EXISTENCE AND UNIQUENESS THEOREMS

FOR N^{TH} ORDER LINEAR AND

NONLINEAR INTEGRAL

EQUATIONS

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CHAPTER I

INTRODUCTION

The purpose of this paper is to study n^{th} order integral equations. The integrals studied in this paper are of the Riemann type. The Riemann integral always exists for a function which is continuous in the closed interval defined by the limits of integration; continuity is here a sufficient condition.

In this study a knowledge of the real number system will be assumed. The following definitions and theorems which are developed in analysis courses will be assumed and used in the following chapters.

Definition 1.1. Suppose a and b are real numbers such that $a < b$. Then the closed interval $[a,b]$ is the set of all real numbers x such that $a \leq x \leq b$.

Definition 1.2. The statement that $f(x)$ is bounded in the interval $[a,b]$ means that there exist constants m and M such that $m \leq f(x) \leq M$. The notation "$|f(x)| \leq P$" will be used to mean that the function, $f(x)$, is bounded.

Definition 1.3. Suppose each of f and g is a function such that there is an element common to their domains. The sum of f and g, indicated by $f + g$, is the function h such that $D_h = D_f \cap D_g$ and if x is an element of D_h (denoted by "$x \in D_h$"), then $h(x) = f(x) + g(x)$.

Definition 1.4. The statement that the function f is continuous at $(x_0, f(x_0))$ means if ε is a positive number, there is a positive number δ such that if $x \in D_f$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Definition 1.5. The statement that f is continuous means if $x \in D_f$, then f is continuous at $(x, f(x))$.

Definition 1.6. A sequence is a function whose domain is the set of positive integers; if a denotes the function, then a_k denotes the k^{th} term. Let $\{a_i\}_{i=1}^{\infty}$ denote the sequence $\{(1,a_1), (2,a_1), \ldots\}$.

Definition 1.7. The statement that the sequence $\{a_i\}_{i=1}^{\infty}$ converges means there is a number a such that if ε is a positive number, there exists a positive integer N such that if $n > N$, then $|a_n - a| < \varepsilon$.

Definition 1.8. The statement that $\{f_i\}_{i=1}^{\infty}$ converges uniformly on S means $S \subseteq D_{fi}$ for all i, and there exists a function f, $S \subseteq D_f$, such that if ε is a positive number, there exists a positive integer N such that if $n > N$ and $x \in S$, then $|f_n(x) - f(x)| < \varepsilon$.

Theorem 1.1. Suppose that $[a,b]$ is a number interval and f is a function with domain $[a,b]$. Suppose that for each positive integer n, g_n is a function continuous on $[a,b]$. Suppose that if $\varepsilon > 0$, then there is a positive integer N such that if $m > N$ and $a \leq x \leq b$, then $|f(x) - g_m(x)| < \varepsilon$. Then f is continuous on [a,b].

Theorem 1.2. Suppose [a,b] is a number interval and for each positive integer n, g_n is a function continuous on [a,b]. Suppose that if $\varepsilon > 0$, then there is a positive integer N, such that if each of p and q is a positive integer $> N$ then $|g_p(x) - g_q(x)| < \varepsilon$ for all x in [a,b]. Then there is a function, f, continuous on [a,b] such that the hypothesis of Theorem 1.1 is satisfied.

Theorem 1.3. If g is a bounded function on [a,b] and $\{f_i\}_{i=1}^{\infty}$ converges uniformly on [a,b], then $\{gf_i\}_{i=1}^{\infty}$ converges uniformly.

Definition 1.9 Suppose [a,b] is a number interval. The statement that $x_0, x_1, \ldots, x_n$ is a subdivision of [a,b] means that $x_0, x_1, \ldots, x_n$ is a sequence of numbers such that $x_0 = a$, $x_n = b$ and $x_{i-1} < x_i$; $i = 1, 2, \ldots, n$. The statement that $t_1, t_2, \ldots, t_n$ is an interpolating sequence for $x_0, \ldots, x_n$ means that $t_1, \ldots, t_n$ is a sequence of numbers such that $x_{i-1} \leq t_i \leq x_i$, $i = 1, 2, \ldots, n$.

Definition 1.10. If f is a function whose domain includes [a,b] then the statement that J is a Reimann integral of f on [a,b] means that J is a number such that if $0 < \varepsilon$, then there is a positive number δ_ε such that if $x_0, \ldots, x_n$ is a subdivision of [a,b] with $x_i - x_{i-1} < \delta_\varepsilon$, $i = 1, 2, \ldots, n$ and $t_1, \ldots, t_n$ is an interpolating sequence for

$x_0, \ldots, x_n$ then

$$|J - \sum_{i=1}^{n} \{f(t_i)(x_i - x_{i-1})\}| < \varepsilon.$$

Theorem 1.4. If f is a function whose domain includes $[a,b]$ then there is not more than one number, J, which is an integral of f on $[a,b]$.

Definition 1.11. Some elementary properties of a Reimann integral are:

(1). Interchanging the limits of integration simply changes the sign of the integral.

(2). For any numbers,

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

(3). If the first two integrals exist, then the other exists and

$$\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b [f(x) + g(x)]dx$$

(4). If the first two integrals exist, then the other exists and

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

(5). $$\int_a^a f(x)dx = 0$$

(6). If $a \leq x \leq b$, $f(x) \leq g(x)$ then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(7). $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$ if $a < b$.

Theorem 1.5. If $[a,b]$ is an interval and f is a function defined on $[a,b]$ and m is a number such that $|f(x)| \leq M$ for $a \leq x \leq b$ and

$$\int_a^b f(x)dx$$

exists then $\left|\int_a^b f(x)dx\right| \leq M(b - a)$.

Theorem 1.6. Suppose f is a function defined on $[a,b]$ and $\{f_i\}_{i=1}^{\infty}$ is a sequence of functions defined on $[a,b]$ such that if n is a positive integer, then

$$\int_a^b f_n(x)dx$$

exists. Suppose that the sequence $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f on $[a,b]$. Then $\int_a^b f(x)\,dx$ exists and $\int_a^b f_n(x)dx$ converges to $\int_a^b f(x)dx$ as n approaches ∞.

Theorem 1.7. If f is continuous on $[a,b]$, then $\int_a^b f(x)dx$ exists.

Theorem 1.8. Suppose f is a function continuous on $[a,b]$ and $a \leq c \leq b$ and A is a number. Then the following two statements are equivalent:

(1). $f'(x) = f(x)$ for $a \leq x \leq b$ and $f(c) = A$

(2). $f(x) = A + \int_c^x f(t)dt$ for $a \leq x \leq b$.

Theorem 1.9. Suppose $[a,b]$ is a number interval and $\{f_i\}_{i=1}^{\infty}$ is a sequence of functions with domain $[a,b]$ such that for each positive integer n there is a number M_n, such that $|f_{n+1}(x) - f_n(x)| \leq M_n$ for $a \leq x \leq b$. Suppose that the series $\sum_{K=1}^{\infty} M_k$ converges. Then the sequence $\{f_i\}_{i=1}^{\infty}$ converges uniformly to some function f on $[a,b]$.

Theorem 1.10. If $a \leq x \leq b$, then $|\int_c^x |t - c|^m dt|$

$$= \frac{|x - c|^{m + 1}}{m + 1}.$$

Theorem 1.11. If each of f and g is a function continuous on $[a,b]$ and $a \leq c \leq b$ and $a \leq x \leq b$, then

$$|\int_c^x [f(t) - g(t)]dt| \leq |\int_c^x |f(t) - g(t)|dt|.$$

Theorem 1.12. Suppose $[a,b]$ is a number interval and f is a bounded function from $[a,b]$ into the numbers such that if $a \leq p < q \leq b$, then

$$\int_p^q f(x)dx$$

exists. Suppose $|f(x)| \leq M$ for all $x \in [a,b]$. Suppose $a \leq c \leq b$ and H is a number. Let

$F(x) = \int_c^x f(t)dt$ for $a \leq x \leq b$. Then F is continuous on $[a,b]$.

Proof: Let $\varepsilon > 0$. Let $\delta = \varepsilon/M$ where M is a bound of $|f(x)|$. Let $x_0 \in [a,b]$. Then if $x \in [a,b]$ and $|x - x_0| < \delta$, then

$$|F(x) - F(x_0)| = |H + \int_c^x f(t)dt - H - \int_c^{x_0} f(t)dt|$$

$$= |\int_c^x f(t)dt - \int_c^{x_0} f(t)dt| = |\int_{x_0}^c f(t)dt + \int_c^x f(t)dt|$$

$$= |\int_{x_0}^x f(t)dt| = M|x - x_0| < M\delta = M\ \varepsilon/M + 1 < \varepsilon.$$

Therefore F(x) is continuous.

<u>Theorem 1.13</u>. Suppose $\{f_i\}_{i=1}^{\infty}$ is a sequence of continuous functions defined on $[a,b]$. If $\{f_i\}_{i=1}^{\infty}$ converges uniformly to some function g on $[a,b]$ then g is continuous.

Proof: Let $x_0 \in [a,b]$. Let $\varepsilon > 0$. Since $\{f_i\}_{i=1}^{\infty}$ converges uniformly to g on $[a,b]$ there exists a positive integer N such that if $n > N$ and $x \in [a,b]$ then $|f_n(x) - g(x)| < \varepsilon/3$. If $n > N$, then since

$$f_{n+1} \in \{f_i\}_{i=1}^{\infty}$$

is continuous at x_0 there exists a $\delta > 0$ such that if $x \in [a,b]$ and $|x - x_0| < \delta$ then $|f_{n+1}(x) - f_{n+1}(x_0)| < \varepsilon/3$. Let $x_0 \in [a,b]$ such that $|x - x_0| < \delta$ and $n > N$. Then

$$
\begin{aligned}
|g(x) - g(x_0)| &= \\
&|g(x) - f_{n+1}(x) + f_{n+1}(x) + f_{n+1}(x_0) - f_{n+1}(x_0) - g(x_0)| \leq \\
&|g(x) - f_{n+1}(x)| + |f_{n+1}(x) - f_{n+1}(x_0)| + |f_{n+1}(x_0) \\
&\qquad - g(x_0)| = \\
&|f_{n+1}(x) - g(x)| + |f_{n+1}(x) - f_{n+1}(x_0)| + \\
&\qquad |f_{n+1}(x_0) - g(x_0)| < \\
&\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}
$$

Therefore g is continuous.

CHAPTER II

UNIQUE SOLUTIONS OF N^{TH} ORDER LINEAR INTEGRAL EQUATIONS

Suppose $[a,b]$ is a number interval and f is a bounded function from $[a,b]$ into the real numbers. The purpose of this chapter is to prove the existence of a unique solution of an n^{th} order linear integral equation. A first degree linear equation is studied to show its existence and uniqueness properties and the second degree equation is studied to show the properties that are analogous to the first degree equation.

Theorem 2.1. Suppose $[a,b]$ is a number interval and f is a bounded function from $[a,b]$ into the numbers such that if $a \leq p < q \leq b$, then

$$\int_p^q f(x)dx$$

exists. Suppose $|f(x)| \leq M$ for all $x \in [a,b]$. Suppose $a \leq c \leq b$, and H is a number. Let the sequence $\{f_i\}_{i=1}^{\infty}$ of functions on $[a,b]$ be defined as follows:

$$f_1(x) = H + \int_c^x f(t)dt$$

$$f_2(x) = H + \int_c^x f_1(t)dt$$

$\vdots$

$$f_{n+1}(x) = H + \int_c^x f_n(t)dt$$

$\vdots$

If m is a positive integer and z is in $[a,b]$, then

$|f_{m+1}(z) - f_m(z)| \leq \frac{M^* \cdot |z - c|^m}{m!}$, where

$|f_1(w) - f(w)| \leq M^*$ for all $w \in [a,b]$.

Proof: Show f(x) is bounded.

$f(x) = H + \int_c^x f(t)dt$. From previous work we have shown that $|\int_a^b f(x)dx| \leq M(b - a)$ where $|f(x)| \leq M$ and $a \leq x \leq b$.

Hence $f_1(x) = H + \int_c^x f(t)dt$, which implies $|f_1(x)| =$

$$|H + \int_c^x f(t)dt| \leq |H| + |\int_c^x f(t)dt| \leq$$

$$|H| + M|x - c| = K.$$

This implies $f_1(x)$ is bounded. Suppose $z \in [a,b]$. We use mathematical induction to show that

$$|f_{m+1}(z) - f_m(z)| \leq \frac{M^* \cdot |z - c|^m}{m!}$$

for $|f_1(w) - f(w)| < M^*$ for all $w \in [a,b]$.

Show true for m = 1.

$$|f_{m+1}(z) - f_m(z)| = |f_2(z) - f_1(z)| =$$

$$|H + \int_c^z f_1(t)dt - H - \int_c^z f(t)dt| = |\int_c^z f_1(t)dt - \int_c^z f(t)dt| =$$

$$|\int_c^z (f_1(t) - f(t))dt| \leq |\int_c^z |f_1(t) - f(t)|dt| \leq |\int_c^z M^* dt| =$$

$$M^*|z - c| = \frac{M^* \cdot |z - c|^m}{m!}$$

Now suppose true for $m = n$. That is, we assume that $|f_{n+1}(z) - f_n(z)| \leq \frac{M^* \cdot |z - c|^n}{n!}$ for all z such that $a \leq z \leq b$. Therefore show true for $m = n+1$.

$$|f_{m+1}(z) - f_m(z)| = |f_{n+2}(z) - f_{n+1}(z)| =$$

$$|H + \int_c^z f_{n+1}(t)dt - H - \int_c^z f_n(t)dt| =$$

$$|\int_c^z f_{n+1}(t)dt - \int_c^z f_n(t)dt| = |\int_c^z (f_{n+1}(t) - f_n(t))dt| \leq$$

$$|\int_c^z |f_{n+1}(t) - f_n(t)|dt| \leq |\int_c^z \frac{M^* \cdot |t - c|^n}{n!} dt| \leq$$

$$\frac{M^* \cdot |z - c|^{n+1}}{(n+1)!} = \frac{M^* \cdot |z - c|^m}{m!}$$

Hence true for all m.

Theorem 2.2. Suppose $[a,b]$ is a number interval and f is a bounded function from $[a,b]$ into the numbers such that if $a \leq p < q \leq b$, then

$$\int_p^q f(x)dx$$

exists. Suppose $|f(x)| \leq M$ for all $x \in [a,b]$. Suppose $a \leq c \leq b$ and H is a number. Let the sequence $\{f_i\}_{i=1}^{\infty}$ of functions on $[a,b]$ be defined as follows:

$$f_1(x) = H + \int_c^x f(t)dt$$

$$f_2(x) = H + \int_c^x f_1(t)dt$$

$$\vdots$$

$$f_{n+1}(x) = H + \int_c^x f_m(t)dt$$

$$\vdots$$

Then there is a function g on $[a,b] \ni f_n(x)$ converges to g uniformly on $[a,b]$, and g has the following properties:

(1). g is continuous, and

(2). $g(x) =$

$$H + \int_c^x g(t)dt$$

for all $x \in [a,b]$.

Proof:

$$|f_1(x) - f(x)| = |H + \int_c^x f(t)dt - f(x)| \leq |H| + |\int_c^x f(t)dt| + |f(x)|$$

$$\leq |H| + M|x - c| + M$$

$$\leq |H| + M(b - a) + M$$

$$= M^*$$

$|f_1(x) - f(x)| \leq M^*$, for all $x \in [a,b]$, and therefore, for m is a positive integer and $x \in [a,b]$

$$|f_{m+1}(x) - f_m(x)| \leq \frac{M^*|x - c|^m}{m!} \leq \frac{M^*|b - a|^m}{m!}$$

Now, $\sum_{m=1}^{\infty} \frac{M^*|b - a|^m}{m!} = M^* \sum_{m=1}^{\infty} \frac{|b - a|^m}{m!}$

Since $\sum_{n=1}^{\infty} \frac{k^n}{n!}$ converges, $M^* \sum_{m=1}^{\infty} \frac{|x - c|^m}{m!}$ converges.

Therefore $\{f_i\}_{i=1}^{\infty}$ converges uniformly to some function g on $[a,b]$. By Theorem 1.11 each $f_{n+1} \in \{f_i\}_{i=1}^{\infty}$ is continuous. By Theorem 1.12 a sequence of continuous functions which converge uniformly convergesto a continuous function. Therefore g is continuous.

Since $\int_c^x f_n(t)dt$ exists for each n and $x \in [a,b]$ and since $\{f_i\}_{i=1}^{\infty}$ converges uniformly to g on $[a,b]$, then $\int_c^x g(t)dt$ for each $x \in [a,b]$ exists and $\int_c^x f_n(t)dt$ converges to $\int_c^x g(t)dt$, as $n \to \infty$. But $\int_c^x f_n(t)dt = f_{n+1}(x) - H \Rightarrow f_{n+1}(x) - H$ converges to $\int_c^x g(t)dt$ as $n \to \infty$, but $f_{n+1}(x)$ converges to $g(x)$. Therefore $g(x) = H + \int_c^x g(t)dt$, for all $x \in [a,b]$.

<u>Theorem 2.3</u>. If G is a continuous function defined

on $[a,b]$ and $a \leq c \leq b$ and $g(x) = \int_c^x g(t)dt$ for all $x \in [a,b]$, then $g(x) = 0$ for all $x \in [a,b]$.

<u>Proof</u>: As in previous theorems, we can define a sequence of functions $\{f_i\}_{i=1}^{\infty}$ on $[a,b]$, as follows:

$$f_1(x) = H + \int_c^x f(t)dt$$

$$f_2(x) = H + \int_c^x f_1(t)dt$$

$$\vdots$$

$$f_{n+1}(x) = H + \int_c^x f_n(t)dt$$

$$\vdots$$

where H is equal to zero, and $f_k = g$ for all k. Now we show by induction that $|f_{n+1}(x)| \leq \frac{M|x - c|^{n+1}}{(n+1)!}$, for all n and all $x \in [a,b]$.

Show true for n = 1.

$$|f_1(x)| = |\int_c^x f(t)dt| \leq M|x - c| \text{ where } |f(x)| \leq M$$

Show true for n = 2.

$$|f_2(x)| = |\int_c^x f_1(t)dt| \leq |\int_c^x M|t - c|dt| = \frac{M|b - a|^2}{2!}$$

Now, assume $|f_n(x)| \leq \frac{M|x - c|^n}{n!}$ for all $x \in [a,b]$.

$$|f_{n+1}(x)| = |\int_c^x f_n(t)dt| \leq |\int_c^x \frac{M|t - c|^n dt}{n!}|$$
$$= \frac{M|x - c|^{n+1}}{n!\ n+1}$$
$$= \frac{M|x - c|^{n+1}}{(n+1)!} \leq \frac{M|b - a|^{n+1}}{(n+1)!}$$

Therefore, for all n $|f_n| \leq \frac{M|b - a|^n}{n!}$.

Since $M \sum_{n=1}^{\infty} \frac{|b - a|^n}{n!}$ converges, the limit of the sequence, $\{\frac{M|b - a|^n}{n!}\}_{n=1}^{\infty}$ is equal to zero. This implies $\{f_i\}_{i=1}^{\infty}$ converges uniformly to 0. Therefore $g(x) = 0$.

<u>Theorem 2.4</u>. If H is a number and $[a,b]$ is a number interval and $a \leq c \leq b$ then there exists one and only one function, g with domain $[a,b]$ such that $\int_c^b g(t)dt$ exists and $g(x) = H + \int_c^x g(t)dt$ for all $x \in [a,b]$.

<u>Proof</u>: By previous theorem we know there exists a continuous function g defined on $[a,b]$ such that $\int_a^b g(t)dt$ exists and $g(x) = H + \int_c^x g(t)dt$ for all $x \in [a,b]$. Suppose there exists a continuous function, $g^* \neq g$, defined on $[a,b]$ such that $\int_a^b g^*(t)dt$ exists and $g^*(x) = H + \int_c^x g^*(t)dt$.
Since each of g and g^* is a continuous function, $(g - g^*)$ is

a continuous function. Now

$$h(x) = g(x) - g^*(x) = H + \int_c^x g(t)dt - H - \int_c^x g^*(t)dt =$$

$$\int_c^x g(t)dt - \int_c^x g^*(t)dt = \int_c^x [g(t) - g^*(t)]dt = \int_c^x h(t)dt$$

Therefore by Theorem 2.3 $h = 0$ on $[a,b]$.

Theorem 2.5. Suppose that each of p, q, r and s is a function continuous on $[a,b]$ and $a \leq c \leq b$ and each of H and K is a number. There is one and only one pair of functions f, g such that if $x \in [a,b]$, then

$$f(x) = H + \int_c^x [p(t)f(t) + q(t)g(t)]dt \text{ and}$$

$$g(x) = H + \int_c^x [r(t)f(t) + s(t)g(t)]dt$$

Proof: Consider the sequence $\{f_i(x), g_i(x)\}_{i=1}^{\infty}$ defined as

$$f_1(x) = H + \int_c^x [p(t)f_0(t) + q(t)g_0(t)]dt$$

$$g_1(x) = K + \int_c^x [r(t)f_0(t) + s(t)g_0(t)]dt$$

$$\vdots$$

$$f_{n+1}(x) = H + \int_c^x [p(t)f_n(t) + q(t)g_n(t)]dt$$

$$g_{n+1}(x) = K + \int_c^x [r(t)f_n(t) + s(t)g_n(t)]dt$$

Suppose $x \in [a,b]$. Use mathematical induction to show for some T, N and all n

$$|f_n(x) - f_{n-1}(x)| \leq \frac{2^{n-2} \cdot T^{n-1} \cdot N|x - c|^{n-1}}{n!} \quad \text{and}$$

$$|g_n(x) - g_{n-1}(x)| \leq \frac{2^{n-2} \cdot T^{n-1} \cdot N|x - c|^{n-1}}{n!}.$$

Since each of r, s, p, and q is a bounded function, there is a T such that T bounds each of r, s, p, and q; since f and g are bounded, $|f_1 - f_0|$ and $|g_1 - g_0|$ are bounded. There is an M and M^* such that $|f_1 - f_0| \leq M$ and $|g_1 - g_0| \leq M^*$. Let $N = M + M^*$.

Show true for n = 2.

$$|f_2(x) - f_1(x)| = |H + \int_c^x [p(t)f_1(t) + q(t)g_1(t)]dt$$

$$- H - \int_c^x [p(t)f_0(t) + q(t)g_0(t)]dt|$$

$$= |\int_c^x [p(t)f_1(t) + q(t)g_1(t)]dt - \int_c^x [p(t)f_0(t) + q(t)g_0(t)]dt|$$

$$\leq |\int_c^x |[p(t)f_1(t) + q(t)g_1(t)] - [p(t)f_0(t) + q(t)g_0(t)]|dt$$

$$= |\int_c^x |p(t)(f_1(t) - f_0(t)) + q(t)(g_1(t) - g_0(t))|dt|$$

$$\leq |\int_c^x [|p(t)(f_1(t) - f_0(t))| + |q(t)(g_1(t) - g_0(t))|]dt$$

$$\leq |\int_c^x TM + TM^* dt|$$

$$= |\int_c^x T(M + M^*) dt|$$

$$= |\int_c^x TN dt| = TN|x - c| \quad \text{and}$$

$$|g_2(x) - g_1(x)| = |K + \int_c^x r(t)f_1(t) + s(t)g_1(t)dt$$

$$- K - \int_c^x r(t)f_0(t) + s(t)g_0(t)dt|$$

$$\leq |\int_c^x |r(t)f_1(t) + s(t)g_1(t) - r(t)f_0(t) - s(t)g_0(t)|dt|$$

$$\leq |\int_c^x |r(t)(f_1(t) - f_0(t))| + |s(t)(g_1(t) - g_0(t))|dt|$$

$$\leq |\int_c^x TM + TM^* dt| \leq |\int_c^x T(M + M^*)dt| = |\int_c^x TN dt| = Tn|x - c|$$

Suppose $|f_n(x) - f_{n-1}(x)| \leq \dfrac{2^{n-2} \cdot T^{n-1} \cdot N \cdot |x - c|^{n-1}}{(n-1)!}$ and

$$|g_n(x) - g_{n-1}(x)| \leq \frac{2^{n-2} \cdot T^{n-1} \cdot N \cdot |x - c|^{n-1}}{(n-1)!}$$

Show true for n + 1.

$$|f_{n+1}(x) - f_n(x)| = |H + \int_c^x [p(t)f_n(t) + q(t)g_n(t)]dt$$

$$- H - \int_c^x [p(t)f_{n-1}(t) + q(t)g_{n-1}(t)]dt|$$

$$= \left|\int_c^x [p(t)f_n(t) + q(t)g_n(t) - p(t)f_{n-1}(t) + q(t)g_{n-1}(t)]dt\right|$$

$$\leq \left|\int_c^x |p(t)(f_n(t) - f_{n-1}(t))| + |q(t)(g_n(t) - g_{n-1}(t))|dt\right|$$

$$\leq \left|\int_c^x \frac{T(2^{n-2}\cdot T^{n-1}\cdot N\cdot|t - c|^{n-1})}{(n-1)!} + \frac{T(2^{n-2}\cdot T^{n-1}\cdot N\cdot|t - c|^{n-1})}{(n-1)!}\right|$$

$$= \left|\int_c^x \frac{T(2^{n-1}\cdot T^{n-1}\cdot N\cdot|t - c|^{n-1})dt}{(n-1)!}\right|$$

$$= \frac{2^{n-1}\cdot T^n\cdot N\cdot|x - c|^n}{n!}$$

Therefore for all n $|f_{n+1}(x) - f_n(x)| \leq \dfrac{2^{n-1}\cdot T^n\cdot N\cdot|b - a|^n}{n!}$

$$|g_{n+1}(x) - g_n(x)| = |K + \int_c^x [r(t)f_n(t) + s(t)g_n(t)]dt$$

$$- K - \int_c^x [r(t)f_{n-1}(t) + s(t)g_{n-1}(t)]dt$$

$$\leq \left|\int_c^x |r(t)(f_n(t) - f_{n-1}(t))| + |s(t)(g_n(t) - g_{n-1}(t))|dt\right|$$

$$\leq \left|\int_c^x \frac{T(2^{n-2}\cdot T^{n-1}\cdot N\cdot|t - c|^{n-1})}{(n-1)!} + \frac{T(2^{n-2}\cdot T^{n-1}\cdot N\cdot|t - c|^{n-1})}{(n-1)!}dt\right|$$

$$= \left|\int_c^x \frac{T(2^{n-1}\cdot T^{n-1}\cdot N\cdot|t - c|^{n-1})}{(n-1)!}dt\right|$$

$$= \frac{2^{n-1}\cdot T^n\cdot N\cdot|x - c|^n}{n!}$$

Therefore for all n $|g_{n+1}(x) - g_n(x)| \leq \dfrac{2^{n-1}\cdot T^n\cdot N\cdot|b - a|^n}{n!}$

Now $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-2}\cdot T^{n-1}\cdot N\cdot|b - a|^{n-1}}{(n-1)!}$ converges; therefore,

$\{f_i(x)_{i=1}^{\infty}$ and $\{g_i(x)\}_{i=1}^{\infty}$ converges uniformly to some function respectively, say f and g, on $[a,b]$. By Theorem 1.11 each $g_i \in \{g_i(x)\}_{i=1}^{\infty}$ and each $f_i \in \{f_i(x)\}_{i=1}^{\infty}$ is continuous. Therefore $f_{n+1}(x)$ is continuous $\Rightarrow \{f_i(x)\}_{i=1}^{\infty}$ is a sequence of continuous functions. Similarly $\{g_i(x)\}_{i=1}^{\infty}$ is a sequence of continuous functions. Since the uniform limit of a sequence of continuous functions is continuous by Theorem 1.12, f and g are continuous. By previous theorem since

$$\int_c^x [p(t)f_n(t) + q(t)g_n(t)]dt$$

exists and $\int_c^x [r(t)f_n(t) + s(t)g_n(t)]dt$ exists for all n and $x \in [a,b]$ and since $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f on $[a,b]$ and $\{g_i\}_{i=1}^{\infty}$ converges uniformly to g then

$$\int_c^x [p(t)f(t) + q(t)g(t)]dt$$

exists for all $x \in [a,b]$; $\int_c^x [r(t)f(t) + s(t)g(t)]dt$ exists for all $x \in [a,b]$. Now $\int_c^x [p(t)f_n(t) + q(t)g_n(t)]dt$ converges to $\int_c^x [p(t)f(t) + q(t)g(t)]dt$ as $n \to \infty$.

$\int_c^x [r(t)f_n(t) + s(t)g_n(t)]dt$ converges to

$$\int_c^x [r(t)f(t) + s(t)g(t)]dt$$

as $n \to \infty$.

But $\int_c^x [p(t)f_n(t) + q(t)g_n(t)]dt = f_{n+1}(x) - H \Rightarrow f_{n+1}(x) - H$ converges to $\int_c^x [p(t)f(t) + q(t)g(t)]dt$ and $\int_c^x [r(t)f_n(t) + s(t)g_n(t)]dt = g_{n+1}(x) - K \Rightarrow g_{n+1}(x) - K$ converges to $\int_c^x [r(t)f(t) + s(t)g(t)]dt$ as $n \to \infty$. However, $f_{n+1}(x)$ converges to $f(x)$ and $g_{n+1}(x)$ converges to $g(x)$. Therefore, $f(x) = H + \int_c^x [p(t)f(t) + q(t)g(t)]dt$ and

$$g(x) = K + \int_c^x [r(t)f(t) + s(t)g(t)]dt.$$

Now, suppose there is a pair of continuous functions f^*, g^*, where $f^* \neq f$ and $g^* \neq g$, defined on $[a,b]$ such that $\int_a^b [p(t)f^*(t) + g(t)g^*(t)]dt$ exists and $f^*(x) =$

$$H + \int_c^x [p(t)f^*(t) + q(t)g^*(t)]dt;$$

$\int_a^b [r(t)f^*(t) + s(t)g^*(t)]dt$ exists and $g^*(x) =$

$$K + \int_c^x [r(t)f^*(t) + s(t)g^*(t)]dt$$

for all $x \in [a,b]$. We want to show

$$|f^*(x) - f(x)| \leq \frac{2^{n-1} \cdot J^n \cdot P \cdot |x - c|^n}{n!}$$

and $|g^*(x) - g(x)| \leq \frac{2^{n-1} \cdot J^n \cdot P \cdot |x - c|^n}{n!}$ for some J, P and all n.

Since each of r, s, p and q is a bounded function there exists a number J such that J bounds each of r, s, p, q; since f, f^*, g, and g^* are bounded, $|f - f^*|$ and $|g - g^*|$ is bounded. There exists a z and z^* such that $|f - f^*| \leq z$ and $|g - g^*| \leq z^*$. Let $P = z + z^*$.

Show true for n = 1.

$$|f(x) - f^*(x)| = |H + \int_c^x [p(t)f(t) + q(t)g(t)]dt$$

$$- H - \int_c^x [p(t)f^*(x) + q(t)g^*(t)]dt|$$

$$= |\int_c^x [p(t)f(t) + q(t)g(t) - p(t)f^*(t) - q(t)g^*(t)]dt|$$

$$\leq |\int_c^x |p(f(t) - f^*(t))| + |q(t)(g(t) - g^*(t))|dt|$$

$$\leq |\int_c^x J(z) + J(z^*)dt| \leq |\int_c^x JPdt| = JP|x - c|.$$

$$|g(x) - g^*(x)| = |K + \int_c^x [r(t)f(t) + s(t)g(t)]dt$$

$$- K - \int_c^x [r(t)f^*(t) + s(t)g^*(t)]dt|$$

$$\leq |\int_c^x |r(t)(f(t) - f^*(t))| + |s(t)(g(t) - g^*(t))|dt|$$

$$\leq |\int_c^x Jz + Jz^*dt| \leq |\int_c^x JPdt| = JP|x - c|.$$

Assume $|f(x) - f^*(x)| \leq \frac{2^{n-1} \cdot J^n \cdot P \cdot |x - c|^n}{n!}$ and

$|g(x) - g^*(x)| \leq \frac{2^{n-1} \cdot J^n \cdot P \cdot |x - c|^n}{n!}$

Show true for n+1.

$$|f(x) - f^*(x)| = |H + \int_c^x [p(t)f(t) + q(t)g(t)]dt$$

$$- H - \int_c^x [p(t)f^*(t) + q(t)g^*(t)]dt|$$

$$\leq |\int_c^x |p(t)(f(t) - f^*(t))| + |s(t)(g(t) - g^*(t))|dt|$$

$$\leq |\int_c^x \frac{J(2^{n-1} \cdot J^n \cdot P \cdot |t - c|^n)}{n!} + \frac{J(2^{n-1} \cdot J^n \cdot P \cdot |t - c|^n)}{n!}|$$

$$= |\int_c^x \frac{2^n \cdot J^{n+1} \cdot P \cdot |t - c|^n}{n!} dt| = \frac{2^n \cdot J^{n+1} \cdot P \cdot |x - c|^{n+1}}{n!}$$

$$|g(x) - g^*(x)| = |K + \int_c^x [r(t)f(t) + s(t)g(t)]dt$$

$$- K - \int_c^x [r(t)f^*(t) + s(t)g^*(t)]dt$$

$$\leq |\int_c^x |r(t)(f(t) - f^*(t))| + |s(t)(g(t) - g^*(t))|dt|$$

$$\leq |\int_c^x \frac{J(2^{n-1} \cdot J^n \cdot P \cdot |t - c|^n}{n!} + \frac{J(2^{n-1} \cdot J^n \cdot P \cdot |t - c|^n)}{n!} dt|$$

$$= |\int_c^x \frac{2^n \cdot J^{n+1} \cdot P \cdot |t - c|^n}{n!} dt| = \frac{2^n \cdot J^{n+1} \cdot P \cdot |x - c|^{n+1}}{n!}$$

Since $\sum_{n=1}^{\infty} \frac{2^{n-1} \cdot J^n \cdot P \cdot |x - c|^n}{n!}$ converges, the limit of the sequence $\{\frac{2^{n-1} J^n P |x - c|^n}{n!}\}$ is zero. This implies that each of $\{g(x) - g^*(x)\}$ and $\{f(x) - f^*(x)\}$ is zero. Therefore $g(x) - g^*(x) = 0$; $f(x) - f^*(x) = 0$ for all $x \in [a,b]$; hence $g(x) = g^*(x)$; hence $f(x) = f^*(x)$ for all $x \in [a,b]$. Hence there exists one and only one pair of functions f, g, such that if $x \in [a,b]$ then:

$$f(x) = H + \int_c^x [p(t)f(t) + q(t)g(t)]dt$$

$$f(x) = K + \int_c^x [r(t)f(t) + s(t)g(t)]dt.$$

<u>Theorem 2.6</u>. If n is a positive integer and for all positive integers i and $j \leq n$, $g_{i,j}$ is a function continuous on $[a,b]$ and A_i is a number and $a \leq c \leq b$, then there is one and only one sequence of functions $f_1, f_2, \ldots, f_n$ satisfying

$$f_k(x) = A_k + \int_c^x \sum_{j=1}^{n} g_{k,j}(t)f_j(t)dt \quad K = 1, 2, \ldots, n.$$

<u>Proof</u>: Consider the sequence $\{h_{i,k}\}_{i=1}^{\infty}$ $(K = 1, 2, \ldots, n)$ defined as:

$$h_{1,k}(x) = A_k + \int_c^x \sum_{j=1}^{n} g_{k,j}(t)h_{0,j}(t)\, dt$$

$$h_{2,k}(x) = A_k + \int_c^x \sum_{j=1}^{n} g_{k,j}(t)h_{1,j}(t)dt$$

$$\vdots$$

$$h_{w+1,k}(x) = A_k + \int_c^x \sum_{j=1}^{n} g_{k,j}(t)h_{w,j}(t)dt$$

$$\vdots$$

where each of $h_{0,1}$, $h_{0,2}$, ..., $h_{0,n}$ is a function continuous on $[a,b]$. We want to show that $\{h_{i,k}\}_{i=1}^{\infty}$ ($k = 1, 2, \ldots, n$) converges to f_k for each k. For any k

$$|h_{2,k}(x) - h_{1,k}(x)| = |A_k + \int_c^x \sum_{j=1}^{n} g_{k,j}(t)h_{1,j}(t)dt$$

$$- A_k - \int_c^x \sum_{j=1}^{n} g_{k,j}(t)h_{0,j}(t)dt|$$

$$= |\int_c^x \sum_{j=1}^{n} g_{k,j}(t) - h_{1,j}(t)dt - \int_c^x \sum_{j=1}^{n} g_{k,j}(t)h(t)dt|$$

$$\leq |\int_c^x |\sum_{j=1}^{n} g_{k,j}(t)(h_{1,j}(t) - h_{0,j}(t))|dt|$$

$$\leq |\int_c^x \sum_{j=1}^{n} |g_{k,j}(t)(h_{1,j}(t) - h_{0,j}(t)|dt|$$

$$\leq |\int_c^x \sum_{j=1}^{n} T \cdot N dt| = nTN|x - c|$$

where T is a bound of each $g_{k,j}$ ($k = 1, 2, \ldots, n$) and $|h_{1,j}(t) - h_{0,j}(t)| \leq n$. Since all of these functions are continuous, we know these are bounded.

Suppose, $|h_{w,k}(x) - h_{w-1,k}(x)| \leq \dfrac{n^{w-1} \cdot T^{w-1} \cdot N \cdot |x - c|^{w-1}}{(w-1)!}$

Show true for w + 1.

$$|h_{w+1,k}(x) - h_{w,k}(x)| = |A_k + \int_c^x \sum_{j=1}^{n} g_{k,j}(t)h_{w,j}(t)dt$$

$$- A_k - \int_c^x \sum_{j=1}^{n} g_{k,j}(t)h_{w-1,j}(t)dt|$$

$$\leq \left|\int_c^x \left| \sum_{j=1}^{n} g_{k,j}(x)(h_{w,j}(t) - h_{w-1,j}(t))\right| dt\right|$$

$$\leq \left|\int_c^x \sum_{j=1}^{n} |g_{k,j}(t)(h_{w,j}(t) - h_{w-1,j}(t))| dt\right|$$

$$\leq \left|\int_c^x \sum_{j=1}^{n} \frac{T(n^{w-1} \cdot T^{w-1} \cdot N \cdot |t - c|^{w-1})}{(w-1)!} dt\right|$$

$$= \frac{n^w \cdot T^w \cdot N \cdot |x - c|^w}{w!} .$$

Therefore, for all w $|h_{w,k}(x) - h_{w-1,k}(x)|$

$$\leq \frac{n^{w-1} \cdot T^{w-1} \cdot N \cdot |b - a|^{w-1}}{(w-1)!}$$

Now $\sum_{j=1}^{n} \frac{n^w \cdot T^w \cdot N \cdot |b - a|^w}{w!}$ converges; therefore, $\{h_{i,k}\}_{i=1}^{\infty}$ $(k = 1, 2, \ldots, n)$ converges uniformly to some function, say f_k, on $[a,b]$. By Theorem 1.11 each $h_{i,k} \in \{h_{i,k}\}_{i=1}^{\infty}$ $(k = 1, 2, \ldots, n)$ is continuous. Therefore $\{h_{i,k}\}_{i=1}^{\infty}$ $(k = 1, 2, \ldots, n)$ is a sequence of continuous functions. Since the uniform limit of a sequence of continuous functions is continuous, Theorem 1.12, $f_k(x)$ is continuous. By previous theorem $\int_c^x \sum_{j=1}^{n} g_{k,j}(t)h_{w,j}(t)dt$ exists for all n and $x \in [a,b]$ and since $\{h_{i,k}(x)\}_{i=1}^{\infty}$ converges uniformly to f_k on $[a,b]$ then $\int_c^x \sum_{j=1}^{n} g_{k,j}(t)f_j(t)dt$ exists for all $x \in [a,b]$ and $\int_c^x \sum_{j=1}^{n} g_{k,j}(t)h_{w,j}(t)dt$ converges to $\int_c^x \sum_{j=1}^{n} g_{k,j}(t)f_j(t)dt$.

But $\int_c^x \sum_{j=1}^n g_{k,j}(t)h_{w,j}(t)dt = h_{w+1,j}(x) - A_k$, which converges to $\int_c^x \sum_{j=1}^n g_{k,j}(t)f_j(t)dt$, which implies $h_{w,j}(x)$ converges to f_w. Therefore

$$f_k(x) = A_k + \int_c^x \sum_{j=1}^n g_{k,j}(t)f_j(t)dt.$$

Suppose there exists a sequence of continuous functions $f_1^*, f_2^*, \ldots, f_n^*$ satisfying $f_k = A_k + \int_c^x \sum_{j=1}^n g_{k,j}(t)f_j(t)dt$.

$$|f_k(x) - f_k^*(x)| = |A_k + \int_c^x \sum_{j=1}^n g_{k,j}(t)f_k(t)dt$$

$$- A_k - \int_c^x \sum_{j=1}^n g_{k,j}(x)f_k^*(t)dt|$$

$$= |\int_c^x \sum_{j=1}^n g_{k,j}(t)(f_k(t) - f_k^*(t))dt|$$

$$\leq |\int_c^x \sum_{j=1}^n g_{k,j}(t)(f_k(t) - f_k^*(t)|dt|$$

$$\leq |\int_c^x \sum_{j=1}^n |g_{k,j}(t)(f_k(t) - f_k^*(t))|dt|$$

$$\leq |\int_c^x \sum_{j=1}^n TM\, dt| = nTM|x - c|$$

where T is the bound of each $g_{k,j}$ and $|f_k^*(t) - f_k(t)| \leq M$.

Suppose $|f_k(x) - f_k^*(x)| \leq \frac{n^{w-1}\, T^{w-1}\, M\, |x - c|^{w-1}}{(w-1)!}$.

Show true for w.

$$|f_k(x) - f_k^*(x)| = |A_k + \int_c^x \sum_{j=1}^n g_{k,j}(t)f_k(t)dt$$

$$- A_k - \int_c^x \sum_{j=1}^n g_{k,j}(t)f_k^*(t)dt|$$

$$= |\int_c^x \sum_{j=1}^n g_{k,j}(t)(f_k(t) - f_k^*(t))dt|$$

$$\leq |\int_c^x \sum_{j=1}^n g_{k,j}(t)(f_k(t) - f_k^*(t))|dt|$$

$$\leq |\int_c^x \sum_{j=1}^n |g_{k,j}(t)(f_k(t) - f_k^*(t))|dt|$$

$$\leq |\int_c^x \sum_{j=1}^n \frac{T(n^{w-1} \cdot T^{w-1} \cdot M \cdot |t - c|^{w-1})}{(w - 1)!}| = \frac{n^w \cdot T^w \cdot M \cdot |x - c|^w}{w!}$$

Since $\sum_{w=1} \frac{n^w \cdot T^w \cdot M \cdot |x - c|^w}{w!}$ converges, the limit of the sequence $\{\frac{n^w \cdot T^w \cdot M \cdot |x - c|^w}{w!}\}$ is zero. This implies

$$\{f_k(x) - f_k^*(x)\}$$

is zero. Therefore $f_k(x) - f_k^*(x) = 0$ hence $f_k(x) = f_k^*(x)$ for all $x \in [a,b]$. Hence there exists one and only one sequence of functions $f_1, \ldots, f_n$ such that if $x \in [a,b]$ then

$$f_k(x) = A_k + \int_c^x \sum_{j=1}^n g_{k,j}(t)f_j(t)dt.$$

CHAPTER III

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF N^{TH} ORDER NONLINEAR INTEGRAL EQUATIONS

In this chapter nonlinear integral equations will be studied. Suppose ϕ is a bounded function defined on $[a,b] \times R$ (where R is the set of all real numbers). The purpose of this chapter will be to show by Picard's method the existence of unique solutions to integral equations of the form $f(x) = f_0(x) + \int_{x_0}^{x} \phi[t,f(t)]dt$. Starting with an initial function, the method consists of making successive substitutions and is used for higher-ordered equations and systems of equations.

<u>Theorem 3.1</u>. Suppose ϕ is a bounded function defined on $[a,b] \times R$, is the set of all numbers such that

(1). if g is a function continuous on $[a,b]$, then the function $h(x) = \phi(x,g(x))$ is continuous on $[a,b]$

(2). there is a number M such that if each of x and y is a number and $z \in [a,b]$ then

$$|\phi(z,x) - \phi(z,y)| \leq M|x - y|.$$

Suppose $a \leq c \leq b$ and $\{h_i\}_{i=1}^{\infty}$ is a sequence of functions and h is a bounded function on [a,b] and T is a number such that $|\phi(x,h(x))| \leq T$ for all $x \in [a,b]$ and $\int_a^b \phi(x,h(x))$ exists.

Suppose further, H is a number and

$$h_1(x) = H + \int_c^x \phi(t,h(t))dt$$

$$h_2(x) = H + \int_c^x \phi(t,h_1(t))dt$$

$$\vdots$$

$$h_{n+1}(x) = H + \int_c^x \phi(t,h_n(t))dt$$

Then $\{h_i\}_{i=1}^{\infty}$ converges uniformly to a function g defined and continuous on [a,b] such that

$$g(x) = H + \int_c^x \phi(t,g(t))dt$$

for all x in [a,b].

Proof: Show h_1 is bounded. If x is in [a,b], then $h_1(x) = H + \int_c^x \phi(t,h(t))dt$. This implies

$$|h_1(x)| = |H + \int_c^x \phi(t,h(t))dt|$$

$$\leq |H| + |\int_c^x \phi(t,h(t))dt|$$

$$\leq |H| + T|x - c| \leq |H| + T|a - b|.$$

Therefore $|h_1(x)| \leq K$. There is a number N, such that $|h(x)| \leq N$ for all x in [a,b]. Then

$$|h_1(x) - h(x)| = |H + \int_c^x \phi(t,h(t))dt - h(x)|$$

$$\leq |H| + |\int_c^x \phi(t,h(t))dt| + |h(x)|$$

$$\leq |H| + T|x - c| + N = M^*.$$

Therefore $|h_1(x) - h(x)| \leq M^*$ for all $x \in [a,b]$.

Suppose $x \in [a,b]$. Use mathematical induction to show

$$|h_{y+1}(x) - h_y(x)| \leq \frac{M^y \cdot M^* \cdot |x - c|^n}{n!}$$

for all x in [a,b].

Show true for y = 1.

$$|h_2(x) - h_1(x)| = |H + \int_c^x \phi(t,h_1(t))dt - H - \int_c^x \phi(t,h(t))dt|$$

$$= |\int_c^x \phi(t,h_1(t))dt - \int_c^x \phi(t,h(t))dt|$$

$$= |\int_c^x [\phi(t,h_1(t)) - \phi(t,h(t))]dt|$$

$$\leq |\int_c^x |\phi(t,h_1(t)) - \phi(t,h(t))|dt|$$

$$\leq |\int_c^x M|h_1(t) - h(t)|dt|$$

$$\leq |\int_c^x M \cdot M^* \, dt| \leq M \cdot M^* |x - c|.$$

Suppose true for $y = n - 1$ or

$$|h_n(x) - h_{n-1}(x)| \le \frac{M^{n-1} \cdot M^{*} \cdot |x - c|^{n-1}}{(n-1)!}$$

for all x in $[a,b]$.

Show true for $y = n$.

$$|h_{n+1}(x) - h_n(x)| = |H + \int_c^x \phi(t,h_n(t))dt - H - \int_c^x \phi(t,h_{n-1}(t))dt|$$

$$= |\int_c^x \phi(t,h_n(t))dt - \int_c^x \phi(t,h_{n-1}(t))dt|$$

$$= |\int_c^x [\phi(t,h_n(t)) - \phi(t,h_{n-1}(t))]dt|$$

$$\le |\int_c^x |\phi(t,h_n(t)) - \phi(t,h_{n-1}(t))|dt|$$

$$\le |\int_c^x M|h_n - h_{n-1}|dt|$$

$$\le |\int_c^x \frac{M \cdot M^{n-1} \cdot M^{*} \cdot |t - c|^{n-1}}{(n-1)!} dt|$$

$$\le \frac{M^n \cdot M^{*} \cdot |x - c|^n}{n!}$$

Therefore $|h_{n+1}(x) - h_n(x)| \le \frac{M^n \cdot M^{*} \cdot |x - c|^n}{n!} \le \frac{M^n \cdot M^{*} \cdot |b - a|^n}{n!}$

for all n.

Now, $\sum_{n=1}^{\infty} \frac{M^n \cdot M^{*} \cdot |x - c|^n}{n!} = M^n \cdot M^{*} \cdot \sum_{n=1}^{\infty} \frac{|x - c|^n}{n!}$

$$\le M^n \cdot M^{*} \cdot \sum_{n=1}^{\infty} \frac{|b - a|^n}{n!}$$

and since $\sum_{n=1}^{\infty} \frac{k^n}{n!}$ converges, $\frac{M^n \cdot M^* \cdot |b - a|^n}{n!}$ converges. Therefore $\{h_i\}_{i=1}^{\infty}$ converges uniformly to some function, g, on $[a,b]$. To show g is continuous, we need to show that each $h \in \{h_i\}_{i=1}^{\infty}$ is continuous. Let $h_{n+1}(x) \in \{h_i\}_{i=1}^{\infty}$.

By Theorem 1.11 $h_{n+1}(x)$ is continuous which implies $\{h_i\}_{i=1}^{\infty}$ is a sequence of continuous functions. Since the uniform limit of a sequence of continuous functions is continuous, g is continuous.

Since $\int_c^x \phi(t,h_n(t))dt$ exists for each n and $x \in [a,b]$ and since $\{h_i\}_{i=1}^{\infty}$ converges uniformly to g on $[a,b]$, then $\int_c^x \phi(t,g(t))dt$ exists for each $x \in [a,b]$. As $n \to \infty$, $\int_c^x \phi(t,h_n(t))dt$ converges to $\int_c^x \phi(t,g(t))dt$ because, if $0 < \varepsilon$, since $\{h_i\}_{i=1}^{\infty}$ converges uniformly to g on $[a,b]$, there is a positive integer N such that if $p > N$, then for all x in $[a,b]$

$$|h_p(x) - g(x)| < \varepsilon/M+1$$

where M is some number such that for numbers x and y, and some $z \in [a,b]$, then $|\phi(z,x) - \phi(z,y)| < M|x - y|$. Let $0 < \varepsilon$. Since $\{h_i\}_{i=1}^{\infty}$ converges uniformly to g on $[a,b]$, there is a positive integer N such that if $q > N$ and x is in $[a,b]$, then

$$|\phi(x,h_q(x)) - \phi(x,g(x))| \leq M|h(x) - g(x)| < M\cdot\varepsilon/M+1 < \varepsilon$$

which implies $\{\phi(x,h_i(x))\}_{i=1}^{\infty}$ converges uniformly to $\{\phi(x,g(x))\}$ on $[a,b]$.

But $\int_c^x \phi(t,h_n(t))dt = h_{n+1}(x) - H$, which implies $h_{n+1}(x) - H$ converges to $\int_c^x \phi(t,g(t))dt$ and $n \to \infty$. However, $h_{n+1}(x)$ converges to $g(x)$ and thus $g(x) = H + \int_c^x \phi(t,g(t))dt$.

Theorem 3.2. If H is a number and $[a,b]$ is a number interval such that $a \leq c \leq b$, then there exists one and only one function g with domain $[a,b]$ such that $\int_a^b \phi(t,g(t))dt$ exists and $g(x) = H + \int_a^b \phi(t,g(t))dt$ for all $x \in [a,b]$.

Proof: By Theorem one, there is a function g defined such that $\int_a^b \phi(t,g(t))$ exists and $g(x) = H + \int_c^x \phi(t,g(t))dt$ for all $x \in [a,b]$.

Suppose there is a continuous function $g^* \neq g$, defined on $[a,b]$ such that $\int_a^b \phi(t,g^*(t))dt$ exists and

$$g^*(x) = H + \int_c^x \phi(t,g^*(t))dt$$

for all $x \in [a,b]$. We want to show

$$|g(x) - g^*(x)| \leq \frac{M^n W |x - c|^n}{n!}$$

for all $x \in [a,b]$.

Show true for $n = 1$.

$$|g(x) - g^*(x)| = |H + \int_c^x \phi(t,g(t))dt - H - \int_c^x \phi(t,g^*(t))dt|$$

$$= |\int_c^x \phi(t,g(t))dt - \int_c^x \phi(t,g^*(t))dt|$$

$$\leq |\int_c^x |\phi(t,g(t)) - \phi(t,g^*(t))|dt|$$

$$\leq |\int_c^x M|g(t) - g^*(t)|dt| \leq |\int_c^x M\cdot W dt|$$

$$\leq M\cdot W|x - c|$$

where $|g(t) - g^*(t)| \leq W$ for all $t \in [a,b]$. Since g is bounded and g^* is bounded, we know there exists a W since $|g(t) - g^*(t)| \leq |g(t)| + |g^*(t)| \leq M^* + M^{**} = W$ for all $t \in [a,b]$. Assume

$$|g(x) - g^*(x)| \leq \frac{M^n\cdot W\cdot |x - c|^n}{n!}$$

for all $x \in [a,b]$.

Show true for n + 1.

$$|g(x) - g^*(x)| = |\int_c^x \phi(t,g(t))dt + H - H - \int_c^x \phi(t,g^*(t))dt|$$

$$\leq |\int_c^x |\phi(t,g(t))dt - \int_c^x \phi(t,g^*(t))|dt|$$

$$\leq |\int_c^x M|g(t) - g^*(t)|dt|$$

$$\leq |\int_c^x \frac{M\cdot M^n\cdot W\cdot |t - c|^n}{n!} dt|$$

$$= \frac{M^{n+1} W |x - c|^{n+1}}{(n+1)!}$$

Therefore if n is a positive integer, then

$$|g(x) - g^{*}(x)| \leq \frac{M^{n+1} \cdot W \cdot |x - c|^{n+1}}{(n+1)!}$$

$$\leq \frac{M^{n+1} \cdot W \cdot |b - a|^{n+1}}{(n+1)!} .$$

Since $\sum_{n=1}^{\infty} \frac{M^{n} \cdot |b - a|^{n}}{n!}$ converges, the limit of the sequence $\{\frac{M^{n} \cdot |b - a|^{n}}{n!}\}_{n=1}^{\infty}$ is equal to zero. This implies $\{g - g^{*}\}$ converges uniformly to zero. Therefore $g(x) - g^{*}(x) = 0$ for all $x \in [a,b]$. Therefore $g(x) = g^{*}(x)$ for all $x \in [a,b]$.

Hence there is one and only one function such that $\int_{a}^{b} \phi(t,g(t))dt$ exists and $g(x) = H + \int_{c}^{x} \phi(t,g(t))dt$.

<u>Theorem 3.3</u>: Suppose that for each positive integer $K \leq n$, ϕ_k is a bounded function defined on $[a,b] \times R \times R \ldots \times R$ such that

(1). if $g_1, \ldots, g_n$ is a sequence of functions continuous on $[a,b]$, then the function $h(x) = \phi_k(x,g_1(x) \ldots, g_n(x))$ is continuous on $[a,b]$,

(2). if k is a positive integer $\leq n$, then there is a number M_k, such that if each of x and y is a number and $q \in [a,b]$ and each of $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ is a number sequence; then

$$|\phi_k(q,x_1, \ldots, x_n) - \phi_k(q,y_1, \ldots, y_n)|$$

$$\leq M_k(\sum_{i=1}^{n} |x_i - y_i|).$$

Suppose $a \leq c \leq b$ and for each positive integer $K \leq n$, $\{h_{k,i}\}_{i=1}^{\infty}$ is a sequence of functions, and h is a function on $[a,b]$, and T is a number such that for each positive integer $k \leq n$, $|\phi_k(x,h_{1,0}(x), \ldots, h_{n,0}(x))| \leq T$ for all $x \in [a,b]$, and $\int_a^b \phi_k(x,h_{1,0}(x), \ldots, h_{n,0}(x))dx$ exists.

Suppose further A_k is a number $(k = 1, 2, \ldots, n)$ and

$$h_{k,1}(x) = A_k + \int_c^x \phi_k(t,h_{1,0}(t), h_{2,0}(t), \ldots, h_{n,0}(t))dt$$

$$h_{k,2}(x) = A_k + \int_c^x \phi_k(t,h_{1,1}(t), h_{2,1}(t), \ldots, h_{n,1}(t))dt$$

$$\vdots$$

$$h_{k,w}(x) = A_k + \int_c^x \phi_k(t,h_{1,w-1}(t), \ldots, h_{n,w-1}(t))dt$$

$$\vdots$$

Then for each $k \in \{h_{k,i}\}_{i=1}^{\infty}$ converges uniformly to a function f_k defined and continuous on $[a,b]$, such that if x is in $[a,b]$; then

$$f_k(x) = A_k + \int_c^x \phi_k(t,f_1(t), f_2(t), \ldots, f_n(t))dt.$$

<u>Proof</u>: For each k, we need to show $h_{k,1}$ is bounded.

$h_{k,1}(x) = A_k + \int_c^x \phi_k(t,h_{1,0}(t), \ldots, h_{n,0}(t))dt.$ This implies

$$|h_{k,1}(x)| = |A_k + \int_c^x \phi_k(t,h_{1,0}(t), \ldots, h_{n,0}(t))dt|$$

$$\leq |A_k| + |\int_c^x \phi_k(t,h_{1,0}(t), \ldots, h_{n,0}(t))dt|$$

$$\leq |A_k| + T|x - c| \leq |A_1| + T|a - b|.$$

This implies $|h_{k,1}(x)| \leq R$, where $k = 1, 2, \ldots, n$.

By mathematical induction we need to show for each w

$$|h_{k,w}(x) - h_{k,w-1}(x)| \leq \frac{M_k^{w-1} \cdot n^{w-1} \cdot P \cdot |b - a|^{w-1}}{(w-1)!}$$

Suppose $w = 2$. Then we have

$$|h_{k,2}(x) - h_{k,1}(x)| = |A_k$$

$$+\int_c^x \phi_k(t, h_{1,1}(t), h_{2,1}(t), \ldots, h_{n,1}(t))dt$$

$$- A_k - \int_c^x \phi_k(t,h_{1,0}(t), \ldots, h_{n,0}(t))dt|$$

$$= |\int_c^x \phi_k(t,h_{1,1}(t), \ldots, h_{n,1}(t))dt$$

$$-\int_c^x \phi_k(t,h_{1,0}(t), \ldots, h_{n,0}(t))dt|$$

$$\leq |\int_c^x |\phi_k(t,h_{1,1}(t), \ldots, h_{n,1}(t))dt$$

$$- \phi_k(t,h_{1,0}(t), \ldots, h_{n,0}(t))|dt|$$

$$\leq |\int_c^x M_k(\sum_{i=1}^{n} |h_{i,1}(t) - h_{i,0}(t)|)dt|$$

$$\leq |\int_c^x M_k \cdot n \cdot P \, dt| = M_k \cdot n \cdot P|x - c|$$

where $P \geq |h_{i,1}(x) - h_{i,0}(x)|$, which exists since each

$\{h_{i,1}(x) - h_{i,0}(x)\}$ is continuous on $[a,b]$ and hence bounded on $[a,b]$ for each $i \leq n$.

Suppose true for $w = y-1$. Show true for $w = y$.

$$|h_{k,y}(x) - h_{k,y-1}(x)| = |A_k + \int_c^x \Phi_k(t, h_{1,y-1}(t), \ldots, h_{n,y-1}(t))dt - A_k - \int_c^x \Phi_k(t, h_{1,y-2}(t), \ldots, h_{n,y-2}(t))dt|$$

$$\leq |\int_c^x |\Phi_k(t, h_{1,y-1}(t), \ldots, h_{n,y-1}(t)) - \Phi_k(t, h_{1,y-2}(t), \ldots, h_{n,y-2}(t))|dt|$$

$$\leq |\int_c^x M_k \sum_{i=1}^{n} |h_{i,y-1}(t) - h_{i,y-2}(x)|dt|$$

$$\leq |\int_c^x M_k \cdot \sum_{i=1}^{n} \frac{M_k^{y-2} \cdot n^{y-2} \cdot P \cdot |t - c|^{y-2}}{(y-2)!} dt|$$

$$= |\int_c^x \frac{M_k^{y-1} \cdot n^{y-1} \cdot P \cdot |t - c|^{y-2}}{(y-2)!} dt|$$

$$= \frac{M_k^{y-1} \cdot n^{y-1} \cdot P \cdot |x - c|^{y-1}}{(y-1)!}$$

Therefore if w is a positive integer, then

$$|h_{k,w}(x) - h_{k,w-1}(x)| \leq \frac{M_k^w \cdot n^w \cdot P \cdot |x - c|^w}{w!}$$

$$\leq \frac{M_k^w \cdot n^w \cdot P \cdot |b - a|^w}{w!}$$

Now, $\sum_{w=1}^{\infty} \frac{M_k^w \cdot n^w \cdot P \cdot |b - a|^w}{w!}$ converges; therefore $\{h_{k,i}\}_{i=1}^{\infty}$

$(k = 1, 2, \ldots, n)$ converges uniformly to some function, f_k, on $[a,b]$. To show f_k is continuous we must show that each $h_{k,i}$ an element of $\{h_{k,i}\}_{i=1}^{\infty}$ $(k = 1, 2, \ldots, n)$ is continuous. Let $h_{k,y} \in \{h_{k,i}\}_{i=1}^{\infty}$. By Theorem 1.11 $h_{k,y}$ is continuous, which implies $\{h_{k,i}\}_{i=1}^{\infty}$ $(k = 1, 2, \ldots, n)$ is a sequence of continuous functions. Since the uniform limit of a sequence of continuous functions is continuous, $f_k(x)$ is continuous.

Since $\int_c^x \phi(t, h_1(t), \ldots, h_n(t))dt$ exists for all k and $x \in [a,b]$ and since $\{h_{i,k}\}_{i=1}^{\infty}$ converges uniformly to f_k on $[a,b]$ then $\int_c^x \phi_k(t, f_1(t), \ldots, f_n(t))dt$ exists for all $x \in [a,b]$. As $n \to \infty$ $\int_c^x \phi(t, h_1(t), \ldots, h_n(t))dt$ converges to $\int_c^x \phi_k(t, f_1(t), \ldots, f_n(t))dt$ because if $0 < \varepsilon$, since $\{h_{i,k}\}_{i=1}^{\infty}$ $(k = 1, 2, \ldots, n)$ converges uniformly to f_k on $[a,b]$, there is a positive integer T such that if $w > T$, $\sum_{p=1}^{n} |h_{p,w}(x) - f_p(x)| < \varepsilon/M+1$ for all x in $[a,b]$ where M is a number such that if each of $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ is a number sequence and q is in $[a,b]$ then

$$|\phi_k(q, x_1, \ldots, x_n) - \phi_k(q, y_1, \ldots, y_n)|$$

$$\leq M_k\left(\sum_{i=1}^{n} |x_i - y_i|\right).$$

Let $\varepsilon > 0$. There is a positive integer T such that if $w > T$ and $t \in [a,b]$, then

$$|\phi_k(t, h_{1,w}(t), \ldots, h_{n,w}(t)) - \phi_k(t, f_1(t), \ldots, f_n(t))|$$

$$\leq M_k \sum_{p=1}^{\infty} |h_{p,w}(x) - f_p(x)| \leq M_k \ \varepsilon/M+1 < \varepsilon.$$

Therefore $\{\phi_k(x, h_{i,k}(x))\}_{i=1}^{\infty}$ $(k = 1, 2, \ldots, n)$ converges uniformly to $\phi_k(x, f_k(x))$ on $[a,b]$.

But $\int_c^x \phi_k(t, h_{1,y}(t), \ldots, h_{n,y}(t))dt = h_{k,y+1}$

$$- A_k \rightarrow \int_c^x \phi_k(t, f_1(t), \ldots, f_n(t))dt$$

which implies $h_{k,w}(x) \rightarrow f_k$.

Therefore $f_k(x) = A_k + \int_c^x \phi_k(t, f_1(t), \ldots, f_n(t))dt.$

Suppose there exists a sequence of continuous functions $f_1^*, f_2^*, \ldots, f_k^*$ satisfying

$$f_k^* = A_k + \int_c^x \phi_k(t, f_1^*(t), \ldots, f_n(t))dt$$

$$|f_k(x) - f_k^*(x)| = A_k + \int_c^x \phi_k(t, f_1(t), \ldots, f_n(t))dt$$

$$- \int_c^x \phi_k(t, f_1^*(t), \ldots, f_n^*(t))dt|$$

$$\leq |\int_c^x |\phi_k(t, f_1(t), \ldots, f_n(t)) - \phi_k(t, f_1^*(t), \ldots, f_n^*(t))|dt|$$

$$\leq |\int_c^x M_k \sum_{i=1}^{n} |f_i(t) - f_i^*(t)|dt| \leq |\int_c^x M_k \cdot n \cdot S \ dt|$$

$$= M_k \cdot n \cdot S|x - c| \leq M_k \cdot n \cdot S|b - a|$$

where $S \geq \sum_{i=1}^{n} |f_i(t) - f_i^*(t)|$ which exists, since each f_i is continuous and hence bounded.

Suppose $|f_k(x) - f_k^*(x)| \leq \frac{n^{w-1} \cdot M_k^{w-1} \cdot S \cdot |x - c|^{w-1}}{(w-1)!}$ for all x in [a,b] and $k \leq n$.

Show true for w.

$$|f_k(x) - f_k^*(x)| = |A_k + \int_c^x \phi_k(t, f_1(t), \ldots, f_n(t))dt - A_k$$

$$- \int_c^x \phi_k(t, f_1^*(t), \ldots, f_n^*(t))dt|$$

$$\leq |\int_c^x |\phi_k(t, f_1(t), \ldots, f_n(t))dt - \phi_k(t, f_1^*(t), \ldots, f_n^*(t))|dt|$$

$$\leq |\int_c^x M_k \sum_{i=1}^{n} |f_i(t) - f_i^*(t)|dt|$$

$$\leq |\int_c^x M_k \sum_{i=1}^{n} \frac{M_k^{w-1} \cdot n^{w-1} \cdot S \cdot |t - c|^{w-1}}{(w-1)!} dt|$$

$$= |\int_c^x \frac{M_k^w \cdot n^w \cdot S \cdot |t - c|^{w-1}}{(w-1)!} dt|$$

$$= \frac{M_k^w \cdot n^w \cdot S \cdot |x - c|^w}{w!}.$$

Therefore, if w is a positive integer, then

$$|f_k(x) - f_k^*(x)| \leq \frac{M_k^w \cdot n^w \cdot S \cdot |x - c|^w}{w!}$$

$$\leq \frac{M_k^w \cdot n^w \cdot S \cdot |b - a|^w}{w!} .$$

Since $\sum_{w=1}^{\infty} \frac{M_k^w \cdot n^w \cdot S \cdot |b - a|^w}{w!}$ converges, the limit of the sequence $\{\frac{M_k^w \cdot n^w \cdot S \cdot |b - a|^w}{w!}\}$ is zero. This implies $\{f_k(x) - f_k^*(x)\}$ is zero. Therefore $f_k(x) - f_k^*(x) = 0$; hence $f_k(x) = f_k^*(x)$ for all $x \in [a,b]$. Hence there is one and only one sequence of functions defined and continuous on $[a,b]$ such that $\{h_{k,i}\}_{i=1}^{\infty}$ $(k = 1, 2, \ldots, n)$ converges uniformly to f_k and

$$f_k(x) = A_k + \int_c^x \phi_k(t, f_1(t), f_2(t), \ldots, f_n(t))dt.$$