ does not intersect $M$, then the proof is complete.

If line $L$ intersects $M$, let $Q$ be a point in the intersection. Let line $L'$ be a line distinct from $L$ which contains $A$. Again if line $L'$ does not intersect $M$, the proof is complete. If $L'$ does intersect $M$ let $P$ denote a point in the intersection. There is a point $R$ such that $PAR [A, Axiom VI]$. Line $\overrightarrow{QR}$ is the only line containing both $Q$ and $R [A, Theorem 12]$. There is a point $Q'$ such that $RQQ' [A, Axiom VI]$. Let set $S_1$ be $\{X : X$ is on ray $QQ'$ or $X$ is in segment $\overrightarrow{QR}$ and line $\overrightarrow{AX}$ intersects set $M$ on the $Q$-side of line $L'\}$. Let set $S_2$ be
\{X \colon X \text{ is in line QR} \cap S_1^c\}$. If a point \(X\) is on ray \(QQ'\), then \(X\) is on the \(Q\)-side of line \(L'\). If a point \(X\) is in segment \(QR\) then \(QXR\) and \(X\) is on the \(Q\)-side of line \(L'\). Any point in set \(S_1\) is on the \(Q\)-side of line \(L'\). By the definition of sets \(S_1\) and \(S_2\), \(S_1 \cup S_2\) is line \(QR\) and \(S_1 \cap S_2\) is empty. Suppose \(FGH\) where points \(F\) and \(H\) are in set \(S_1\) and \(G\) is a point of set \(S_2\). Since ray \(QQ'\) is a convex set, if \(F\) and \(H\) are in the ray, then interval \(FH\) is in the ray and hence point \(G\) is in ray \(QQ'\) and in set \(S_1\), a contradiction. Then either \(RFGQ\) or \(RHGQ\) is true. Assume \(RHGQ\). Point \(H\) is in set \(S_1\) and hence line \(AH\) intersects \(M\) on the \(Q\)-side of line \(L'\). Let \(W\) be a point in the intersection. Point \(W\) and point \(H\) are on the \(Q\)-side of line \(L'\) and interval \(QH\) must also be on the \(Q\)-side of line \(L'\). Then \(WHA\). \(W\) is \(H\), or \(HWA\). Suppose \(WHA\). Set \(M\) is an open convex set and hence \(M\) is \(M^0\) and every point of interval \(AW\), except possibly \(A\), is in \(M\) [Theorem 2.4]. Point \(H\) is in \(M\) and interval \(QH\) is in \(M\). Then line \(AG\) intersects set \(M\) on the \(Q\)-side of line \(L'\), that is, \(G\) is a point of \(M\) and a point of segment \(QR\) and hence on the \(Q\)-side of line \(L'\). Point \(G\) is then in set \(S_1\), a contradiction. If \(W\) is \(H\), then interval \(HQ\) is in \(M\) and once again point \(G\) would be in \(M\) and line \(AG\) would intersect \(M\) on the \(Q\)-side of line \(L'\), a contradiction. Suppose \(HWA\). Since \(H\), \(Q\), and \(A\) are non-collinear points, consider triangle \(\triangle HQA\). Points \(W\) and \(G\) are on the triangle, and line \(AG\), the only line containing both \(A\) and \(G\), intersects the triangle union its interior. Point \(W\) is not in line \(AG\) as that would imply
that H, Q, and A are collinear. Since HQG, H is on the non-Q-side of line AG. If there is a point Z such that WZH and Z is on line \( \overrightarrow{AG} \), then A and Z are points of line \( \overrightarrow{AG} \) and line \( \overrightarrow{AH} \) which implies the lines are the same. Then there is no point of line \( \overrightarrow{AG} \) between points W and H, and W is on the non-Q-side of line \( \overrightarrow{AG} \). There is a point Z on line \( \overrightarrow{AG} \) between Q and W [A, Theorem 32]. Since points Q and W are on the triangle, point Z is in the interior of triangle \( \triangle HQA \) [A, Theorem 39]. This implies that point Z is in segment \( \overrightarrow{AG} \). Since Q and W are in \( M \), point Z is in \( M \) and line \( \overrightarrow{AG} \) intersects set \( M \) on the Q-side of line \( L' \), a contradiction. Then there is no point of set \( S_2 \) between two points of set \( S_1 \). Suppose FGH where F and H are in set \( S_2 \) and G is a point of set \( S_1 \). If G is on ray \( QQ' \), then F or H must be on the ray and in set \( S_1 \), a contradiction. Then either \( Q'QFG \) or \( Q'QHG \). Assume \( Q'QHG \). Point G is in set \( S_1 \) and thus line \( \overrightarrow{AG} \) intersects \( M \) on the Q-side of line \( L' \). Let Y be in the intersection of line \( \overrightarrow{AG} \) and \( M \). The YGA, Y is G, or GYA. These are the same three situations as were in the previous assumption. Each will imply that H is in set \( S_1 \). By Axiom K there is a point N of line \( \overrightarrow{QR} \) such that N is not between two points of set \( S_1 \) or two points of set \( S_2 \). If line \( \overrightarrow{AN} \) intersects \( M \) on the Q-side of line \( L' \) it is possible to show that N is between two points of set \( S_1 \). If line \( \overrightarrow{AN} \) intersects \( M \) on the non-Q-side of line \( L' \) it is possible to show that N is between two points of set \( S_2 \). Therefore line \( \overrightarrow{AN} \) does not intersect \( M \).
Consider a point \( A \) which is not in \( M \) and is not a frontier point of \( M \). Let \( L \) be a line containing \( A \). If \( L \) does not intersect \( M \), then the proof is complete. If \( L \) does intersect \( M \), then the intersection is a collinear convex set which is contained in both \( L \) and \( M \) [Theorem 2.1]. The intersection may be a ray without the endpoint or a segment. If an endpoint is included, it would be a frontier point of \( M \) as every open disk with the point as its center would intersect \( M \) and \( M^c \). Thus the endpoint or endpoints are not in the intersection. Let \( P \) and \( R \) represent two such points, that is, frontier points of \( M \) on line \( L \). Note if the intersection is a ray only one of the points exists. Assume \( P \) is the frontier point of \( M \) on line \( L \) such that \( P \) is between any point of \( M \) and point \( A \). There is a line \( L' \) containing point \( P \) which does not intersect \( M \). All points of \( M \) are on the non-\( A \)-side of line \( L' \). There is only one line \( K \) containing point \( A \) and parallel to line \( L' \) \([A, \text{ Axiom P}]\). Since line \( K \) is parallel to line \( L' \), it does not intersect line \( L' \) and hence does not intersect set \( M \) on the non-\( A \)-side of line \( L' \). Therefore is \( A \) is a point not contained in open convex set \( M \), there is a line \( K \) containing \( A \) which does not intersect \( M \).

**Definition 2.1.** A line \( L \) separates set \( S \) from set \( T \) if the intersection of \( S \) and \( T \) is empty, \( S^o \) is on one side of line \( L \) and \( T^o \) is on the other side of the line, and if line \( L \) contains a point of \( S\sim S^o \) there is no point of \( T \) on line \( L \).
The following theorem is a weak version of the geometric form of one of the most important theorems in convex analysis, Hahn-Banach Theorem.

Lemma 2.5. Let $A$ and $B$ be convex sets with non-empty interiors, that is, non-collinear sets, such that $A^o \cap B^o$ is empty. If $M$ is a maximal convex set containing $A$ such that $M \cap B^o$ is empty, then the complement $M^c$ is convex.

Proof. Set $M$ is a maximal convex set containing $A$ and $M \cap B^o$ is empty. Suppose $M^c$ is not convex. Then there are points $X$ and $Y$ in $M^c$ such that interval $XY$ is not contained in $M^c$. That is, there is a point $Z$ such that $XZY$ and $Z$ is in $M$. Consider the convex hull $H(M \cup X)$ and the convex hull $H(M \cup Y)$. Set $M$ is in $H(M \cup X)$ and $H(M \cup Y)$, and hence set $A$ is in both sets. Since set $M$ is the maximal convex set containing $A$ and not intersecting $B^o$, $H(M \cup X)$ and $H(M \cup Y)$ intersect $B^o$. There is a point $H$ in $M$ such that interval $XH$ intersects $B^o$, and a point $K$ in $M$ such that interval $YK$ intersects $B^o$. Let $P$ and $R$ be in the intersections of interval $XH$ and $B^o$ and interval $YK$ and $B^o$, respectively. Points $P$ and $R$ are in $B^o$ and hence interval $PR$ is in $B^o$ [Corollary 2.3]. Points $P$ and $R$ are not in $M^o$. There is a line $L$ containing $P$ which does not intersect $M^o$, and a line $L'$ containing $R$ which does not intersect $M^o$ [Theorem 2.13]. Line $L$ or line $L'$ may be line $PR$ or both distinct form line $PR$. Suppose line $L$ is line $PR$ and point $Z$ is in $M^o$, that is, not on line $L$. Either $X$ and $Z$ are on the same side of line $L$ or $Z$ and $Y$ are on the
same side of line $L$. If both $X$ and $Y$ are on the non-$Z$-side of line $L$ then there would be a point of the line between both $X$ and $Z$ and $Y$ and $Z$ [A, Theorem 32]. Line $L$ and line $XY$ would be the same line, a contradiction to the assumption that $Z$ is not on line $L$. Then assume $X$ and $Z$ are on the same side of line $L$ which is line $PR$. Since $XPH$, $H$ is on the non-$X$-side of line $L$, that is, $H$ is on the non-$Z$-side of the line. There is a point $C$ of line $L$ between $Z$ and $H$ [A, Theorem 32]. Since $H$ is in $M$ and $Z$ is in $M^\circ$, point $C$ is in $M^\circ$ [Theorem 2.3]. Line $L$ does not intersect $M^\circ$. Therefore if line $L$ is line $PR$, then $Z$ is not in $M^\circ$. Then suppose line $L$ is line $PR$ and $Z$ is in line $L$. Either $PRZ$ or $ZPR$. Assume $ZPR$. Line $XY$ is distinct from line $L$, otherwise points $X$, $Z$, $Y$, $P$, $R$, $H$, and $K$ would be collinear and each of the possible arrangements of the points would result in a contradiction. Since $XZY$, $X$ is on the non-$Z$-side of line $L$. Then $XPH$ implies that $H$ is on the $Y$-side of line $L$. Point $K$ is on the $X$-side of line $L$ since $YRK$. There is a point $D$ of line $L$ between $H$ and $K$ [A, Theorem 32]. Interval $HK$ is in $M$ and hence $D$ is a point of $M$. Points $Z$, $H$, and $K$ are non-collinear points of $M$; thus triangle $\triangle ZHK$ union its interior is a convex subset of $M$ [Lemma 2.1]. Points $D$ and $Z$ are on the triangle and one line $L$. A point $E$ such that $DEZ$ is in the interior of triangle $\triangle ZHK$ [A, Theorem 39]. There is an open disk $J$ with center $E$ which is contained in the triangle union its interior and hence in $M$. 
Point $E$ is in the interior of $M$ and in line $L$, a contradiction. Therefore if line $L$ is line $PR$, point $Z$ is not on line $L$ and not in $M^o$. Then line $L$ is not line $PR$. Suppose line $L$ is distinct from line $PR$ and $Z$ is in $M^o$. If $Z$ is in $M^o$ and $X$ is on line $L$ or on the non-$Z$-side of line $L$, then $Z$ and $Y$ are on the same side of the line. Otherwise line $L$ would be line $XY$. If point $R$ is on the non-$Z$-side of line $L$, then $YRK$ implies $K$ is on the non-$Z$-side of line $L$. Since $K$ is in $M$ and $Z$ is in $M^o$, the point between $Z$ and $K$ is in $M^o$ [A, Theorem 32, Theorem 2.3]. Line $L$ would intersect $M^o$. If $R$ is on the $Z$-side of line $L$, then triangle $\Delta ZHK$ union its interior may be shown to intersect interval $PR$, and interval $PR$ would contain a point of $M$. If $X$ and $Z$ are in line $L$ or $Z$ only is in line $L$, the possible arrangements of points $P$, $R$, $H$, and $K$ would imply that triangle $\Delta ZHK$ union its interior intersects interval $PR$. Then there is no line containing point $P$ or point $R$ which does not intersect $M^o$. However, Theorem 2.13 says there do exist two such lines. Then the assumption that $Z$ is in $M$ is false. Therefore $M^c$ is convex.

Lemma 2.6. If $S$ is a closed convex set with a non-empty interior and $S^c$ is convex and has a non-empty interior, then $S \cap S^c$ is a line.

Proof. Sets $S$ and $S^c$ have no point in common. Since $S^c$ is convex, $\overline{S^c}$ is convex [Corollary 2.2]. Point $P$ which is a frontier point of $S$ is by definition a frontier point
of $S^c$ and hence in $S^c$. Then $S \cap S^c$ is non-empty. The intersection contains more than one point since both sets are non-collinear. Let $X$ and $Y$ be two distinct points in $S \cap S^c$. Since the intersection of convex sets is empty or convex, this intersection is convex [Theorem 2.1].

Interval $XY$ is in the intersection. Let $Z$ be a point such that $XYZ$ [A, Axiom VI]. If $Z$ is in $S^o$, then $Y$ is in $S^o$ [Theorem 2.4]. If $Z$ is in $(S^c)^o$, then $Y$ is in $(S^c)^o$ [Theorem 2.4]. The interior of $S^c$ is the interior of $S^c$ [Theorem 2.5]. Then point $Z$ is in $S \sim S^o$ and $S^c \sim S^o$. Point $Z$ is in $S \cap S^c$.

If $ZXY$, point $Z$ would also be in the intersection. Then line $XY$ is in the intersection. Since $S$ and $S^c$ are non-collinear sets, $S \cap S^c$ is not the plane and hence the intersection is a line [Theorem 1.15].

Theorem 2.14. Let $M_1$ be an open convex set and $M_2$ be a convex set which does not intersect $M_1$. There is a line $L$ such that $L$ separates $M_1$ from $M_2$.

Proof. Set $M_1$ is open and hence is $M_1^o$. For every point of $M_1$ there is an open disk which has the point as its center which is contained in $M_1$. Since an open disk is a set of non-collinear points $M_1$ is a set of non-collinear points.

If $M_2$ is a single point, then there is a line $L$ containing $M_2$ which does not intersect $M_1$ [Theorem 2.13]. Line $L$ separates $M_1$ from $M_2$.

If set $M_2$ is a line, then $M_2$ separates $M_1$ from itself. If set $M_2$ is a set of collinear points which is not a line, then the linear manifold $L(M_2)$ is a line [Theorem 1.18].
The linear manifold \( L(M_2) \) may not intersect \( M_1 \) and hence it would be a line which separates \( M_1 \) from \( M_2 \) [Theorem 2.13]. If the linear manifold \( L(M_2) \) does intersect \( M_1 \), then there is at least one frontier point of \( M_1 \) on \( L(M_2) \) as was shown in Theorem 2.13. If \( F \) is such a frontier point, \( P \) is a point of \( M_1 \), on \( L(M_2) \), and \( R \) is in \( M_2 \), then \( PFR \) or \( PQF \). If \( PQF \), then there is a frontier point \( F' \) such that \( PF'Q \). Through frontier point \( F \) or \( F' \) there is a line \( L \) which does not intersect \( M_1 \) [Theorem 2.13]. Line \( L \) is distinct from the linear manifold \( L(M_2) \) and hence intersects \( L(M_2) \) only at point \( F \) or \( F' \). Line \( L \) separates \( M_1 \) from \( M_2 \).

If \( M_2 \) is a set of non-collinear points, then \( M_2^\circ \) is an open convex set [Corollary 2.3]. Let \( C \) be the collection of all convex sets \( S \) containing \( M_1 \) such that \( S \cap M_2^\circ \) is empty. Collection \( C \) is not an empty collection as \( M_1 \) is in \( C \), and collection \( C \) contains a monotonic subcollection of convex sets which contain \( M_1 \) and do not intersect \( M_2^\circ \). Let \( T \) denote the monotonic subcollection of \( C \). Let \( U \) be the union of the sets in \( T \). If \( P \) and \( R \) are two points in \( U \), then \( P \) is in some convex set \( A \) and \( R \) is in a convex set \( B \) where \( A \) and \( B \) are in \( U \), that is, \( A \) and \( B \) are in \( T \). Either \( A \) contains \( B \) or \( B \) contains \( A \). Assume \( A \) contains \( B \). Points \( P \) and \( R \) are both in \( A \) and interval \( PR \) is in \( A \) and hence in \( U \). Then \( U \) is a convex set. Since each element of \( U \) contains \( M_1 \) and does not intersect \( M_2^\circ \), \( U \) contains \( M_1 \) and does not intersect \( M_2^\circ \). Hence \( U \) is in \( T \). For any monotonic subcollection in \( C \), the union of the elements of the
subcollection is in C. By Zorn's Lemma, collection C contains at least one maximal convex set containing $M_1$ and not intersecting $M_2^\circ$. Let $M$ denote such a maximal convex set. Consider a frontier point $F$ of $M$. If $F$ is $M_2^\circ$, then there is an open disk $I$ with center $F$ which is completely contained in $M_2^\circ$. Since every open disk with center $F$ intersects $M$, $I$ intersects $M$ and $M$ intersects $M_2^\circ$, a contradiction. Then every frontier point of $M$ is in a set which does not intersect $M_2^\circ$. Set $M$ is the maximal convex set which does not intersect $M_2^\circ$; hence $M$ contains all its frontier points and is closed. Then $M^c$ is a convex set [Lemma 2.5]. Also $M \cap M^c$ is a line $L$ [Lemma 2.6].

Line $L$ intersects $M \sim M^\circ$ and since $M_1$ is open $M_1$ is contained in $M^\circ$ and line $L$ does not intersect $M_1^\circ$. Line $L$ intersects $M^c \sim (M^c)^\circ$ and $M_2^\circ$ is contained in $(M^c)^\circ$. Hence line $L$ does not intersect $M_2^\circ$. All points of $M$ are on line $L$ and on one side of line $L$. There are no points of $M^c$ on the same side of line $L$ as set $M$. Otherwise there would be a point of $(M^c)^\circ$ on line $L$. Then line $L$ defines a closed half-plane containing $M_1$ which does not intersect $M_2^\circ$. Sets $M_1$ and $M_2^\circ$ are on opposite sides of line $L$. Therefore line $L$ separates $M_1$ from $M_2$.

Definition 2.3. If $A$ and $B$ are two disjoint convex sets, line $L$ strictly separates $A$ and $B$ if $A$ is contained in one of the open half-planes determined by line $L$ and $B$ is contained in the other open half-plane determined by line $L$. 

Lemma 2.7. If interval $XY$ is covered by a collection $C$ of open disks $J$ with centers $P$ where $P$ is in interval $XY$, then a finite number of open disks $J$ in $C$ may be chosen which cover interval $XY$.

Proof. Line $XY$ is the only line containing both points $X$ and $Y$ [A, Theorem 12]. There is a point $W$ such that $WXY$ [A, Axiom VI]. Let set $S_1$ be $\{P : P$ is on ray $\overrightarrow{XW}$ or interval $XP$ can be covered by a finite number of open disks $J$ in collection $C\}$. Let set $S_2$ be $\{P : P$ is in $S_1 \cap \text{line } \overrightarrow{XY}\}$. Line $\overrightarrow{XY}$ is $S_1$ union and $S_2$ and $S_1$ intersected with $S_2$ is empty. If $Y$ is not in set $S_2$, then $Y$ is in set $S_1$ and interval $XY$ can be covered by a finite number of open disks $J$ in collection $C$. Suppose $Y$ is in $S_2$. Suppose $ABC$ where $A$ and $C$ are in set $S_1$ and $B$ is in set $S_2$. If $A$ and $C$ are in ray $\overrightarrow{XW}$, then since ray $\overrightarrow{XW}$ is convex, point $B$ is in ray $\overrightarrow{XW}$ and in set $S_1$, a contradiction. Then $WXBC$ or $WXBA$. Assume $WXBC$. Point $C$ is in set $S_1$ and not in ray $\overrightarrow{XW}$; hence interval $XC$ may be covered by a finite number of open disks. Thus interval $XB$ may be covered by a finite number of open disk and $B$ is in set $S_1$, a contradiction. Then no point of set $S_2$ is between two points of set $S_1$. Suppose $ABC$ where $A$ and $C$ are in set $S_2$ and $B$ is in set $S_1$. If $B$ is in ray $\overrightarrow{XW}$ either $A$ or $C$ is in the ray and hence in set $S_1$, a contradiction. Either $WXAB$ or $WXCB$. Assume $WXAB$. Point $B$ is in set $S_1$ and not in ray $\overrightarrow{XW}$; interval $XB$ may be covered by a finite number of open disks. Then interval $XA$ may be covered by a finite.
number of open disks and A is in set $S_1$, a contradiction. Therefore no point of set $S_1$ is between two points of set $S_2$.

By Axiom K, there is a point D in line $\overline{XY}$ which is not between two points of set $S_1$ or set $S_2$. Point D is not in ray $\overrightarrow{XW}$ as that would imply that D is between two points of set $S_1$. Suppose $\overrightarrow{XY}$. If a finite collection of open disks does not cover interval $XD$, then D is between two points of set $S_2$.

Then there is a finite collection of open disks covering interval $XD$. Let $J$ be an open disk with center D from the collection C. The intersection of line $\overleftrightarrow{XY}$ and $J$ is a segment $\overrightarrow{EF}$ (Lemma 2.4). Assume $\overrightarrow{XEDF}$. A point $G$ such that $EGD$ is in $J$ and interval $XG$ may be covered by a finite collection of open disks. A point $H$ such that $DHF$ is in $J$ and hence interval $XH$ may be covered by a finite number of open disks. Point $H$ is in $S_1$ and hence D is between two points of $S_1$. Then D is not between X and Y. If $\overrightarrow{XYD}$, then D is between two points of set $S_2$. Then point D is Y and interval $XY$ may be covered by a finite number of open disks $J$ with centers $P$ where $P$ is in interval $XY$.

Theorem 2.15. If A and B are non-empty disjoint closed bounded convex sets, there is a line $L$ which strictly separates A and B.

Proof. If A and B are two points, line $\overleftrightarrow{AB}$ is the only line containing both A and B $[A', \text{ Theorem 12}]$. There is a point $P$ such that $APB$ $[A, \text{ Theorem 16}]$. There is only one
line L containing P and perpendicular to line AB
[A, Theorem 63]. Line L strictly separates A and B.

If A is a point and B is an interval XY and both are
in line XY, then AXY ≡ XYA. Let P be a point between A and X
or A and Y, and let line L be the perpendicular line to
line XY containing point P. Line L strictly separates A and B.
If A is not on line XY, then let line K be the line containing
A and perpendicular to line XY [A, Theorem 77]. Let P be
the point of intersection and R be a point such that ARP
[A, Theorem 16]. There is only one line L containing R and
parallel to line XY [A, Axiom P]. Line L strictly separates
A and B.

If A and B are both intervals, then consider the linear
manifold L(A) and the linear manifold L(B). Both are lines
[Theorem 1.18]. If L(A) is parallel to L(B), then it is
possible to find a line L between L(A) and L(B) which is
parallel to both lines and strictly separates A and B.
If L(A) intersects L(B), then let P be the point of inter-
section and R be a point of interval A. Point R is between
endpoints X and Y of interval A or is one of the endpoints.
Then XRYP or XRP. Note that point P may be in A or B,
but cannot be in both. The assumption is that P is not in A.
Then let Q be a point such that UQP or RQP if R is an
endpoint [A, Theorem 16]. There is only one line containing
Q and parallel to L(B). This line strictly separates A and B.
Suppose A and B are non-collinear sets. There is a line K which separates A and B°, and a line K' which separates B and A° [Theorem 2.14]. Line K or line K' may strictly separate A and B. If not, line K contains a point of A and line K' contains a point of B. If line K is parallel to line K', then it is possible to find a line L between K and K' which is parallel to both and hence strictly separates A and B. If lines K and K' intersect and the point of intersection is not in A or B, it may be shown that the point of intersection and a point in one of the vertical angles which does not contain either determines a line L which strictly separates A and B. If the point of intersection is in one of the sets, there are several arrangements of the points in the sets which are on the lines. By using open disks, parallel lines, perpendicular lines, and Lemma 2.7 if is possible to find a line L which strictly separates A and B.

Theorem 2.16. Point Y is a frontier point of a convex set M which is not a single point if, and only if, there is a point X in M and a point Z in M° such that segment XY is in M and segment YZ is in M°.

Proof. Suppose set M is a set of collinear points and Y is a frontier point of M. Let J be an open disk with center Y. Since J and M are convex sets, J and M are convex sets, J ∩ M is convex [Lemma 2.3, Theorem 2.1]. Let X be a point in J ∩ M and distinct from Y if Y is in the intersection. If Y is in J ∩ M, interval XY is in J ∩ M and hence segment XY is in J ∩ M and in M. If Y is not in J ∩ M, then segment XY
union endpoint $X$ is in $J \cap M$ and hence segment $\overline{XY}$ is in $J \cap M$ and in $M$. Let $Z$ be a point in $M^c$ such that $X$, $Y$, and $Z$ are non-collinear points. Consider segment $\overline{YZ}$. If point $Z'$ such that $YZ'Z$ is in $M$, then $Y$, $Z'$, and $Z$ are collinear and hence $X$, $Y$, and $Z$ are collinear, a contradiction. Therefore segment $\overline{YZ}$ is in $M^c$.

Suppose $M$ is in a non-collinear set and $X$ is a frontier point of $M$. Since $M$ is a non-collinear set, $M^o$ is non-empty, open, and convex [Corollary 2.4]. The closure of $M^o$ is $\overline{M}$ [Theorem 2.5]. Since $Y$ is a frontier point, $Y$ is in $\overline{M}$. Then every open disk with center $Y$ intersects $M^o$. The intersections are convex [Theorem 2.1]. Let $X$ be in $J \cap M^o$ where $J$ is an open disk with center $Y$. As before segment $\overline{XY}$ is in $J \cap M$ and hence in $M$. Let $Z$ be a point such that $XYZ$ [A, Axiom VI]. If $Z$ is in $M$, then interval $XZ$, except possibly $Z$, is in $M^o$ [Theorem 2.3]. Point $Y$ would be in $M^o$, a contradiction. Then point $Z$ is in $M^c$. Likewise if any point $Z'$ such that $ZZ'Y$ is in $M$, $Y$ would be in $M^o$. Therefore segment $\overline{YZ}$ is in $M^c$.

Suppose $M$ is a convex set and $Y$ is a point such that segment $\overline{XY}$ is in $M$ and segment $\overline{YZ}$ is in $M^c$ when $X$ is in $M$ and $Z$ is in $M^c$. Every open disk with center $Y$ intersects segment $\overline{XY}$ and segment $\overline{XZ}$, and hence intersects $M$ and $M^c$. Therefore point $Y$ is a frontier point of $M$.

Definition 2.4. A line $L$ that intersects the closure of a convex set $M$ and does not cut $M$ is said to be a support line of $M$. 
Theorem 2.17. Through every point in the frontier of a convex set \( M \), there passes at least one support line of \( M \).

Proof. If \( M \) is a single point, then \( M = \overline{M} \) and any line containing \( M \) intersects \( \overline{M} \) and does not cut \( M \). Every line containing \( M \) is a support line.

If \( M \) is a set of collinear points, then every point is a frontier point and in \( \overline{M} \). The linear manifold \( L(M) \) intersects \( \overline{M} \) and does not cut \( M \). The linear manifold \( L(M) \) passes through every frontier point of \( M \) and \( L(M) \) is a support line of \( M \).

If \( M \) is a set of non-collinear points, then \( M^o \) is non-empty, open, and convex [Corollary 2.4]. If \( P \) is a frontier point of \( M \), \( P \) is not in \( M^o \). There is a line \( L \) containing \( P \) which does not intersect \( M^o \) [Theorem 2.13]. Since \( M \) is a set of non-collinear points the relative interior of \( M \) is \( M^o \), and hence line \( L \) does not intersect the relative interior of \( M \) [Theorem 2.9]. Then line \( L \) does not cut \( M \) [Theorem 2.11]. Point \( P \) is a frontier point and hence in \( \overline{M} \). Therefore line \( L \) intersects \( \overline{M} \) and does not cut \( M \). Line \( L \) is a support line of \( M \).

Lemma 2.8. Let \( X \) be an interior point of set \( M \) and \( Y \) be a point distinct from \( X \). If a point \( P \) of segment \( \overrightarrow{XY} \) is not in \( M \), then there is a frontier point of \( M \) on segment \( \overrightarrow{XY} \).

Proof. Point \( X \) is in \( M^o \) means there is an open disk \( I \) with center \( X \) such that \( I \) is contained in \( M \). Open disk \( I \) has a radius \( XX' \). On ray \( \overrightarrow{XY} \) there is only one point \( W \)
such that interval $XW \equiv$ interval $XX'$ [A, Theorem 44]. Since $P$ is not in $M$ and $P$ is in segment $\overrightarrow{XY}$, $XWP$. There is a point $Z$ such that $PXZ$ [A, Axiom VI]. To show there is at least one frontier point in the segment Axiom $K$ must be used. The sets are $S_1 = \{B: B$ is in ray $\overrightarrow{XZ}$ or $XAB$ implies $A$ is in $M^o\}$ and $S_2 = \{B: B$ is in $S_1^c$ line $\overrightarrow{XY}\}$. The steps to showing the necessary conditions of Axiom $K$ are the ones used before in previous proofs. By Axiom $K$ there is a point $F$ which is not between two points of set $S_1$ and not between two points of set $S_2$. Point $F$ is not in ray $\overrightarrow{XZ}$ as that would imply that $D$ is between two points of $S_1$. If $XYF$ or $FY$, point $F$ is between two points of set $S_2$. Then $XFP$. All points in segment $\overrightarrow{FX}$ are in $M^o$ and hence in $M$. Every open disk with $F$ as its center intersects $M$. Consider a point $G$ such that $FGP$. Point $G$ is such that segment $\overrightarrow{XG}$ is not in $M^o$, that is, there is a point $H$ in segment $\overrightarrow{XG}$ which is in $M^c$ or $M^o$. Either implies that there is a frontier point in segment $\overrightarrow{XY}$.

Theorem 2.18. If the closed set $M$ has a non-empty interior and if through every point of its frontier there passes a support line to $M$, then $M$ is convex.

Proof. Set $M$ has a non-empty interior implies there is a point $S$ and an open disk $I$ with center $S$ such that $I$ is contained in $M$. An open disk is a non-collinear set and hence $M$ is a non-collinear set. Let $P$ and $R$ be two points of $M$ such that $P$, $R$, and $S$ are non-collinear. Triangle $\triangle PRS$ is the union of interval $PR$, $PS$, and $RS$. Let $Q$ be a point
such that $PQR$ [A, Theorem .6]. Point $Q$ is in triangle $\triangle PRS$ and any point $X$ such that $SXQ$ is in the interior of the triangle [A, Theorem 39]. Clearly $S$ is not a frontier point of $M$ as $S$ is in the interior. If any point of segment $SQ$ is not in $M$, there is a frontier point $X$ such that $SXQ$ [Lemma 2.8]. There is a line $L$ containing $X$ which is a support line of $M$. Since $X$ is in the interior of the triangle there are two points $W$ and $Y$ of the triangle such that $WXY$ and $W$ and $Y$ are on line $L$. Neither $W$ nor $Y$ is point $S$ since $S$ is in $M^o$ and a support line does not cut the set. Then either $PWS$ and $RYS$ or $PWR$ and $RYS$. Since $P$, $R$, and $S$ are in $M$, each of the four implies that line $L$ cuts $M$. Therefore there is no point of the frontier of set $M$ on segment $QS$. Then segment $QS$ union point $S$ is in $M$, and every open disk with center $Q$ intersects $M$. Point $Q$ is in $\overline{M}$ which is $M$ since $M$ is closed. For any two points $P$ and $R$ of $M$, every point $Q$ such that $PQR$ is in $M$. Therefore $M$ is a convex set.

Theorem 2.19. If $N$ is closed and bounded, the convex hull $H(N)$ is closed and bounded.

Proof. Set $N$ is bounded implies there is an open disk $I$ such that $N$ is contained in $I$. An open disk is convex and hence the convex hull $H(N)$ is contained in $I$. The convex hull $H(N)$ is bounded.

If $N$ is a single point, then the convex hull $H(N)$ is $N$ and hence closed and bounded.
If \( N \) is a set of collinear points, then \( N \) is contained in some line \( L \). Open disk \( I \) intersected with line \( L \) is a segment \( \overline{AB} \) [Lemma 2.4]. Set \( N \) is in segment \( \overline{AB} \) and likewise the convex hull \( H(N) \) is in segment \( \overline{AB} \). The bounded subsets of a line have been established as a segment, a segment union an endpoint, and an interval. If the convex hull \( H(N) \) is a segment \( \overline{XY} \) or a segment \( \overline{XY} \) union endpoint \( Y \), then \( X \) is not in the convex hull \( H(N) \) and not in \( N \). Then there is an open disk \( J \) with center \( X \) such that \( J \cap N \) is empty. Let the radius of \( J \) be denoted by interval \( XC \). On ray \( \overrightarrow{XY} \) there is only one point \( D \) such that interval \( XD = \text{interval } XC \) [A, Theorem 44].

Since \( D \) is in ray \( \overrightarrow{XY} \), \( XDY \), \( D \) is \( Y \), or \( XYD \). If \( XDY \), then every point \( E \) of \( N \) is such that \( DEY \) or \( D \) is \( E \). Then \( N \) is contained in segment \( \overline{DY} \), segment \( \overline{DY} \) union endpoint \( D \), or interval \( DY \). Segment \( \overline{DY} \), segment \( \overline{DY} \) union endpoint \( D \), or interval \( \overline{DY} \) is the convex hull of set \( N \), a contradiction. If \( D \) is \( Y \), then \( N \) is contained in segment \( \overline{XD} \) or segment \( \overline{XD} \) union endpoint \( D \); and \( J \cap N \) is non-empty, a contradiction.

If \( XYD \), then \( N \) is contained in segment \( \overline{XD} \) or segment \( \overline{XD} \) union endpoint \( D \); and \( J \cap N \) is non-empty, a contradiction. Then the convex hull \( H(N) \) is the interval \( XY \) and \( H(N) \) is closed and bounded.

If \( N \) is a non-collinear set, the convex hull \( H(N) \) is a non-collinear set and \( H(N)^\circ \) is non-empty, open, and convex [Corollary 2.4]. Let \( P \) be a frontier point of \( H(N) \). There is
a support line \( L \) through \( P \) [Theorem 2.17]. Since the convex hull \( H(N) \) is bounded, open disk \( I \) which bounds \( N \) contains the convex hull \( H(N) \). The line \( L \) intersected with \( I \) is a segment \( \overline{XY} \) [Lemma 2.4]. Let \( A \) be a point of set \( N \) that is not in line \( L \). Since \( N \) is non-collinear, there are points \( B \) and \( C \) such that \( A, B, \) and \( C \) are non-collinear. Points \( A, B, \) and \( C \) are in \( N \) and hence in the convex hull \( H(N) \). Triangle \( \triangle ABC \) union its interior is a convex subset of \( H(N) \). Suppose \( B \) is on the non-\( A \)-side of line \( L \). There is a point \( D \) of line \( L \) such that \( ADB \) [A, Theorem 32]. Point \( C \) is on the \( A \)-side of line \( L \); \( C \) is on line \( L \); or \( C \) is on the non-\( A \)-side of line \( L \). There is a point \( E \) of line \( L \) such that \( DEC, AEC, \) or \( C \) is \( E \) [A, Theorem 32]. Points \( D \) and \( E \) are in the triangle and hence in the convex hull \( H(N) \). A point \( F \) such that \( DFE \) is in line \( L \) and in the interior of the triangle [A, Theorem 39]. There is an open disk \( J \) with center \( F \) contained in the interior of the triangle. Then \( J \) is contained in \( H(N) \) and point \( F \) is in \( H(N)^{0} \). Since \( H(N) \) is non-collinear the interior is the relative interior and line \( L \) intersects the relative interior. Further the linear manifold is the plane and line \( L \) does not contain the plane. Line \( L \) cuts \( H(N) \), a contradiction [Theorem 2.11]. Therefore all points of \( N \) are on the \( A \)-side of line \( L \) or on line \( L \). Suppose there is no point of \( N \) on line \( L \). The open disk \( I \) which bounds \( N \) and \( H(N) \) has a diameter \( DD' \). On the two rays of line \( L \) determined by point \( P \) find points \( W \) and \( Z \) such that \( WPZ \) and
interval \( WP = interval DD' \) and interval \( PZ = interval DD' \) [A, Theorem 44]. There is only one line \( H \) containing \( X \) and perpendicular to line \( L \), and there is only one line \( H' \) containing \( Z \) and perpendicular to line \( L \) [A, Theorem 63].

Line \( H \) is parallel to line \( H' \) since they are both perpendicular to line \( L \) [A, Theorem 68]. The set \( N, H(N) \), and \( I \) are between line \( H \) and line \( H' \). That is, there are no points of \( N \) or \( H(N) \) on the non-\( Z \)-side of line \( H \) or the non-\( W \)-side of line \( H' \).

Since no points of \( N \) are in line \( L \), none are in the interval \( WZ \). For every point \( R \) in interval \( WZ \), there is an open disk with \( R \) as its center which does not intersect \( N \). Let \( C^* \) be the collection of such open disks. Collection \( C^* \) covers interval \( WZ \). There is a finite number of open disks in \( C^* \) which cover interval \( WZ \) [Lemma 2.7]. Let \( J_1 (1 = 1, 2, \ldots, n) \) denote the collection of open disks. None of the open disks intersects \( N \); then the intersections of the open disks will not intersect set \( N \). The disks will intersect; otherwise interval \( WZ \) will not be covered. The intersection of two open disks is the intersection of the interiors of two circles. The circles will intersect in two points \( T \) and \( S \) [A, Theorem 89]. Segment \( \overline{TS} \) is in the intersection of the interiors. The line determined by the centers of the circles is the perpendicular bisector of interval \( TS \) and hence of segment \( \overline{TS} \) [A, Theorem 90]. Then the midpoint \( M \) of segment \( \overline{TS} \) is on the line determined by the centers. Since there are a finite number of intersections, there is a finite number of segments \( \overline{MS} \). Let segment \( MM' \)
be the minimum of the segments $\overline{MS}$. There is only one line $K$ containing $P$ and perpendicular to line $L$ [A, Theorem 63].

Let $P'$ be a point of line $K$ of the A-side of line $L$. On ray $PP'$ there is only one point $Q$ such that interval $PQ$ is congruent to interval $MS$ [A, Theorem 44]. There is only one line $K'$ containing $Q$ and parallel to line $L$. The method of constructing line $K'$ implies that $P$ is on the non-A-side of line $K'$ and all the points of set $N$ are on the A-side of the line. Line $K'$ determines a closed half-plane which contains $N$ and hence $H(N)$. A disk with center $P$ and radius $MS$ will be on the P-side of line $K'$ and will not intersect $H(N)$. Point $P$ will not be a frontier point of $H(N)$. Therefore there is a point $B$ of $N$ on line $L$ and contained in segment $\overline{N}$. Assume $APBY$. If every point of segment $\overline{PB}$ is in $N$, $P$ is a frontier point of $N$ and hence in $N$ and $H(N)$. Since $P$ is assumed not to be in $N$, there is a point $B'$ such that $\overline{PB'B}$ and segment $\overline{PB}$ does not intersect $N$. Since $P$ is not in $N$ and if there is not a point $C$ of $N$ such that $CPB$, interval $WP$ may be covered with a collection of open disks which do not intersect $N$. In the same manner as before a finite cover may be found for the interval. On line $H$ which is perpendicular to line $L$ at point $W$, find the point $W'$ on the A-side of line $L$ such that interval $WW'$ is congruent to the minimum segment in the intersections of the open disks. Line $L'$ determined by $W'$ and $B'$ will separate some open disk with center $P$ from $H(N)$ and point $P$ will not be a frontier point of $H(N)$. 
Therefore either point \( P \) is in \( N \) or there are two points \( C \) and \( B \) of \( N \) such that \( CPB \). Point \( P \) is in the convex hull \( H(N) \) and the convex hull \( H(N) \) is closed as well as bounded.

The theorem is not true if the bounded condition is removed. Consider the coordinate plane and the set of all points such that \( Y \geq 1/X \) when \( X \) is positive and \( Y \) is zero when \( X \) is not positive. This is a closed set which is not bound. The convex hull of the set is the upper half-plane determined by the \( X \)-axis union the set of points such that \( Y \) is zero when \( X \) is not positive.

Example 2.1.

![Diagram](image)

**Fig. 1**—Closed set with non-closed convex hull

**Theorem 2.20.** If \( N \) is an open set, the convex hull \( H(N) \) is open.

**Proof.** Set \( N \) is open implies that \( N \) is \( N^\circ \). Set \( N \) does contain an open disk and hence is a non-collinear set. Then the convex hull \( H(N) \) is a non-collinear set and \( H(N)^\circ \) is a non-empty, open convex set [Corollary 2.4]. The convex hull contains \( H(N)^\circ \) and \( H(N)^\circ \) is a convex set containing \( N \). Hence \( H(N)^\circ \) contains \( H(N) \) and is the convex hull. Therefore the convex hull is open.
Lemma 2.9. If set $N$ is bounded, then $\overline{N}$ is bounded.

Proof. If set $N$ is bounded there is an open disk $I$ which contains $N$. The open disk $I$ is the interior of some circle $C$ which has a center $0$ and a radius which can be denoted by interval $OX$ where $X$ is a point of the circle. If $P$ is in $\overline{N}$, then every open disk which has $P$ as a center intersects $N$ and hence intersects $I$. Point $P$ is in $\overline{N}$ then implies that $P$ is in $\overline{I}$ which is $C$ union $I$. The closure $\overline{I}$ may be bounded by an open disk which has center $0$ and a radius greater than interval $OX$. Both $\overline{I}$ and $\overline{N}$ are contained in the new open disk.

Theorem 2.21. If set $N$ is bounded, then $H(\overline{N})$ is the intersection of all closed half-planes containing $N$.

Proof. Since set $N$ is bounded, $\overline{N}$ is bounded [Lemma 2.9]. The convex hull $H(\overline{N})$ is closed and bounded since $\overline{N}$ is closed and bounded [Theorem 2.19]. Let $M$ be the intersection of all closed half-planes $S$ which contain $N$. Each set $S$ is convex and hence $M$ is convex [Theorem 2.1]. Since set $N$ is in each $S$, $N$ is contained in $M$. Further if point $P$ is in $\overline{N}$, then every open disk with $P$ as its center intersects $N$ and hence intersects $M$. Then $\overline{N}$ is contained in $M$ and $H(\overline{N})$ is contained in $M$. Suppose there is a point $P$ in $M$ which is not in $H(\overline{N})$. Since $H(\overline{N})$ is closed, there is an open disk $I$ with center $P$ such that $I \cap H(\overline{N})$ is empty. There is a line $L$ which separates $I$ and $H(\overline{N})$ [Theorem 2.14]. Line $L$ determines a closed half-plane containing $H(\overline{N})$ which does not contain
open disk I and hence does not contain P, a contradiction. Therefore point P is in \( H(\overline{N}) \) which implies that \( M \) is contained in \( H(\overline{N}) \). Thus \( M \) is \( H(\overline{N}) \).

Definition 2.5. A point \( X \) is said to be an extreme point of a convex set \( M \) if \( X \) is in \( M \) and there are not two points \( Y \) and \( Z \) in \( M \) such that \( X \) is in segment \( \overline{YZ} \).

Theorem 2.22. A support line to a closed bounded convex set \( M \) contains at least one extreme point of \( M \).

Proof. Let \( L \) be a support line of \( M \). Line \( L \) does not intersect \( M \) and does not cut \( M \), that is, if \( A \) and \( B \) are two points of \( M \) which are not on line \( L \), then \( A \) and \( B \) are on the same side of line \( L \). Set \( M \) is closed implies that \( M \) is the closure \( \overline{M} \); set \( M \) is bounded implies there is an open disk \( J \) containing \( M \). Then line \( L \) intersects \( M \) and hence intersects open disk \( J \). Therefore \( M \cap L \) is contained in \( J \cap L \). Let segment \( \overline{XY} \) be \( J \cap L \) [Lemma 2.4]. Since \( M \cap L \) is contained in segment \( \overline{XY} \) and \( M \cap L \) is convex, \( M \cap L \) is a point, a segment, a segment union an endpoint, or an interval.

If \( M \cap L \) is a point \( P \), then \( P \) is an extreme point of set \( M \). There is no other point of \( M \) on line \( L \) and points of \( M \) are not on both sides of line \( L \).

Suppose \( M \cap L \) is a segment \( \overline{UV} \) or segment \( UV \) union point \( V \). If \( U \) is not in set \( M \), then every open disk with center \( U \) will intersect segment \( \overline{UV} \) and hence intersect \( M \). Hence \( U \) is in \( M \) and likewise point \( V \) is in \( M \). Therefore \( M \cap L \) is interval \( UV \). A point \( Z \) such that \( Z \overline{UV} \) is not in the interval
and hence not in $M$. The points of $M$ are not on both sides of line $L$. Therefore there do not exist two points of $M$ such that $U$ is between them. Point $U$ is an extreme point. Thus line $L$ does contain at least one extreme point of $M$.

The following is the Krein-Milman Theorem for the plane.

**Theorem 2.23.** A closed bounded convex set is in the closure of the convex hull of its extreme points.

**Proof.** Let $M$ be a closed bounded convex set and let $N$ be the set of extreme points of $M$. Let $F$ be a frontier point of $M$. There is a support line $L$ to $M$ containing $F$ [**Theorem 2.17**]. Every support line to a closed bounded convex set $M$ contains at least one extreme point [**Theorem 2.22**].

Then set $N$ is a non-empty set. Since it is contained in convex set $M$, the convex hull $H(N)$ is contained in $M$. The closure of $H(N)$ is convex [**Corollary 2.2**]. Further if any closed set contains $H(N)$, then the set contains the closure of $H(N)$. The closure of $H(N)$ is contained in closed set $M$.

Let $P$ be a point of $M$. Point $P$ is in set $N$ or not in set $N$. If $P$ is a frontier point which is not an extreme point, then there is a support line $L'$ of $M$ containing $P$ [**Theorem 2.17**]. In **Theorem 2.22** it was shown that $L'(\cap)M$ is an interval or a point. Since $P$ is in $L'(\cap)M$ and not an extreme point, the intersection is an interval which may be denoted interval $AB$. Both $A$ and $B$ were shown to be extreme points in **Theorem 2.22**. Interval $AB$ is in the convex hull $H(N)$ and hence point $P$ is
in the convex hull \( H(N) \) and in the closure of \( H(N) \). Suppose P is a point of set \( M \) which is not a frontier point. This assumption implies \( M \) is a non-collinear convex set and point P is in the interior \( M^\circ \). Let \( K \) be a line containing point P. Line \( K \) intersects convex set \( M \). The intersection is a closed bounded collinear convex set, that is, an interval \( CD \) [Theorem 1.16, 1.17]. Points C and D are frontier points and hence in the closure of \( H(N) \). Then point P is in the closure of \( H(N) \) and set \( M \) is contained in the closure of \( H(N) \). Therefore \( M \) is the closure of the convex hull \( H(N) \) where \( N \) is the set of extreme points of \( M \).

Definition 2.6. A set \( S \) is starlike from a point \( C \) in \( S \) if for each point \( X \) in \( S \), interval \( XC \) is contained in \( S \).

Definition 2.7. The kernel \( K(S) \) of a set \( S \) is the set of points in \( S \) from which \( S \) is starlike.

Theorem 2.24. The kernel of a set is empty or convex.

Proof. Let \( S \) be a point set and suppose the kernel \( K(S) \) is non-empty. If the kernel \( K(S) \) is a single point, then \( K(S) \) is convex. Let \( P \) and \( R \) be two points in the kernel \( K(S) \). The kernel \( K(S) \) is contained in set \( S \) and both points \( P \) and \( R \) are in \( S \). Then interval \( PR \) is in set \( S \). Let \( Q \) be a point in interval \( PR \). If set \( S \) is collinear, then for every point \( T \) is set \( S \) such that \( TPR \), interval \( TQ \) is contained in \( S \). Since \( TPR \) and \( PQR \) imply \( TQR \) [A, Theorem 4], interval \( TQ \) is contained in interval \( TR \) and hence interval \( TQ \) is contained in set \( S \). Similarly if \( PRT \), interval \( QT \) is contained in
interval PT and hence in set S. Then set S is convex and the kernel $K(S) = S$ and is convex. If S is non-collinear, let T be a point in S such that P, R, and T are non-collinear. Intervals PR, RT, and TP are in S. Triangle $\triangle PRT$ is in S and point Q is in $\triangle PRT$. Let U be a point of segment TQ; point U is in the interior of the triangle [A, Theorem 39]. Ray RU intersects segment PT at a point V and RUV [Lemma 2.2]. Point V is in interval PT and hence in set S. Then interval RV is in S and point U is in S. Therefore interval TQ is contained in S. Point Q is then in the kernel $K(S)$. For P and R in the kernel, interval PR is contained in the kernel. Therefore the kernel is convex.

Theorem 2.25. If S is a closed starlike set, then the kernel of S is closed.

Proof. Let S be a closed starlike set with a non-empty kernel $K(S)$. If S is a single point, then the kernel $K(S)$ is a single point and closed. If S is a set of collinear points, then in Theorem 2.24 it was shown that S is the kernel of S; and hence the kernel is closed. Consider a set S which is non-collinear. Suppose $K(S)$ is not a single point. Let P be a point of the closure of $K(S)$ and R be a point of $K(S)$. Every open disk with center P intersects $K(S)$ and hence intersects S. Point P is in the closure of S which is S. Segment $\overline{PR}$ is in the kernel, and interval PR is in the set S. Let X be a point of S distinct from P and R. If X, P, and R are collinear, then XPR, PXR, or PRX [A, Theorem 1]. If XPR, then interval XP is contained in interval XR.
which is contained in S since R is in the kernel. If PXR, then interval PX is contained in interval PR which is contained in S. If PRX, then interval PX is interval PR union interval RX which are both in S. Then interval PX is contained in S if X is on line PR. Suppose X, P, and R are non-collinear and consider triangle \( \triangle XPR \). Let Z be a point such that XZP. A triangle is the set of frontier points of the interior of the triangle. Every open disk with center Z intersects the interior of triangle \( \triangle XPR \). Let J be such an open disk. Let Y be a point of the intersection of J and the interior of the triangle. Since Y is in the interior of the triangle, ray XY intersects segment \( PR \) at a point V and XYV [Lemma 2.2]. Since V is in segment \( PR \), V is in the kernel \( K(S) \). Interval XV is in S and Y is a point of S. Then every open disk with center Z intersects S. Point Z is in the closure \( \overline{S} \) which is S. Interval XP is in S and point P is in the kernel. Therefore the closure of the kernel is the kernel and the kernel is closed.

The kernel of an open starlike set is not necessarily open. Consider the interior of a star. The kernel includes some of its frontier points.

Example 2.2.

Fig. 2—Open starlike set with non-open kernel
Carathéodory's Theorem which follows states that the convex hull of a set is the union of all points of the set with all intervals which have endpoints in the set and triangles union interiors which have vertices in the set. The complete proof is not present due to its length. The cases presented are for a single point, a collinear set, a closed and bounded non-collinear set, and a non-collinear set without restrictions. The case for a bounded non-collinear set depends on the closed and bounded case and the fact that every open disk which has a point of the closure of the set as its center intersects the set. There are two lemmas which will be applied in the proof of the theorem.

Lemma 2.10. Let $M$ be a bounded convex set with a non-empty interior. If $P$ is an interior point of $M$, then every line containing $P$ contains frontier points $X$ and $Y$ of $M$ such that $XPY$.

Proof. Set $M$ is bounded means there is an open disk $I$ which contains $M$. Point $P$ is in the interior $M^°$ and hence is in $I$. Let $L$ be a line containing point $P$. Line $L$ intersects open disk $I$, and the intersection is a segment $\overline{AB}$ which is a diameter of $I$ [Lemma 2.4]. If for every point $R$ such that $ARP$, $R$ is in $M$, then every open disk with center $P$ intersects $M$. Point $A$ is a frontier point of $M$. Similarly if every point of segment $PB$ is in $M$, $B$ is a frontier point. If there is a point $R$ such that $ARP$ which is not in $M$, then there is a frontier point $X$ such that $AXP$ [Lemma 2.8]. If there is a
point S between P and B which is not in M, there is a frontier point Y between P and B [Lemm 2.8]. Then APB, APY, XPB, or XPY according to whether A or X and B or Y are frontier points.

Lemma 2.11. If C* is a collection of sets such that if sets S_a and S_b are in C* there is a third set S_c in C* which contains both S_a and S_b, then \( \bigcup_{S \in C^*} H(S) = H\left(\bigcup_{S \in C^*} S\right) \).

Proof. Set S_a is contained in \( H\left(\bigcup_{S \in C^*} S\right) \). The \( H(S_a) \) is contained in \( H\left(\bigcup_{S \in C^*} S\right) \). For every set S in \( H\left(\bigcup_{S \in C^*} S\right) \), \( H(S) \) is contained in \( H\left(\bigcup_{S \in C^*} S\right) \). Hence \( \bigcup_{S \in C^*} H(S) \) is contained in \( H\left(\bigcup_{S \in C^*} S\right) \).

For any two sets S_a and S_b in C*, there is a set S_c which contains both sets. Then \( H(S_c) \) contains \( H(S_a) \) and \( H(S_b) \). Let P and R be two points of \( \bigcup_{S \in C^*} H(S) \). If P and R are in the same convex hull, then interval PR is in the convex hull and in the union. If P is in \( H(S_a) \) and \( H(S_b) \), then P and R are in \( H(S_c) \) where \( S_c \) contains both \( S_a \) and \( S_b \). Interval PR is in the union of the convex hulls of sets in C*. Then \( \bigcup_{S \in C^*} H(S) \) is convex and contains \( H\left(\bigcup_{S \in C^*} S\right) \).

Therefore \( \bigcup_{S \in C^*} H(S) \) is \( H\left(\bigcup_{S \in C^*} S\right) \).
Theorem 2.26. The convex hull of a point set $N$ is the set $M$ where $M$ is

\[
\{ A : A \text{ is in } N \} \cup \{ \text{interval } AB : A \text{ and } B \text{ are in } N \} \cup \{ \text{triangle } \triangle ABC \cup \text{ interior triangle } \triangle ABC : A, B, \text{ and } C \text{ are in } N \}.
\]

Proof. If $N$ is a single point, then the convex hull is $\{ N \} \cup \emptyset \cup \emptyset$ which is $N$.

If $N$ is a set of collinear points, then $N$ is contained in some line $L$ and contains no triangles. Then by the definition of convex hull, set $M$ which is $\{ A : A \text{ is in } N \} \cup \{ \text{interval } AB : A \text{ and } B \text{ are in } N \} \cup \emptyset$ is contained in $H(N)$. Let $Y$ be a point of the convex hull. Point $Y$ is in line $L$. Let $X$ and $Z$ be two points of line $L$ such that $XYZ$. If $Y$ is in $N$, then $Y$ is in set $M$. Suppose $Y$ is not in $N$ and for every point $G$ in $N$, $XYG$ or $GYZ$. Assume $XYG$. If there is an open disk $I$ with center $Y$ such that $I$ does not intersect $N$, then there are no points of $N$ in segment $\overrightarrow{YW}$, the intersection of $I$ and $L$ [Lemma 2.4]. Since $N$ is contained in ray $\overrightarrow{YG}$ and not in segment $\overrightarrow{YW}$, $N$ is contained in ray $\overrightarrow{WG}$ and the convex hull of $N$ is in the ray. Point $Y$ is not in the convex hull. Then every open disk with center $Y$ intersects $N$. Since $Y$ is not in $N$, $N$ is contained in the open ray $\overrightarrow{YW}$. Again the convex hull is in the open ray and $Y$ is not, a contradiction. Thus if $Y$ is not in $N$, there are two points $G$ and $H$ of $N$ such that $GYH$. Point $Y$ is in interval $GH$ and in set $M$. The convex hull is in set $M$ and the sets are identical.
If $N$ is a non-collinear point set, then the convex hull is non-collinear and the interior $H(N)^\circ$ is non-empty, open, and convex [Corollary 2.4]. Set $M$ is clearly contained in the convex hull.

Consider the case where $N$ is closed and bounded. The convex hull of $N$ is closed and bounded [Theorem 2.19]. Let $J$ denote an open disk containing the convex hull of $N$ and set $N$. Let $Z$ be a point in the convex hull. If $Z$ is in $N$, then $Z$ is in set $M$. Suppose $Z$ is not in $N$ and $Z$ is a frontier point of $H(N)$. In Theorem 2.19 it was established that frontier point $Z$ is either in set $N$ or there are two points of $N$ such that $Z$ is between them. Then point $Z$ is in an interval which has endpoints in $N$. A frontier point of $H(N)$ which is not in $N$ is in set $M$. Suppose $Z$ is not in $N$ and $Z$ is in $H(N)^\circ$. Let line $L$ be a line containing $Z$ and hence intersects $H(N)$. There are points $X$ and $Y$ in the frontier of $H(N)$ such that $XZY$ [Lemma 2.10]. Points $X$ and $Y$ are in $N$ or between two points of $N$. Then it is possible to show that point $Z$ is between two points of $N$ or in the interior of a triangle which has points of $N$ as vertices. Point $Z$ in the interior of $H(N)$ is set $M$. If a point is in the convex hull is in set $M$. If $N$ is a closed and bounded non-collinear set, the convex hull identical with set $M$.

The case where $N$ is bounded may be shown by use of Lemma 2.10 and the case for a closed and bounded set.
Suppose $N$ is a non-collinear set. Let $X$ be a point of $N$ and consider the set of open disks $J$ with center $X$. Each open disk intersects $N$ and the intersection is a bounded subset of $N$. Then $\bigcup_{J} H(N \cap J)$ is $H\bigcup_{J} (N \cap J)$ [Lemma 2.11].

Since $N \cap J$ is in $N$ for each $J$, the union is of the intersections is $N$ and the convex hull of the unions is the convex hull of the closure of $N$. If $P$ is in the convex hull of $N$, $P$ is in $\bigcup_{J} H(N \cap J)$. There is an open disk $J$ such that $P$ is in $H(N \cap J)$. Since $N \cap J$ is bounded by $J$, the previous case implies there are two points in $N \cap J$ such that $P$ is between them or three points in $N \cap J$ such that $P$ is in the interior of the triangle which has the points as vertices. The points in $J \cap N$ are in $N$ and hence point $P$ is in the set $M$ when $P$ is a point of the convex hull of $N$. Therefore the convex hull is identical with the set $M$.

The concluding theorem is concerned with the intersection of convex sets. Lemma 2.13 and Theorem 2.24 which is Helly's Theorem are from Convex Sets, by F. A. Valentine. The proof of the theorem is parallel to Helly's proof, which is presented in Convex Sets. Because of the number of sets in each proof which need to be denoted by subscripts there will be a notation change. In place of $F_{i}$ will be written $F(i)$, and instead of $F_{i,j}$ the set will be denoted $F(i,j)$. 
Lemma 2.12. Let $L$ be a line and $F(i)$, $(i = 1, 2, \ldots, s)$, be a collection of closed and bounded convex sets in $L$ which contain at least two points and such that the intersection of any two is non-empty. Then $\bigcap_{i=1}^{s} F(i)$ is non-empty.

Proof. Closed and bounded convex sets in a line are intervals or points [Theorem 1.16, Théorème 1.17] Suppose $\bigcap_{i=1}^{s} F(i)$ is empty. Let $k$ be the smallest positive integer such that some intersection of $k$ sets in $F(i)$, $(i = 1, 2, \ldots, s)$, is empty. Note that $k \geq 3$. Let $F(i,1), F(i,2), \ldots, F(i,k)$ denote such a collection. Select $F(i,j)$ where $1 \leq j \leq k$. Then $F(i,1) \cap F(i,2) \cap \ldots \cap F(i,j-1) \cap F(i,j+1) \cap \ldots \cap F(i,k)$ is non-empty. The sets are points or intervals; hence the intersection is a point or interval. Each of the members of the intersection are disjoint from $F(i,j)$ which is a point or an interval. Let the intersection be point $P$ or interval $PR$. Set $F(i,j)$ is point $X$ or interval $XY$. Since $F(i,j) \cap F(i,1) \cap \ldots \cap F(i,j-1) \cap F(i,j+1) \cap \ldots \cap F(i,k)$ is empty, Points $P$, $R$, $X$, and $Y$ are distinct. The possible arrangements of the single points and interval are that $P$ and $X$ are two distinct points in the line, $PXY$ or $PRX$, and $PRXY$ or $XYPR$. There is a point $H$ such that $PHX$, $PHXY$ or $PRHX$, and $RHXY$ or $XYHP$ [2, Theorem 16]. Point $H$ is not in $F(i,m)$, $(i = 1, 2, \ldots, k)$. However every $F(i,m) \cap F(i,j)$ is non-empty and point $H$ is in each of these intersections.
Then point $H$ is in each $F(i,m)$ and the assumption that there is a positive integer $k$ such that the intersection of $k$ of the sets is empty is false. Hence \( \bigcap_{i=1}^{s} F(i) \) is non-empty.

**Lemma 2.13.** Let $K(i), (i = 1, 2, \ldots, s)$ be convex sets in the plane such that every $s-1$ of them have a non-empty intersection; then there is a set $M(i), (i = 1, \ldots, 2)$, which is the convex hull of a finite set of points in $K(i)$ and hence contained in $K(i), (i = 1, \ldots, s)$.

**Proof.** Let $N(j)$ be the set of positive integers 
\[
\{1, 2, \ldots, j-1, j+1, \ldots, s\}
\] where $j \geq i$ in the set
\[
\{1 = 1, 2, \ldots, j-1, j+1, \ldots, s\}.
\] Then $P(i)$ is a point in $\bigcap_{i \in N(j)} K(i)$. Then $\bigcup_{i \in N(j)} P(i)$ is contained in $K(j)$ and $H\left(\bigcup_{i \in N(j)} P(i)\right)$ is contained in $K(j)$. Let $M(j)$ be \( H\left(\bigcup_{i \in N(j)} P(i)\right) \). If any two of the sets $M(i)$ are the same consider only one, that is, consider only the distinct convex hulls $M(i), (i = 1, \ldots, s)$. The property does hold for one set. Suppose the set $M(1), \ldots, M(k), K(k+1), \ldots, K(s)$ is a collection of convex sets such that $iM(1)$ is contained in $K(1)$, $M(2)$ is contained in $K(2)$, and $M(k)$ is contained in $K(k)$; and the intersection of $s-1$ of the sets in the collection is non-empty. Let point $R(1)$ be in
\[
M(2) \bigcap M(3) \bigcap \ldots \bigcap M(k) \bigcap K(k+1) \bigcap \ldots \bigcap K(s),
\] $R(2)$ be in
\[
M(1) \bigcap M(3) \bigcap \ldots \bigcap M(k) \bigcap K(k+1) \bigcap \ldots \bigcap K(s),
\] \ldots
\( R(k) \) be in
\[ M(1) \cap M(2) \cap \ldots \cap M(k-1) \cap K(k+1) \cap \ldots \cap K(s), \]
\( R(k+1) \) be in
\[ M(1) \cap \ldots \cap M(k) \cap K(k+2) \cap K(k+3) \cap \ldots \cap K(s), \]
\[ \ldots \]
\( R(k+(s-2)) \) be in
\[ M(1) \cap M(2) \cap \ldots \cap M(k) \cap K(k+1) \cap \ldots \cap K(s-1). \]
Then \( M(k+1) \) is the convex hull of the set of points
\[ \{ R(1), R(2), \ldots, R(k+(s-2)) \} \]
and is contained in \( K(k+1) \). Therefore by the Principle of Mathematical Induction there is a set \( M(1), M(2), \ldots, M(s) \) with the property that \( M(i), (i = 1, \ldots, s) \) is the convex hull of a finite number of points in \( K(i) \) and is contained in \( K(i), (i = 1, \ldots, s). \)

Theorem 2.27. Let \( F \) be a finite family of convex sets in the plane containing at least three non-collinear members. A necessary and sufficient condition that all members of \( F \) have a point in common is that every three members have a point in common.

Proof. Suppose \( \bigcap_{i=1}^{s} F(i) \) is empty and \( k \) is the smallest positive integer such that the intersection of \( k \) sets is empty. Then \( 3 < k \leq n \). Let \( F(i,1), F(i,2), \ldots, F(i,k) \) be such a collection. By Lemma 2.13, there are sets \( M(1), M(2), \ldots, M(k) \) which are convex hulls of a finite number of points in \( F(i,1), F(i,2), \ldots, F(i,k) \), respectively.
Select $M(j)$ and let
\[ M(1) \cap M(2) \cap \ldots \cap M(j-1) \cap M(j+1) \cap \ldots \cap M(k) \]
be set: $M$. A finite number of points is closed and bounded; each $M(i)$ is closed and bounded [Theorem 2.19]. Then $M(j)$ and $M$ are closed and bounded sets which are disjoint. There is a line $L$ which strictly separates $M(j)$ and $M$ [Theorem 2.15]. For each $i \neq j$ $M(i) \cap M(j)$ is non-empty; hence $M(i) \cap L$ is non-empty. Then $\bigcap_{i \neq j, i \neq m} M(i)$ where $j$ and $m$ are in the set $\{1, 2, \ldots, k\}$ is the intersection of $k-2$ sets and is non-empty. Then $\bigcap_{i \neq j, i \neq m} M(i) \cap L$ is non-empty and contained in $L$. By Lemma 2.12

\[ \bigcap_{i \neq j} M(i) \cap L \text{ is non-empty and line } L \text{ does not strictly separate } M \text{ and } M(j). \]

Then $\bigcap_{i \neq 1} M(i)$ is non-empty and $\bigcap_{i=1}^{s} F(i)$ is non-empty. If the intersection of any three sets in the family is non-empty, then $\bigcap_{i=1}^{s} F(i)$ is non-empty. If a finite family contains three or more convex sets and the intersection of all the sets in the family is non-empty, then the intersection of any three is non-empty.
APPENDIX

The following axioms and theorems are those assumed from *Synthetic Geometry*, by R. F. Jolly. They are listed in the same order as they are found in the text and are numbered as they are in the book.

The terms which will be left undefined are point and interval.

Axiom I. Every interval is a point set.

Axiom II. If A and B are two distinct points, then there is an interval which contains both A and B and is a subset of every interval containing both A and B.

Axiom III. If I is an interval, then there exist two points A and B, such that I is the interval AB.

Axiom IV. If A, B, and C are three distinct points, and B is an element of the interval AC, then the interval AB is a proper subset of the interval AC, and interval AC is the union of the interval AB and the interval BC.

Theorem 1. If ABC is true, then CBA is true, ACB is not true and A, B, and C are three distinct collinear points.

Theorem 2. If M is a set of just three distinct points and AB is an interval containing M then there is a point P in M such that M is contained in AP.

Theorem 3. Of any three distinct collinear points, one of them is between the other two.
Theorem 4. If \( ABC \) and \( ACD \), then \( ABD \).

Theorem 5. If \( ABC \) and \( ACD \), then \( BCD \).

Theorem 6. If \( ABC \) and \( BCD \), then \( ABD \).

Axiom V. If \( A, B, \) and \( C \) are three distinct collinear points, and \( B, C, \) and \( D \) are three distinct collinear points, then the points \( A, B, C, \) and \( D \) are collinear.

Theorem 7. If \( A, B, \) and \( C \) are collinear points, \( B, C, \) and \( D \) are collinear points, and \( B \neq C \), then \( A, B, C, \) and \( D \) are collinear.

Theorem 8. If \( ABD \) and \( ACD \), then either \( B \) is \( C \), \( ABCD \), or \( ACBD \).

Theorem 9. If \( ABC, ABD, \) and \( C \) is not \( D \), then either \( ABCD \) or \( ABDC \).

Theorem 10. If \( A, B, C, \) and \( D \) are four distinct collinear points, then there exist four distinct points \( W, X, Y, \) and \( Z \) such that \( WXYZ \) and each of the points \( W, X, Y, \) and \( Z \) is one of the points \( A, B, C, \) and \( D \).

Theorem 11. If the intersection of two intervals is nondegenerate, then their union is an interval.

Theorem 12. Every line is a point set and for any two distinct points \( A \) and \( B \), there is one, and only one, line which contains both \( A \) and \( B \).

Axiom VI. If \( A \) and \( B \) are two distinct points, then there is a point \( C \), such that \( ABC \).

Axiom VII. There exist three noncollinear points.
Theorem 13. If A and B are two distinct points, then there is a point C such that A, B, and C are noncollinear.

Theorem 14. If A, B, and C are noncollinear points and X is a point of the line $\overrightarrow{AC}$, then X is the only point on both the line $\overrightarrow{AC}$ and the line $\overrightarrow{BX}$.

Theorem 15. If A, B, and C are noncollinear points, and the point D is collinear with A and B, and $B \neq D$, then B, C, and D are three distinct noncollinear points.

Axiom VIII. If A, B, and C are noncollinear points, and D is a point such that $\overrightarrow{ABD}$, and E is a point such that $\overrightarrow{BEC}$, then there is a point $F$ on the line $\overrightarrow{DE}$, such that $\overrightarrow{AFC}$.

Theorem 16. Every interval contains three points.

Theorem 17. If A, B, and C are noncollinear points, $\overrightarrow{ABD}$, $\overrightarrow{BEC}$, and F is a point on the line $\overrightarrow{DE}$, such that $\overrightarrow{AFC}$, then $\overrightarrow{DEF}$.

Theorem 18. If A, B, and C are noncollinear points, then there do not exist collinear points X, Y, and Z, such that $\overrightarrow{AXB}$, $\overrightarrow{BYC}$, and $\overrightarrow{CZA}$.

Theorem 19. If $\overrightarrow{ABD}$, $\overrightarrow{AFC}$, and A, B, and C are noncollinear, then there is a point E, such that $\overrightarrow{BEC}$ and $\overrightarrow{DEF}$.

Theorem 20. Suppose that A, B, and C are three distinct points, the line L intersects the line $\overrightarrow{AB}$ at a point not in the interval AB, and L also intersects the segment $\overrightarrow{AC}$. Then L intersects the segment $\overrightarrow{BC}$. 
Theorem 21. Suppose that \( L \) is a line, and \( A \) is a point not on \( L \). Then the \( A \)-side of \( L \) and the null-\( A \)-side of \( L \) both exist, as point sets, they do not intersect and each point not on \( L \) belongs to one of them.

Theorem 22. If \( X \) and \( Y \) are points on the null-\( A \)-side of \( L \), then no point of \( L \) is between them.

Theorem 23. The ray \( \overrightarrow{AB} \) consists of all points of the line \( AB \), except those points \( X \), such that \( X \in AB \); moreover if \( C \) is a point different from \( A \) on the ray \( \overrightarrow{AB} \), then ray \( AC \) is the ray \( \overrightarrow{AB} \).

Theorem 24. If \( A \) is a point of the line \( L \), and \( B \) is a point not on \( L \), then the ray \( \overrightarrow{AB} \), except for \( A \), is on the \( B \)-side of \( L \).

Theorem 25. Suppose that \( A, B, \) and \( C \) are noncollinear points, \( AOB \), and \( X \) is a point collinear with \( A \) and some other point of the triangle \( \triangle ABC \). The \( X \) is collinear with \( O \) and some other point \( G \) of the triangle \( \triangle ABC \).

Theorem 26. Suppose that \( A, B, \) and \( C \) are noncollinear points, \( AOB \), \( E \) is a point of the interval \( AB \), and \( X \) is a point collinear with \( E \) and some other point, \( F \), of the triangle \( \triangle ABC \). Then \( X \) is collinear with \( O \) and some other point, \( G \), of the triangle \( \triangle ABC \).

Theorem 27. If \( O \) is a point of the edge \( AB \) of the triangle \( \triangle ABC \), then the plane \( \overline{ABC} \) is the union of all lines which contain \( O \) and some other point of the triangle \( \triangle ABC \).
Theorem 28. If \( X, Y, \) and \( Z \) are noncollinear points in the plane \( \mathbb{ABC} \), then the plane \( \mathbb{XYZ} \) is the plane \( \mathbb{ABC} \).

Theorem 29. Suppose that \( \mathbb{T} \) is a plane, and \( A, B, \) and \( C \) are noncollinear points in \( \mathbb{T} \), and the line \( L \) contains both a point between \( A \) and \( B \) and some other point of \( \mathbb{T} \). Then \( L \) is a subset of \( \mathbb{T} \) which intersects either the interval \( BC \), or the interval \( AC \).

Axiom IX. If \( A, B, \) and \( C \) are noncollinear points, and the line \( L \) contains a point between \( A \) and \( B \), then \( L \) intersects either the interval \( BC \), or the interval \( AC \).

Theorem 30. All points are coplanar.

Theorem 31. If \( X \) and \( Y \) are points on the A-side of \( L \), then the segment \( \overline{XY} \) is on the A-side of \( L \).

Theorem 32. If \( X \) is a point on the A-side of \( L \), and \( Y \) is a point on the null-A-side of \( L \), then there is a point of \( L \) between \( X \) and \( Y \).

Theorem 33. Every line has just two sides.

Theorem 34. If two convex points sets intersect, their intersection is convex.

Theorem 35. Suppose that \( A, B, \) and \( C \) are noncollinear points, and \( M \) is either the angle \( \angle ABC \) or the triangle \( \triangle ABC \). The (1) the interior and exterior of \( M \) exist, as point sets, and are disjoint; (2) the union of the interior and exterior of \( M \) is the complement of \( M \), that is, every point not in \( M \) is either in the interior or the exterior of \( M \); (3) the interior of \( M \) is convex; and (4) the exterior of \( M \) is not convex.
Theorem 36. If A, B, and C are noncollinear points, then there is a point Z such that if X and Y are points in the exterior of the angle $\angle$ ABC, then both of the intervals XZ and YZ are subsets of the exterior of the angle $\angle$ ABC.

Theorem 37. Assume the hypothesis of Theorem 35. Then, if X is a point of the interior of M, and Y is a point of the exterior of M, then there is one, and only one, point of M between X and Y.

Theorem 38. If the points X and Y are in the exterior, E, of the triangle $\triangle$ ABC, then there is a point Z such that both of the intervals XZ and YZ lie in E, that is, $XZ \cup YZ$ is contained in E.

Theorem 39. Assume the hypothesis of Theorem 35. Then P is a point of the interior of M if, and only if, there exist points X and Y of M such that XPY.

Theorem 40. If D is a point of the interior of the angle $\angle$ ABC, then (1) the ray $\overrightarrow{BD}$ lies, except for B, wholly in the interior of the angle $\angle$ ABC; (2) the interior of the angle $\angle$ ABD is a proper subset of the interior of the angle $\angle$ ABC; and (3) the exterior of the angle $\angle$ ABC is a proper subset of the exterior of the angle $\angle$ ABD.

Theorem 41. If D is a point of the interior of the angle $\angle$ ABC, then the ray $\overrightarrow{BD}$ intersects the segment $\overline{AC}$.

Theorem 42. Assume the hypothesis of Theorem 35. If S is either the interior or the exterior of M, and P is a point of S, then there exists a triangle, T, whose interior contains P, and which, together with its interior, lies wholly in S.
Theorem 43. Suppose that \( M \) is a point set. Then the following two conditions are equivalent:

1. For every point \( P \) in \( M \), there exists a triangle \( T \) with \( P \) in its interior such that \( T \) and the interior of \( T \) are both subsets of \( M \).

2. Every point of \( M \) belongs to two noncollinear segments lying wholly in \( M \).

Axiom X(a). If \( A \) and \( B \) are two distinct points, and \( C \) and \( D \) are two distinct points, then there exists a point \( E \) such that \( CDE \), and \( AB \) is congruent to \( DE \).

Axiom X(b). If \( CDE, CDF, AB \equiv DE \), and \( AB \equiv DF \), then \( E = F \).

Theorem 44. If \( P \) is a point and \( AB \) is an interval, then on each ray starting from \( P \), that is, with endpoint \( P \), there is one, and only one, point \( Q \) such that \( AB \equiv PQ \).

Axiom X. If \( A \) and \( B \) are two distinct points, and \( C \) and \( D \) are two distinct points, then there exists one, and only one, point \( E \), such that \( CDE \) and \( AB \equiv DE \).

Axiom XI. If \( ABC, DEF, AB \equiv DE \), and \( BC \equiv EF \), then \( AC \equiv DF \).

Theorem 45. If \( ABC, AB \equiv DE, AC \equiv DF \), and \( F \) is on the ray \( DE \), then \( DEF \).

Axiom XII. If \( AB \equiv CD \) and \( CD \equiv EF \), then \( AB \equiv EF \).

Theorem 46. Every interval is congruent to itself.

Theorem 47. If \( AB \equiv CD \), then \( CD \equiv AB \).

Theorem 48. If \( ABC \) and \( AC \equiv DF \), then there exists a point \( E \), such that \( DEF, AB \equiv DE \), and \( BC \equiv EF \).
Theorem 49. If $\triangle ABC$, $\triangle DEF$, $AB \equiv DE$, and $AC \equiv DF$, then $BC \equiv EF$.

Theorem 50. If $AB \equiv BC$ and $A$, $B$, and $C$ are three distinct, collinear points, then $ABC$.

Theorem 51. If $AB \equiv DE$ and $DEF$, then there is a point $X$, such that $DXF$ and $AB \equiv FX$.

Theorem 52. No interval has two midpoints.

Axiom XIII. If $A$, $B$, and $C$ are noncollinear points, $W$, $X$, and $Y$ are noncollinear points, $ABD$, $WXZ$, $AB \equiv WX$, $BC \equiv XY$, $AC \equiv WY$, and $BD \equiv XZ$, then $CD \equiv YZ$.

Theorem 53. If $A$, $B$, and $C$ are noncollinear points $W$, $X$, and $Y$ are noncollinear points, $ABD$, $WXZ$, $AD \equiv WZ$, $AB \equiv WX$, $AC \equiv WY$, and $CD \equiv YZ$, then $BC \equiv XY$.

Theorem 54. The angle $\angle ABC$ is congruent to the angle $\angle DEF$ if, and only if, there exists a point $W$ on the ray $BA$, and a point $X$ on the ray $BC$, and there exists a point $Y$ on the ray $ED$, and a point $Z$ on the ray $EF$ such that $BW \equiv EY$, $BX \equiv EZ$, and $WX \equiv YX$.

Theorem 55. If $\triangle ABC \equiv \triangle DEF$, $AB \equiv DE$, and $BC \equiv EF$, then $AC \equiv DF$.

Theorem 56. If $A$, $B$, and $C$ are noncollinear points and $AC \equiv BC$, then $\triangle ABC \equiv \triangle BAC$. 
Theorem 57. Suppose \( \triangle ABC \cong \triangle DEF \). Then each of the following statements is true:

1. The vertical angle of \( \triangle ABC \) is congruent to the vertical angle of \( \triangle DEF \).

2. The vertical angle of \( \triangle ABC \) is congruent to \( \angle ABC \), that is, vertical angles are congruent.

3. Each supplement to \( \triangle ABC \) is congruent to each supplement to \( \triangle DEF \).

4. Each supplement to \( \triangle ABC \) is congruent to each other supplement to \( \triangle ABC \), that is, supplementary angles are congruent.

Theorem 58. The angle \( \alpha \) is congruent to the angle \( \beta \) if, and only if, for any interval \( PQ \), there exist points \( A, B, C, D, E, \) and \( R \) such that \( \alpha \) is \( \angle ABC \), \( \beta \) is \( \angle DEF \), \( PQ \equiv AB \), \( AB \equiv BC \), \( BC \equiv DE \), \( DE \equiv EF \), and \( AC \equiv DF \).

Theorem 59. If \( \triangle ABC \cong \triangle DEF \), and \( X \) is a point in the interior of \( \triangle ABC \), then there exists a point \( Y \) in the interior of \( \triangle DEF \) such that \( \angle ABX \equiv \angle DEF \) and \( \angle CBX \equiv \angle FEY \).

Theorem 60. If \( \triangle ABC \equiv \triangle BAC \), then \( AC \equiv BC \).

Theorem 61. If \( A, B, \) and \( C \) are noncollinear points, then there exists a point \( X \) in the interior of \( \triangle ABC \) such that \( \angle ABX \equiv \angle XBC \).

Theorem 62. If \( A, B, \) and \( C \) are noncollinear points and \( AB \equiv BC \), then the interval \( AC \) has a midpoint, \( M \), and the line \( MB \) is perpendicular to line \( AC \).

Theorem 63. If \( P \) is a point of the line \( L \), then there do not exist two lines both containing \( P \) and perpendicular to \( L \).
Theorem 64. If AMB, AM ≡ MB, and P is a point distinct from M, such that AP ≡ PB, then AX ≡ XB if, and only if, X is on the line MP.

Theorem 65. If A and B are points, then there do not exist two points, X and Y, on the same side of line AB, such that AX ≡ AY and BX ≡ BY.

Theorem 66. If D is a point in the interior of 4- ABC, and Z is a point in the interior of 4- WXY and 4- ABD ≡ 4- WXY, then 4- DBC ≡ 4- ZXY.

Theorem 67. If ABC and the line BE is perpendicular to the line BC, then there is a point X on the ray BC such that AE ≡ EX.

Theorem 68. If L is a line, then no two lines perpendicular to L intersect.

Theorem 69. If 4- ABC ≡ 4- DEF, 4- BAC ≡ 4- EDF, and AB ≡ DE, then AC ≡ DF, BC ≡ EF, and 4- ACB ≡ 4- DFE.

Theorem 70. No two of the following statements are true simultaneously:

\[ \angle ABC < \angle DEF. \]
\[ \angle ABC = \angle DEF. \]
\[ \angle ABC > \angle DEF. \]

Theorem 71. Suppose that WXY, Z is a point not on the line WZ, 4- YXZ < 4- ABC, and 4- WZX < 4- BAD. Then the ray AD does not intersect the ray BC.

Theorem 72. If A, B, and C are noncollinear points, and AC < BC, then 4- ABC < 4- BAC.
Theorem 73. If D is in the interior of $\triangle ABC$, Z is in the interior of $\triangle WXY$, $\triangle ABD \cong \triangle WXZ$, and $\triangle DBC \cong \triangle ZXY$, then $\triangle ABC \cong \triangle WXY$.

Theorem 74. Suppose that there is some line that is perpendicular to the line L. Then for every point P of L, there is one, and only one, line containing P and perpendicular to L.

Theorem 75. Suppose that there is a line perpendicular to the line L. Then for every point P, there is one, and only one, line containing P and perpendicular to L.

Theorem 76. If two lines intersect and there is a perpendicular to one of them, then there is a perpendicular to the other.

Theorem 77. If L is a line and P is a point, then there is one, and only one, line containing P and perpendicular to L.

Theorem 78. If the line $\overrightarrow{AB}$ is perpendicular to the line $\overrightarrow{AD}$, and the line $\overrightarrow{BC}$ is perpendicular to the line $\overrightarrow{CE}$, then $\angle BAD \cong \angle BCE$.

Theorem 79. Every interval has a midpoint.

Theorem 80. If A, B, and C are noncollinear points, and M is the midpoint of the interval AB, then there is a point D on the C-side of the line $\overrightarrow{AB}$ such that AC = DM and CM = BD.

Theorem 81. All right angles are congruent.

Theorem 82. Suppose that A, B, and C are noncollinear points, and D, E, and R are noncollinear points. Then there exists a point Q on the F-side of line $\overrightarrow{DE}$ such that $\angle ABC \cong \angle DEG$. 
Theorem 83. Suppose that $\alpha$ is an angle and $\beta$ is an angle. Then one, and only one, of the following statements is true:

\[ \alpha < \beta, \]
\[ \alpha \equiv \beta, \]
\[ \alpha > \beta. \]

Theorem 84. Suppose that $A$, $B$, and $C$ are noncollinear points. Then $\angle ABC < \angle BAC$ if, and only if, $AC < BC$.

Theorem 85. If $\angle ABC$ is greater than or congruent to some right angle, then $AB < AC$.

Theorem 86. If $A$, $B$, and $C$ are noncollinear points, $DEF$, $AB \equiv DE$, and $AC \equiv EF$, then $BC < DF$.

Theorem 87. If $A$, $B$, and $C$ are noncollinear points, $W$, $X$, and $Y$ are noncollinear points, and $AB \equiv XY$, then there exists one, and only one, point $Z$, on the $W$-side of the line $XY$, such that $AC \equiv XZ$ and $BC \equiv YZ$.

Theorem 88. Suppose that $A$ and $B$ are points of a circle $C$ with center $O$, and $M$ is a point of the line $\overline{AB}$ distinct from $O$. Then the line $\overline{AB}$ is perpendicular to the line $\overline{MO}$ if, and only if, $M$ is the midpoint of $AB$.

Theorem 89. The intersection of two circles cannot contain three points.

Theorem 90. If two distinct circles with centers $O$ and $P$, respectively, have the two distinct points, $A$ and $B$ in common, then $O \neq P$, and the line $OP$ is the perpendicular bisector of the interval $AB$.

Theorem 91. The intersection of a line and a circle cannot contain three points.
Theorem 92. If the line $L$ intersects both the circle $C$, and the interior of $C$, then the intersection of $C$ and $L$ is nondegenerate.

Theorem 93. If the intersection of the line $\overline{AB}$ and the circle $C$ contains $B$, then the line $\overline{AB}$ is tangent to $C$ if, and only if, the line $\overline{AB}$ does not intersect the interior of $C$.

Theorem 94. If $A$ and $B$ are points of a circle $C$, then the segment $\overline{AB}$ is a subset of the interior of $C$.

Theorem 95. If $B$ is a point of a circle $C$ with center $O$, then the angle $\angle{ABO}$ is a right angle if, and only if, the line $\overline{AB}$ is tangent to $C$.

Theorem 96. If $B$ is a point of the circle $C$, then there is one, and only one, line tangent to $C$ at $B$.

Axiom S. If $ABCD$, and $M$ is the midpoint of the interval $BD$, then there is a point $E$ such that $AC = AE$ and $MD = ME$.

Theorem 97. If one circle intersects both the interior and the exterior of another circle, then the circles intersect in just two points, one on each side of the line containing their centers.

Theorem 98. If $ABC$, and $D$ is a point not on the line $\overline{AC}$, then there is a point $E$ on the ray $\overrightarrow{BD}$ such that $AC = AE$.

Axiom P. If $L$ is a line and $P$ is a point not on $L$, then there do not exist two lines containing $P$ and parallel to $L$.

Axiom A. If $AB < CD$, then there is a finite collection, $K$, of nonoverlapping intervals, such that $K$ covers $CD$ and each interval in $K$ is congruent to $AB$. 

Axiom K. If the line E is the union of two disjoint points sets L and R, and no point of either one of these sets is between two points of the other, then there is a point of E that is neither between two points of L nor between two points of R.

Theorem 99. Suppose that A and B are two distinct points, and C is a monotonic collection of intervals, each containing both A and B. The intersection of all of the members of C is an interval.

Theorem 100. If C is a monotonic collection of intervals, then there is a point P that is an element of every member of C, that is, there is a point common to all members of C.

Theorem 101. Suppose that E is a line, I is an interval of E, and C is a collection of segments of E. Then, if C covers I, some finite subcollection of C covers I.
BIBLIOGRAPHY

