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Solution of Elliptic Partial Differential Equations**

**Maithili Sharan
E.J. Kansa
Suman Gupta**

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Application of Multiquadric Method for Numerical Solution of
Elliptic Partial Differential Equations.

Maithili Sharan^{*}, E.J. Kansa^{**} and Suman Gupta^{***}

*Centre for Atmospheric Sciences,
Indian Institute of Technology,
Hauz Khas, New Delhi -110016, India.

**Department of Physics,
Lawrence Livermore National Laboratory,
Livermore, CA, USA.

*** Govt Girls Sr. Sec. School I,
Madangir, New Delhi, India.

Abstract

We have used the multiquadric (MQ) approximation scheme for the solution of elliptic partial differential equations with Dirichlet and/or Neumann boundary conditions. The scheme has the advantage to use the data points in arbitrary locations with an arbitrary ordering. Two dimensional Laplace, Poisson and Biharmonic equations describing the various physical processes, have been taken as the test examples. The agreement is found to be very good between the computed and exact solutions. The method also provides an excellent approximation with curve boundary.

Introduction

In general, mathematical description of physical processes leads to the partial differential equations (PDE). In some cases, an exact solution to PDE can be obtained using analytical tools such as method of reflection and superposition, separation of variables and integral transforms etc. The approximate analytical solution of PDE can be found by using perturbation methods (Nayfeh(1973), Van Dyke (1975), Kevorkian & Cole (1981)), method of successive approximation (Parlange (1971)) and orthogonal functions (Tsang (1960)).

The solution of equations which more closely model experimental and practical situations is possible by the methods of numerical analysis. The equations can be solved by using finite difference, finite element and boundary element methods. When the boundary conditions and domains are irregular, the finite difference scheme becomes complex. It can be overcome to a greater extent by using the method of finite elements.

It is difficult to use finite difference methods on arbitrarily scattered data points. Hardy in 1968 (Hardy (1990)) introduced for the first time multiquadric (MQ) method for the construction of approximate 2-dimensional surfaces through field data. The importance and usefulness of the MQ method is recognised from the fact that it has been applied in geology, geophysics, survey & mapping, photogrammetry, remote sensing and signal processing, geography and digital terrain models, and hydrology. Hardy (1990) has presented an excellent review summary on the various applications of MQ method during last twenty years.

Based on the comparison of different scattered data schemes, Franke (1982) has concluded that MQ performs the best in

accuracy and ease of implementation even against various finite element schemes. Kansa (1990 a) has explored MQ scattered data scheme in interpolation of data points to approximate surfaces and estimating partial derivatives. Kansa (1990b) has pointed out that the interpolation scheme is intimately connected with the solution of PDEs and he applied successfully this novel technique for the solution of PDEs in computational fluid dynamics. Dubal et al (1992) and Dubal (1992) have used the MQ scheme for the solution of a three dimensional elliptic partial differential equation describing the initial state of the collision of two-black holes. Makroglou (1992) has used MQ method to solve Fredholm integral and Integro-differential equations.

The MQ method replaces the classical finite difference and finite element spatial discretization schemes by a exponentially convergent, grid free scattered data approximation scheme.

Finite difference and finite element methods use low order polynomial basis functions. MQ in contrast is very high order (Moridis & Kansa (1993)). They have discussed the advantages/disadvantages associated with finite difference, finite element and MQ methods.

In the present paper, we have applied MQ scheme to solve the two-dimensional elliptic partial differential equations. Laplace, Poisson and Biharmonic equations have been taken as the test examples. The method is also applied successfully in a problem with the curve boundary.

Application of MQ method.

Consider an elliptic PDE of the form.

$$\nabla^2 f = s(\vec{x}) \quad (1)$$

where ∇^2 is the Laplacian operator, $\vec{x}$ is the space vector, s is a known function of space coordinates and f is an unknown function of $\vec{x}$.

The equation (1) is subject to Dirichlet and/or Neumann boundary conditions:

$$(a) \quad f = f_1(\vec{x}) \quad \text{on } \Gamma_1$$

$$(b) \quad \hat{n} \cdot \nabla f = f_2(\vec{x}) \quad \text{on } \Gamma_2 \quad (2)$$

where $\hat{n}$ is the outward unit normal and Γ_1 and Γ_2 are the boundaries of the domain, f_1 and f_2 are known functions of $\vec{x}$.

MQ method assumes that any function can be expanded as a finite series of basis functions (Kansa (1990b) & Hardy (1990)). The expansion used is the Madych & Nelson's (1989) expansion with an appended constant. i.e.,

$$f(\vec{x}) = a_1 + \sum_{j=2}^N a_j \hat{g}(\vec{x} - \vec{x}_j) \quad (3)$$

where N is the total number of data points under consideration, a_j 's are the coefficients and the functions $\hat{g}$ are defined by

$$\hat{g}(\vec{x} - \vec{x}_j) = g_j - q \quad (4)$$

in which

$$g_j = g(\vec{x} - \vec{x}_j) = [(\vec{x} - \vec{x}_j)^2 + r_j^2]^{1/2} \quad (5)$$

where $(\vec{x} - \vec{x}_j)^2$ is the square of the Euclidean distance R in R^d and $r_j^2 > 0$ is the shape function. Hagen & Kansa (1993) have studied

the role of the parameter (r_j^2) in the multiquadric function. Carlson & Foley (1991) have used a constant value of r_j in the equation (5). Kansa (1988,90a&b) has found numerically that the following power law function increases accuracy significantly:

$$r_j^2 = r_m^2 \left(\frac{r_n^2}{r_m^2} \right)^{\frac{j-1}{N-1}} ; j= 1,2,\dots,N \quad (6)$$

in which r_m and r_n are input parameters.

Spatial partial derivatives of the function in (3) are obtained by differentiating the spatial basis functions. Substitution of the function (3) and its partial derivatives in equation (1) together with the boundary conditions (2a & b), yields a system of linear algebraic equations which can be solved for the coefficients a_j .

We illustrate the method for a two dimensional problem. The derivatives are given by :

$$\frac{\partial f}{\partial x} \bigg|_{(x_i, y_i)} = \sum_{j=2}^N a_j \left[\frac{x_i - x_j}{g_{ij}} - \frac{x_i - x_1}{g_{i1}} \right]$$

$$\frac{\partial f}{\partial y} \bigg|_{(x_i, y_i)} = \sum_{j=2}^N a_j \left[\frac{y_i - y_j}{g_{ij}} - \frac{y_i - y_1}{g_{i1}} \right] \quad (7)$$

$$\frac{\partial^2 f}{\partial x^2} \bigg|_{(x_i, y_i)} = \sum_{j=2}^N a_j \left[\frac{1}{g_{ij}} - \frac{(x_i - x_j)^3}{g_{ij}^3} - \frac{1}{g_{i1}} + \frac{(x_i - x_1)^3}{g_{i1}^3} \right]$$

$$\frac{\partial^2 f}{\partial y^2} (x_i, y_i) = \sum_{j=2}^N a_j \left[\frac{1}{g_{ij}} - \frac{(y_i - y_j)^2}{g_{ij}^3} - \frac{1}{g_{i1}} + \frac{(y_i - y_1)^2}{g_{i1}^3} \right]$$

in which

$$g_{ij} = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + r_j^2 \right]^{1/2}, \quad 1 \leq i, j \leq N. \quad (8)$$

Substituting the expansion of f (equation 3) and its derivatives (equation 7) in the two dimensional version of equation (1), we find for each of the interior node i ,

$$\sum_{j=1}^N p_{ij} a_j = b_i \quad (9)$$

in which

$$p_{i1} = 0$$

$$p_{ij} = 2 (g_{ij}^{-1} - g_{i1}^{-1}) - g_{ij}^{-3} \left\{ (x_i - x_j)^2 + (y_i - y_j)^2 \right\} + g_{i1}^{-3} \left\{ (x_i - x_1)^2 + (y_i - y_1)^2 \right\} \quad ; \quad 2 \leq j \leq N \quad (10)$$

$$b_i = s(x_i, y_i)$$

If a point i lies on the boundary, it may satisfy either Dirichlet or a Neumann boundary condition.

Case I : With Dirichlet condition $f = f_1$, coefficients in i -th equation (9) will be

$$\begin{cases} p_{i1} = 1 \\ p_{ij} = g_{ij} - g_{i1} \\ b_i = f_1(x_i, y_i) \end{cases} \quad 2 \leq j \leq N \quad (11)$$

Case II: Neumann condition $\hat{n} \cdot \nabla f = f_2$.

(a) If the point i lies on the boundary on which the normal is in the direction of x , coefficients in i -th equation (9) are given by :

$$\begin{aligned} p_{i1} &= 0 \\ p_{ij} &= (x_i - x_j) / g_{ij} - (x_i - x_1) / g_{i1} \quad 2 \leq j \leq N \\ b_i &= f_2(x_i, y_i) \end{aligned} \quad (12)$$

(b) If the point i lies on the boundary on which the normal is in the direction of y , coefficients in i -th equation (9) are given by :

$$\begin{aligned} p_{i1} &= 0 \\ p_{ij} &= (y_i - y_j) / g_{ij} - (y_i - y_1) / g_{i1} \quad 2 \leq j \leq N \\ b_i &= f_2(x_i, y_i) \end{aligned} \quad (13)$$

For all the grid points, equation (9) forms a system of linear algebraic equations of the form :

$$P A = B \quad (14)$$

in which $P = (p_{ij})$ is the square matrix of order N .

$A = (a_1, a_2, \dots, a_N)^T$ is a column vector of unknown coefficients and $B = (b_1, b_2, \dots, b_N)^T$. The elements of P & B are defined by the relations (10-13).

Kansa (1990) and Dubal (1992) have discussed the nature of the system of equations (14). Here, we have solved it successfully using Gauss elimination method with total pivoting.

The mean square error (E) is computed from:

$$E = \frac{1}{N} \sum_{i=1}^N (f_i - \hat{f}_i)^2 \quad (15)$$

where f_i is the computed value of f using MQ method and $\hat{f}_i$ is the known value of f at the point i . It may be known either analytically or computationally by means of some other methods such as finite differences, finite elements etc.

We have used the double precision FORTRAN on ICL 3980 computing machine at IIT Delhi.

Examples:

In the following, we present the various examples for determining the solution of elliptic PDEs using MQ method.

(A) Laplace equation in a square :

Consider a Laplace equation

$$f_{xx} + f_{yy} = 0 \quad (16)$$

in a square domain R ($0 < x < 1, 0 < y < 1$) with the conditions:

(i) $f = 0$ along the edges $x = 0$; $y = 1$ & $x = 1$

(ii) $f = T_0$ along $y = 0$ (17)

In (16), subscripts "xx" & "yy" denote the second partial derivatives with respect to x & y respectively. T_0 is a constant. The equation (16) with the boundary conditions (17) describes the steady-state temperature distribution in a square plate, one side of which is maintained at a constant temperature T_0 with the other three sides are maintained at 0°

An analytical solution of (16) with the boundary conditions (17) is given by (Carslaw & Jeager (1976)) :

$$f(x,y) = \frac{4T_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \frac{\sin(2n+1)\pi x \sinh(1-y)(2n+1)\pi}{\cosh(2n+1)\pi} \quad (18)$$

The problem described by equation (16) with the boundary conditions (17) becomes a particular case of the problem (1-2) with $s(x,y) = 0$ and $f_1 = T_0$ with Dirichlet condition only. We use the MQ method discussed in the earlier section. The computation has been done with $T_0 = 100$ for different number of grid points (N). For 81 grid points at a uniform spacing $\Delta x = \Delta y = 0.125$, we found the mean square error = 0.682 with a maximum error |3.441|. In the next case, we have removed the grid points lying on the lines $y = 0.375$ and $y = 0.625$ resulting in $N = 63$. Further, the grid points are reduce to 49 in the domain by deleting the points on the vertical lines $x = 0.375$ & $x = 0.625$. By removing the additional grid points from the line $y = 0.75$, we have 42 points in the domain.

Also, we have obtained the solution for scattered data points 30, 33 & 35 respectively, and for 25 (i.e., 5×5) uniform grid

points. Table 1 provides the information about the number of boundary points, number of interior points, the value of the input parameters R_m & R_n , mean square error & the maximum absolute error for each of the test run. Table 1 shows that mean square errors are not significantly different in all the cases. As an illustration, the computed results for $N=42$ are given in Table 2 & Fig.1. There is a reasonably good agreement between the computed & exact values, considering that over the unit square the values varying 0 to 100, except at two points where the maximum error is about $|3.68|$.

The maximum error is relatively higher and is introduced at the grid point near the point of intersection of the sides with temperatures $T_0 = 100^\circ$ & $T = 0^\circ$. Thus, the large error is because of discontinuity in temperatures $T = 100^\circ$ & $T = 0^\circ$. It remains of the same order even with the increase in the number of grid points (table 1).

(B) Poisson equation in an elliptic domain :

Consider a Poisson equation

$$f_{xx} + f_{yy} = -2 \quad (19)$$

in an ellipse with major & minor semiaxes α and β .

Equation (19) is subject to the condition that f vanishes on the surface of the ellipse.

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \quad (20)$$

Since the solution of (19) with (20) is expected to be symmetric with respect to x and y , it is sufficient to solve the equation (19) in the first quadrant of the ellipse and accordingly, we consider the equation (19) with the following

Neumann & Dirichlet boundary conditions :

- (i) $f_x = 0$ at $x = 0$
- (ii) $f_y = 0$ at $y = 0$
- (iii) $f = 0$ on the curve surface (21)

The equation (19) provides a function f from which the angle of twist of a cylindrical shaft of an elliptical cross-section under torsion can be calculated (Hughes & Gaylord (1964)).

An analytical solution of (19) with (20) is

$$f(x,y) = - \left[\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 \right] \left[\frac{\alpha^2 \beta^2}{\alpha^2 + \beta^2} \right] \quad (22)$$

The solution of the problem (19) with (21) can be found as a particular case of the problem (1-2) with $s(x,y) = -2$, $f_1 = 0$ and $f_2 = 0$.

For $\alpha = 10$, $\beta = 8$ and $N = 28$ (figure 2), the results are presented in table 3. The function varies from 39 to 0 in the domain with $E = 2.92 \times 10^{-8}$. The maximum error is found to be $|3.574 \times 10^{-4}|$ which is 0.0025% of the exact functional value. The computed values are close to the exact values of the function. MQ method provides the accuracy upto five significant decimal places. Computations have also been done with different sets of data points and the results are found to be very good. For $N = 87$, the maximum error is found to be $|6.68 \times 10^{-5}|$ which is 0.00027 %.

For $\alpha = 2$, $\beta = 1$, the function varies from 0.8 to 0. Again, we obtain an accurate solution with $N=28$ not only at the interior points as well as on the boundary points with $E = 2.96 \times 10^{-10}$ and maximum error = $|4.8 \times 10^{-5}|$.

The MQ method provides an excellent approximate solution. The curve boundary is accounted in this method in a natural way through the location of the boundary points whereas in finite difference method, an interpolation technique is required for expressing boundary points in terms of grid points.

(C) Poisson Equation in a rectangle with a constant right hand side :

Consider a Poisson equation

$$f_{xx} + f_{yy} = Q \quad (23)$$

where Q is a constant.

The equation (23) describes the steady state transport of oxygen in a slab of tissue with a constant oxygen consumption rate (Sharan et al (1991)). This also governs the steady state temperature distribution in a rectangular region with the heat loss (production) at a constant rate Q ($-Q$) (Carslaw & Jeager (1976)). If $Q = \frac{q}{D}$ in which D is the flexural rigidity of a thin plate subject to a normal load q per unit area, the equation (23) arises by splitting a biharmonic equation (see example(E)) into two successive Poisson equations.

The boundary conditions for equation (23) are

$$f = 0 \quad \text{along the four edges } (x = \pm a, y = \pm b) \quad (24)$$

of the rectangle with sides a & b .

The solution of (23) with boundary conditions (24) will be symmetric with respect to x & y and thus, we solve the equation (23) in the region $(0 < x < a, 0 < y < b)$ with the boundary conditions :

$$(i) \quad f_x = 0 \quad \text{at } x = 0$$

$$\begin{aligned}
\text{(ii)} \quad f &= 0 \quad \text{at } x = a \\
\text{(iii)} \quad f_y &= 0 \quad \text{at } y = 0 \\
\text{(iv)} \quad f &= 0 \quad \text{at } y = a
\end{aligned} \tag{25}$$

An analytical solution of the problem (23-24) is given by (Carslaw & Jeager(1976)) :

$$f(x,y) = \frac{16qa^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n \cos[(2n+1)\pi x/2a] \cosh[(2n+1)\pi y/2a]}{(2n+1)^3 \cosh[(2n+1)\pi b/2a]} - \frac{q}{2} (a^2 - x^2) \tag{26}$$

The solution of (23) with (25) can be found similar to the problem (1-2) with $s(x,y) = q$, $f_1 = 0$ and $f_2 = 0$. The domain has been normalized to a unit square with a & b . For the computation, we have taken $q = 20$, $D = 2.1978 \times 10^4$, $a = 20$ and $b = 20$. We have performed a series of calculations with 25 (i.e., 5×5) grid points with a uniform spacing $\Delta x = \Delta y = 0.25$ & with 35 scattered data points. Values of the input parameters, mean square error, and maximum absolute error are summarized in Table 4. Table 5 shows that computed solution compares well with the exact solution for 35 scattered data points with $E = 2.03 \times 10^{-7}$. The function varies from -0.11 to 0 . The maximum error is found to be 1.35×10^{-3} which is less than 2.3 %. We have also obtained a good solution with $N = 35$ for $q = 1$.

(D) Poisson Equation in a rectangle with right hand side as a function of space coordinates :

Consider a Poisson equation in a rectangular plate

$(-a < x < a, -b < y < b)$:

$$f_{xx} + f_{yy} = g(x,y) \tag{27}$$

where g is a known function of x & y . Important applications occur : (a) in fluid dynamics with f = stream function and g = -vorticity, (b) in electrostatics, with f = electric potential and $-g$ represents the ratio of charge density to dielectric constant, (c) in heat transfer, with f = temperature and g is the loss/generation of heat, (d) in mass transfer with f = concentration and g is the source/sink term.

Here we consider g of the form of (26). The equation (27) arises as the second Poisson equation in the splitting of biharmonic equation (see, example (E)) describing the transverse deflection of a thin plate.

Again, with the boundary conditions (24), the solution of (27) is expected to be symmetric in x and y and thus, we consider the region $(0 < x < a, 0 < y < b)$ with the boundary conditions (25).

An analytical solution of (27) with boundary conditions (24) is obtained in form :

$$f(x,y) = \frac{b}{a} \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha_n^5} \left[2 + \alpha_n y \frac{\sinh \alpha_n y}{\cosh \alpha_n b} - \left(2 + \frac{\alpha_n b \sinh \alpha_n b}{\cosh \alpha_n b} \right) \frac{\cosh \alpha_n y}{\cosh \alpha_n b} \right] \cos \alpha_n x \quad (28)$$

in which

$$\alpha_n = \frac{(2n+1)\pi}{2a}$$

We normalize the domain to a unit square with a & b . The values

of q , D , a , & b are the same as in the previous example. Table 6 contains the values of the input parameters, mean square error and maximum absolute error for $q = 20$ and $q = 1$ with different N .

The results presented in table 7a for $q=20$ with $N = 35$ are very good, considering the points are scattered. In most of the runs (table 6), the relative percentage error

$$(100 \times \max |f_j - \hat{f}_j| / \hat{f}_j)$$

is less than one percent.

The table 7b shows that the computed values are close to the analytical values for $q = 1$ with $N = 30$. The maximum error is 6.6×10^{-4} which is approximately 0.02 % of its analytical value.

Thus, the MQ method provides an accurate solution to the equation (27) with the boundary conditions (24).

(E) Biharmonic Equation :

The transverse deflection (f) of a thin rectangular plate with sides $2a$ and $2b$, of flexural rigidity D subject to normal load q per unit area is governed by

$$\nabla^4 f = f_{xxxx} + 2 f_{xxyy} + f_{yyyy} = \frac{q}{D}, \quad (-a < x < a, -b < y < b) \quad (29)$$

The equation (29) is subject to the boundary conditions $f=0$ and $f_{\eta\eta} = 0$ along its four edges, where η denotes the normal to the boundary.

By introducing the variable $u = \nabla^2 f$, the problem (29) amounts to solving Poisson's equation twice in succession:

$$(i) \quad u_{xx} + u_{yy} = \frac{q}{D}; \quad -a < x < a, -b < y < b \quad (30)$$

with $u = 0$ along the four edges,

$$\text{and (ii) } f_{xx} + f_{yy} = u(x,y) ; -a < x < a, -b < y < b \quad (31)$$

with $f = 0$ along the four edges $x = \pm a$ & $y = \pm b$.

Notice that the problems (30) and (31) are the same to the problems (23) & (27). In order to obtain f from (31), the solution of (30) will be used as the input. The relation (28) is the analytical solution of the biharmonic equation (29). The values of q , D , a & b are the same as given in examples (C) & (D). As in the previous cases, the domain was normalized to a unit square with reference to a and b . The values of the inputs parameters, mean square error and the maximum absolute error for different N and q are given in table 8.

The solution of the biharmonic equation (29) for $q = 20$ is presented in table 9 for 35 scattered data points and in figure 3 for 25 uniform points. There is a good agreement between the computed and analytical values (table 9 & fig 3).

The computed values are found to be closed (Table 10) to the exact values for $q = 1$ with 25 uniform grid points. Notice that the accuracy of the solution of biharmonic equation computed in two stages using equations (30 & 31) and in single stage from equation (27) is found to be almost same.

Conclusions :

We have applied successfully MQ method to compute the solution of elliptic partial differential equations with Dirichlet and/or Neumann boundary conditions. The data points in arbitrary locations with an arbitrary ordering allows us to obtain an accurate solution. The method is shown to work well for Laplace,

Poisson and Biharmonic equations. When comparing the MQ computed values with the exact solution, the agreement is found to be very good. For the summation of the series in (18), (26) and (28), sufficiently large number of terms (n) are considered such that the absolute value of n th term is less than or equal to 10^{-10} .

In the first example, the error was relatively higher on the grid points near the point of intersection of the sides with different temperatures.

MQ method performs well for the problems with curve boundary. The curve boundary is accounted, here, in a natural way through the location of the data points unlike finite difference method in which an interpolation technique is required to express a boundary point in terms of grid points.

We have shown MQ is an excellent approximation scheme for solving a biharmonic equation. The solution of this equation obtained in two stages is approximately the same as obtained in single stage.

Because the coefficient matrix is full, Kansa (1990b) has suggested domain decomposition and blending (Foley (1992)). However, this will increase the computational complexities. Dubal et al (1992) have suggested the use of singular value decomposition (Press et al(1986)) for solving the resulting system of algebraic equations. However, we have observed that in the examples considered here the Gaussian elimination with total pivoting gives the better results than the singular value decomposition.

The accuracy of the computed solution depends on the choice of

the parameters R_m & R_n . In the present study, their values are chosen by experimentation. Madych (1992) has shown that multiquadrics converge very quickly and Baxter (1992) has found that the MQ scheme converges rapidly for a constant value of R^2 if R^2 goes to infinity. Such large values forces severe ill-conditioning. Carlson & Foley (1991) have estimated a good value for the constant R^2 in the MQ equation. Kansa (1990a& b) has shown that the power law seems to work very well. Hager & Kansa (1993) use brent algorithm to find the optimum constant R or the shape parameters R_m & R_n . We have also made an attempt to obtain these shape parameters using nonlinear optimization technique suggested by Marquardt (1963). However, the determination of these parameters requires the prior knowledge of the exact solution. Recently, Galperin and Zheng (1993) and Galperin (1990) have suggested a method based on global optimization techniques which may be used for the estimation of shape parameters in case the exact solution is not known.

Quite often, the solution is very sensitive to R_m & R_n and a slight variation in R_m & R_n may change significantly the mean square errors. Further, there is no systematic way to determine the choice of R_m & R_n like the finite difference methods where for consistent & stable approximation scheme, a decrease in the grid spacing yields to a more accurate results. Thus, it is necessary to develop a suitable procedure for estimating R_m & R_n in terms of function values, N and location of grid points.

Recently, Galperin and Zheng (1993) have developed a new method for the solution and control of PDEs via global optimization methods (Galperin(1990) and Galperin and Zheng (1990)). By formulating any PDE problem as an optimization one

(Galperin and Zheng (1993)), even if the exact solution is not known, one can readily optimize the choices of the MQ shape parameters, and also can deal more effectively with the nonlinear PDEs. Such an approach would eliminate the guess work in dealing with the shape parameters and the delta algorithm (Galperin (1990)) is expected to provide fast convergence with a minimum number of evaluations. Thus, one of the future direction of the research is to apply global optimization techniques for the solution of PDEs with MQ.

In the subsequent studies, we propose to apply MQ scheme to more complicated physical problems involving arbitrary geometrical configuration, interface conditions and nonlinearities.

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Table 1. Values of input parameters, mean square error and maximum absolute error for Laplace equation.

S.No.	Total number of points	Number of boundary Points	Number of interior points	Rm	Rn	Mean square error	Maximum Absolute error
1.	81	32	49	.22	11	.682	3.44
2.	63	28	35	.12	121	.57	4.02
3.	49	24	25	.14	12	.832	3.65
4.	42	22	20	.14	10	1.06	3.68
5.	25	16	9	0.4	50	1.88	3.88
6.	30	18	12	1.3	10	2.68	4.71
7.	33	19	14	0.068	50	3.53	5.23
8.	35	15	20	.7	10	3.35	6.0

TABLE 2: COMPUTED AND EXACT SOLUTION OF THE LAPLACE
EQUATION IN A SQUARE.

$N = 42$, $R_m = 0.14$, $\alpha = 10$.

I	X	Y	COM.	EXACT	ERROR
*1	0.000	0.000	0.0000	0.0000	0.0000
*2	0.125	0.000	100.0000	100.1664	-0.1664
*3	0.250	0.000	100.0000	99.9100	0.0900
*4	0.500	0.000	100.0000	99.9363	0.0637
*5	0.750	0.000	100.0000	99.9100	0.0900
*6	0.875	0.000	100.0000	100.1664	-0.1664
*7	1.000	0.000	0.0000	0.0000	0.0000
*8	0.000	0.125	0.0000	0.0000	0.0000
9	0.125	0.125	51.9729	48.2909	3.6820
10	0.250	0.125	68.8953	66.8953	1.9999
11	0.500	0.125	74.9900	75.4269	-0.4369
12	0.750	0.125	68.8720	66.8953	1.9767
13	0.875	0.125	51.9473	48.2909	3.6564
*14	1.000	0.125	0.0000	0.0000	0.0000
*15	0.000	0.250	0.0000	0.0000	0.0000
16	0.125	0.250	26.9965	26.2587	0.7379
17	0.250	0.250	44.2271	43.2028	1.0243
18	0.500	0.250	54.1177	54.0529	0.0648
19	0.750	0.250	44.2031	43.2028	1.0002
20	0.875	0.250	26.9641	26.2587	0.7054
*21	1.000	0.250	0.0000	0.0000	0.0000
*22	0.000	0.500	0.0000	0.0000	0.0000
23	0.125	0.500	11.0281	10.0708	0.9573
24	0.250	0.500	19.4364	18.2028	1.2336
25	0.500	0.500	26.1726	25.0000	1.1726
26	0.750	0.500	19.4055	18.2028	1.2026
27	0.875	0.500	10.9899	10.0708	0.9191
*28	1.000	0.500	0.0000	0.0000	0.0000
*29	0.000	0.875	0.0000	0.0000	0.0000
30	0.125	0.875	1.8004	1.7091	0.0914
31	0.250	0.875	3.1869	3.1478	0.0391
32	0.500	0.875	4.4374	4.4316	0.0058
33	0.750	0.875	3.1680	3.1478	0.0202
34	0.875	0.875	1.7750	1.7091	0.0659
*35	1.000	0.875	0.0000	0.0000	0.0000
*36	0.000	1.000	0.0000	0.0000	0.0000
*37	0.125	1.000	0.0000	0.0000	0.0000
*38	0.250	1.000	0.0000	0.0000	0.0000
*39	0.500	1.000	0.0000	0.0000	0.0000
*40	0.750	1.000	0.0000	0.0000	0.0000
*41	0.875	1.000	0.0000	0.0000	0.0000
*42	1.000	1.000	0.0000	0.0000	0.0000

*Boundary points.

TABLE 3: COMPUTED AND EXACT SOLUTION OF THE POISSON EQUATION IN AN ELLIPSE .

$\alpha = 10$, $\beta = 8$, $N = 28$, $R_m = 55$, $R_n = 4$.

I	X	Y	COMP	EXACT	ERROR
*1	0.000	0.000	39.0247	39.0244	0.0003
*2	0.000	1.600	37.4635	37.4634	0.0001
*3	0.000	4.000	29.2681	29.2683	-0.0001
*4	0.000	6.400	14.0484	14.0488	-0.0004
*5	0.000	8.000	0.0000	0.0000	0.0000
*6	2.000	0.000	37.4637	37.4634	0.0003
*7	4.000	0.000	32.7808	32.7805	0.0003
*8	6.000	0.000	24.9759	24.9756	0.0003
*9	8.000	0.000	14.0490	14.0488	0.0002
*10	10.000	0.000	0.0000	0.0000	0.0000
11	2.000	1.600	35.9026	35.9024	0.0001
12	4.000	1.600	31.2197	31.2195	0.0002
13	6.000	1.600	23.4148	23.4146	0.0002
14	8.000	1.600	12.4879	12.4878	0.0001
15	2.000	4.000	27.7072	27.7073	-0.0001
16	4.000	4.000	23.0244	23.0244	0.0000
17	6.000	4.000	15.2195	15.2195	0.0000
18	8.000	4.000	4.2927	4.2927	0.0000
19	2.000	5.600	18.3412	18.3415	-0.0002
20	4.000	5.600	13.6584	13.6585	-0.0001
21	6.000	5.600	5.8536	5.8537	0.0000
*22	9.798	1.600	0.0000	0.0000	0.0000
*23	8.660	4.000	0.0000	0.0000	0.0000
*24	7.141	5.600	0.0000	0.0000	0.0000
*25	2.000	7.838	0.0000	0.0000	0.0000
*26	4.000	7.332	0.0000	0.0000	0.0000
*27	6.000	6.400	0.0000	0.0000	0.0000
*28	8.000	4.800	0.0000	0.0000	0.0000

* Boundary points.

Table 4: Values of the input parameters, Mean Square error (E), absolute maximum error for different N for $\nabla^2 f = 0$.

S. No.	q	N	R_m	R_n	E	Max error
1.	20	25	. 5	. 2	1.59×10^{-7}	8.2×10^{-4}
2.	20	35	. 6	. 1	2.03×10^{-7}	1.35×10^{-3}
3.	1	25	. 5	. 2	3.98×10^{-10}	4.1×10^{-5}
4.	1	35	. 7	. 1	1.28×10^{-9}	1.46×10^{-4}

TABLE 5: COMPUTED AND EXACT SOLUTION OF THE POISSON
EQUATION $f_{xx} + f_{yy} = q/D$ IN WHICH q IS A
CONSTANT. $q = 20$, $N = 35$, $R_m = 0.9$, $R_n = 0.1$.

I	X	Y	COMP	EXACT	ERROR
*1	0.000	0.000	-0.10644	-0.10727	0.00077
*2	0.000	0.220	-0.10222	-0.10282	0.00060
*3	0.000	0.500	-0.08387	-0.08348	-0.00039
*4	0.000	0.700	-0.06006	-0.05871	-0.00135
*5	0.000	1.000	0.00000	0.00000	0.00000
*6	0.250	0.000	-0.10045	-0.10151	0.00107
*7	0.500	0.000	-0.08334	-0.08348	0.00014
*8	0.810	0.000	-0.04167	-0.04048	-0.00119
*9	0.180	1.000	0.00000	0.00000	0.00000
*10	0.500	1.000	0.00000	0.00000	0.00000
*11	1.000	0.000	0.00000	0.00000	0.00000
*12	1.000	1.000	0.00000	0.00000	0.00000
*13	1.000	0.500	0.00000	0.00000	0.00000
*14	1.000	0.700	0.00000	0.00000	0.00000
*15	0.800	1.000	0.00000	0.00000	0.00000
16	0.130	0.260	-0.09909	-0.09961	0.00052
17	0.310	0.420	-0.08330	-0.08349	0.00018
18	0.770	0.580	-0.03505	-0.03496	-0.00009
19	0.220	0.730	-0.05261	-0.05216	-0.00046
20	0.930	0.180	-0.01642	-0.01599	-0.00044
21	0.860	0.780	-0.01545	-0.01538	-0.00007
22	0.420	0.910	-0.01801	-0.01816	0.00015
23	0.510	0.570	-0.05946	-0.05961	0.00015
24	0.680	0.820	-0.02430	-0.02435	0.00005
25	0.720	0.620	-0.03827	-0.03824	-0.00003
26	0.640	0.330	-0.06183	-0.06152	-0.00030
27	0.840	0.370	-0.03185	-0.03147	-0.00039
28	0.970	0.680	-0.00481	-0.00477	-0.00004
29	0.270	0.930	-0.01555	-0.01554	-0.00001
30	0.530	0.110	-0.07978	-0.07963	-0.00015
31	0.150	0.210	-0.10062	-0.10126	0.00064
32	0.690	0.950	-0.00766	-0.00763	-0.00003
33	0.480	0.440	-0.07151	-0.07163	0.00012
34	0.210	0.570	-0.07358	-0.07332	-0.00026
35	0.590	0.940	-0.01061	-0.01058	-0.00003

*Boundary points.

Table 6 : Values of the input parameters, Mean Square error (E), absolute maximum error for different N for $\nabla^2 f = g(x,y)$.

S.No.	q	N	R_m	R_n	E	Max error
1.	20	25	1.52	.15	4.86×10^{-4}	5.65×10^{-2}
2.	20	30	2.8	.1	5.06×10^{-5}	1.25×10^{-2}
3.	20	35	2.8	.1	2.11×10^{-5}	.0148
4.	1	25	1.5	.3	1.3×10^{-6}	2.94×10^{-3}
5.	1	30	2.8	.1	1.27×10^{-7}	6.6×10^{-4}
6.	1	35	2.8	.1	5.28×10^{-8}	7.38×10^{-4}

TABLE 7a: COMPUTED AND EXACT SOLUTION OF THE POISSON
EQUATION $f_{xx} + f_{yy} = g(x,y)$.
 $q = 20$, $N = 35$, $R_m = 2.8$, $R_n = 0.1$.

I	X	Y	COMP	EXACT	ERROR
* 1	0.000	0.000	9.46105	9.46366	-0.00261
* 2	0.000	0.220	8.94240	8.94669	-0.00428
* 3	0.000	0.500	6.83381	6.84478	-0.01097
* 4	0.000	0.700	4.45365	4.46840	-0.01476
* 5	0.000	1.000	0.00000	0.00000	0.00000
* 6	0.250	0.000	8.79259	8.79692	-0.00434
* 7	0.500	0.000	6.83966	6.84478	-0.00512
* 8	0.810	0.000	2.91293	2.91812	-0.00519
* 9	0.180	1.000	0.00000	0.00000	0.00000
*10	0.500	1.000	0.00000	0.00000	0.00000
*11	1.000	0.000	0.00000	0.00000	0.00000
*12	1.000	1.000	0.00000	0.00000	0.00000
*13	1.000	0.500	0.00000	0.00000	0.00000
*14	1.000	0.700	0.00000	0.00000	0.00000
*15	0.800	1.000	0.00000	0.00000	0.00000
16	0.130	0.260	8.57076	8.57616	-0.00540
17	0.310	0.420	6.78130	6.78721	-0.00591
18	0.770	0.580	2.22804	2.22966	-0.00162
19	0.220	0.730	3.83535	3.84322	-0.00787
20	0.930	0.180	1.05650	1.05684	-0.00035
21	0.860	0.780	0.78539	0.78637	-0.00098
22	0.420	0.910	1.13562	1.13662	-0.00101
23	0.510	0.570	4.35235	4.35595	-0.00360
24	0.680	0.820	1.40578	1.40709	-0.00131
25	0.720	0.620	2.46047	2.46234	-0.00187
26	0.640	0.330	4.61329	4.61689	-0.00361
27	0.840	0.370	2.10201	2.10327	-0.00126
28	0.970	0.680	0.23959	0.23987	-0.00028
29	0.270	0.930	1.00667	1.00844	-0.00177
30	0.530	0.110	6.43731	6.44232	-0.00501
31	0.150	0.210	8.75920	8.76403	-0.00482
32	0.690	0.950	0.38783	0.38836	-0.00052
33	0.480	0.440	5.53565	5.53995	-0.00429
34	0.210	0.570	5.78024	5.78855	-0.00831
35	0.590	0.940	0.58868	0.58942	-0.00074

*Boundary points.

TABLE 7b: COMPUTED AND EXACT SOLUTION OF THE POISSON
EQUATION $\{xx + \{yy = g(x,y)$.
 $q = 1, N = 30, R_m = 2.8, R_n = 0.1$.

I	X	Y	COMP	EXACT	ERROR
* 1	0.000	0.000	0.47259	0.47318	-0.00059
* 2	0.000	0.250	0.43932	0.43985	-0.00052
* 3	0.000	0.500	0.34184	0.34224	-0.00040
* 4	0.000	0.750	0.18864	0.18907	-0.00043
* 5	0.000	1.000	0.00000	0.00000	0.00000
* 6	1.000	0.000	0.00000	0.00000	0.00000
* 7	1.000	0.250	0.00000	0.00000	0.00000
* 8	1.000	0.500	0.00000	0.00000	0.00000
* 9	1.000	0.750	0.00000	0.00000	0.00000
*10	1.000	1.000	0.00000	0.00000	0.00000
*11	0.200	0.000	0.45114	0.45177	-0.00063
*12	0.200	1.000	0.00000	0.00000	0.00000
*13	0.400	0.000	0.38796	0.38862	-0.00065
*14	0.400	1.000	0.00000	0.00000	0.00000
*15	0.600	0.000	0.28628	0.28695	-0.00066
*16	0.600	1.000	0.00000	0.00000	0.00000
*17	0.800	0.000	0.15261	0.15323	-0.00062
*18	0.800	1.000	0.00000	0.00000	0.00000
19	0.050	0.050	0.46991	0.47051	-0.00059
20	0.130	0.260	0.42831	0.42881	-0.00050
21	0.460	0.160	0.35106	0.35154	-0.00048
22	0.310	0.420	0.33905	0.33936	-0.00031
23	0.070	0.580	0.29663	0.29698	-0.00035
24	0.120	0.730	0.19946	0.19977	-0.00031
25	0.420	0.910	0.05679	0.05683	-0.00004
26	0.510	0.570	0.21766	0.21780	-0.00014
27	0.680	0.820	0.07033	0.07035	-0.00003
28	0.840	0.370	0.10504	0.10516	-0.00012
29	0.970	0.680	0.01199	0.01199	0.00000
30	0.170	0.930	0.05292	0.05305	-0.00013

*Boundary points.

Table 8: Values of the input parameters, Mean Square error (E), absolute maximum error for different N, for $\gamma^4 = 0$.

S. No.	q	N	R_m	R_n	E	Max error
1.	20	25	2.2	.15	3.89×10^{-4}	5.2×10^{-2}
2.	20	35	2.8	.1	1.58×10^{-4}	3.49×10^{-2}
3.	1	25	1.5	.3	1.20×10^{-6}	2.85×10^{-3}
4.	1	35	1.5	.3	1.72×10^{-7}	1.26×10^{-3}

TABLE 9: COMPUTED AND EXACT SOLUTION OF THE BIHARMONIC EQUATION. $q=20$, $N = 35$, $R_m = 2.8$, $R_n = 0.1$.

I	X	Y	COMP	EXACT	ERROR
* 1	0.000	0.000	9.42875	9.46366	-0.03491
* 2	0.000	0.220	8.91823	8.94669	-0.02845
* 3	0.000	0.500	6.82939	6.84478	-0.01539
* 4	0.000	0.700	4.45867	4.46840	-0.00973
* 5	0.000	1.000	0.00000	0.00000	0.0000
* 6	0.250	0.000	8.76810	8.79692	-0.02882
* 7	0.500	0.000	6.82994	6.84478	-0.01484
* 8	0.810	0.000	2.91433	2.91812	-0.00379
* 9	0.180	1.000	0.00000	0.00000	0.00000
*10	0.500	1.000	0.00000	0.00000	0.00000
*11	1.000	0.000	0.00000	0.00000	0.00000
*12	1.000	1.000	0.00000	0.00000	0.00000
*13	1.000	0.500	0.00000	0.00000	0.00000
*14	1.000	0.700	0.00000	0.00000	0.00000
*15	0.800	1.000	0.00000	0.00000	0.00000
16	0.130	0.260	8.55057	8.57616	-0.02559
17	0.310	0.420	6.77241	6.78721	-0.01480
18	0.770	0.580	2.22803	2.22966	-0.00163
19	0.220	0.730	3.83735	3.84322	-0.00587
20	0.930	0.180	1.05840	1.05684	0.00156
21	0.860	0.780	0.78568	0.78637	-0.00070
22	0.420	0.910	1.13516	1.13662	-0.00147
23	0.510	0.570	4.34914	4.35595	-0.00680
24	0.680	0.820	1.40524	1.40709	-0.00186
25	0.720	0.620	2.45991	2.46234	-0.00243
26	0.640	0.330	4.61198	4.61689	-0.00492
27	0.840	0.370	2.10377	2.10327	0.00050
28	0.970	0.680	0.23966	0.23987	-0.00021
29	0.270	0.930	1.00727	1.00844	-0.00117
30	0.530	0.110	6.43005	6.44232	-0.01227
31	0.150	0.210	8.73667	8.76403	-0.02736
32	0.690	0.950	0.38784	0.38836	-0.00051
33	0.480	0.440	5.53038	5.53995	-0.00956
34	0.210	0.570	5.77787	5.78855	-0.01068
35	0.590	0.940	0.58848	0.58942	-0.00093

*Boundary points.

TABLE 10: COMPUTED AND EXACT SOLUTION OF BIHARMONIC
EQUATION. $q = 1$, $N = 25$, $R_m = 1.5$, $R_n = 0.3$.

I	X	Y	COMP	EXACT	ERROR
* 1	0.000	0.000	0.47075	0.47318	-0.00243
* 2	0.250	0.000	0.43843	0.43985	-0.00142
* 3	0.500	0.000	0.34288	0.34224	0.00064
* 4	0.750	0.000	0.19193	0.18907	0.00285
* 5	1.000	0.000	0.00000	0.00000	0.00000
* 6	0.000	0.250	0.43843	0.43985	-0.00142
7	0.250	0.250	0.40835	0.40895	-0.00060
8	0.500	0.250	0.31876	0.31839	0.00037
9	0.750	0.250	0.17707	0.17606	0.00101
*10	1.000	0.250	0.00000	0.00000	0.00000
*11	0.000	0.500	0.34288	0.34224	0.00064
12	0.250	0.500	0.31876	0.31839	0.00037
13	0.500	0.500	0.24853	0.24836	0.00018
14	0.750	0.500	0.13783	0.13772	0.00011
*15	1.000	0.500	0.00000	0.00000	0.00000
*16	0.000	0.750	0.19193	0.18907	0.00285
17	0.250	0.750	0.17707	0.17606	0.00101
18	0.500	0.750	0.13783	0.13772	0.00011
19	0.750	0.750	0.07627	0.07675	-0.00048
*20	1.000	0.750	0.00000	0.00000	0.00000
*21	0.000	1.000	0.00000	0.00000	0.00000
*22	0.250	1.000	0.00000	0.00000	0.00000
*23	0.500	1.000	0.00000	0.00000	0.00000
*24	0.750	1.000	0.00000	0.00000	0.00000
*25	1.000	1.000	0.00000	0.00000	0.00000

*Boundary points.

Fig1. Variation of temperature along the Y - axis for different values of X. The continuous lines from the top to bottom denote the MQ solution for $X = 0.5, 0.25$ and 0.125 respectively. The symbols denote the corresponding exact solution.
 $N=42, R_m=0.14, R_n=10, E=1.06$

Fig.2. Location of grid points in an ellipse.

Fig.3. Computed (continuous line) and exact (symbols) solution of the biharmonic equation versus X for $Y=0$ and $Y=0.5$. $\Delta Y=0.5$, Here $q=20$, $N=25$, $Rm=2.2$, $Rn=0.15$, $E=3.89 \times 10^4$.

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