INTEGRALS DEFINED ON A FIELD OF SETS

APPROVED:

William D. Epling
Major Professor

Robert W. Crawford
Minor Professor

John J. Mahal
Director of the Department of Mathematics

Robert B. Tondreau
Dean of the Graduate School
INTEGRALS DEFINED ON A FIELD OF SETS

THESIS

Presented to the Graduate Council of the North Texas State University in Partial Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

Grady W. Troute, B. A.
Denton, Texas
August, 1968
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Basic Lemmas and Theorems</td>
<td></td>
</tr>
<tr>
<td>II. DEVELOPMENT OF THE INTEGRAL</td>
<td>9</td>
</tr>
<tr>
<td>III. EXISTENCE THEOREMS</td>
<td>27</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

The purpose of this paper is to define an integral for real-valued functions which are defined on a field of sets and to demonstrate several properties of such an integral. Chapter I is devoted to the definition and investigation of the functions for which the integral will be defined. In Chapter II the integral is developed, and several of its basic properties are demonstrated. Chapter III is dedicated almost entirely to theorems concerning the existence of particular integrals.

Definition 1-1. If $U$ is a set, then the statement that $F$ is a field of subsets of $U$ means that

1. $F$ is a collection of subsets of $U$, and $U$ is in $F$,
2. if $X$ is in $F$ and is a proper subset of $U$, then $U - X$, denoted $X'$, is in $F$, and
3. if each of $X$ and $Y$ is in $F$, then $X \cup Y$ is in $F$.

The remainder of the work in this paper will be based on the supposition that $U$ is a set and $F$ is a field of subsets of $U$.

Definition 1-2. If $\{A_i\}_{i=1}^n$ is a sequence of elements of $F$, then $\bigcup_{i=1}^n A_i$ is the set of all elements $a$ such that $a$ is in $A_i$ for some $i$ such that $1 \leq i \leq n$. Similarly, if there exists
an element $a'$ such that $a'$ is in $A_i$ for each $i$ such that $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} A_i$ is the set of all elements $a$ such that $a$ is in $A_i$ for each $i$ such that $1 \leq i \leq n$.

**Definition 1-3.** If $f$ is a real-valued function such that the domain of $f$ is $F$, then the statement that $f$ is finitely additive means that if $S_1$ and $S_2$ are mutually exclusive sets of $F$, then

$$f(S_1 \cup S_2) = f(S_1) + f(S_2).$$

**Definition 1-4.** Notations for function sets:

1. $R(F)$ is the set of all real-valued functions $f$ such that the domain of $f$ is $F$,
2. $R_b(F)$ is the set of all bounded elements of $R(F)$,
3. $R_a(F)$ is the set of all finitely additive elements of $R(F)$,
4. $R^+(F)$ is the set of all nonnegative-valued elements of $R(F)$,
5. $R^+_a(F) = R_a(F) \cap R^+(F)$, and
6. $R_{ab}(F) = R_a(F) \cap R_b(F)$.

**Definition 1-5.** If $V$ is in $F$, then the statement that $D$ is a subdivision of $V$ means that $D$ is a finite collection of sets of $F$ with union $V$. 
Definition 1-6. If \( V \) is in \( F \) and \( D \) is a subdivision of \( V \), then the statement that \( E \) is a refinement of \( D \) means:

1. \( E \) is a subdivision of \( V \), and
2. if \( I \) is an element of \( E \), then \( I \) is a subset of \( I' \) for some \( I' \) in \( D \).

Definition 1-7. Suppose \( V \) is in \( F \). If \( D \) is a subdivision of \( V \) and \( E \) is a refinement of \( D \), then for each \( I \) in \( D \), \( E_1 \) denotes the collection of sets of \( E \) having union \( I \).

Basic Lemmas and Theorems

Lemma 1-1. Suppose each of \( X \) and \( Y \) is in \( F \). If there exists an element \( b \) such that \( b \) is in each of \( X \) and \( Y \), then \( X \cap Y \) is in \( F \).

Proof. If \( X = U \), then \( X \cap Y = U \cap Y = Y \), which is in \( F \). Similarly, if \( Y = U \), then \( X \cap Y = X \cap U = X \), which is in \( F \). Therefore, suppose each of \( X \) and \( Y \) is a proper subset of \( U \). Since each of \( X \) and \( Y \) is in \( F \), then each of \( \overline{X} \) and \( \overline{Y} \) is in \( F \), which implies that \( \overline{X} \cup \overline{Y} \) is in \( F \). Since \( b \) is not in \( \overline{X} \) and \( b \) is not in \( \overline{Y} \), then \( \overline{X} \cup \overline{Y} \) is not \( U \). Therefore, \( (X \cap Y) \), which is \( X \cap Y \), is in \( F \).

Lemma 1-2. If \( \{A_i\}_{i=1}^{n} \) is a sequence of elements of \( F \), then \( \bigcup_{i=1}^{n} \{A_i\} \) is in \( F \); and if for some element \( a \), \( a \) is in \( A_i \) for each \( i \) such that \( 1 \leq i \leq n \), then \( \bigcap_{i=1}^{n} \{A_i\} \) is in \( F \).

Proof. Let \( m \) denote the least upper bound of the set \( S \) such that
\[ S = \left\{ j \mid \bigcup_{i=1}^{j} A_i \text{ is in } F, 1 \leq j \leq n \right\}. \]

Suppose by way of contradiction that \( m < n \). Then \( m + 1 \leq n \), which implies that \( A_{(m+1)} \) is in \( F \). Since \( S \) is an integer set, then \( m \) is in \( S \); and, consequently, \( \bigcup_{i=1}^{m} A_i \) is in \( F \).

Therefore, by Definition 1-1, \( \bigcup_{i=1}^{m} A_i \bigcup A_{(m+1)} \), which is \( \bigcup_{i=1}^{m+1} A_i \), is in \( F \). This implies that \( m + 1 \) is in \( S \); and hence, \( m \) is not the least upper bound of \( S \). This is contradictory. Therefore, the supposition that \( m < n \) is false, requiring that \( m = n \), which satisfies the first part of the lemma. An analogous proof holds for the remainder of the lemma by substitution of the intersection symbol where the union symbol occurs and use of Lemma 1-1 in place of Definition 1-1.

**Lemma 1-3.** \( R_a^+ (F) \subseteq R_{ab} (F) \).

**Proof.** Since \( R_a^+ (F) \subseteq R_a (F) \), it remains only to be established that \( R_a^+ (F) \subseteq R_b (F) \). Suppose \( f \) is in \( R_a^+ (F) \). Since \( f \) is additive, then for each \( I \) in \( F \) such that \( I \neq U \),

\[ f(I \cup \overline{I}) = f(I) + f(\overline{I}) = f(U). \]

Since each of \( f(I), f(\overline{I}), \) and \( f(U) \) is positive or zero, then \( f(I) \leq f(U) \). Therefore, if \( f \) is in \( R_a^+ (F) \), then \( f \) is bounded by \( f(U) \), and consequently, \( f \) is in \( R_b (F) \). Hence, \( R_a^+ (F) \subseteq R_b (F) \), and the proof is complete.
Lemma 1-4. Suppose $V$ is in $F$ and $f$ is in $R_a(F)$. If $D$ is a subdivision of $V$, then
\[ \sum_{D} f(I) = f(V). \]

Proof. This lemma is easily established with the use of Definition 1-3.

Lemma 1-5. Suppose $V$ is in $F$. If each of $D_1$ and $D_2$ is a subdivision of $V$, then there is a subdivision $E$ of $V$ such that

(1) $E$ is a refinement of each of $D_1$ and $D_2$, and

(2) if $E'$ is a refinement of each of $D_1$ and $D_2$, then $E'$ is a refinement of $E$.

Proof. Suppose $V$ is in $F$ and each of $D_1$ and $D_2$ is a subdivision of $V$. Let $E$ denote the set
\[ \{ x \mid x = I_1 \cap I_2, I_1 \text{ is in } D_1, I_2 \text{ is in } D_2, I_1 \cap I_2 \neq \emptyset \}. \]

Suppose each of $J_1$ and $J_2$ is in $E$. Then $J_1 = I_1 \cap I_2$ for some $I_1$ in $D_1$ and some $I_2$ in $D_2$, and $J_2 = I_3 \cap I_4$ for some $I_3$ in $D_1$ and some $I_4$ in $D_2$. Suppose there exists an element $a$ such that $a$ is in each of $J_1$ and $J_2$. Then $a$ is of necessity in each of $I_1$, $I_2$, $I_3$ and $I_4$; but since each of $I_1$ and $I_3$ is in $D_1$ and each of $I_2$ and $I_4$ is in $D_2$, the existence of the common element $a$ implies that $I_1 = I_3$ and $I_2 = I_4$. Consequently, $I_1 \cap I_2 = I_3 \cap I_4$, which means $J_1 = J_2$. Hence, if each of
J_1 and J_2 is in E and J_1 \cap J_2 \neq \emptyset, then J_1 = J_2; that is, the sets of E are mutually exclusive.

Suppose x is in V. Then x is in I_1 for some I_1 in D_1, and x is in I_2 for some I_2 in D_2. Therefore, x is in I_1 \cap I_2, which is an element of E. Suppose x is in J for some J in E. Then x is in I_1 \cap I_2 for some I_1 in D_1 and some I_2 in D_2. All elements of each of I_1 and I_2 are in V. Therefore, x is in V. Hence, the union of the sets of E is V.

Obviously E is a finite collection. Therefore, E is a subdivision of V.

Suppose E' is a refinement of each of D_1 and D_2. Suppose J is in E'. Then J \subseteq I_1 for some I_1 in D_1, and J \subseteq I_2 for some I_2 in D_2. This implies that J \subseteq I_1 \cap I_2 which is an element of E. Hence, if J is in E', then J \subseteq I for some I in E, which completes the proof.

**Definition 1-8.** Suppose V is in F and each of D_1 and D_2 is a subdivision of V. The subdivision E of V which satisfies Lemma 1-5 with respect to D_1 and D_2 is called the greatest common refinement of D_1 and D_2.

**Theorem 1-1.** Suppose V is in F and f is in R_a(F). If D is a subdivision of V and E is a refinement of D, then
\[ \sum_{D} f(I) \leq \sum_{E} f(J). \]
Proof. For each I in D, $E_I$ is a subdivision of I, and since $f$ is additive, $f(I)$ is $\sum f(J)$. Using this property, together with the triangle inequality,

$$\sum_{D} |f(I)| = \sum_{E_I} \sum_{D} f(J) \leq \sum_{E_I} \sum_{D} |f(J)| = \sum_{E_I} |f(J)|.$$  

**Theorem 1-2.** If $V$ is in $F$ and $f$ is in $R_{ab}(F)$, then there exists a number $m$ such that for each subdivision $D$ of $V$

$$\sum_{D} |f(I)| \leq m.$$  

Proof. Suppose $D$ is a subdivision of $V$. Since $f$ is bounded, there exists a number $b$ such that

$$|f(I)| \leq b$$

for each $I$ in $D$. Suppose $f$ is everywhere nonnegative on $D$. Then

$$0 \leq \sum_{D} |f(I)| = \sum_{D} f(I) = f(V) \leq b.$$  

Similarly, if $f$ is everywhere negative on $D$, then

$$0 \geq \sum_{D} f(I) = -\sum_{D} |f(I)| = -|f(V)| \geq -b.$$  

Therefore, if $f$ is of constant sign on $D$, then

$$\sum_{D} |f(I)| \leq b.$$  

Suppose $f$ is not of constant sign on $D$. Consider the subsets $D_1$ and $D_2$ of $D$ such that

$$D_1 = \{ I \mid I \text{ is in } D, f(I) \geq 0 \},$$
$$D_2 = \{ I \mid I \text{ is in } D, f(I) < 0 \}.$$
Let $V_1$ denote the union of the sets of $D_1$, and let $V_2$ denote the union of the sets of $D_2$. Now, the union of $V_1$ and $V_2$ is $V$, and the union of $D_1$ and $D_2$ is $D$. Also, $D_1$ constitutes a subdivision of $V_1$, and $D_2$ constitutes a subdivision of $V_2$. Therefore, applying Lemma 1-4,

$$
\sum_{D_1} |f(I)| = \sum_{D_1} f(I) - \sum_{D_2} f(I) = f(V_1) - f(V_2) \leq 2b.
$$

Therefore, if $D$ is any subdivision of $V$, then

$$
\sum_{D} |f(I)| \leq 2b.
$$
CHAPTER II

DEVELOPMENT OF THE INTEGRAL

Definition 2-1. Suppose \( h \) is in \( R(F) \) and \( J \) is in \( F \). The statement that \( X \) is an integral of \( h \) on \( J \) means that \( X \) is a number and for each positive number \( c \) there exists a subdivision \( D \) of \( J \) such that if \( E \) is a refinement of \( D \), then

\[
\left| \sum_{E} h(I) - X \right| < c.
\]

Theorem 2-1. Suppose \( h \) is in \( R(F) \) and \( V \) is in \( F \). There exists at most one number \( X \) such that \( X \) is an integral of \( h \) on \( V \).

Proof. Suppose \( X \) is an integral of \( h \) on \( V \). Assume for purposes of contradiction that \( Y \) is also an integral of \( h \) on \( V \), and \( Y \neq X \). Let \( c \) denote \( |X - Y|/2 \). Since \( X \neq Y \), then \( c \) is a positive number. In accordance with Definition 2-1, there exists a subdivision \( D_1 \) of \( V \) such that if \( E \) is any refinement of \( D_1 \), then

\[
\left| X - \sum_{E} h(I) \right| = \left| \sum_{E} h(I) - X \right| < c.
\]

Similarly, there exists a subdivision \( D_2 \) of \( V \) such that if \( E \) is a refinement of \( D_2 \), then

\[
\left| Y - \sum_{E} h(I) \right| < c.
\]
Let $E^*$ denote the greatest common refinement of $D_1$ and $D_2$.

By use of the triangle inequality

$$|X - Y| = \left| \left[ X - \sum_{E^*} h(I) \right] + \left[ \sum_{E^*} h(I) - Y \right] \right| \leq \left| X - \sum_{E^*} h(I) \right| + \left| \sum_{E^*} h(I) - Y \right| < \frac{\alpha}{\alpha} + \frac{\alpha}{\alpha} = |X - Y|,$$

which is contradictory. Hence, if $X$ is an integral of $h$ on $V$, then $X$ is the only integral of $h$ on $V$.

**Definition 2-2.** Suppose $h$ is in $R(F)$ and $V$ is in $F$. If $h$ has an integral on $V$, then it is denoted $\int_V h(I)$.

**Definition 2-3.** Suppose $h$ is in $R(F)$ and $V$ is in $F$. The statement that $X$ is an upper integral of $h$ on $V$ means that $X$ is a number and for each positive number $c$ there exists a subdivision $D$ of $V$ such that if $E$ is any refinement of $D$, then

$(1)$ $\sum E h(I) - X < c$, and

$(2)$ there exists a refinement $E'$ of $E$ such that

$$\left| \sum_{E'} h(I) - X \right| < c.$$

Similarly, the statement that $X$ is a lower integral of $h$ on $V$ means that $X$ is a number and for each positive number $c$ there exists a subdivision $D$ of $V$ such that if $E$ is a refinement of $D$, then

$(1)$ $X - \sum E h(I) < c$, and

$(2)$ there exists a refinement $E'$ of $E$ such that

$$\left| \sum_{E'} h(I) - X \right| < c.$$
Theorem 2-2. Suppose \( h \) is in \( R(F) \) and \( V \) is in \( F \). There exists at most one number \( X \) such that \( X \) is an upper integral of \( h \) on \( V \). Similarly, there exists at most one number \( Y \) such that \( Y \) is a lower integral of \( h \) on \( V \).

Proof. Suppose \( X \) is an upper integral of \( h \) on \( V \). Assume for purposes of contradiction that \( X' \) is also an upper integral of \( h \) on \( V \), and \( X \neq X' \). Let \( Y \) denote the maximum of \( X \) and \( X' \), and let \( Y' \) denote the minimum of \( X \) and \( X' \). Now, each of \( Y \) and \( Y' \) is an upper integral of \( h \) on \( V \), and \( Y > Y' \). Let \( c \) denote the positive number \( (Y - Y')/2 \). In accordance with Definition 2-3, there exists a subdivision \( D_1 \) of \( V \) such that if \( E \) is any refinement of \( D_1 \), then

(1) \[ \sum_{I \in E} h(I) - Y < c, \]
(2) there exists a refinement \( E' \) of \( E \) such that \[ \left| \sum_{I \in E'} h(I) - Y \right| < c. \]

Similarly, there exists a subdivision \( D_2 \) of \( V \) such that if \( E \) is any refinement of \( D_2 \), then

(3) \[ \sum_{I \in E} h(I) - Y' < c, \]
(4) there exists a refinement \( E' \) of \( E \) such that \[ \left| \sum_{I \in E'} h(I) - Y' \right| < c. \]

Let \( D \) denote the greatest common refinement of \( D_1 \) and \( D_2 \).

Since \( D \) is a refinement of \( D_1 \), statement (2) above implies
that there exists a subdivision $D^*$ of $D$ such that

$$ (5) \quad Y - \sum_{D^*} h(I) \leq |Y - \sum_{D^*} h(I)| = |\sum_{D^*} h(I) - Y| < c. $$

Also, it is easily shown that $D^*$ is a refinement of $D_2$; and, consequently,

$$ (6) \quad \sum_{D^*} h(I) - Y' < c. $$

Using inequalities (5) and (6),

$$ Y - Y' = Y - \sum_{D^*} h(I) + \sum_{D^*} h(I) - Y' < c + c = Y - Y', $$

which is the contradiction sought. Hence, the upper integral of $h$ on $V$ is unique. A similar proof exists for the lower integral.

**Definition 2-4.** Suppose $h$ is in $R(F)$ and $V$ is in $F$. If $h$ has an upper integral on $V$, then it is denoted $\int^+ h(I)$.

Also, if $h$ has a lower integral on $V$, then it is denoted $\int^! h(I)$.

**Lemma 2-1.** Suppose $h$ is in $R(F)$ and $V$ is in $F$. If each of $\int^+ h(I)$ and $\int^! h(I)$ exists, then $\int^+ h(I) \geq \int^! h(I)$.

**Proof.** Suppose each of $\int^+ h(I)$ and $\int^! h(I)$ exists.

Assume for the purpose of contradiction that $\int^+ h(I) < \int^! h(I)$. Let $c$ denote the positive number $\int^! h(I) - \int^+ h(I)$. Then $c/2$ is also a positive number. In accordance with Definition 2-3, there exists a subdivision $D_1$ of $V$ such that if $E$ is any refinement of $D_1$, then
Similarly, there exists a subdivision $D_2$ of $V$ such that if $E$ is a refinement of $D_2$, then

$$\sum_{E} h(I) - \int_{V} h(I) < c/2.$$ 

Let $D$ denote the greatest common refinement of $D_1$ and $D_2$.

With the use of inequalities (1) and (2),

$$\int_{V} h(I) - \sum_{D} h(I) = \int_{V} h(I) - \sum_{D} h(I) + \sum_{D} h(I) - \int_{D} h(I)$$

$$< c/2 + c/2 = c,$$

a contradiction. Therefore,

$$\int_{V} h(I) \leq \int_{D} h(I).$$

**Definition 2-5.** Suppose $h$ is in $R(F)$ and $V$ is in $F$. The statement that $h$ is summation bounded on $V$ with respect to $D$ means that $D$ is a subdivision of $V$, and there exists a nonnegative number $m$ such that if $E$ is a refinement of $D$, then

$$|\sum_{E} h(I)| \leq m.$$ 

**Theorem 2-3.** Suppose $h$ is in $R(F)$ and $V$ is in $F$. Each of $\int_{V} h(I)$ and $\int_{D} h(I)$ exists if and only if there exists a subdivision $D$ of $V$ such that $h$ is summation bounded on $V$ with respect to $D$.

**Proof.** First, let it be supposed that each of $\int_{V} h(I)$ and $\int_{D} h(I)$ exists. In accordance with the definitions of the upper and lower integrals, there exist subdivisions $D_1$
and $D_2$ of $V$ such that if $E$ is a refinement of $D_1$, then
\[ \sum_{E} h(I) - \int_{E} h(I) < 1, \]
and if $E$ is a refinement of $D_2$, then
\[ \int_{E} h(I) - \sum_{E} h(I) < 1. \]

Let $D$ denote the greatest common refinement of $D_1$ and $D_2$.

Now, if $E$ is a refinement of $D$, then $E$ is also a refinement of each of $D_1$ and $D_2$. Consequently, if $E$ is a refinement of $D$, then
\[ \sum_{E} h(I) - \int_{E} h(I) < 1, \text{ and } \int_{E} h(I) - \sum_{E} h(I) < 1, \]
which implies
\[ \int_{E} h(I) - 1 < \sum_{E} h(I) < \int_{E} h(I) + 1. \]

Therefore, if $m$ denotes the maximum of the two numbers $|\int_{E} h(I) - 1|$ and $|\int_{E} h(I) + 1|$, then for each subdivision $E$ of $V$ such that $E$ is a refinement of $D$
\[ |\sum_{E} h(I)| < m. \]

Hence, if $h$ has an upper and a lower integral on $V$, then there exists a subdivision $D$ of $V$ such that $h$ is summation bounded on $V$ with respect to $D$. This completes the first part of the proof.

Now suppose that there exists a subdivision $D$ of $V$ such that $h$ is summation bounded on $V$ with respect to $D$. In accordance with Definition 2-5, there exists a positive number $m$ such
that if $E$ is any refinement of $D$, then

$$|\sum_{E} h(I)| < m.$$ 

For each refinement $E$ of $D$, let $S_E$ denote the set defined as follows:

$$S_E = \{ X \mid X = \sum_{E'} h(I), E' \text{ is a refinement of } E \}.$$ 

If $E$ is a refinement of $D$, then each refinement $E'$ of $E$ is also a refinement of $D$. Consequently, $S_E$ is a subset of $S_D$ for each refinement $E$ of $D$. Since the set $S_D$ is bounded above by $m$ and below by $-m$, then each set $S_E$, where $E$ is a refinement of $D$, is bounded above by $m$ and below by $-m$. Therefore, if $E$ is a refinement of $D$, then $S_E$ has a least upper bound and a greatest lower bound. For each refinement $E$ of $D$, let $G(E)$ denote the greatest lower bound of $S_E$, and let $L(E)$ denote the least upper bound of $S_E$. From the preceding discussion, the following inequality holds for each refinement $E$ of $D$:

$$-m \leq G(D) \leq G(E) \leq L(E) \leq L(D) \leq m.$$ 

Similarly, if $E$ is a refinement of $D$ and $E'$ is a refinement of $E$, then the preceding inequality may be extended to the following:

$$-m \leq G(D) \leq G(E) \leq G(E') \leq L(E') \leq L(E) \leq L(D) \leq m.$$ 

Let $X$ denote the least upper bound of the set $S$ such that

$$S = \{ x \mid x = G(E), E \text{ is a refinement of } D \}.$$ 

Similarly, let $Y$ denote the greatest lower bound of the set $\overline{S}$ such that

$$\overline{S} = \{ y \mid y = L(E), E \text{ is a refinement of } D \}.$$
It will now be shown that $X$ is the lower integral of $h$ on $V$ and $Y$ is the upper integral of $h$ on $V$.

Suppose $c$ is any positive number. Since $X$ is the least upper bound of $S$, then there must exist an element $x'$ of $S$ such that $X - c < x' \leq X$. Now $x'$ is $G(E)$ for some refinement $E$ of $D$. In order to satisfy Definition 2-3 with respect to the lower integral, it will be established that if $E'$ is any refinement of $E$, then:

1. $X - \sum_{E'} h(I) < c$, and
2. there exists a refinement $E^*$ of $E'$ such that

$$\left| \sum_{E^*} h(I) - X \right| < c.$$

Suppose $E'$ is a refinement of $E$. Since $G(E)$ is the greatest lower bound of the set $S_{E'}$ of which $\sum_{E'} h(I)$ is an element, then

$$X - c < G(E) \leq \sum_{E'} h(I).$$

This satisfies condition (1) of Definition 2-3. Since $G(E')$ is the greatest lower bound of $S_{E'}$, then there exists an element $z$ of $S_{E'}$, such that $G(E') \leq z < G(E') + c$. Now, $z$ is $\sum_{E^*} h(I)$ for some refinement $E^*$ of $E'$. Therefore,

$$X - c < G(E) < G(E') < \sum_{E^*} h(I) < G(E') + c < X + c.$$

This is sufficient to satisfy condition (2) of Definition 2-3. Hence, $X$ is the lower integral of $h$ on $V$. Similarly, it may be demonstrated that $Y$ is the upper integral of $h$ on $V$. 
Theorem 2-4. Suppose $h$ is in $R(F)$, $V$ is in $F$, and $X$ is a number. The following two statements are equivalent:

1. Each of $\int_V h(I)$ and $\int_V h(I)$ exists, and
   $$\int_V h(I) = \int_V h(I) = X.$$

2. The integral of $h$ on $V$ is $X$.

Proof. For the first part of the proof, let it be supposed that each of $\int_V h(I)$ and $\int_V h(I)$ exists and has value $X$. Let $c$ denote any positive number. Since $X$ is the upper integral of $h$ on $V$, there exists a subdivision $D_1$ of $V$ such that if $E$ is any refinement of $D_1$, then
   $$\sum_{I \in E} h(I) - X < c.$$

Similarly, since $X$ is the lower integral of $h$ on $V$, then there exists a subdivision $D_2$ of $V$ such that if $E$ is any refinement of $D_1$, then
   $$X - \sum_{I \in E} h(I) < c.$$

Let $D$ denote the greatest common refinement of $D_1$ and $D_2$. Now if $E$ is any refinement of $D$, then $E$ is also a refinement of each of $D_1$ and $D_2$. Therefore, if $E$ refines $D$, then
   $$X - c < \sum_{I \in E} h(I) < X + c,$$

which is equivalent to
   $$\left| \sum_{I \in E} h(I) - X \right| < c.$$

Therefore, by Definition 2-1, $X$ is the integral of $h$ on $V$. 

Now let it be supposed that $X$ is the integral of $h$ on $V$. Let $c$ denote any positive number. There exists a subdivision $D$ of $V$ such that if $E$ is any refinement of $D$, then

$$|\sum_{E} h(I) - X| < c.$$

Therefore, $X$ is the upper integral of $h$ on $V$, since

1. $\sum_{E} h(I) - X < c$, and

2. there exists a refinement $E'$ of $E$ such that

$$|\sum_{E'} h(I) - X| < c.$$

Any refinement of $E$ will suffice for $E'$. $X$ is the lower integral of $h$ on $V$ by similar argument.

Theorem 2-5. Suppose $h$ is in $R(F)$ and each of $V_1$ and $V_2$ is in $F$, with $V_1$ and $V_2$ mutually exclusive. If each of $\int_{V_1} h(I)$ and $\int_{V_2} h(I)$ exists, then $(\int_{V_1 \cup V_2} h(I))$ exists; and, furthermore,

$$\int_{V_1 \cup V_2} h(I) = \int_{V_1} h(I) + \int_{V_2} h(I).$$

Proof. Suppose $c$ is a positive number. In accordance with Definition 2-1, there exists a subdivision $D_1$ of $V_1$ such that if $E$ is any refinement of $D_1$, then

$$|\sum_{E} h(I) - \int_{V_1} h(I)| < c/2.$$

Similarly, there exists a subdivision $D_2$ of $V_2$ such that if $E$ is any refinement of $D_2$, then
\[ \left| \sum_{E} h(I) - v_2 \int h(I) \right| < c/2. \]

Consider the subdivision \( D \) of \( V_1 \cup V_2 \) which is formed by the union of \( D_1 \) and \( D_2 \). Suppose \( E \) is any refinement of \( D \). Then \( E_{v_1} \) is a refinement of \( D_1 \) and \( E_{v_2} \) is a refinement of \( D_2 \).

Therefore,

\[
\left| \sum_{E} h(I) - \left[ v_1 \int h(I) + v_2 \int h(I) \right] \right| =
\left| \sum_{E_{v_1}} h(I) - v_1 \int h(I) \right| + \left| \sum_{E_{v_2}} h(I) - v_2 \int h(I) \right| < c/2 + c/2 = c.
\]

Therefore, by Definition 2-1,

\[
(V_1 \cup V_2) \int h(I) = v_1 \int h(I) + v_2 \int h(I).
\]

**Theorem 2-6.** Suppose \( h \) is in \( R(F) \) and each of \( V_1 \) and \( V_2 \) is in \( F \), with \( V_1 \) and \( V_2 \) mutually exclusive. If each of \( v_1 \int h(I) \) and \( v_2 \int h(I) \) exists, then \( (V_1 \cup V_2) \int h(I) \) exists; and, furthermore,

\[
(V_1 \cup V_2) \int h(I) = v_1 \int h(I) + v_2 \int h(I).
\]

The analogous statement holds for the lower integral.

**Proof.** Suppose \( c \) is a positive number. Since \( v_1 \int h(I) \) exists, there is a subdivision \( D_1 \) of \( V_1 \) such that if \( E \) is a refinement of \( D_1 \), then
(1) $\Sigma \ h(I) - \bigvee_{1} \ h(I) < c/2$, and

(2) there exists a refinement $E'$ of $E$ such that

$$|\Sigma \ h(I) - \bigvee_{1} \ h(I)| < c/2.$$ 

Similarly, since $\bigvee_{2} \ h(I)$ exists, there is a subdivision $D_{2}$ of $V_{2}$ such that if $E$ is any refinement of $D_{2}$, then

(3) $\Sigma \ h(I) - \bigvee_{2} \ h(I) < c/2$, and

(4) there exists a refinement $E'$ of $E$ such that

$$|\Sigma \ h(I) - \bigvee_{2} \ h(I)| < c/2.$$ 

Let $D$ denote the union of $D_{1}$ and $D_{2}$. Suppose $E$ is any refinement of $D$. Then $E_{V_{1}}$ is a refinement of $D_{1}$, and $E_{V_{2}}$ is a refinement of $D_{2}$; and, consequently,

$$\Sigma \ h(I) - [\bigvee_{1} \ h(I) + \bigvee_{2} \ h(I)] =$$

$$[\Sigma \ h(I) - \bigvee_{1} \ h(I)] + [\Sigma \ h(I) - \bigvee_{2} \ h(I)] < c/2 + c/2 = c.$$ 

This satisfies condition (1) of Definition 2-3. Since $E_{V_{1}}$ is a refinement of $D_{1}$, then statement (2) above implies that there exists a refinement $E_{1}$ of $E_{V_{1}}$ such that

$$|\Sigma \ h(I) - \bigvee_{1} \ h(I)| < c/2.$$
Also, since $E_{V_2}$ is a refinement of $D_2$, then statement (4) above implies that there exists a refinement $E_2$ of $E_{V_2}$ such that

$$\left| \sum_{E_2} h(I) - \int_{V_2} h(I) \right| < c/2.$$ 

Let the union of $E_1$ and $E_2$ be denoted $E*$. $E*$ is a refinement of $E$. Therefore, there exists a refinement, namely, $E*$, of $E$ such that

$$\left| \sum_{E*} h(I) - \int_{V_1} h(I) - \int_{V_2} h(I) \right| =$$

$$\left| \sum_{E_1} h(I) - \int_{V_1} h(I) + \sum_{E_2} h(I) - \int_{V_2} h(I) \right| \leq$$

$$\left| \sum_{E_1} h(I) - \int_{V_1} h(I) \right| + \left| \sum_{E_2} h(I) - \int_{V_2} h(I) \right| < c/2 + c/2 = c.$$ 

This satisfies condition (2) of Definition 2-3, and completes the proof with respect to the upper integral. A similar procedure may be used in the case of the lower integral.

**Theorem 2-7.** Suppose $h$ is in $R(F)$ and each of $V$ and $V'$ is in $F$ with $V'$ a subset of $V$. If each of the upper and lower integrals of $h$ on $V$ exist, then each of the upper and lower integrals of $h$ on $V'$ exist.

**Proof.** The equivalence established in Theorem 2-3 implies that it is sufficient to establish only that there exists a subdivision $D$ of $V'$ such that $h$ is summation bounded on $V'$ with respect to $D$. Since each of the upper and lower integrals of $h$ on $V$ exist, there is a subdivision $D*$ of $V$ such that $h$ is
summation bounded on \( V \) with respect to \( D^* \). Therefore, there exists a positive real number \( M \) such that if \( E \) is any refinement of \( D^* \), then

\[
|\sum_{E} h(I)| \leq M.
\]

Let \( V'' \) denote the element of \( F \) such that \( V'' = V - V' \). Now, the union of \( V' \) and \( V'' \) is \( V \). Let \( E \) denote the greatest common refinement of \( D^* \) and the subdivision consisting of the two elements \( V' \) and \( V'' \). It is now asserted that \( h \) is summation bounded on \( V' \) with respect to \( E_{V'} \). Consider the positive real number \( M + |\sum_{E_{V''}} h(I)| \). Suppose \( E' \) is any refinement of \( E_{V''} \). Then the union of \( E' \) and \( E_{V''} \) is a refinement of \( D^* \); and, consequently,

\[
|\sum_{E'} h(I) + \sum_{E_{V''}} h(I)| \leq M.
\]

This implies that

\[
|\sum_{E'} h(I)| \leq M + |\sum_{E_{V''}} h(I)|.
\]

Therefore, there exists a subdivision \( E_{V'} \) of \( V' \) and a positive number, namely, \( M + |\sum_{E_{V''}} h(I)| \), such that if \( E' \) is any refinement of \( E_{V''} \), then

\[
|\sum_{E_{V'}} h(I)| < M + |\sum_{E_{V''}} h(I)|.
\]
Therefore, h is summation bounded on V' with respect to E_V; and, consequently, each of the upper and lower integrals of h on V' exists.

**Theorem 2-8.** Suppose h is in R(F) and each of V and V' is in F with V' a subset of V. If \( V \int h(I) \) exists, then \( V' \int h(I) \) exists.

**Proof.** Suppose that \( V \int h(I) \) exists. The theorem holds trivially if V' is V. Therefore, suppose V' is a proper subset of V. Since \( V \int h(I) \) exists, then each of \( V \int h(I) \) and \( V' \int h(I) \) exists; and, furthermore,

\[
V \int h(I) - V' \int h(I) = 0.
\]

In accordance with Theorem 2-7, the upper and lower integrals for the sets V' and V - V' exist. Suppose for the purpose of contradiction that \( V' \int h(I) \) does not exist. Then, applying Theorem 2-4 and Lemma 2-1,

\[
V' \int h(I) - V' \underline{\int} h(I) > 0.
\]

Applying Lemma 2-1 with respect to the set V - V',

\[
(V - V') \int h(I) - (V - V') \underline{\int} h(I) \geq 0.
\]

Using the two preceding inequalities and the additive properties for upper and lower integrals established in Theorem 2-6,

\[
V \int h(I) - V \underline{\int} h(I) = V' \int h(I) - V' \underline{\int} h(I) + (V - V') \int h(I) - (V - V') \underline{\int} h(I) > 0,
\]

a contradiction. Therefore, \( V \int h(I) \) exists.
Theorem 2-9. If \( h \) is in \( R(F) \), \( V \) is in \( F \), and \( \int_V h(I) \) exists, then

\[
\int_V |h(I) - \int_I h(J)|
\]

exists and is zero.

Proof. Suppose \( c \) is a positive number. Since \( \int_V h(I) \) exists, then there is a subdivision \( D \) of \( V \) such that if \( E \) is any refinement of \( D \), then

\[
|\int_E h(I) - \int_V h(J)| < \frac{c}{4}.
\]

Since the integral is additive, then for any refinement \( E \),

\[
|\int_E h(I) - \int_V h(J)| = |\int_E h(I) - \sum_{E} I \int_J h(J)|
\]

\[
= |\sum_{E} [h(I) - I \int_J h(J)]| < \frac{c}{4}.
\]

Now, suppose \( E \) is a refinement of \( D \). Let \( E_p \) denote the set defined as follows:

\[
E_p = \{ I \mid I \text{ is in } E, h(I) - \int_I h(J) \geq 0 \}.
\]

Similarly, let \( E_m \) denote the set defined as follows:

\[
E_m = \{ I \mid I \text{ is in } E, h(I) - \int_I h(J) < 0 \}.
\]

The union of \( E_p \) and \( E_m \) is \( E \); and, consequently,

\[
\left|\sum_{E_p} [h(I) - \int_I h(J)] + \sum_{E_m} [h(I) - \int_I h(J)]\right|
\]

\[
= \left|\sum_{E} [h(I) - I \int_J h(J)]\right| < \frac{c}{4}.
\]

Also, the following two equalities result from definition of \( E_p \) and \( E_m \),
\begin{align*}
|\sum_{E_p} [h(I) - \int h(J)]| &= \sum_{E_p} |h(I) - \int h(J)|, \text{ and} \\
|\sum_{E_m} [h(I) - \int h(J)]| &= \sum_{E_m} |h(I) - \int h(J)|.
\end{align*}

It will be shown that
\[|\sum_{E_p} [h(I) - \int h(J)]| < c/2, \text{ and } |\sum_{E_m} [h(I) - \int h(J)]| < c/2.\]

Let \(V_p\) denote the union of the sets of \(E_p\), and let \(V_m\) denote the union of the sets of \(E_m\). Since \(\int h(I)\) exists and \(V_p \cup V_m\) is \(V\), then each of \(\int h(I)\) and \(\int h(I)\) exists, and
\[\int h(I) + \int h(I) = \int h(I).\]

There exists a subdivision \(D_1\) of \(V_p\) such that if \(E_1\) is any refinement of \(D_1\), then
\[|\sum_{E_1} h(I) - \int h(J)| < c/4.\]

Let \(E'\) denote the greatest common refinement of \(D_1\) and \(E_p\).

Since \(E' \cup E_m\) is a refinement of \(D\), then
\[|\sum_{E'} h(I) + \sum_{E_m} h(I) - \int h(I)| = \left|\sum_{E'} h(I) - \int h(I)| < c/4.\]

Since \(E'\) is a refinement of \(D_1\), then
\[|\sum_{E'} h(I) - \int h(I)| < c/4.\]

Adding the two preceding inequalities,
\begin{align*}
\left| \sum_{E_m} h(I) - \nu \int_{E'} h(I) + \nu \int_{E'} h(I) \right| \\
\leq \left| \sum_{E_m} h(I) + \sum_{E_m} h(I) - \nu \int_{E'} h(I) \right| + \left| \nu \int_{E'} h(I) - \sum_{E_m} h(I) \right| \\
< \frac{c}{4} + \frac{c}{4} = \frac{c}{2}.
\end{align*}

Since the integral is an additive function,
\[-\nu \int_{E} h(I) + \nu \int_{E'} h(I) = -\nu \int_{E} h(I).\]

Therefore, substituting,
\[\left| \sum_{E_m} [h(I) - I \int h(J)] \right| = \left| \sum_{E_m} h(I) - \nu \int h(I) \right| < \frac{c}{2}.\]

A similar argument may be used to establish that
\[\left| \sum_{E_p} [h(I) - I \int h(J)] \right| < \frac{c}{2}.\]

Therefore,
\[\sum_{E} \left| h(I) - I \int h(J) \right| = \sum_{E_p} \left| h(I) - I \int h(J) \right| + \sum_{E_m} \left| h(I) - I \int h(J) \right|\]
\[= \left| \sum_{E_p} [h(I) - I \int h(J)] \right| + \left| \sum_{E_m} [h(I) - I \int h(J)] \right| < \frac{c}{2} + \frac{c}{2} = c.\]

Therefore, by Definition 2-1,
\[-\nu \int \left| h(I) - I \int h(J) \right| = 0.\]
CHAPTER III

EXISTENCE THEOREMS

The primary intention of Chapter III is to demonstrate the conditions under which certain integrals exist. Emphasis is placed upon functions which are finitely additive.

Theorem 3-1. Suppose $H$ is in $R(F)$ and $V$ is in $F$. If $m$ is a number and $\int_V h(I)$ exists, then $\int_V m[h(I)]$ exists; and, furthermore,

$$\int_V m[h(I)] = m[\int_V h(I)].$$

Proof. Suppose $c$ is a positive number. Suppose $m$ is zero. Then for all subdivisions $D$ of $V$,

$$|\sum_D m[h(I)] - m[\int_V h(I)]| = 0 < c.$$

Therefore, suppose $m$ is not zero. Then $c/m$ is a positive number; and, consequently, there exists a subdivision $D$ of $V$ such that if $E$ is any refinement of $D$, then

$$|\sum_E h(I) - \int_V h(I)| < c/|m|.$$

If both sides of the inequality are multiplied by $|m|$, then

$$|\sum_E m[h(I)] - m[\int_V h(I)]| = m|\sum_E h(I) - \int_V h(I)| < c.$$

Therefore, by Definition 2-1,

$$\int_V m[h(I)] = m[\int_V h(I)].$$
Theorem 3-2. Suppose each of \( h \) and \( k \) is in \( R(F) \) and \( V \) is in \( F \). If each of \( \int_{V} h(I) \) and \( \int_{V} k(I) \) exists, then

\[ \int_{V} [h(I) + k(I)] \text{ exists}; \text{ and, furthermore,} \]

\[ \int_{V} [h(I) + k(I)] = \int_{V} h(I) + \int_{V} k(I). \]

Proof. Suppose \( c \) is a positive number. Since \( \int_{V} h(I) \) exists, there is a subdivision \( D_1 \) of \( V \) such that if \( E \) is any of \( D_1 \), then

\[ \left| \sum_{E} h(I) - \int_{V} h(I) \right| < \frac{c}{2}. \]

Similarly, there exists a subdivision \( D_2 \) of \( V \) such that if \( E \) is any refinement of \( D_2 \), then

\[ \left| \sum_{E} k(I) - \int_{V} k(I) \right| < \frac{c}{2}. \]

Let \( D \) denote the greatest common refinement of \( D_1 \) and \( D_2 \). Now, if \( E \) is a refinement of \( D \), then \( E \) is also a refinement of each of \( D_1 \) and \( D_2 \). Therefore, suppose \( E \) is a refinement of \( D \). Then

\[ \left| \sum_{E} [h(I) + k(I)] - [\int_{V} h(I) + \int_{V} k(I)] \right| \]

\[ = \left| \sum_{E} h(I) - \int_{V} h(I) \right| + \left| \sum_{E} k(I) - \int_{V} k(I) \right| \]

\[ \leq \left| \sum_{E} h(I) - \int_{V} h(I) \right| + \left| \sum_{E} k(I) - \int_{V} k(I) \right| \]

\[ < \frac{c}{2} + \frac{c}{2} = c. \]

Therefore, by Definition 2-1,

\[ \int_{V} [h(I) + k(I)] = \int_{V} h(I) + \int_{V} k(I). \]
Lemma 3-1. Suppose $m$ and $n$ are numbers. Then

$$2mn \leq m^2 + n^2.$$ 

Proof. Let $c$ denote the number $n-m$. Then $n = m + c$; and, substituting,

$$2mn = 2m(m + c) = 2m^2 + 2mc \leq 2m^2 + 2mc + c^2 = m^2 + (m + c)^2 = m^2 + n^2.$$ 

Lemma 3-2. Suppose $a, b, k_1$, and $k_2$ are numbers. If each of $k_1$ and $k_2$ is positive, then

$$\frac{(a + b)^2}{k_1 + k_2} \leq \frac{a^2}{k_1} + \frac{b^2}{k_2}.$$ 

Proof. In accordance with Lemma 3-1,

$$2(ak_2)(bk_1) \leq (ak_2)^2 + (bk_1)^2.$$ 

Adding $k_1k_2(a^2 + b^2)$ to each side of the above inequality, the following result is obtained:

$$k_1k_2(a + b)^2 \leq (k_1 + k_2)(a^2k_2 + b^2k_1).$$

The proof is completed by dividing both sides of the preceding inequality by the positive number $k_1k_2(k_1 + k_2)$, which yields

$$\frac{(a + b)^2}{k_1 + k_2} \leq \frac{a^2}{k_1} + \frac{b^2}{k_2}.$$ 

Lemma 3-3. Suppose each of $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ is a sequence and $b_1$ is positive for each $i$ such that $1 \leq i \leq n$. Then
\[
\frac{\left( \sum_{i=1}^{n} a_i \right)^2}{\left( \sum_{i=1}^{n} b_i \right)} \leq \sum_{i=1}^{n} \frac{(a_i)^2}{b_i}.
\]

Proof by Induction. If \( n \) is 1, then the inequality holds trivially. Suppose the inequality holds for the case where \( n \) is \( k \) for some positive integer \( k \). Then applying Lemma 3-2,

\[
\frac{\left( \sum_{i=1}^{k+1} a_i \right)^2}{\left( \sum_{i=1}^{k+1} b_i \right)} = \frac{\left( \sum_{i=1}^{k} a_i + a_{k+1} \right)^2}{\left( \sum_{i=1}^{k} b_i + b_{k+1} \right)} \leq \frac{\left( \sum_{i=1}^{k} a_i \right)^2}{\left( \sum_{i=1}^{k} b_i \right)} + \frac{(a_{k+1})^2}{(b_{k+1})}.
\]

Therefore, by mathematical induction, the inequality holds for all positive integers \( n \).

Definition 3-1. Suppose each of \( a \) and \( b \) is a number. The notation \( a/b \) means:

1. \( a \) divided by \( b \) if \( b \) is not zero, and
2. zero if each of \( a \) and \( b \) is zero.

Theorem 3-3. Suppose \( h \) is in \( R_a(F) \), \( k \) is in \( R_a^+(F) \), and \( V \) is in \( F \). Suppose, furthermore, that \( h(I) \) is zero whenever \( k(I) \) is zero for each \( I \) in \( F \) such that \( I \) is a subset of \( V \). If \( D \) is a subdivision of \( V \), then for each refinement \( E \) of \( D \),

\[
\sum_{D} \left[ \frac{h(I)}{k(I)} \right]^2 \leq \sum_{E} \left[ \frac{h(J)}{k(J)} \right]^2.
\]
Proof. Let D denote an arbitrary subdivision of V. Suppose E is a refinement of D. Let I denote an arbitrary element of D. Since \( k \) is in \( R_+^a(F) \), then

\[
0 \leq k(J) \leq k(I)
\]

for each element J of E such that J is a subset of I. Suppose k(I) is zero. Then k(J) is zero for each J in \( E_I \). Also, from the hypothesis of the theorem, h is zero on \( E_I \), and h(I) is zero. Therefore, applying Definition 3-1,

\[
\frac{[h(I)]^2}{k(I)} = \sum_{E_I} \frac{[h(J)]^2}{k(J)} = 0.
\]

Suppose k(I) is not zero. Then there must exist at least one set J of \( E_I \) such that k(J) is positive. Let \( E_p \) denote the collection of sets of \( E_I \) such that k(J) is positive for each J in \( E_p \). Let \( I' \) denote the union of the sets of \( E_p \). Since k is in \( R_+^a(F) \), then k(J) is zero for each J in \( E_I \) such that J is not in \( E_p \). Also, h is zero by hypothesis for these sets. Therefore,

\[
k(I) = k(I') = \sum_{E_p} k(J) = \sum_{E_I} k(J), \quad \text{and}
\]

\[
h(I) = h(I') = \sum_{E_p} h(J) = \sum_{E_I} h(J).
\]

Each term of the form \([h(J)]^2/k(J)\) is 0 if J is in \( E_I \) but J is not in \( E_p \).
Ep is a finite set; and, therefore, Lemma 3-3 may be applied in the following manner:

\[
\left[ \frac{h(I)}{k(I)} \right]^2 = \frac{\left[ \sum_{E_I} h(J) \right]^2}{\sum_{E_I} k(J)} = \frac{\left[ \sum_{E_p} h(J) \right]^2}{\sum_{E_p} k(J)}
\]

\[
\leq \sum_{E_p} \left[ \frac{h(J)}{k(J)} \right]^2 = \sum_{E_I} \left[ \frac{h(J)}{k(J)} \right]^2.
\]

Since I was arbitrarily chosen in D, then

\[
\left[ \frac{h(I)}{k(I)} \right]^2 \leq \sum_{E_I} \left[ \frac{h(J)}{k(J)} \right]^2
\]

for each I in D. This implies that

\[
\sum_D \left[ \frac{h(I)}{k(I)} \right]^2 \leq \sum_D \sum_{E_I} \left[ \frac{h(J)}{k(J)} \right]^2 = \sum_E \left[ \frac{h(J)}{k(J)} \right]^2,
\]

which completes the proof.

**Theorem 3-4.** Suppose h is in R_a(F), k is in R^+_a(F), and V is in F. Suppose, furthermore, that h(I) is zero whenever k(I) is zero for each I in F such that I is a subset of V.

If there exists a number m such that

\[
\sum_D \left[ \frac{h(I)}{k(I)} \right]^2 \leq m
\]

for each subdivision D of V, then \( \int_V \left[ \frac{h(I)}{k(I)} \right]^2 \) exists.
Proof. Let \( B \) denote the least upper bound of the set \( S \) such that

\[
S = \left\{ X \mid X = \sum_D \frac{[h(I)]^2}{k(I)}, \text{D is a subdivision of} \ V \right\}.
\]

It will be shown that \( B \) is the integral of the function defined by \( h^2/k \) on \( V \). Let \( c \) denote any positive number. Since \( B \) is the least upper bound of \( S \), there is an element \( X \) of \( S \) such that \( B - c < X \leq B \). From the definition of \( S \),

\[
X = \sum_D \frac{[h(I)]^2}{k(I)}
\]

for some subdivision \( D \) of \( V \). Therefore, substituting,

\[
B - c < \sum_D \frac{[h(I)]^2}{k(I)} \leq B.
\]

Suppose \( E \) is any refinement of \( D \). Then by Theorem 3-3,

\[
\sum_D \frac{[h(I)]^2}{k(I)} \leq \sum_E \frac{[h(J)]^2}{k(J)}.
\]

The right hand summation of the preceding inequality is also an element of \( S \); and, consequently,

\[
\sum_E \frac{[h(J)]^2}{k(J)} \leq B.
\]

Therefore,

\[
B - c < \sum_D \frac{[h(I)]^2}{k(I)} \leq \sum_E \frac{[h(J)]^2}{k(J)} \leq B,
\]
which implies that

$$\left| \sum_{E} \frac{[h(J)]^2}{k(J)} - B \right| < c$$

for each refinement $E$ of $D$. By Definition 2-1, the integral of $h^2/k$ on $V$ exists and is $B$.

**Theorem 3-5.** Suppose each of $h$ and $k$ is in $R_b(F)$ and $q$ is in $R_a^+(F)$. If $V$ is in $F$ and each of $\int f h(I)q(I)$ and $\int f k(I)q(I)$ exists, then $\int f h(I)k(I)q(I)$ exists; and, furthermore,

$$\int h(I)k(I)q(I) = \int \frac{\int h(J)q(J) [\int k(J)q(J) - q(I)]}{q(I)}.$$

**Proof.** Suppose $f$ is in $R_b(F)$ and $\int f(I)q(I)$ exists. Suppose $I$ is in $F$ and $I$ is a subset of $V$. Since $q$ is in $R_a^+(F)$, then

$$0 \leq q(J) \leq q(I)$$

for each $J$ such that $J$ is a subset of $I$. Using this inequality, it is easily established that $\int f(J)q(J)$ is zero whenever $q(I)$ is zero. Since $f$ is in $R_b(F)$, there exists a number $m'$ such that $|f(I)| \leq m'$ for each $I$ in $F$. Let $m$ denote the positive number $m' + 1$. We easily see that

$$\left| \int f(J)q(J) \right| \leq \left| \int m[q(J)] \right| = m[q(I)].$$

for each $I$ in $F$ such that $I$ is a subset of $V$. Therefore, if $D$ is any subdivision of $V$, then
This implies by Theorem 3-4 that
\[ \int_V \left( \frac{\int f(J)q(J)}{q(I)} \right)^2 \]
exists.

Suppose \( c \) is a positive number. From Definition 2-1, there exists a subdivision \( D_1 \) of \( V \) such that if \( E \) is any refinement of \( D_1 \), then

\[ \sum_E \left[ \frac{\int f(J)q(J)}{q(I)} \right]^2 \leq \frac{m^2}{q(I)} = \sum_D \frac{m^2}{q(I)} = m^2 \left[ q(V) \right]. \]

This implies by Theorem 3-4 that
\[ \int_V \left( \frac{\int f(J)q(J)}{q(I)} \right)^2 \]
exists.

Since \( \int f(I)q(I) \) exists, Theorem 2-9 implies that there is a subdivision \( D_2 \) of \( V \) such that if \( E \) is any refinement of \( D_2 \), then

\[ \sum_E \left| f(I)q(I) - \int f(J)q(J) \right| < c/4m. \]

Therefore, if \( E \) is a refinement of \( D_2 \), then

\[ \sum_E \left[ f(I) \right]^2 q(I) - \sum_E f(I) \left[ \int f(J)q(J) \right] \]
\[ \leq \sum_E \left[ f(I) \right]^2 q(I) - f(I) \left[ \int f(J)q(J) \right] \]
\[ = \sum_E \left| f(I) \right| \left| f(I)q(I) - \int f(J)q(J) \right| \]
\[ \leq \sum_E \left| f(I)q(I) - \int f(J)q(J) \right| < c/4. \]
Also, since

\[ \left| \int_I f(J)q(J) \right| \leq m[q(I)] \]

for each I in F such that I is a subset of V, then

\[ \left| \int_I \frac{f(J)q(J)}{q(I)} \right| \leq m \]

for each I in F such that I is a subset of V. Therefore, if E is a refinement of D_2, then

\[(3) \quad \left| \sum_{E} f(I) \left[ \int_I f(J)q(J) \right] - \sum_{E} \left[ \int_I \frac{f(J)q(J)}{q(I)} \right]^2 \right| \]

\[ \leq \sum_{E} \left| f(I) \left[ \int_I f(J)q(J) \right] - \frac{\left[ \int_I f(J)q(J) \right]^2}{q(I)} \right| \]

\[ = \sum_{E} \left| \int_I \frac{f(J)q(J)}{q(I)} \right| \cdot \left| f(I)q(I) - \int_I f(J)q(J) \right| \]

\[ \leq \sum_{E} m \left| f(I)q(I) - \int_I f(J)q(J) \right| < \frac{c}{4}. \]

Therefore, using (2) and (3),

\[(4) \quad \left| \sum_{E} [f(I)]^2 q(I) - \sum_{E} \left[ \int_I \frac{f(J)q(J)}{q(I)} \right]^2 \right| < \frac{c}{2}. \]

Let D denote the greatest common refinement of D_1 and D_2.

Now if E is any refinement of D, then E is also a refinement of each of D_1 and D_2. Therefore, from (1) and (4)

\[ \left| \sum_{E} [f(I)]^2 q(I) - \int \frac{\left[ \int_I f(J)q(J) \right]^2}{q(I)} \right| < c. \]
By Definition 2-1, \(\nu \int [f(I)]^2 q(I)\) exists; and, furthermore,

\[
\nu \int [f(I)]^2 q(I) = \frac{\int [I \int f(J)q(J)]^2}{q(I)}.
\]

Since each of \(\nu \int h(I)q(I)\) and \(\nu \int h(I)q(I)\) exists, then \(\nu \int [h(I) + h(I)]q(I)\) exists. From the preceding discussion

\[
\nu \int [h(I) + k(I)]^2 q(I)\]

and \(\nu \int \left\{ \frac{[I \int h(J)q(J)]^2}{q(I)} + 2\frac{[I \int h(J)q(J)][I \int k(J)q(J)]}{q(I)} + \frac{[I \int h(J)q(J)]^2}{q(I)} \right\}.
\]

Also, from the preceding discussion,

\[
\nu \int [h(I)]^2 q(I) = \nu \int \frac{[I \int h(J)q(J)]^2}{q(I)}, \text{ and}
\]

\[
\nu \int [k(I)]^2 q(I) = \nu \int \frac{[I \int k(J)q(J)]^2}{q(I)}.
\]

With use of Theorems 3-1 and 3-2, it is easily established that

\[
\nu \int h(I)k(I)q(I)\]

and \(\nu \int \frac{[I \int h(J)q(J)][I \int k(J)q(J)]}{q(I)}\)

exist and are equal, which completes the proof.

**Theorem 3-6.** Suppose \(h\) is in \(R_b(F)\) and \(q\) is in \(R_A^+(F)\).

If \(V\) is in \(F\) and \(\nu \int h(I)q(I)\) exists, then each of \(\nu \int |h(I)|q(I)\)
and \( \int_I h(I)q(I) \) exists; and, furthermore,

\[ \int_I h(I)q(I) = \int_I h(I)q(I). \]

Proof. Suppose \( h \) is in \( R_b(F) \) and \( q \) is in \( R^+_A(F) \). Suppose, furthermore, that \( V \) is in \( F \) and \( \int_I h(I)q(I) \) exists. Since \( h \) is in \( R_b(F) \), then there exists a number \( m \) such that \( |h(I)| \leq m \) for each \( I \) in \( F \). Therefore, if \( I \) is in \( F \) and \( I \) is a subset of \( V \), then

\[ \left| \sum_D h(J)q(J) \right| \leq \sum_D |h(J)q(J)| \leq \sum_D m[q(J)] = m[q(I)]. \]

This implies that

\[ \left| \int_I h(I)q(I) \right| \leq m[q(I)] \]

for each \( I \) in \( F \) such that \( I \) is a subset of \( V \). Therefore, if \( D \) is any subdivision of \( V \), then

\[ \sum_D \left| \int_I h(J)q(J) \right| \leq \sum_D m[q(I)] = m[q(V)]. \]

Let \( B \) denote the least upper bound of the set \( S \) such that

\[ S = \{ X \mid X = \sum_D \left| \int_I h(J)q(J) \right| , \ D \text{ is a subdivision of } V \}. \]

Using Theorem 1-1, it is easily demonstrated that

\[ \int_I h(J)q(J) = B. \]

Let \( c \) denote any positive number. There exists a subdivision \( D_1 \) of \( V \) such that if \( E \) is any refinement of \( D_1 \), then

\[ (1) \quad \left| \sum_E \left| \int_I h(I)q(I) \right| - \int_I h(J)q(J) \right| < c/2. \]

Since \( \int_I h(I)q(I) \) exists, Theorem 2-9 implies there exists a
a subdivision $D_2$ of $V$ such that if $E$ is any refinement of $D_2$, then

$$
(2) \quad \left| \sum_E |h(I)|q(I) - S \right| - \sum_E \left| \int h(J)q(J) \right| \\
= \left| \sum_E |h(I)q(I)| - \sum E \left| \int h(J)q(J) \right| \right| \\
\leq \sum E \left| |h(I)q(I)| - \left| \int h(J)q(J) \right| \right| \\
\leq \sum E \left| h(I)q(I) - \int h(J)q(J) \right| < c/2.
$$

Let $D$ denote the greatest common refinement of $D_1$ and $D_2$. Then, by (1) and (2),

$$
\left| \sum_E |h(I)|q(I) - \int h(J)q(J) \right| < c,
$$

for each refinement $E$ of $D$. This satisfies Definition 2-1, and completes the proof.

**Theorem 3-7.** Suppose $h$ is in $R^+(F)$ and $q$ is in $R^+_A(F)$. Suppose, furthermore, that $V$ is in $F$ and $\int h(I)q(I)$ exists.

If there exists a positive number $p$ such that $p \leq h(I)$ for each $I$ in $F$ such that $I$ is a subset of $V$, then $\int [1/h(I)]q(I)$ exists.

**Proof.** Since $h$ is in $R^+_b$, there exists a number $m$ such that $0 \leq p \leq h(I) \leq m$ for each $I$ in $F$ such that $I$ is a subset of $V$.

Suppose $I$ is in $F$, $I$ is a subset of $V$, and $D$ is a subdivision of $I$. Then
0 \leq p[\{q(I)\}] = \sum_{D} p[q(J)] \leq \sum_{D} h(J)q(J) \leq \sum_{D} m[q(J)] = m[\{q(I)\}].

This implies that
\[ p[\{q(I)\}] \leq \int_{I} h(J)q(J) \leq m[\{q(I)\}]. \]

Therefore,
\[
\frac{q(I)}{m} = \frac{[q(I)]^2}{m[q(I)]} \leq \frac{[q(I)]^2}{\int_{I} h(J)q(J)} \leq \frac{[q(I)]^2}{p[q(I)]} = \frac{q(I)}{p}.
\]

Suppose D is any subdivision of V. Then
\[
\frac{q(V)}{m} = \sum_{D} \frac{[q(I)]^2}{m[q(I)]} \leq \sum_{D} \frac{[q(I)]^2}{\int_{I} h(J)q(J)} \leq \sum_{D} \frac{[q(I)]^2}{p[q(I)]} = \frac{q(V)}{p}.
\]

Therefore, by Theorem 3-4,
\[
\int_{I} \frac{[q(I)]^2}{h(I)q(I)}
\]
exists. Since h(I) \geq p and \int_{I} h(J)q(J) \geq p[q(I)] \geq 0 for each I in F such that I is a subset of V, then
\[
0 \leq \frac{q(I)}{h(I)\int_{I} h(J)q(J)} \leq \frac{q(I)}{p[\{q(I)\}]} \leq \frac{1}{p^2}
\]
for each I in F such that I is a subset of V. Let c denote and positive number. Since \int_{I} h(I)q(I) exists, Theorem 2-9
implies that there exists a subdivision $D_1$ of $V$ such that if $E$ is any refinement of $D_1$, then

$$E \sum_E \left| h(I)q(I) - \int_I h(I)q(I) \right| \leq \left( \frac{p^2c}{2} \right).$$

Therefore, if $E$ is a refinement of $D_1$, then

$$E \left| \sum_E \frac{[q(I)]^2}{h(J)q(J)} - \sum_E \frac{q(I)}{h(I)} \right| \leq \sum_E \left| \frac{[q(I)]^2}{h(J)q(J)} - \frac{q(I)}{h(I)} \right|$$

$$= \sum_E \left| \frac{q(I)}{h(I)h(J)q(J)} \right| \left| h(I)q(I) - \int_I h(J)q(J) \right|$$

$$\leq \sum_E \frac{1}{p^2} \left| h(I)q(I) - \int_I h(J)q(J) \right| < \frac{c}{2}.$$

Also, since

$$\int_V \int_I \frac{[q(I)]^2}{h(J)q(J)}$$

exists, Definition 2-1 implies that there exists a subdivision $D_2$ of $V$ such that if $E$ is a refinement of $D_2$, then

$$(2) \quad \left| \sum_E \frac{[q(I)]^2}{h(J)q(J)} - \frac{[q(I)]^2}{\int_I h(J)q(J)} \right| < \frac{c}{2}.$$
\[ \left| \sum_{E} \frac{q(I)}{h(I)} - \sqrt{\int_{V} \frac{[q(I)]^2}{h(J)q(J)}} \right| < c. \]

This is sufficient to satisfy Definition 2-1 in such a way that

\[ \sqrt{\int_{V} \frac{q(I)}{h(I)}} = \int_{I} \frac{[q(I)]^2}{h(J)q(J)}. \]

**Definition 3-2.** Suppose \( q \) is in \( R_{ab}(F) \). The variation of \( q \), denoted \( \text{Var}_q \), is defined as follows:

\( \text{Var}_q(I) \) is the least upper bound of the set

\[ \left\{ X \mid X = \sum_{D} |q(J)|, \text{D is a subdivision of I} \right\} \]

for each \( I \) in \( F \).

**Theorem 3-8.** Suppose \( q \) is in \( R_{ab}(F) \) and \( V \) is in \( F \).

Then

\[ \sqrt{\int_{V} |q(I)|} = \text{Var}_q(V). \]

**Proof.** Let \( S \) denote the set

\[ \left\{ X \mid X = \sum_{D} |q(I)| , \text{D is a subdivision of V} \right\}. \]

Let \( c \) denote any positive number. Since \( \text{Var}_q(V) \) is the least upper bound of the set \( S \), then there must exist an element \( X' \) of \( S \) such that

\[ 0 \leq \text{Var}_q(V) - X' < c. \]

Since \( X' \) is \( \sum_{D} q(I) \) for some subdivision \( D \) of \( V \),

\[ 0 \leq \text{Var}_q(V) - \sum_{D} |q(I)| < c. \]
Suppose $E$ is a refinement of $D$. Then \( \sum_{D} |q(I)| \) is also an element of $S$; and, consequently,

\[
\sum_{E} |q(I)| \leq \text{Var}_q(V).
\]

By Theorem 1-1,

\[
\sum_{D} |q(I)| \leq \sum_{E} q(J).
\]

Therefore,

\[
0 \leq \text{Var}_q(V) - \sum_{E} |q(I)| < c
\]

for each refinement $E$ of $D$. By Definition 2-1, it follows that

\[
\int q(I) = \text{Var}_q(V).
\]

**Definition 3-3.** Suppose $h$ is in $R(F)$. The sign function of $h$, denoted $\text{Sign}^h_I$, is defined as follows:

1. $\text{Sign}^h_I(I)$ is minus one if $h(I)$ is negative, and
2. $\text{Sign}^h_I(I)$ is one otherwise.

**Theorem 3-9.** Suppose $q$ is in $R_{ab}(F)$. If $V$ is in $F$, then $\int \text{Sign}^q_I \text{Var}_q(I)$ exists and is $q(V)$.

**Proof.** Let $c$ denote any positive number. In accordance with Theorem 3-8,

\[
\int |q(I)| = \text{Var}_q(I).
\]

Therefore, by Theorem 2-9, there exists a subdivision $D$ of $V$ such that if $E$ is any refinement of $D$, then

\[
\sum_{E} |\int q(I) - \text{Var}_q(I)| < c.
\]
Since $\text{Sign}_q(I)$ is one for each $I$ in $F$, then

$$\left| \sum_{E} \text{Sign}_q(I)|q(I)| - \sum_{E} \text{Sign}_q(I)|\text{Var}_q(I)| \right|$$

$$\leq \sum_{E} \left| \text{Sign}_q(I)|q(I)| - \text{Sign}_q(I)|\text{Var}_q(I)| \right|$$

$$= \sum_{E} \left| \text{Sign}_q(I)|q(I)| - \text{Var}_q(I) \right|$$

$$= \sum_{E} \left| q(I) - \text{Var}_q(I) \right| < c.$$  

Using the preceding inequality and the fact that $\text{Sign}_q(I)|q(I)|$ is $q(I)$ for each $I$ in $F$,

$$\left| \sum_{E} q(I) - \sum_{E} \text{Sign}_q(I)|\text{Var}_q(I)| \right| < c,$$

for each refinement $E$ of $D$. Since $\sum_{E} q(I)$ is $q(V)$ for each refinement $E$ of $D$, then

$$\left| q(V) - \sum_{E} \text{Sign}_q(I)|\text{Var}_q(I)| \right| < c.$$  

This satisfies Definition 2-1 in such a manner that 

$$\forall I \: \text{Sign}_q(I)|\text{Var}_q(I) = q(V).$$

**Theorem 3-10.** Suppose $h$ is in $R_b(F)$ and $q$ is in $R_{ab}(F)$.

If $V$ is in $F$, then $\forall I \: h(I)|q(I)$ exists if and only if

$\forall I \: h(I)\text{Sign}_q(I)|\text{Var}_q(I)$ exists; and, furthermore,

$$\forall I \: h(I)|q(I) = \forall I \: h(I)\text{Sign}_q(I)|\text{Var}_q(I).$$

**Proof.** First, let it be supposed that $\forall I \: h(I)|q(I)$ exists.

Since $h$ is in $R_b(F)$, there exists a number $m$ such that

$$|h(I)| \leq m$$

for each $I$ in $F$. By Theorem 3-9
\[ \int \text{Sign}_q(J) \text{Var}_q(F) = q(I) \]

for each I in F. Therefore, by Theorem 2-9, there exists a subdivision $D_1$ of V such that if E is any refinement of $D_1$, then

\[ \sum_{E} \left| \text{Sign}_q(I) \text{Var}_q(I) - q(I) \right| < \frac{c}{2^{(m+1)}}. \]

Therefore,

\begin{align*}
(1) & \quad \left| \sum_{E} h(I) \text{Sign}_q(I) \text{Var}_q(I) - \sum_{E} h(I) q(I) \right| \\
& \leq \sum_{E} \left| h(I) \text{Sign}_q(I) \text{Var}_q(I) - h(I) q(I) \right| \\
& = \sum_{E} \left| h(I) \right| \left| \text{Sign}_q(I) \text{Var}_q(I) - q(I) \right| \\
& < \frac{mc}{2^{(m+1)}} \leq \frac{c}{2}
\end{align*}

for each refinement E of $D_1$. Also, Definition 2-1 implies that there exists a subdivision $D_2$ of V such that if E is any refinement of $D_2$, then

\begin{align*}
(2) & \quad \left| \sum_{E} h(I) q(I) - \int h(I) q(I) \right| < \frac{c}{2}.
\end{align*}

Let D denote the greatest common refinement of $D_1$ and $D_2$. If E is any refinement of D, then statements (1) and (2) imply that

\[ \left| \sum_{E} h(I) \text{Sign}_q(I) \text{Var}_q(I) - \int h(I) q(I) \right| < c, \]

so that by Definition 2-1,

\[ \int h(I) \text{Sign}_q(I) \text{Var}_q(I) = \int h(I) q(I). \]
Now, let it be supposed that $\int h(I)\text{Sign}_q(I)\text{Var}_q(I)$ exists. Suppose $c$ is a positive number. Since $h$ is in $R_b(F)$, there exists a number $m$ such that $|h(I)| \leq m$ for each $I$ in $F$.

Theorem 3-9 implies that

$$\int \text{Sign}_q(I)\text{Var}_q(I) = q(V).$$

Again, a subdivision $D_1$ is chosen such that

$$\sum_{E} h(I)\text{Sign}_q(I)\text{Var}_q(I) - \sum_{E} h(I)q(I) < c/2$$

for each refinement $E$ of $D_1$. Since $\int h(I)\text{Sign}_q(I)\text{Var}_q(I)$ exists, then there is a subdivision $D_2$ of $V$ such that if $E$ is a refinement of $D_2$, then

$$\sum_{E} h(I)\text{Sign}_q(I)\text{Var}_q(I) - \int h(I)\text{Sign}_q(I)\text{Var}_q(I) < c/2.$$  

Now if $D$ is the greatest common refinement of $D_1$ and $D_2$, then statements (3) and (4) imply that

$$\sum_{E} h(I)q(I) - \int h(I)\text{Sign}_q(I)\text{Var}_q(I) < c$$

for each refinement $E$ of $D$. Therefore,

$$\int h(I)q(I) = \int h(I)\text{Sign}_q(I)\text{Var}_q(I),$$

which completes the proof.

**Theorem 3-11.** Suppose $h$ is in $R_b(F)$ and $q$ is in $R_{ab}(F)$.

If $V$ is in $F$, then $\int h(I)q(I)$ exists if and only if $\int h(I)\text{Var}_q(I)$ exists.
Proof. First, suppose that $\forall \int h(I)q(I)$ exists. Theorem 3-9 implies that $\forall \int \text{Sign}_q(I)\text{Var}_q(I)$ exists and is $q(V)$. Theorem 3-10 implies that $\forall \int h(I)\text{Sign}_q(I)\text{Var}_q(I)$ exists and is $\forall \int h(I)q(I)$. Therefore, by Theorem 3-5

$$\forall \int [\text{Sign}_q(I)]^2 h(I)\text{Var}_q(I)$$

exists. Since $[\text{Sign}_q(I)]^2$ is one for each $I$ in $F$, then

$$\forall \int [\text{Sign}_q(I)]^2 h(I)\text{Var}_q(I) = \forall \int h(I)\text{Var}_q(I).$$

Now suppose that $\forall \int h(I)\text{Var}_q(I)$ exists. Since $\forall \int \text{Sign}_q(I)\text{Var}_q(I)$ also exists, Theorem 3-5 implies that $\forall \int h(I)\text{Sign}_q(I)\text{Var}_q(I)$ exists; but $\forall \int h(I)\text{Sign}_q(I)\text{Var}_q(I)$ is $\forall \int h(I)q(I)$, which completes the proof.