SOME PROPERTIES OF TOPOLOGICAL SPACES

APPROVED:

\[ \text{George Copp} \]
Major Professor

\[ \text{John T. Mahaf} \]
Minor Professor

\[ \text{Robert B. Tomlous} \]
Director of the Department of Mathematics

\[ \text{Dean of the Graduate School} \]
SOME PROPERTIES OF TOPOLOGICAL SPACES

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Bayard M. Smith, Jr., B. S.
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CHAPTER I

SOME FUNDAMENTAL PROPERTIES OF TOPOLOGICAL SPACES

This thesis presents a development of some useful concepts concerning topological spaces. Most of the theorems given apply to the most general form of topological space. The first chapter introduces some fundamental definitions and theorems. Chapter II deals with functions, and Chapter III develops the important notion of connectedness. Chapters IV and V explore some relationships between sets in a topological space, while the sixth chapter provides a short discussion of compactness. The seventh chapter adds material useful in the investigation of restricted topological spaces, and the final chapter gives a brief outline of a few common topological spaces.

We assume the well-known definitions, theorems, and notations of naive set theory. Some definitions and theorems, however, which are not standard or not in frequent use will be presented. By a space we will mean a nonempty set of elements which has been defined or assumed, and by a subspace of a space $X$ we will mean a space which is a subset of $X$. When we refer to a set we will mean a subset of some space under consideration.

The intersection of a class of sets. Suppose that $\mathcal{A}$ is a class of subsets of a space $X$. The statement that $Q$ is the intersection of $\mathcal{A}$, $\bigcap \mathcal{A}$, means that $Q$ is the set such that $y$
belongs to \( Q \) if and only if \( y \) is an element of \( X \) and \( y \) is in \( A \) for every \( A \) in \( \mathcal{A} \). (By the statement that \( y \) is in \( A \) for every \( A \) in \( \mathcal{A} \), we mean that if \( A \) is in \( \mathcal{A} \), then \( y \) is in \( A \).)

The union of a class of sets. Suppose that \( \mathcal{A} \) is a class of subsets of a space \( X \). The statement that \( S \) is the union of \( \mathcal{A} \), \( \bigcup \mathcal{A} \), means that \( S \) is the set such that \( y \) belongs to \( S \) if and only if \( y \) is an element of \( X \) and \( y \) is in \( A \) for at least one set \( A \) of \( \mathcal{A} \).

**Theorem 1-1.** Suppose that \( \mathcal{A} \) is the empty class of subsets of a space \( X \). Then the intersection \( Q \) of \( \mathcal{A} \) is \( X \).

**Proof.** Clearly \( Q \) is a subset of \( X \). Suppose that \( z \) is an element of \( X \). It follows vacuously that if \( A \) is in \( \mathcal{A} \), then \( z \) is an element of \( A \). Hence \( z \) is in \( Q \). Thus \( X \) is a subset of \( Q \), and consequently \( Q \) is \( X \).

**Theorem 1-2.** Suppose that \( \mathcal{A} \) is the empty class of subsets of a space \( X \). Then the union \( S \) of \( \mathcal{A} \) is \( \emptyset \).

**Proof.** Suppose that \( S \) contains an element \( z \). Then there is a set \( A \) in \( \mathcal{A} \) with the property that \( z \) is in \( A \). But the existence of a set in \( \mathcal{A} \) is a contradiction of the hypothesis that \( \mathcal{A} \) is empty. Hence \( S \) does not contain an element; that is, \( S = \emptyset \).

A topology for \( X \). The statement that \( \mathcal{T} \) is a topology for a space \( X \) means that \( \mathcal{T} \) is a class of subsets of \( X \) for which the following two postulates hold:

(a) the union of any number of sets of \( \mathcal{T} \) is in \( \mathcal{T} \);
(b) the intersection of any finite number of sets of \( \mathcal{T} \) is in \( \mathcal{T} \).
Theorem 1-3. A class $\mathcal{T}$ of subsets of a space $X$ is a topology for $X$ if and only if $\mathcal{T}$ has the following three properties:

1. $\emptyset$ and $X$ are in $\mathcal{T}$;
2. if $\mathcal{A}$ is a subspace of $\mathcal{T}$, then the union of $\mathcal{A}$ is in $\mathcal{T}$;
3. if $\mathcal{A}$ is a finite subspace of $\mathcal{T}$, then the intersection of $\mathcal{A}$ is in $\mathcal{T}$.

Proof. Suppose $\mathcal{T}$ is a topology for $X$. By Theorem 1-1, Theorem 1-2, and Postulates (a) and (b) we see that $\emptyset$ and $X$ are in $\mathcal{T}$; so Property (1) holds. Properties (2) and (3) follow immediately from Postulates (a) and (b).

Suppose that $\mathcal{T}$ has Properties (1), (2), and (3). From Property (1), Theorem 1-1, and Theorem 1-2, we see that the intersection and union of the empty subset of $\mathcal{T}$ are in $\mathcal{T}$. Suppose $\mathcal{A}$ is a subspace of $\mathcal{T}$. By Property (2) the union of $\mathcal{A}$ is in $\mathcal{T}$; and if $\mathcal{A}$ is finite, the intersection of $\mathcal{A}$ is in $\mathcal{T}$ by Property (3). Hence Postulates (a) and (b) are satisfied.

Topological space. The statement that $(X, \mathcal{T})$ is a topological space means that $X$ is a space and $\mathcal{T}$ is a topology for $X$. By a set in $(X, \mathcal{T})$ we will mean a subset of $X$. The members of $\mathcal{T}$ will be called the open sets of $(X, \mathcal{T})$.

Discrete topology for $X$. The topology of all subsets of a space $X$ is called the discrete topology for $X$. 

Point. The statement that \( y \) is a point means that \( y \) is an element of a space.

For the remainder of this chapter we assume the existence of a topological space \((X, \mathcal{T})\). Each set referred to will either be a subset of \( X \) (represented by a capital letter) or a class of subsets of \( X \) (represented by a capital script letter). In particular we assume that each of \( S \) and \( T \) is a subset of \( X \).

**Theorem 1-4.** Suppose that \( y \) an element of \( S \) implies that there is an open set \( V \) such that \( y \) is in \( V \) and \( V \) is a subset of \( S \). Then \( S \) is an open set.

**Proof.** Let \( U \) be the union of all open subsets of \( S \). Then \( U \) is a subset of \( S \), and by Postulate (a) \( U \) is open. We note that if \( S = \emptyset \), \( S \) is open by Property (1) of Theorem 1-3. Suppose \( S \) is not empty and \( y \) is in \( S \). Then, by hypothesis, there is an open set \( V \) such that \( y \) is in \( V \) and \( V \) is a subset of \( S \), so that \( y \) is in \( U \). Hence \( S \) is a subset of \( U \). Thus \( S \) is \( U \); and since \( U \) is open, \( S \) is an open set.

Closed set. The statement that a subset \( A \) of \( X \) is closed means that the complement \( A^C \) of \( A \) is open, that is, \( A^C = X - A \) is open.

**Theorem 1-5.** \( \emptyset \) and \( X \) are both open and both closed.

**Proof.** By Property (1) in Theorem 1-3, \( X \) and \( \emptyset \) are open sets. Now \( X^C = \emptyset \) and \( \emptyset^C = X \). Thus \( \emptyset \) and \( X \) are both closed.

**Theorem 1-6.** If \( S \) is open, then \( S^C \) is closed.
Proof. Suppose $S$ is open and consider $S^C$. Since $(S^C)^C = S$ and $S$ is open, $S^C$ is closed.

Theorem 1-7. Suppose $\mathcal{A}$ is a class of subsets of $X$ and $\mathcal{B}$ is the class such that $B$ belongs to $\mathcal{B}$ if and only if $B = A^C$ for some $A$ in $\mathcal{A}$. Then the complement of the union of $\mathcal{A}$ is the intersection of $\mathcal{B}$ $(\bigcup \mathcal{A})^C = \bigcap \mathcal{B}$ and the complement of the intersection of $\mathcal{A}$ is the union of $\mathcal{B}$ $(\bigcap \mathcal{A})^C = \bigcup \mathcal{B}$.

Proof. If $\mathcal{A}$ is empty, $\mathcal{B}$ is empty; and by Theorems 1-1 and 1-2 we have $\bigcap \mathcal{A} = \emptyset = X$ and $\bigcup \mathcal{A} = \bigcup \mathcal{B} = \emptyset$. But $X^C = \emptyset$ and $\emptyset^C = X$, so $(\bigcup \mathcal{A})^C = \bigcap \mathcal{B}$ and $(\bigcap \mathcal{A})^C = \bigcup \mathcal{B}$.

Suppose $\mathcal{A}$ is not empty. Then $\mathcal{B}$ is not empty. Suppose $x$ is in the complement of the union of $\mathcal{A}$. Then $x$ is not in the union of $\mathcal{A}$, so no set of $\mathcal{A}$ contains $x$. Consequently $x$ is in each set of $\mathcal{B}$, so that $x$ is an element of the intersection of $\mathcal{B}$. Thus $(\bigcup \mathcal{A})^C$ is a subset of $\bigcap \mathcal{B}$. Suppose next that $y$ is in the intersection of $\mathcal{B}$. Then $y$ is in each set of $\mathcal{B}$, so $y$ is not in any set of $\mathcal{A}$. Thus $y$ is not in the union of $\mathcal{A}$. Therefore $y$ is in the complement of the union of $\mathcal{A}$. It follows that $\bigcap \mathcal{B}$ is a subset of $(\bigcup \mathcal{A})^C$. Hence $(\bigcup \mathcal{A})^C = \bigcap \mathcal{B}$.

Suppose $z$ is in the complement of the intersection of $\mathcal{A}$. Then $z$ is not in the intersection of $\mathcal{A}$. Therefore there is a set $A$ of $\mathcal{A}$ which does not contain $z$ as an element. Now $z$ not in $A$ implies that $z$ is in $A^C$. But $A^C$ is in $\mathcal{B}$, so $z$ is in the union of $\mathcal{B}$. Thus $(\bigcap \mathcal{A})^C$ is a subset of $\bigcup \mathcal{B}$. Suppose $w$ is an element of the union of $\mathcal{B}$. Then $w$ is in some set $B$ of
Consequently \( w \) does not belong to \( B^c \) and is not in the intersection of \( A \). Hence \( w \) is contained as an element by the complement of the intersection of \( A \). Thus \( \cup A \) is a subset of \( (\bigcap A)^c \). Therefore \( (\bigcap A)^c = \bigcup A \) and the theorem is established.

**Theorem 1-8.** Any intersection of closed sets is a closed set.

**Proof.** Suppose that \( A \) is a class of closed sets. Consider the class \( B \) of all sets which are complements of the sets of \( A \). Since each set of \( A \) is closed, each set of \( B \) is open. Thus the union of \( B \) is an open set by Postulate (a). Now by Theorem 1-7, \( (\bigcap A)^c = \bigcup B \). Hence \( (\bigcap A)^c \) is open, and \( [(\bigcap A)^c]^c = \bigcap A \) is closed.

**Theorem 1-9.** Any finite union of closed sets is closed.

**Proof.** Suppose that \( A \) is a finite class of closed sets. Consider the class \( B \) of all sets which are complements of the sets of \( A \). Since each set of \( A \) is closed, each set of \( B \) is open; and since \( A \) is finite, \( B \) is also finite. Hence by Postulate (b), the intersection of \( B \) is open. But from Theorem 1-7 we have \( (\bigcup A)^c = \bigcap B \). Therefore \( (\bigcup A)^c \) is open and \( [(\bigcup A)^c]^c = \bigcup A \) is closed.

**Accumulation point.** The statement that \( y \) is an accumulation point of \( S \) means that \( y \) is a point of \( X \) such that if \( V \) is an open set containing \( y \), then there is a point \( z \) in \( S \) such that \( z \) is in \( V \) and \( z \neq y \).
Neighborhood of a point. The statement that \( V \) is a neighborhood of a point \( y \) means that \( V \) is an open set containing \( y \).

Derived set. The set \( S' \) of all accumulation points of \( S \) is called the derived set (or the derivative) of \( S \).

Closure of \( S \). The set \( \overline{S} = S \cup S' \) is called the closure of \( S \).

**Theorem 1-10.** \( S \) is closed if and only if \( S' \) is a subset of \( S \).

**Proof.** Suppose \( S \) is closed. Then \( S^c \) is an open set containing no point of \( S \), so that \( S^c \) does not contain an accumulation point of \( S \). Hence \( S' \) is a subset of \( S \).

Suppose \( S' \) is a subset of \( S \). Then \( S^c \cap S' = \emptyset \). Now suppose that \( y \) is in \( S^c \). Then \( y \) is not in \( S' \). Hence there is an open set \( V \) such that \( y \) is in \( V \) and \( V \cap S \) does not contain a point \( z \) such that \( z \neq y \). But since \( y \) is an element of \( S^c \), \( y \) is not in \( S \). Therefore \( V \cap S = \emptyset \), so that \( V \) is a subset of \( S^c \). Hence, by Theorem 1-4, \( S^c \) is open. Thus \( S \) is closed.

**Theorem 1-11.** A point \( y \) is in \( \overline{S} \) if and only if each open set \( V \) which contains \( y \) is such that \( V \cap S \neq \emptyset \).

**Proof.** Suppose \( y \) is in \( \overline{S} \). Then \( y \) is in \( S \) or \( y \) is in \( S' \). Suppose \( V \) is an open set containing \( y \). Then if \( y \) is in \( S' \), \( V \cap S \neq \emptyset \) by definition of \( S' \); and if \( y \) is in \( S \), \( y \) is an element of \( V \cap S \), so \( V \cap S \neq \emptyset \).
Suppose $y$ is a point such that if $V$ is an open set containing $y$, then $V \cap S \neq \emptyset$. Suppose $z$ is in $(\overline{S})^c$. Then $z$ is not a point of $S$ and not a point of $S'$. Now since $z$ is not in $S'$, there is a neighborhood $U$ of $z$ with the property that $U \cap S$ does not contain a point $w$ such that $w \neq z$; and since $z$ is not in $S$, we have $U \cap S = \emptyset$. Hence $y$ is not in $(\overline{S})^c$. Thus $y$ is a point of $\overline{S}$.

**Theorem 1-12.** $(\overline{S})'$ is a subset of $\overline{S}$.

**Proof.** Suppose $y$ is in $(\overline{S})'$ and $V$ is a neighborhood of $y$. Then $V \cap \overline{S}$ contains a point $z$ such that $z \neq y$. Now $z$ in $\overline{S}$ implies that $z$ is in $S'$ or $z$ is in $S$. If $z$ is a point of $S$, $V \cap S$ contains $z$, so $V \cap S \neq \emptyset$. Suppose $z$ is a point of $S'$. Then $V \cap S$ contains a point $w$ such that $w \neq z$, so $V \cap S \neq \emptyset$.

By Theorem 1-11, then, $y$ is in $\overline{S}$. Therefore $(\overline{S})'$ is a subset of $\overline{S}$.

**Corollary 1-13.** $\overline{S}$ is closed.

**Proof.** Since $(\overline{S})'$ is a subset of $\overline{S}$, it follows from Theorem 1-10 that $\overline{S}$ is closed.

**Theorem 1-14.** $S$ is closed if and only if $S = \overline{S}$.

**Proof.** Suppose $S$ is closed. Then, by Theorem 1-10, $S'$ is a subset of $S$. Hence $S = S \cup S'$, that is, $S = \overline{S}$.

Suppose $S = \overline{S}$. Then $S = S \cup S'$, so $S'$ is a subset of $S$. Thus, by Theorem 1-10, $S$ is closed.

**Theorem 1-15.** $\overline{S}$ is the intersection of all closed sets containing $S$ as a subset.
Proof. Consider the class $\mathcal{A}$ of all closed set containing $S$ as a subset and the intersection $G$ of $\mathcal{A}$. By Theorem 1-8 $G$ is closed, and by Theorem 1-14 $G = \overline{G}$. Now, since $\overline{S}$ is a closed set containing $S$, $G$ is a subset of $\overline{S}$. Suppose $y$ is in $\overline{S}$. Then $y$ is either in $S$ or in $S'$. If $y$ is in $S$ it is also in $G$, since $S$ is contained in $G$. Suppose $y$ is a point of $S'$ and $Q$ is a neighborhood of $y$. Then $Q \cap S$ contains a point $z$ such that $z \neq y$. But since $S$ is a subset of $G$, $z$ is also in $Q \cap G$. Therefore $y$ is in $G'$, that is, $y$ is a point of $\overline{G}$. Then, since $G = \overline{G}$, $\overline{S}$ is a subset of $G$. Consequently $\overline{S} = G$ and the theorem is established.

Theorem 1-16. If $S$ is a subset of $T$, then $S'$ is a subset of $T'$.

Proof. Suppose $S$ is a subset of $T$, $y$ is a point of $S'$, and $V$ is a neighborhood of $y$. Then $V \cap S$ contains a point $z$ such that $z \neq y$. But $S$ a subset of $T$ implies that $V \cap S$ is a subset of $V \cap T$, so $z$ is in $T$. Therefore $y$ is an accumulation point of $T$, and it follows that $S'$ is a subset of $T'$.

Theorem 1-17. If $S$ is a subset of $T$, then $\overline{S}$ is a subset of $\overline{T}$.

Proof. Suppose $S$ is a subset of $T$. Then, by Theorem 1-16, $S'$ is a subset of $T'$. Hence $S \cup S'$ is contained in $T \cup T'$, that is, $\overline{S}$ is a subset of $\overline{T}$.

Theorem 1-18. If $S$ is a subset of $T$ and $T$ is closed, then $\overline{S}$ is a subset of $T$. 
**Proof.** Suppose $S$ is a subset of $T$ and $T$ is closed. By Theorem 1-17 $\overline{S}$ is a subset of $\overline{T}$. But $T$ closed implies that $T = \overline{T}$. Hence $\overline{S}$ is a subset of $T$.

**Theorem 1-19.** $(S \cup T)' = S' \cup T'$.

**Proof.** Suppose $x$ is a point of $(S \cup T)'$. Then $x$ is not a point of $S'$ and not a point of $T'$. Hence there is a neighborhood $U$ of $x$ and a neighborhood $V$ of $x$ such that neither of $U \cap S$ and $V \cap T$ contains a point which is not $x$. Consider $A = U \cap V$, which is also a neighborhood of $x$. $A \cap S$ is contained in $U \cap S$ and $A \cap T$ is contained in $V \cap T$. Hence neither of $A \cap S$ and $A \cap T$ contains a point other than possibly $x$. Consequently $A \cap (S \cup T) = (A \cap S) \cup (A \cap T)$ contains no point which is not $x$. Thus $x$ is not in $(S \cup T)'$, that is, $x$ is in $[(S \cup T)']^C$. Therefore $(S \cup T)'^C$ is a subset of $[(S \cup T)']^C$, or equivalently, $(S \cup T)'$ is a subset of $S' \cup T'$.

Suppose $y$ is in $S'$. Then there is a neighborhood $W$ of $y$ such that $W \cap S$ contains a point $z$ which is not $y$. But $W \cap S$ is contained in $W \cap (S \cup T)$. Hence $y$ is an accumulation point of $S \cup T$. Similarly if $w$ in $T$ implies that $w$ is an accumulation point of $S \cup T$. Thus $S' \cup T'$ is a subset of $(S \cup T)'$. The theorem follows.

**Corollary 1-20.** $S \cup T = S \cup T'$.

**Proof.** Using Theorem 1-19 we have

$$S \cup T = (S \cup T) \cup (S \cup T)'$$
$$= (S \cup T) \cup (S' \cup T')$$
$$= (S \cup S') \cup (T \cup T')$$
$$= S \cup T'.$$
The interior of $S$. The interior $S^\circ$ of $S$ is defined by

$$S^\circ = \{ y \mid \text{there is a neighborhood } V \text{ of } y \text{ such that } V \subseteq S \}.$$

Theorem 1-21. $S^\circ$ is the union of all open subsets of $S$.

Proof. Consider the class $\mathcal{A}$ of all open subsets of $S$. Since each set of $\mathcal{A}$ is open, the union $G$ of $\mathcal{A}$ is open. Suppose $y$ is a point of $S^\circ$. Then there is an open set $V$ such that $V$ contains $y$ and $V$ is a subset of $S$, so $y$ is in $G$; that is, $S^\circ$ is a subset of $G$. Suppose $z$ is a point of $G$. Then, by definition of $G$, $z$ is in some open subset $U$ of $S$. Hence $z$ is a point of $S^\circ$, so $G$ is a subset of $S$. Therefore $S^\circ = G$ and the theorem is established.

Corollary 1-22. The interior of $S$ is open.

Proof. The corollary follows immediately from Theorem 1-21 and Postulate (a).

Theorem 1-23. $S$ is open if and only if $S = S^\circ$.

Proof. If $S = S^\circ$, then $S$ is open by Corollary 1-22. Suppose $S$ is open. Then by Theorem 1-21, $S$ is a subset of $S^\circ$. But from the definition of $S^\circ$ we see that $S^\circ$ is a subset of $S$. Hence $S = S^\circ$ and we are through.

The boundary of $S$. The boundary $\partial(S)$ of $S$ is given by

$$\partial(S) = \overline{S} \cap \overline{S^c}.$$

Theorem 1-24. $X = S^\circ \cup \partial(S) \cup (S^c)^\circ$.

Proof. Let $A = S^\circ \cup \partial(S) \cup (S^c)^\circ$. Clearly $A$ is a subset of $X$, so the theorem will follow if $X$ is a subset of $A$. Suppose $y$ is a point of $X$. If $y$ is in $S^\circ \cup (S^c)^\circ$ we are through. Suppose $y$ is not in $S^\circ \cup (S^c)^\circ$. Then $y$ is not in $S^\circ$.
and not in \((S^c)^0\). Suppose \(V\) is an open set containing \(y\) (\(X\) is one such set). Since \(y\) is not in \(S^0\), \(V\) is not a subset of \(S\), so \(V \cap S^c \neq \emptyset\); and since \(y\) is not in \((S^c)^0\), \(V\) is not a subset of \(S^c\), so \(V \cap S^c \neq \emptyset\). Hence, by Theorem 1-11, \(y\) is in each of \(S^c\) and \(S\); that is, \(y\) is in \(\mathcal{B}(S)\). Thus \(y\) is in \(A\), and \(X\) is therefore a subset of \(A\).

**Theorem 1-25.** \(S^0 = S - \mathcal{B}(S)\).

**Proof.** Suppose \(y\) is in \(S\) but not in \(\mathcal{B}(S)\). Since \(y\) is in \(S\), \(y\) is not in \(S^c\). But since \((S^c)^0\) is a subset of \(S^c\), \(y\) not in \(S^c\) implies that \(y\) is not a point of \((S^c)^0\). Hence we have \(y\) in \(S\) but not in \(\mathcal{B}(S)\) or \((S^c)^0\). Thus from Theorem 1-24 we see that \(y\) is a point of \(S^0\). Therefore \(S - \mathcal{B}(S)\) is a subset of \(S^0\).

Suppose \(z\) is a point of \(S^0\). Then \(z\) is in \(S\) and there is a neighborhood \(V\) of \(z\) such that \(V\) is contained in \(S\). But \(V\) a subset of \(S\) implies that \(V \cap S^c = \emptyset\). Hence \(z\) is not an accumulation point of \(S^c\) and not a point of \(S^c\), so \(z\) is not an element of \(S^c\). Thus \(z\) cannot be in \(\mathcal{B}(S)\), and \(S^0\) is therefore a subset of \(S - \mathcal{B}(S)\). Consequently \(S^0 = S - \mathcal{B}(S)\).

**Corollary 1-26.** \(S^0 = S \cap [\mathcal{B}(S)]^c\).

**Proof.** Since \(S - \mathcal{B}(S)\) is \(S \cap [\mathcal{B}(S)]^c\), it follows from Theorem 1-25 that \(S^0 = S \cap [\mathcal{B}(S)]^c\).

**Theorem 1-27.** \(\mathcal{B}(S) = [S \cap (S^c)^\prime] \cup [S^c \cap S^\prime] \) (2, p. 78).

**Proof.** By definition of the boundary of \(S\) we have \(\mathcal{B}(S) = \overline{S} \cap \overline{S^c}\). Now \(X = S \cup S^c\) and \(\mathcal{B}(S) = X \cap \mathcal{B}(S)\). Hence we have
\[ \mathcal{C}(S) = [S \cup S^c] \cap [\overline{S} \cap \overline{S^c}] \]

\[ = \{S \cap [\overline{S} \cap \overline{S^c}]\} \cup \{S^c \cap [\overline{S} \cap \overline{S^c}]\} \]

\[ = \{[S \cap \overline{S}] \cap \overline{S^c}\} \cup \{[S^c \cap \overline{S}] \cap \overline{S^c}\} \]

\[ = \{S \cap \overline{S^c}\} \cup \{S^c \cap \overline{S}\} \]

\[ = \{S \cap [S^c \cup (S^c)'\}] \cup \{S^c \cap [S \cup S']\} \]

\[ = \{[S \cap S^c] \cup [S \cap (S^c)']\} \cup \{[S^c \cap S] \cup [S^c \cap S']\} \]

\[ = \emptyset \cup [S \cap (S^c)'] \cup \emptyset \cup [S^c \cap S'] \]

\[ = [S \cap (S^c)'] \cup [S^c \cap S']. \]

**Theorem 1-28.** \( \mathcal{C}(S) \) is closed.

**Proof.** By Corollary 1-13 \( \mathcal{C}(S) \) is the intersection of two closed sets. It follows, then, from Theorem 1-8, that \( \mathcal{C}(S) \) is closed.

**S dense in \( T \).** The statement that \( S \) is dense in \( T \) means that \( T \) is a subset of \( \overline{S} \).

**Theorem 1-29.** \( S \) is dense in \( X \) if and only if \( X = \overline{S} \).

**Proof.** Suppose that \( S \) is dense in \( X \). Then \( X \) is a subset of \( \overline{S} \). But \( \overline{S} \) is also a subset of \( X \), so \( X = \overline{S} \). Also if \( X = \overline{S} \), it is true that \( X \) is a subset of \( \overline{S} \), and the proof is complete.

**Theorem 1-30.** \( (\overline{S})^C = (S^C)^C \).

**Proof.** Consider the class \( \mathcal{C} \) of all closed sets which contain \( S \) and the class \( \mathcal{A} \) of all open subsets of \( S^C \). From Theorems 1-15 and 1-21 we see that the intersection of \( \mathcal{C} \) is \( \overline{S} \) and the union of \( \mathcal{A} \) is \( (S^C)^C \).

We show next that each set of \( \mathcal{C} \) is the complement of a set in \( \mathcal{A} \) and each set of \( \mathcal{A} \) is the complement of a set in \( \mathcal{C} \).
Suppose first that $B$ is in $\mathbb{A}$, and consider $B^c$. Since $S$ is a subset of $B$, $B^c$ is a subset of $S^c$; and since $B$ is closed, $B^c$ is open. Hence $B^c$ is in $\mathbb{A}$. Now suppose that $A$ is a set in $\mathbb{A}$. Consider $A^c$. Since $A$ is a subset of $S^c$, $(S^c)^c = S$ is a subset of $A^c$; and since $A$ is open, $A^c$ is closed. Hence $A^c$ is in $\mathbb{B}$.

Now we can use Theorem 1-7 to write
$$\overline{S} = \bigcap \mathbb{B} = (\bigcup \mathbb{A})^c.$$ 
Thus
$$(S^c)^c = \bigcup \mathbb{A} = (S^c)^0,$$
and the proof is completed.

**Corollary 1-31.** $\overline{S} \cap (S^c)^0 = \emptyset$.

**Proof.** Since $\overline{S}$ is the complement of $(S^c)^0$, $\overline{S}$ and $(S^c)^0$ have no point in common; that is, $\overline{S} \cap (S^c)^0 = \emptyset$.

**Corollary 1-32.** $\overline{S} = X$ if and only if $(S^c)^0 = \emptyset$.

**Proof.** Suppose $\overline{S} = X$. Then $(S^c)^c = X^c = \emptyset$. But $(S^c)^c$ is $(S^c)^0$ by Theorem 1-30, so $(S^c)^0 = \emptyset$.

Suppose $(S^c)^0 = \emptyset$. Then $[(S^c)^0]^c = \emptyset^c = X$. But $[(S^c)^0]^c = [(S^c)^c]^c$, so that $\overline{S} = X$.

**Corollary 1-33.** $S$ is dense in $X$ if and only if $(S^c)^0 = \emptyset$.

**Proof.** The corollary follows immediately from Corollary 1-32 and Theorem 1-29.

$S$ **dense-in-itself.** The statement that $S$ is dense-in-itself means that $S$ is contained in $S'$.

$S$ **perfect.** The statement that $S$ is perfect means that $S$ is closed and dense-in-itself.
Theorem 1-34. \( S \) is perfect if and only if \( S = S' \).

Proof. \( S = S' \) if and only if \( S \) is a subset of \( S' \) and \( S' \) is a subset of \( S \). But \( S \) a subset of \( S' \) means that \( S \) is dense-in-itself, and \( S' \) is a subset of \( S \) if and only if \( S \) is closed. Thus \( S \) is closed and dense-in-itself if and only if \( S = S' \), that is, \( S \) is perfect if and only if \( S = S' \).

Theorem 1-35. The closure of a set \( E \) which is dense-in-itself is perfect.

Proof. Since \( E \) is dense-in-itself, \( E \) is contained in \( E' \). Hence we have

\[
(1) \quad E \cup E' = E',
\]

But (1) can be written as

\[
(2) \quad \overline{E} = E'.
\]

Also \( E \) is a subset of \( 
\overline{E} \), so that

\[
(3) \quad E' \subseteq (\overline{E})'.
\]

Then using (2) and (3) we have \( \overline{E} \subseteq (\overline{E})' \). Hence \( \overline{E} \) is dense-in-itself. Then, since \( \overline{E} \) is closed, \( \overline{E} \) is perfect.

Theorem 1-36. Suppose \( \mathcal{A} \) is a class of sets each of which is dense-in-itself. Then the union \( U \) of \( \mathcal{A} \) is dense-in-itself.

Proof. Let \( \mathcal{B} \) denote the class of sets such that \( B \) is in \( \mathcal{B} \) if and only if \( B = A' \) for some \( A \) in \( \mathcal{A} \). Consider the union of \( \mathcal{B} \). If \( p \) is in the union of \( \mathcal{B} \) and \( V \) is a neighborhood of \( p \), then for some \( A \) of \( \mathcal{A} \), \( V \cap A \) contains a point \( x \) such that \( x \neq p \). But \( x \) in \( A \) implies that \( x \) is in \( U \). Hence \( p \) is a point of \( U' \).
Suppose \( q \) is a point which is not in \( U' \). Then there is a neighborhood \( W \) of \( q \) such that \( W \cap U \) contains no point of \( U \) which is not \( q \). Hence \( q \) is not in \( A' \) for any \( A \) in \( \mathcal{A} \). Consequently \( q \) is not in the union of \( \mathcal{B} \). Therefore \( U' \) is the union of \( \mathcal{B} \).

Consider \( \mathcal{A} \). Since every set in \( \mathcal{A} \) is contained in some set of \( \mathcal{B} \), the union \( U \) of \( \mathcal{A} \) is a subset of the union \( U' \) of \( \mathcal{B} \); that is, \( U \) is dense-in-itself.

Theorem 1-37. Suppose \( S \) is contained in \( T \) and \( T \) is contained in \( S' \). Then \( T \) is dense-in-itself.

Proof. Since \( S \) is contained in \( T \), \( S' \) is a subset of \( T' \). But \( T \) is a subset of \( S' \). Hence \( T \) is contained in \( T' \), that is, \( T \) is dense-in-itself.

Corollary 1-38. If \( S \) is dense-in-itself, then \( S' \) is dense-in-itself.

Proof. Suppose \( S \) is dense-in-itself. Then \( S \) is contained in \( S' \). But \( S' \) is a subset of itself. Therefore, by Theorem 1-37, \( S' \) is dense-in-itself.

Nucleus of \( S \). The union \( N \) of all subsets of \( S \) which are dense-in-themselves is called the nucleus of \( S \) (1, p. 13).

Remark. By Theorem 1-36, the nucleus of a set is dense-in-itself. Also the nucleus \( N \) of \( S \) is the largest subset of \( S \) which is dense-in-itself, in the sense that any subset of \( S \) which is dense-in-itself is contained in \( N \).

Scattered set. A set is said to be scattered if and only if its nucleus is empty.
Theorem 1-39. Suppose $N$ is the nucleus of $S$. Then $S$ can be expressed as the union of $N$ and a scattered set $Q$.

Proof. Denote $S - N$ as $Q$. Then since $N$ is a subset of $S$, $S = N \cup Q$. Suppose, by way of contradiction, that $Q$ is not scattered. Then there is a subset of $Q$ which is dense-in-itself and therefore contained in $N$, which is contrary to the definition of $Q$. Hence $Q$ is scattered and the theorem follows.

CHAPTER II

FUNCTIONS

Cartesian product. Suppose that each of $X$ and $Y$ is a space. We define the cartesian product $X \times Y$ of $X$ by $Y$ to be the set of all ordered pairs $(x,y)$ such that $x$ is in $X$ and $y$ is in $Y$.

Relation. The statement that $g$ is a relation means that, for some space $X$ and some space $Y$, $g$ is a subspace of $X \times Y$.

Function. The statement that $f$ is a function means that $f$ is a relation such that no two ordered pairs of $f$ have the same first element.

Domain and range. Suppose that $g$ is a relation. The statement that $\mathcal{D}$ is the domain of $g$ means that $\mathcal{D}$ is the set such that $x$ belongs to $\mathcal{D}$ if and only if there is an ordered pair in $g$ having $x$ as its first element; and the statement that $\mathcal{R}$ is the range of $g$ means that $\mathcal{R}$ is the set such that $y$ belongs to $\mathcal{R}$ if and only if there is an ordered pair in $g$ having $y$ as its second element.

Inverse relation. Suppose $g$ is a relation. Then the statement that $g^{-1}$ is the inverse of the relation $g$ means that $g^{-1}$ is the set of all $(y,x)$ such that $(x,y)$ is in $g$. 

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**Reversible function.** The statement that \( f \) is a reversible function means that each of \( f \) and \( f^{-1} \) is a function.

**Mapping of \( A \) into \( B \).** The statement that \( f \) is a mapping from a set \( A \) into a set \( B \) means that \( f \) is a function with domain \( A \) and range a subset of \( B \).

**Mapping of \( A \) onto \( B \).** The statement that \( f \) is a mapping from \( A \) onto \( B \) means that \( f \) is a mapping of \( A \) into \( B \) and \( B \) is the range of \( f \).

**One-to-one mapping of \( A \) into \( B \).** The statement that \( f \) is a one-to-one mapping of \( A \) into \( B \) means that \( f \) is a reversible function with domain \( A \) and range a subset of \( B \).

**One-to-one mapping of \( A \) onto \( B \).** The statement that \( f \) is a one-to-one mapping of \( A \) onto \( B \) means that \( f \) is a one-to-one mapping of \( A \) into \( B \) having \( B \) as its range.

**Images.** Suppose \( f \) is a function with domain \( \mathcal{D} \) and range \( \mathcal{E} \). We use the notation \( y = f(x) \) to designate that \((x,y)\) is in \( f \), and we call \( y \) the image of \( x \) under \( f \). If \( S \) is a subset of \( \mathcal{D} \) we define \( f(S) \) to be the set of all \( f(x) \) such that \( x \) is in \( S \) and refer to \( f(S) \) as the image of \( S \) under \( f \) (1, p.70).

**Remark.** In particular we have \( \mathcal{E} = f(\mathcal{D}) \) and \( f(\emptyset) = \emptyset \).

**Inverse images.** Suppose \( f \) is a function with domain \( \mathcal{D} \) and range \( \mathcal{E} \). Then if \( T \) is a subset of \( \mathcal{E} \) we define \( f^{-1}(T) \) to be the set of all \( x \) such that \( x \) is in \( \mathcal{D} \) and \( f(x) \) is in \( T \), and we call \( f^{-1}(T) \) the inverse image of \( T \) under \( f \).
Remark. We may have \( f^{-1}(T) \) defined when \( f^{-1} \) is not a function.

Throughout the remainder of this chapter we assume that each of \( (X, \mathcal{T}) \) and \( (Y, \mathcal{U}) \) is a topological space and \( f \) is a mapping of \( X \) onto \( Y \).

**Continuity.** The statement that \( f \) is continuous with respect to \( \mathcal{T} \) and \( \mathcal{U} \) at a point \( x_0 \) of \( X \) means that if \( U \) is a neighborhood of \( f(x_0) \) in \( (Y, \mathcal{U}) \), then there is a neighborhood of \( V \) of \( x_0 \) in \( (X, \mathcal{T}) \) such that \( f(V) \) is a subset of \( U \). If \( f \) is continuous with respect to \( \mathcal{T} \) and \( \mathcal{U} \) at each point of \( X \), \( f \) is said to be continuous with respect to \( \mathcal{T} \) and \( \mathcal{U} \). For convenience we will say that \( f \) is continuous at \( x_0 \) or \( f \) is continuous, the dependence on \( \mathcal{T} \) and \( \mathcal{U} \) being understood.

Theorem 2-1. The function \( f \) is continuous if and only if \( S \) open in \( (Y, \mathcal{U}) \) implies that \( f^{-1}(S) \) is open in \( (X, \mathcal{T}) \).

**Proof.** Suppose that if \( S \) is an open set in \( (Y, \mathcal{U}) \), then \( f^{-1}(S) \) is open in \( (X, \mathcal{T}) \). Assume also that \( x \) is a point of \( X \) and \( U \) is a neighborhood of \( f(x) \). Then \( f^{-1}(U) \) is open by hypothesis. And since \( x \) is in \( f^{-1}(U) \), \( f^{-1}(U) \) is a neighborhood of \( x \). But \( f[f^{-1}(U)] = U \), so that \( f[f^{-1}(U)] \) is a subset of \( U \). Thus \( f \) is continuous at \( x \), and it follows that \( f \) is continuous.

Suppose that \( f \) is continuous and \( S \) is an open set in \( (Y, \mathcal{U}) \). If \( S = \emptyset \) we have \( f^{-1}(S) = \emptyset \), which is open.
Suppose \( S \neq \emptyset \). Then \( f^{-1}(S) \) is not empty. Consider any point \( z \) in \( f^{-1}(S) \). Since \( S \) is open, \( S \) is a neighborhood of \( f(z) \); and since \( f \) is continuous at each point of \( X \), there is a neighborhood \( V \) of \( z \) such that \( f(V) \) is a subset of \( S \). But \( f(V) \) a subset of \( S \) implies that \( V \) is a subset of \( f^{-1}(S) \). Hence, by Theorem 1-4, \( f^{-1}(S) \) is an open set in \((X, \mathcal{T})\). The theorem follows.

**Theorem 2-2.** Suppose \( A \) is a subset of \( Y \). Then \( f^{-1}(A^c) \) is \([f^{-1}(A)]^c\).

**Proof.** Suppose \( x \) is in \( f^{-1}(A^c) \). Then there is a point \( p \) of \( A^c \) such that \( f(x) = p \). Suppose, by way of contradiction, that \( x \) is in \( f^{-1}(A) \). Then there is a point \( q \) of \( A \) such that \( f(x) = q \). But \( p \neq q \) since \( p \) is in \( A^c \) and \( q \) is in \( A \), which contradicts the assumption that \( f \) is a function. Hence \( x \) is in \([f^{-1}(A)]^c\) and \( f^{-1}(A^c) \) is a subset of \([f^{-1}(A)]^c\).

Suppose \( z \) is in \([f^{-1}(A)]^c\). Now there is a point \( t \) of \( Y \) such that \( f(z) = t \). But \( t \) is not in \( A \); for \( t \) in \( A \) implies that \( z \) is in \( f^{-1}(A) \), which contradicts the assumption that \( z \) is in \([f^{-1}(A)]^c\). Hence \( t \) is in \( A^c \). Consequently \( t \) is in \( A^c \), so \( z \) is in \( f^{-1}(A^c) \) and \([f^{-1}(A)]^c\) is a subset of \( f^{-1}(A^c) \). It follows that \( f^{-1}(A^c) = [f^{-1}(A)]^c\).

**Homeomorphism.** The statement that \( f \) is a homeomorphism or a homeomorphic mapping means that \( f \) is one-to-one and each of \( f \) and \( f^{-1} \) is continuous. Under these conditions \((X, \mathcal{T})\) is said to be homeomorphic to \((Y, \mathcal{G})\) and \((X, \mathcal{T})\) and \((Y, \mathcal{G})\) are said to be homeomorphic.
Theorem 2-3. Suppose that $f$ is continuous and one-to-one and $S$ is a subset of $X$. Then $f(S') = [f(S)]'$. 

Proof. Suppose $z$ is a point of $f(S')$ and $V$ is a neighborhood of $z$. Since $f$ is one-to-one, $f^{-1}$ is a function and $f^{-1}(z)$ is a uniquely determined point in $f^{-1}(V)$. Then since $f$ is continuous, Theorem 2-1 implies that $f^{-1}(V)$ is a neighborhood of $f^{-1}(z)$. Also, since $z$ is in $f(S')$, $f^{-1}(z)$ is a point of $S'$. Therefore $f^{-1}(V) \cap S$ contains a point $w$ which is not $f^{-1}(z)$. Consequently $f(w)$ is in each of $V$ and $f(S)$. Now since $f^{-1}$ is a function and $f^{-1}(z) \neq w$, it follows that $z \neq f(w)$. Otherwise two ordered pairs of $f^{-1}$ would have the same first element. Hence $z$ is an element of $[f(S)]'$. Thus $f(S')$ is a subset of $[f(S)]'$. 

Suppose $x$ is a point of $[f(S)]'$ and $U$ is a neighborhood of $x$. Then $U \cap f(S)$ contains a point $y$ such that $y \neq x$. Now since $f$ is a function and $y \neq x$, $f^{-1}(x)$ and $f^{-1}(y)$ are distinct and unique points of $X$ such that $f^{-1}(x)$ is in $f^{-1}(U)$ and $f^{-1}(y)$ is in each of $S$ and $f^{-1}(U)$. Also $f^{-1}(U)$ is a neighborhood of $f^{-1}(x)$ as a result of Theorem 2-1. Consequently $f^{-1}(x)$ is a point of $S'$. Hence $x$ is an element of $f(S')$. Therefore $[f(S)]'$ is a subset of $f(S')$, and the proof is complete.

Theorem 2-4. Suppose that $f$ is a homeomorphism. Then $A$ a closed set in $(X, \mathcal{T})$ implies $f(A)$ is closed.

Proof. Suppose $A$ is a closed set in $(X, \mathcal{T})$. Then $A^c$ is open; and since $f^{-1}$ is a continuous function, Theorem 2-1
implies that \( f(A^c) \) is open. Also \( f(A^c) = [f(A)]^c \) as a result of Theorem 2-2. Hence \( [f(A)]^c \) is open and \( f(A) \) is closed.

**Theorem 2-5.** Suppose \( S \) is a subset of \( X \), \( f \) is continuous, and \( A \) a closed set in \( (X, \mathcal{T}) \) implies that \( f(A) \) is closed. Then \( f(S) = f(S) \).

**Proof.** Since \( S \) is a subset of \( S \), \( f(S) \) is a subset of \( f(S) \). Also, since \( S \) is closed, \( f(S) \) is closed by hypothesis. Thus Theorem 1-18 guarantees that \( f(S) \) is a subset of \( f(S) \).

Suppose \( z \) is a point of \( f(S) \). If \( z \) is in \( f(S) \) it is also in \( f(S) \), and we are through. Assume, then, that \( z \) is not an element of \( f(S) \). Now since \( z \) is in \( f(S) \), there is a point \( x \) of \( S \) such that \( f(x) = z \); and \( x \) in \( S \) implies that \( x \) is in \( S \) or \( S' \). But \( x \) is not in \( S \), for \( x \) in \( S \) would imply that \( f(x) = z \) is in \( f(S) \). Hence \( x \) is a point of \( S \). Suppose next that \( U \) is a neighborhood of \( z \). Then since \( f \) is continuous, there is a neighborhood \( V \) of \( x \) such that \( f(V) \) is a subset of \( U \). Now \( x \) in \( S \) and \( V \) a neighborhood of \( x \) imply the existence of a point \( w \) in \( V \cap S \) such that \( w \neq x \). Thus we have \( f(w) \) in \( f(V) \cap f(S) \); and since \( f(V) \) is a subset of \( U \), \( f(w) \) is an element of \( U \cap f(S) \). Furthermore \( f(w) \neq z \); otherwise \( z \) would be in \( f(S) \). Consequently \( z \) is an element of \( [f(S)]' \) and therefore a point of \( f(S) \). Thus \( f(S) \) is a subset of \( f(S) \) and the proof is complete.

**Corollary 2-6.** Suppose \( f \) is a homeomorphism and \( S \) is a subset of \( X \). Then \( f(S) = f(S) \).

**Proof.** By Theorem 2-4 we have the conditions of Theorem 2-5, so \( f(S) = f(S) \).
Theorem 2-7. Suppose $f$ is continuous and one-to-one, $S$ is a subset of $X$ and $S$ is dense-in-itself. Then $f(S)$ is dense-in-itself.

Proof. Since $S$ is dense-in-itself, $S$ is a subset of $S'$. Hence $f(S)$ is contained in $f(S')$. Thus, by Theorem 2-3, $f(S)$ is a subset of $[f(S)]'$; that is, $f(S)$ is dense-in-itself.

Corollary 2-8. Suppose $f$ is homeomorphic and $S$ is a perfect subset of $X$. Then $f(S)$ is perfect.

Proof. Since $S$ is perfect it is closed and dense-in-itself. Now $S$ dense-in-itself implies, by Theorem 2-7, that $f(S)$ is dense-in-itself. Also, by Theorem 2-4, $S$ closed implies that $f(S)$ is closed. Consequently $f(S)$ is perfect.
CHAPTER BIBLIOGRAPHY

CHAPTER III

CONNECTEDNESS

In this chapter we assume that \((X, \mathcal{T})\) is a topological space; and unless otherwise specified, each set of points will be a subset of \(X\).

**Separated sets.** Suppose that \(A\) and \(B\) are sets of points. Then the statement that \(A\) and \(B\) are separated means that \(A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset, A' \cap B = \emptyset,\) and \(A \cap B' = \emptyset\) (2, p. 93).

**Connectedness.** A set of points \(S\) which cannot be expressed as the union of two separated sets is said to be connected.

**Theorem 3-1.** Suppose \(S\) and \(T\) are sets of points, \(S\) is connected, \(S \cap T \neq \emptyset\), and \(S \cap T^C \neq \emptyset\). Then \(S \cap \mathcal{B}(T) \neq \emptyset\).

**Proof.** By Theorem 1-27 we have

\[
(1) \quad \mathcal{B}(T) = [T \cap (T^C)'] \cup [T^C \cap T'].
\]

Let \(A = S \cap T\) and \(B = S \cap T^C\). Then \(A \cup B = S\) and \(A \cap B = \emptyset\).

By hypothesis, \(A \neq \emptyset\) and \(B \neq \emptyset\). Therefore, since \(S\) is connected, it follows that \(A\) and \(B\) are not separated. Thus \(A \cap B' \neq \emptyset\) or \(A' \cap B \neq \emptyset\). Hence we have

\[
(2) \quad (A \cap B') \cup (A' \cap B) \neq \emptyset.
\]

Now \(A \subseteq T\), so \(A' \subseteq T'\) by Theorem 1-16. Similarly, from \(B \subseteq T^C\), we get \(B' \subseteq (T^C)'\). Therefore we can write
(3) \( A' \cap B \subseteq T' \cap B = T' \cap (S \cap T^C) \)
\[ = S \cap [T' \cap T^C], \]
and
(4) \( A \cap B' = (S \cap T) \cap B' \subseteq (S \cap T) \cap (T^C)' \)
\[ = S \cap [T \cap (T^C)']. \]

Thus, from (3) and (4), we have
(5) \( (A' \cap B) \cup (A \cap B') \)
\[ \subseteq \{ S \cap [T' \cap T^C] \} \cup \{ S \cap [T \cap (T^C)'] \} \]
\[ = S \cap \{ [T' \cap T^C] \cup [T \cap (T^C)'] \}. \]

But (2) and (5) imply that
(6) \( S \cap \{ [T' \cap T^C] \cup [T \cap (T^C)'] \} \neq \emptyset. \)

Then using (6) and (1), we see that \( S \cap \emptyset(T) \neq \emptyset \), and the
theorem is established.

**Theorem 3-2.** \( X \) is connected if and only if the only two
subsets of \( X \) which are both open and closed are \( \emptyset \) and \( X \).

**Proof.** Suppose \( A \) is a subset of \( X \) such that \( A \neq X, \)
\( A \neq \emptyset \), and \( A \) is both open and closed. Since \( A \neq X \) and \( A \neq \emptyset \),
\( A^c \neq X \) and \( A^c \neq \emptyset \). Now \( A \) closed implies that \( A' \)
is a subset of \( A \), so \( A' \cap A^c = \emptyset \). But \( A \) is also open, so \( A^c \)
is closed. Hence \( (A^c)' \) is a subset of \( A^c \). Therefore \( (A^c)' \cap A = \emptyset \). It
follows that \( A \) and \( A^c \) are separated, so \( X \) is not connected.

Suppose \( \emptyset \) and \( X \) are the only two subsets of \( X \) which are
both open and closed. Suppose, by way of contradiction, that
\( X \) is not connected. Then there are two sets \( E \) and \( F \) such
that \( E \) and \( F \) are separated and \( E \cup F = X \). By definition of
separated sets we have \( E \neq \emptyset, F \neq \emptyset, E \cap F = \emptyset, E' \cap F = \emptyset, \)
and $E \cap F' = \emptyset$. But since $E \cap F = \emptyset$ and $E \cup F = X$, $F = E^c$.

Consider $E' \cap E^c$. Since $E' \cap F = \emptyset$ and $F = E^c$, we have $E' \cap E^c = \emptyset$, so that $E'$ is a subset of $E$. Thus $E$ is closed.

Now consider $E \cap (E^c)'$. Since $E \cap F = \emptyset$ and $F = E^c$, we have $E \bigcap (E^c)' = \emptyset$, so that $(E^c)'$ is a subset of $E^c$. Hence $E^c$ is closed and $E$ is open. But $E \neq \emptyset$ by hypothesis; and since $E^c = F$ is not $\emptyset$, $E$ is not $X$. Thus we have the contradiction sought. It follows that $X$ is connected, and the proof is complete.

**Theorem 3-3.** Suppose that $(Y, \mathcal{F})$ is a topological space, $f$ is a continuous mapping of $X$ onto $Y$, and $X$ is connected. Then $Y$ is connected.

**Proof.** Suppose, by way of contradiction, that $Y$ is not connected. Then by Theorem 3-2, there is a subset $A$ of $Y$ such that $A \neq \emptyset$, $A \neq Y$, and $A$ is both open and closed. Consider $A^c$. Since $A$ is both open and closed, $A^c$ is both open and closed. Now since $A$ is open and $f$ is continuous, $f^{-1}(A)$ is open as a result of Theorem 2-1. Similarly, since $A^c$ is open, $f^{-1}(A^c)$ is open. But by Theorem 2-2, $f^{-1}(A^c) = [f^{-1}(A)]^c$. Hence $[f^{-1}(A)]^c$ is open, which implies that $f^{-1}(A)$ is closed. Also $A \neq \emptyset$ implies $f^{-1}(A) \neq \emptyset$, and $A \neq Y$ implies $f^{-1}(A) \neq X$. Thus $f^{-1}(A)$ is a proper subspace of $X$ that is both open and closed, which is contrary to Theorem 3-2. Therefore $Y$ is connected.
Theorem 3-4. Suppose that $A$ and $B$ are separated sets and $A_1$ and $B_1$ are sets such that $A_1 \neq \emptyset$, $B_1 \neq \emptyset$, $A_1$ is a subset of $A$, and $B_1$ is a subset of $B$. Then $A_1$ and $B_1$ are separated.

Proof. Since $A$ and $B$ are separated, we have $A \cap B = \emptyset$, $A' \cap B = \emptyset$, and $A \cap B' = \emptyset$. Now by Theorem 1-16, $A_1$ a subset of $A$ and $B_1$ a subset of $B$ imply that $A_1'$ is a subset of $A'$ and $B_1'$ is a subset of $B'$. Hence

- $A_1 \cap B_1 \subset A \cap B = \emptyset$,
- $A_1' \cap B_1' \subset A' \cap B = \emptyset$, and
- $A_1 \cap B_1' \subset A \cap B' = \emptyset$.

Thus $A_1$ and $B_1$ are separated, since $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$.

Theorem 3-5. Suppose that $E$ is a connected set, $A$ and $B$ are separated sets, and $E$ is contained in the union of $A$ and $B$. Then $E$ is contained in one of $A$ and $B$.

Proof. Since $E$ is contained in the union of $A$ and $B$, we can write

$$E = (A \cap E) \cup (B \cap E).$$

Let $A_1$ denote $A \cap E$ and $B_1$ denote $B \cap E$. Then $E = A_1 \cup B_1$.

Now since $A$ and $B$ are separated, $A_1$ is a subset of $A$, and $B_1$ is a subset of $B$, it follows that $A_1$ and $B_1$ are disjoint. Hence either $A_1$ is empty or $B_1$ is empty. Otherwise, by Theorem 3-4, $A_1$ and $B_1$ would be separated and $E$ would not be connected. But $A_1$ empty implies that $E$ is contained in $B$, and $B_1$ empty implies that $E$ is contained in $A$. Hence $E$ is contained in one of $A$ and $B$. 
Theorem 3-6. Suppose that $E$ is a connected set and $T$ is a set such that $E$ is contained in $T$ and $T$ is contained in the closure of $E$. Then $T$ is connected.

Proof. Suppose, by way of contradiction, that $T$ is not connected. Then there are separated sets $A$ and $B$ such that $T$ is the union of $A$ and $B$. Now $E$ is a subset of $A \cup B$ and $E$ is connected. Hence, by Theorem 3-5, $E$ is contained in one of $A$ and $B$.

Suppose that $E$ is a subset of $A$. Then $E'$ is contained in $A'$. Thus $E' \cap B$ is a subset of $A' \cap B$. But since $A$ and $B$ are separated, $A' \cap B$ is $\emptyset$. Therefore $E' \cap B$ is $\emptyset$. Also since $A$ and $E$ are separated, $A \cap B = \emptyset$, so that $E$ a subset of $A$ implies that $E \cap B$ is empty. Thus we have

$$\begin{align*}
(1) \quad B &= \emptyset \cup B \\
&= (A \cap B) \cup B \\
&= (A \cup B) \cap (B \cup B) \\
&= (A \cup B) \cap B \\
&= T \cap B;
\end{align*}$$

and since $T$ is contained in $E$, we can write

$$\begin{align*}
(2) \quad T \cap B &\subseteq \bar{E} \cap B = (E \cup E') \cap B \\
&= (E \cap B) \cup (E' \cap B) \\
&= \emptyset \cup \emptyset \\
&= \emptyset.
\end{align*}$$

Then, from (1) and (2), we get

$$\begin{align*}
(3) \quad B &= \emptyset.
\end{align*}$$
But (3) is a contradiction since \( A \) and \( B \) separated implies that \( B \neq \emptyset \). We reach a similar contradiction by assuming that \( E \) is a subset of \( B \). Therefore \( T \) is connected.

**Corollary 3-7.** The closure of a connected set is connected.

**Proof.** Suppose that \( E \) is a connected set. Now \( E \) is contained in \( \overline{E} \) and \( \overline{E} \) is contained in itself. Hence Theorem 3-6 guarantees that \( \overline{E} \) is connected.

**Theorem 3-8.** Suppose \( E \) is a set such that every two points of \( E \) belong to some connected subset of \( E \). Then \( E \) is connected.

**Proof.** Suppose, by way of contradiction, that \( E \) is not connected. Then there are separated sets \( A \) and \( B \) such that \( E \) is the union of \( A \) and \( B \); and since \( A \) and \( B \) are separated, \( A \) and \( B \) are nonempty. Let \( p \) denote an element of \( A \) and \( q \) an element of \( B \). Also let \( E_1 \) be a subset of \( E \) which contains \( p \) and \( q \). Consider \( A_1 \) and \( B_1 \), where \( A_1 = A \cap E_1 \) and \( B_1 = B \cap E_1 \). Now \( p \) is in \( A_1 \) and \( q \) in \( B_1 \), so that \( A_1 \neq \emptyset \) and \( B_1 \neq \emptyset \). Also \( A_1 \) is contained in \( A \) and \( B_1 \) is contained in \( B \). Hence, by Theorem 3-4, \( A_1 \) and \( B_1 \) are separated. Therefore the set

\[
E_1 = (A \cup B) \cap E_1 = (A \cap E_1) \cup (B \cap E_1) = A_1 \cup B_1
\]

is not connected. It follows that no subset of \( E \) containing \( p \) and \( q \) is connected, which contradicts the hypothesis. Consequently \( E \) is connected.
Lemma 3-9. Suppose \( E \) is a set which is the union of two connected sets, \( E_1 \) and \( E_2 \), and \( E \) is not connected. Then neither \( E_1 \) nor \( E_2 \) is empty.

Proof. If \( E_1 \) is empty, \( E = E_2 \), which is impossible since \( E_2 \) is connected and \( E \) is not connected. Hence \( E_1 \) is not empty. Similarly \( E_2 \) is not empty.

Theorem 3-10. Two connected sets whose union is not connected are separated.

Proof. Suppose \( E_1 \) and \( E_2 \) are two connected sets and the union \( E \) of \( E_1 \) and \( E_2 \) is not connected. Then \( E \) can be expressed as the union of two separated sets \( A \) and \( B \). But \( E_1 \) is a subset of \( A \cup B \), so by Theorem 3-5, \( E_1 \) must be contained in one of \( A \) and \( B \). Similarly \( E_2 \) is contained in one of \( A \) and \( B \).

Suppose \( E_1 \) is a subset of \( A \). If \( E_2 \) is also contained in \( A \), then \( E = E_1 \cup E_2 \) is a subset of \( A \), which is impossible since \( E \) a subset of \( A \) would imply \( B = \emptyset \). Hence \( E_2 \) is contained in \( B \). Similarly if \( E_1 \) is a subset of \( B \), then \( E_2 \) is contained in \( A \).

Also, by Lemma 3-9, \( E_1 \) and \( E_2 \) are nonempty. Therefore, by Theorem 3-4, \( E_1 \) and \( E_2 \) are separated. The theorem follows.

Corollary 3-11. Suppose that each of \( E_1 \) and \( E_2 \) is a connected set and the intersection \( F \) of \( E_1 \) and \( E_2 \) is not empty. Then the union \( E \) of \( E_1 \) and \( E_2 \) is connected.

Proof. If \( E_1 = E_2 \) we are through. Assume, then, that \( E_1 \neq E_2 \). Since \( F \) is not empty, \( E_1 \) and \( E_2 \) are not separated; and by the contrapositive of Theorem 3-10, \( E_1 \) and \( E_2 \) not separated implies that either \( E \) is connected or \( E_1 \) and \( E_2 \) are
not two connected sets. But \( E_1 \) and \( E_2 \) are two connected sets by hypothesis. Hence \( E \) is connected.

**Theorem 3-12.** Suppose \( \mathcal{A} \) is a class of connected sets every pair of which has an element in its intersection. Then the union \( S \) of \( \mathcal{A} \) is connected.

**Proof.** Suppose \( p \) and \( q \) are two elements of \( S \). Then there is a set \( E_1 \) of \( \mathcal{A} \) and a set \( E_2 \) of \( \mathcal{A} \) such that \( p \) is in \( E_1 \) and \( q \) belongs to \( E_2 \). By hypothesis each of \( E_1 \) and \( E_2 \) is connected and the intersection of \( E_1 \) and \( E_2 \) is not empty. Hence, by Corollary 3-11, the union \( E \) of \( E_1 \) and \( E_2 \) is connected. In addition \( E \) contains both \( p \) and \( q \); and since each of \( E_1 \) and \( E_2 \) is a connected subset of \( S \), \( E \) is also a subset of \( S \). Therefore \( E \) is a connected subset of \( S \) containing \( p \) and \( q \). Thus the theorem follows as a consequence of Theorem 3-8.

**Theorem 3-13.** Suppose \( S \) and \( P \) are separated sets. Then \( \overline{S} \) is a subset of \( P^c \).

**Proof.** Since \( S \) and \( P \) are disjoint, \( S \) is a subset of \( P^c \). Also since \( S^c \) and \( P \) are disjoint, \( S^c \) is a subset of \( P^c \). Hence \( S \cup S^c = \overline{S} \) is a subset of \( P^c \).

**Theorem 3-14.** Suppose \( S \) and \( P \) are nonempty, disjoint, open sets. Then \( S \) and \( P \) are separated.

**Proof.** Suppose \( y \) is in \( S \). Then, since \( S \) is open, \( S = S^o \) and \( y \) is in \( S^o \). Now \( y \) in \( S^o \) implies that there is a neighborhood \( V \) of \( y \) such that \( V \) is a subset of \( S \). Hence \( V \cap P = \emptyset \), so \( y \) cannot be in \( P' \). Thus \( S \cap P' = \emptyset \). Similarly \( S' \cap P = \emptyset \). Therefore \( S \) and \( P \) are separated.
Property \( p(E) \). The statement that \( A \) has property \( p(E) \) means that \( E \) is a set and \( A \) is a nonempty subset of \( E \) such that \( x \) in \( A \cup (A' \cap E) \) implies there is a neighborhood \( V \) of \( x \) such that \( V \cap E \) is contained in \( A \).

Condition \( W \). The statement that a set \( E \) satisfies condition \( W \) means that if \( A \) has property \( p(E) \), then \( A = E \) (1, p. 17).

Theorem 3-15. Suppose \( E = P \cup Q \), where \( P \) and \( Q \) are separated sets. Then \( P \) is a proper subset of \( E \) having property \( p(E) \).

Proof. Clearly \( P \) is a nonempty proper subset of \( E \).

Suppose, for contradiction, that \( P \) does not have property \( p(E) \). Then there is a point \( t \) of \( P \cup (P' \cap E) \) such that if \( V \) is a neighborhood of \( t \), then \( V \cap E \) is not contained in \( P \). Choose a neighborhood \( T \) of \( t \). Since \( T \cap E \) is not contained in \( P \) and \( Q = E - P \), there is a point \( z \) of \( Q \) in \( T \cap E \). Now \( t \) in \( P \cup (P' \cap E) \) implies that \( t \) is in \( P \) or \( P' \). But \( P \) and \( Q \) are separated, so that \( P \cap Q = P' \cap Q = \emptyset \). Then since \( z \) belongs to \( Q \) and \( t \) is in \( P \) or \( P' \), \( z \neq t \). Hence \( t \) is a point of \( Q \).

Thus \( t \) is in \( P' \cap E \) since \( P \cap Q' \) is empty. Hence \( t \) is in each of \( P' \) and \( E \) but not in \( P \). Now \( t \) in \( E \) and not in \( P \) implies that \( t \) belongs to \( Q \). Consequently \( t \) is an element of \( P' \cap Q \), which contradicts the hypothesis that \( P \) and \( Q \) are separated. Hence \( P \) is a proper subset of \( E \) having property \( p(E) \).

Lemma 3-16. Suppose that \( A \) has property \( p(E) \). Then \( A' \cap E \) is a subset of \( A \).
Proof. Suppose \( t \) is an element of \( A' \cap E \). Then since \( A \) has property \( p(E) \), there is a neighborhood \( V \) of \( t \) such that \( V \cap E \) is contained in \( A \). Thus, since \( t \) is in each of \( V \) and \( E \), \( t \) is an element of \( A \). Consequently \( A' \cap E \) is a subset of \( A \).

Theorem 3-17. Suppose \( A \) is a proper subset of \( E \) having property \( p(E) \). Then \( A \) and \( E - A \) are separated.

Proof. Let \( B \) denote \( E - A \). Since \( A \) is a proper subset of \( E \) and \( A \) has property \( p(E) \),

\[
(1) \quad A \neq \emptyset \text{ and } B \neq \emptyset.
\]

Also, by definition of \( B \),

\[
(2) \quad A \cap B = \emptyset.
\]

Now if \( z \) is a point of \( A \), then \( z \) is in \( A \cup (A' \cap E) \); and since \( A \) has property \( p(E) \), there is a neighborhood \( V \) of \( z \) such that \( V \cap E \) is contained in \( A \). Hence \( B' \) does not contain any point of \( A \), that is,

\[
(3) \quad A \cap B' = \emptyset.
\]

Suppose, by way of contradiction, that there is a point \( w \) in \( A' \cap B \). Since \( w \) is in \( B \) it is also in \( E \). Thus \( w \) is in \( A' \cap E \) and therefore, by Lemma 3-16, in \( A \), which is impossible since \( w \) is an element of \( B \). Hence

\[
(4) \quad A' \cap B = \emptyset.
\]

Thus, from (1) through (4), we see that \( A \) and \( B \) are separated; that is \( A \) and \( E - A \) are separated.

Theorem 3-18. A set \( E \) is connected if and only if there does not exist a proper subset \( A \) of \( E \) which has property \( p(E) \).
Proof. Suppose $E$ is a connected set and $A$ is a proper subset of $E$. Then $A$ does not have property $p(E)$. Otherwise, by Theorem 3-17, $A$ and $E - A$ would be separated sets and $E$ would not be connected.

Suppose $E$ is not connected. Then there exist separated sets $P$ and $Q$ such that $E = P \cup Q$. Hence, by Theorem 3-15, $P$ is a proper subset of $E$ having property $p(E)$. The theorem follows.

Corollary 3-19. A set $E$ is connected if and only if it satisfies condition $W$.

Proof. Suppose $E$ is a connected set and $A$ is a set having property $p(E)$. Then Theorem 3-18 guarantees that $A$ is not a proper subset of $E$, so $A = E$. Hence $E$ connected implies that $E$ satisfies condition $W$.

Suppose $E$ is not connected. Then by Theorem 3-18, there is a proper subset $Q$ of $E$ having property $p(E)$, which implies that $E$ does not satisfy condition $W$. The corollary follows.
CHAPTER BIBLIOGRAPHY

CHAPTER IV
RELATIVE TOPOLOGIES

Theorem 4-1. Suppose $X$ is a space, $H$ is a subset of $X$, $\mathcal{F}$ is a class of subsets of $X$, and $\mathcal{A}$ is the class of sets such that $A$ belongs to $\mathcal{A}$ if and only if $A = H \cap T$ for some set $T$ in $\mathcal{F}$. If $A$ is in $\mathcal{A}$ let one set $A(\mathcal{F})$ such that $A = H \cap A(\mathcal{F})$ be chosen from $\mathcal{F}$. Let $\mathcal{B}$ denote the set of all $A(\mathcal{F})$'s chosen. Then

1. $\bigcup A = H \cap (\bigcup \mathcal{B})$, and
2. $\bigcap A = H \cap (\bigcap \mathcal{B})$.

Proof. Suppose $x$ is in the union of $\mathcal{A}$. Then $x$ is in $H$ and in some set of $\mathcal{B}$. Hence $x$ is in each of $H$ and the union of $\mathcal{B}$, that is, $x$ is an element of $H \cap (\bigcup \mathcal{B})$. Suppose $y$ is in $H \cap (\bigcup \mathcal{B})$. Then $y$ is in $H$ and also in some set of $\mathcal{B}$. Hence there is a set of $\mathcal{A}$ which contains $y$ as an element, so $y$ is in the union of $\mathcal{A}$. Thus (1) is established.

Suppose $z$ is an element of the intersection of $\mathcal{A}$. Then $z$ is in $H \cap B$ for every $B$ in $\mathcal{B}$. Thus $z$ is in each of $H$ and the intersection of $\mathcal{B}$, that is, $z$ is an element of $H \cap (\bigcap \mathcal{B})$. Now suppose that $w$ is in $H \cap (\bigcap \mathcal{B})$. Then $w$ is in $H$ and also in each set of $\mathcal{B}$. Hence $w$ is in $H \cap B$ for every $B$ in $\mathcal{B}$.
Thus w is in each set of $\mathcal{A}$. Therefore w is an element of the intersection of $\mathcal{A}$, and (2) is established.

**Lemma 4-2.** Suppose $X$ is a space, $\mathcal{T}$ is a topology for $X$, $H$ is a subspace for $X$, and

$$U = \{ Q \mid Q = H \cap T \text{ for some } T \text{ in } \mathcal{T} \}.$$  

Then $U$ is a topology for $H$.

**Proof.** Suppose $A$ is a subset of $U$. If $A$ is a set of $\mathcal{A}$ choose one set $A(T)$ from $\mathcal{T}$ such that $A = H \cap A(T)$. Let $\mathcal{B}$ denote the set of all $A(T)$'s chosen. Since $\mathcal{B}$ is a subset of $\mathcal{T}$ and $\mathcal{T}$ is a topology for $X$, the union of $\mathcal{B}$ is in $\mathcal{T}$. But by Theorem 4-1, $\bigcup A = H \cap (\bigcup B)$. Hence the union of $A$ is in $U$, and Postulate (a) is satisfied.

Suppose $A$ is finite. Then $\mathcal{B}$ is finite, and the intersection of $\mathcal{B}$ is in $\mathcal{T}$ since $\mathcal{T}$ is a topology for $X$. Now by Theorem 4-1, $\bigcap A = H \cap (\bigcap B)$. Hence the intersection of $A$ is in $U$, and Postulate (b) holds. Therefore $U$ is a topology for $H$.

**Relative topology.** Under the conditions of Lemma 4-2, $U$ is called the relative topology for $H$ induced by (or inherited from) $\mathcal{T}$ (1, p. 95).

Throughout the remainder of this chapter, we assume that $(X, \mathcal{T})$ is a topological space, $H$ is a subspace of $X$, and $U$ is the relative topology for $H$ induced by $\mathcal{T}$. If $S$ is a subset of $H$ we denote by $(S')_R$ the set of accumulation points of $S$ with respect to $(H, U)$ and define $(\overline{S})_R$ to be $(S')_R \cup S$. 
Lemma 4-3. Suppose \( E \) is a subset of \( X \), \( B \) is a subset of \( H \), and \( Q = E \cap H \). Then \( Q \cap B = E \cap B \).

Proof. Since \( Q \) is the intersection of \( E \) and \( H \), \( Q \) is a subset of \( E \). Hence \( Q \cap B \) is a subset of \( E \cap B \). Now suppose \( x \) is an element of \( E \cap B \). Then \( x \) is in each of \( E \) and \( B \). But since \( B \) is a subset of \( H \), \( x \) in \( B \) implies that \( x \) is in \( H \). Thus \( x \) is in each of \( E \) and \( H \), so that \( x \) is in \( Q \). Hence we have \( x \) in \( Q \cap B \). Therefore \( E \cap B \) is a subset of \( Q \cap B \), and the lemma follows.

Theorem 4-4. Suppose \( B \) is a subset of \( H \). Then \( (B')_R = B' \cap H \).

Proof. Suppose \( y \) is a point of \( (B')_R \). Then \( y \) is in \( H \) by definition of \( (B')_R \). Consider \( B' \). Suppose \( T \) is a set of \( \mathcal{U} \) which contains \( y \). By definition of \( \mathcal{U} \), there is a set \( V \) of \( \mathcal{U} \) such that \( V = H \cap T \). Now \( V \) contains the point \( y \); and since \( y \) is in \( (B')_R \), \( V \cap B \) contains a point \( z \) such that \( z \neq y \). But \( V \) is a subset of \( T \), so \( z \) is in \( T \cap B \). Therefore \( y \) is a point of \( B' \). Consequently \( y \) is in \( B' \cap H \). Thus \( (B')_R \) is a subset of \( B' \cap H \).

Suppose \( x \) is a point of \( B' \cap H \). Then \( x \) is in each of \( B' \) and \( H \). Suppose \( Q \) is a set of \( \mathcal{U} \) containing \( x \). Then there is a set \( E \) of \( \mathcal{N} \) such that \( Q = E \cap H \). Now since \( x \) is in \( B' \), \( E \cap B \) contains a point \( w \) such that \( w \neq x \). But by Lemma 4-3, \( E \cap B = Q \cap B \). Thus \( w \) is a point of \( Q \cap B \) such that \( w \neq x \). Hence \( x \) is in \( (B')_R \). Therefore \( B' \cap H \) is a subset of \( (B')_R \), and the proof is complete.
Corollary 4-5. Suppose $B$ is a subset of $H$ and $B$ is closed with respect to $(X, \mathcal{T})$. Then $(B')_R = B'$.

Proof. Since $B$ is closed with respect to $(X, \mathcal{T})$, $B'$ is a subset of $B$. But $B$ is also a subset of $H$. Hence $B'$ is a subset of $H$, so $B' \cap H = B'$. Thus $(B')_R = B'$ as a consequence of Theorem 4-4.

Corollary 4-6. Under the hypothesis of Corollary 4-5, $(\overline{B})_R = \overline{B}$.

Proof. Since $(B')_R = B'$, $B \cup (B')_R = B \cup B'$; that is $(\overline{B})_R = \overline{B}$.

Corollary 4-7. Suppose each of $A$ and $B$ is a subset of $H$. Then $(B')_R \cap A = B' \cap A$.

Proof. By Theorem 4-4,

$$(B')_R = B' \cap H.$$ 

Hence

$$(B')_R \cap A = (B' \cap H) \cap A = B' \cap (H \cap A).$$

But $A$ is a subset of $H$, so $H \cap A = A$. Therefore

$$(B')_R \cap A = B' \cap A,$$

and the corollary is established.

Theorem 4-8. $H$ is connected with respect to $(X, \mathcal{T})$ if and only if $H$ is connected with respect to $(H, \mathcal{U})$.

Proof. If there do not exist two nonempty disjoint sets whose union is $H$, then $H$ is connected with respect to each of $(X, \mathcal{T})$ and $(H, \mathcal{U})$. Suppose $A \neq \emptyset$, $B \neq \emptyset$, and $H = A \cup B$. 

Then by Corollary 4-7,
\[ A \cap B' = (B')_R \cap A \]
and
\[ B \cap A' = (A')_R \cap B. \]
Hence each of \( A \cap B' \) and \( B \cap A' \) is empty if and only if each of \( (B')_R \cap A \) and \( (A')_R \cap B \) is empty, that is, \( A \) and \( B \) are separated sets in \( (H, \mathcal{U}) \) if and only if \( A \) and \( B \) are separated sets in \( (X, \mathcal{T}) \). Consequently \( H \) is connected with respect to \( (H, \mathcal{U}) \) if and only if \( H \) is connected with respect to \( (X, \mathcal{T}) \).

**Theorem 4-9.** Suppose \( A \) is a subset of \( H \). Then
\[ (\overline{A})_R = A \cup (A' \cap H). \]

*Proof.* \( (\overline{A})_R = A \cup (A')_R \); and by Theorem 4-4,
\[ (A')_R = A' \cap H. \] Therefore \( (\overline{A})_R = A \cup (A' \cap H). \)

**Theorem 4-10.** Suppose \( A \) is a subset of \( H \). Then \( A \) is dense in \( H \) with respect to \( (H, \mathcal{U}) \) if and only if \( H = A \cup (A' \cap H) \).

*Proof.* By Theorem 1-29, \( A \) is dense in \( H \) with respect to \( (H, \mathcal{U}) \) if and only if \( H = (\overline{A})_R \). Thus \( A \) is dense in \( H \) with respect to \( (H, \mathcal{U}) \) if and only if \( H = A \cup (A' \cap H) \) as a result of Theorem 4-9.

**Theorem 4-11.** Suppose \( A \) is a subset of \( H \), \( B \) is a subset of \( X \), and \( A \) is dense in \( B \) with respect to \( (H, \mathcal{U}) \). Then \( A \) is dense in \( B \) with respect to \( (X, \mathcal{T}) \).

*Proof.* By hypothesis \( B \) is a subset of \( (\overline{A})_R \). Also from Theorem 4-4 we see that \( (A')_R \) is contained in \( A' \), which implies that \( (\overline{A})_R \) is a subset of \( \overline{A} \). Thus \( B \) is contained in \( \overline{A} \), that is, \( A \) is dense in \( B \) with respect to \( (X, \mathcal{T}) \).
**Theorem 4-12.** Suppose $A$ is a subset of $H$ and $H$ is closed with respect to $(X, \mathcal{T})$. Then $(\overline{A})_R = \overline{A}$.

**Proof.** Since $A$ is a subset of $H$, $\overline{A}$ is contained in $\overline{H}$. Also $H$ closed with respect to $(X, \mathcal{T})$ implies that $H = \overline{H}$. Thus $\overline{A}$ is a subset of $H$ which is closed with respect to $(X, \mathcal{T})$. Therefore, by Corollary 4-6, the closure of $(\overline{A})_R$ with respect to $(H, \mathcal{U})$ is the closure of $\overline{A}$ with respect to $(X, \mathcal{T})$. But the closure of a closed set in any topological space is the set itself. Hence $(\overline{A})_R = \overline{A}$.

**Theorem 4-13.** Suppose $A$ is a subset of $H$, $B$ is a subset of $X$, and one of $A$ and $H$ is closed with respect to $(X, \mathcal{T})$. Then $A$ is dense in $B$ with respect to $(H, \mathcal{U})$ if and only if $A$ is dense in $B$ with respect to $(X, \mathcal{T})$.

**Proof.** Suppose $A$ is dense in $B$ with respect to $(X, \mathcal{T})$. Then $B$ is a subset of $\overline{A}$. If $A$ is closed with respect to $(X, \mathcal{T})$, then $(\overline{A})_R = \overline{A}$ by Corollary 4-6; and if $H$ is closed with respect to $(X, \mathcal{T})$, then $(\overline{A})_R = \overline{A}$ as a consequence of Theorem 4-12. In either case $B$ is contained in $(\overline{A})_R$, that is, $A$ is dense in $B$ with respect to $(H, \mathcal{U})$. The converse is guaranteed by Theorem 4-11.
CHAPTER BIBLIOGRAPHY

CHAPTER V

CLOSURE IN A SET

In this chapter we assume that each of \( E \) and \( F \) is a set of points in a topological space \((X, \mathcal{T})\).

\( E \) closed in \( F \). The statement that \( E \) is closed in \( F \) means that \( E' \cap F \) is a subset of \( E \) (1, p. 15).

**Theorem 5-1.** Suppose \( E \) is closed in \( F \) and \( G \) is a subset of \( F \). Then \( E \) is closed in \( G \).

**Proof.** Since \( E \) is closed in \( F \), \( E' \cap F \) is a subset of \( E \). But \( G \) is a subset of \( F \), so \( E' \cap G \) is contained in \( E' \cap F \). It follows that \( E' \cap G \) is a subset of \( E \). Hence \( E \) is closed in \( G \).  

**Theorem 5-2.** Suppose \( F \) is closed. Then \( E \cap F \) is closed in \( E \).

**Proof.** Let \( A \) denote \( E \cap F \). Then \( A \) is a subset of \( F \) and \( A' \) is contained in \( F' \). Also since \( F \) is closed, \( F' \) is contained in \( F \). Hence \( A' \) is a subset of \( F \). Thus \( A' \cap E \) is contained in \( F \cap E = A \), that is, \((E \cap F)' \cap E \) is a subset of \( E \cap F \). Therefore \( E \cap F \) is closed in \( E \).

**Theorem 5-3.** Suppose \( F \) is closed, \( E \) is a subset of \( F \), and \( E \) is closed in \( F \). Then \( E \) is closed.

**Proof.** Since \( E \) is a subset of \( F \), \( E' \) is contained in \( F' \). But \( F \) closed implies \( F' \) is a subset of \( F \), so \( E' \) is a subset of \( F \). Consequently \( E' = E' \cap F \). Now \( E \) is closed in \( F \), so \( E' \cap F \)
is a subset of $E$. Therefore $E'$ is a subset of $E$, which implies that $E$ is closed.

**Theorem 5-4.** $E$ is connected if and only if $E$ is not the union of two nonempty disjoint sets each closed in $E$.

**Proof.** Suppose $E$ is the union of $A$ and $B$, where $A$ and $B$ are nonempty disjoint sets each closed in $E$. Then we have

(1) $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$,

and

(2) $A' \cap E \subseteq A$ and $B' \cap E \subseteq B$.

Now since $E = A \cup B$, it follows that $B = E \cap B$ and $A' \cap B = A' \cap (E \cap B)$. Then using (1) and (2) we get

(3) $A' \cap B = A' \cap (E \cap B) = (A' \cap E) \cap B \subseteq A \cap B = \emptyset$.

Similarly we can write

(4) $A \cap B' = (E \cap A) \cap B' = (B' \cap E) \cap A \subseteq B \cap A = \emptyset$.

Then from (1), (3), and (4) we see that $A$ and $B$ are separated. Therefore $E$ is not connected.

Suppose $E$ is not connected. Then there are two nonempty disjoint sets $S$ and $T$ such that $S' \cap T = \emptyset$, $S \cap T' = \emptyset$, and $S \cup T = E$. Consider $S' \cap E$.

$$S' \cap E = S' \cap (S \cup T)$$
$$= (S' \cap S) \cup (S' \cap T)$$
$$= (S' \cap S) \cup \emptyset$$
$$= S' \cap S \subseteq S.$$
Similarly
\[ T' \cap E = T' \cap (T \cup S) \]
\[ = (T' \cap T) \cup (T' \cap S) \]
\[ = (T' \cap T) \cup \emptyset \]
\[ = T' \cap T \subseteq T. \]

Hence each of $S$ and $T$ is closed in $E$. The theorem follows.

**Theorem 5-5.** Suppose $A$ is a subset of $E$ and $E$ is closed. Then $A$ is closed if and only if $A$ is closed in $E$.

**Proof.** Suppose $A$ is closed. Then $A'$ is a subset of $A$, so $A' \cap E$ is contained in $A \cap E$. But $A \cap E = A$, since $A$ is a subset of $E$; so $A' \cap E$ is contained in $A$. Hence $A$ is closed in $E$.

Suppose $A$ is closed in $E$. Then $A' \cap E$ is a subset of $A$. Now since $A$ is a subset of $E$, $A'$ is contained in $E'$; and since $E$ is closed, $E'$ is a subset of $E$. Thus $A'$ is a subset of $E$, so $A' \cap E$ is $A'$. Therefore, since $A' \cap E$ is contained in $A$, $A'$ is a subset of $A$. Hence $A$ is closed.

**Theorem 5-6.** Suppose $E$ is closed. Then $E$ is connected if and only if $E$ is not the union of two nonempty disjoint closed sets.

**Proof.** Suppose $E$ is the union of two nonempty disjoint closed sets $A$ and $B$. Consider $A \cap B'$. Since $B$ is closed $B'$ is a subset of $B$. But $A \cap B = \emptyset$ and $B'$ is contained in $B$ imply that $A \cap B' = \emptyset$. Similarly $A' \cap B = \emptyset$. Hence $A$ and $B$ are separated and $E$ is not connected. Therefore if $E$ is connected, then $E$ cannot be the union of two nonempty disjoint closed sets.
Suppose $E$ is not connected. Then by Theorem 5-4, there are two nonempty disjoint sets $S$ and $T$ such that each of $S$ and $T$ is closed in $E$ and $E = S \cup T$. Consider $S$. Since $S$ is a subset of $E$, we see from Theorem 5-5 that $S$ is closed. Similarly $T$ is closed. Hence $E$ is the union of two nonempty disjoint closed sets. The theorem follows.

$S$ open in $E$. The statement that $S$ is open in $E$ means that $S$ is a subset of $E$ such that $E - S$ is closed in $E$.

Theorem 5-7. Suppose $S$ is a subset of $E$ and $S$ is closed in $E$. Then $E - S$ is open in $E$.

Proof. Since $S$ is a subset of $E$, $E - (E - S) = S$. But $S$ is closed in $E$. Hence $E - S$ is a subset of $E$ such that $E - (E - S)$ is closed in $E$. Therefore $E - S$ is open in $E$.

Theorem 5-8. $E$ is connected if and only if there does not exist a nonempty proper subset $S$ of $E$ such that $S$ is both open in $E$ and closed in $E$.

Proof. Suppose $S$ is a nonempty proper subset of $E$ and $S$ is both open in $E$ and closed in $E$. Since $S$ is open in $E$, it follows that $E - S$ is closed in $E$. Also since $S$ is a nonempty proper subset of $E$, $E - S$ is a nonempty proper subset of $E$. But $E$ is the union of $S$ and $E - S$. Therefore $E$ is not connected as a result of Theorem 5-4.

Suppose $E$ is not connected. Then by Theorem 5-4, there exist two nonempty disjoint sets $A$ and $B$ such that each of $A$ and $B$ is closed in $E$ and $E$ is the union of $A$ and $B$. But $E$ the union of $A$ and $B$ implies that $B = E - A$; and since $A$ is
a subset of \( E \) which is closed in \( E \), \( B \) is open in \( E \). Also \( B \) is a proper subset of \( E \) since \( A \) is not empty. Thus \( B \) is a nonempty proper subset of \( E \) which is both open in \( E \) and closed in \( E \). The theorem follows.

**Lemma 5-9.** Suppose that \( H \) is a subset of \( X \) and \( S \) is a subset of \( H \). Then \((H - S)^C \cap H = S\).

**Proof.** Suppose \( y \) is an element of \((H - S)^C \cap H\). Then \( y \) is in each of \( H \) and \((H - S)^C\). Now \( y \) in \((H - S)^C\) implies that \( y \) is not in \( H - S \), so that either \( y \) is not in \( H \) or \( y \) is in \( S \). But we already have \( y \) in \( H \). Hence \( y \) is an element of \( S \).

Therefore \((H - S)^C \cap H \) is a subset of \( S \).

Suppose \( z \) is in \( S \). Then \( z \) is not in \( H - S \), so \( z \) is a point of \((H - S)^C\). Now \( S \) is a subset of \( H \), so \( z \) is also an element of \( H \). Hence \( z \) is in \((H - S)^C \cap H\). Thus \( S \) is contained in \((H - S)^C \cap H\), and the lemma is established.

**Lemma 5-10.** Suppose each of \( T \) and \( H \) is a subset of \( X \) and \( S = T \cap H \). Then \( T^C \cap H = H - S \).

**Proof.** Suppose \( y \) is an element of \( T^C \cap H \). Then \( y \) is in each of \( T^C \) and \( H \). But \( y \) in \( T^C \) implies that \( y \) is not in \( T \), so \( y \) does not belong to \( T \cap H \); that is, \( y \) is not in \( S \). Thus \( y \) is in \( H \) but not in \( S \). Hence \( y \) is a point of \( H - S \).

Suppose \( z \) is an element of \( H - S \). Then \( z \) is in \( H \) and not in \( S \). Now \( z \) not in \( S \) implies that \( z \) is not in \( T \) or not in \( H \). But we have already seen that \( z \) is a point of \( H \). Hence \( z \) is not in \( T \), that is, \( z \) belongs to \( T^C \). Thus \( z \) is in each of \( H \) and \( T^C \), so \( z \) is an element of \( T^C \cap H \). The lemma follows.
In the remainder of this chapter, when a relative topology is being considered, dependence on \((X, \mathcal{F})\) will be assumed when discussing closed sets, open sets, sets closed in a set, and sets open in a set unless otherwise specified.

**Theorem 5-11.** Suppose \(H\) is a subspace of \(X\), \(\mathcal{U}\) is the relative topology for \(H\) induced by \(\mathcal{F}\), and \(S\) is a subset of \(X\). Then if \(S\) is in \(\mathcal{U}\), \(S\) is open in \(H\).

**Proof.** Suppose \(S\) is in \(\mathcal{U}\). Then there is a set \(T\) of \(\mathcal{F}\) such that \(S = T \cap H\). Consider \(T^c\). Since \(T\) is in \(\mathcal{F}\), \(T^c\) is closed. Now by Theorem 5-2, \(T^c\) closed implies \(T^c \cap H\) is closed in \(H\). But \(T^c \cap H = H - S\) as a result of Lemma 5-10. Hence \(H - S\) is closed in \(H\). Therefore, since \(S\) is a subset of \(H\), \(S\) is open in \(H\).

**Theorem 5-12.** Under the hypothesis of Theorem 5-11, if \(S\) is open in \(H\), then \(S\) is an open set in \((H, \mathcal{U})\).

**Proof.** Suppose \(S\) is open in \(H\). Then \(K = H - S\) is closed in \(H\); that is, \(K \cap H \subseteq K\). But by Theorem 4-4 (using the notation of Chapter IV) \(K \cap H = (K')_R\). Thus \((K')_R \subseteq K\). Hence \(K\) is a closed set in \((H, \mathcal{U})\). Also since \(S \subseteq H\), \(S = H - K\). Consequently \(S\) is an open set in \((H, \mathcal{U})\).

**Theorem 5-13.** Under the hypothesis of Theorem 5-11, if \(S\) is a closed set in \((H, \mathcal{U})\), then \(S\) is closed in \(H\).

**Proof.** Suppose \(S\) is a closed set in \((H, \mathcal{U})\). Then \(S\) is a subset of \(H\) and \(H - S\) (the complement of \(S\) relative to \(H\)) is in \(\mathcal{U}\). Consider \(H - S\). Since \(H - S\) is in \(\mathcal{U}\), \(H - S\) is open in \(H\) as a consequence of Theorem 5-11. But \(S\) a subset of \(H\) implies that \(S = H - (H - S)\). Hence \(S\) is closed in \(H\).
Theorem 5-14. Under the hypothesis of Theorem 5-11, if $S \subseteq H$ and $S$ is closed in $H$, then $S$ is a closed set in $(H, \mathcal{U})$.

Proof. Suppose $S \subseteq H$ and $S$ is closed in $H$. Then $K = H - S$ is open in $H$. Hence by Theorem 5-12, $K$ is an open set in $(H, \mathcal{U})$. Therefore $S = H - K$ is a closed set in $(H, \mathcal{U})$.

Lemma 5-15. Suppose $\mathcal{A}$ is a class of sets. If $A$ is a set of $\mathcal{A}$, choose one set $B$ such that $B$ is a subset of $A$. Let $\mathcal{B}$ denote the set of all the $B$'s chosen. Then the intersection of $\mathcal{B}$ is a subset of the intersection of $\mathcal{A}$.

Proof. Suppose $x$ is an element of the intersection of $\mathcal{B}$. Then $x$ is in $B$ for every $B$ in $\mathcal{B}$. Hence $x$ is in $A$ for every $A$ in $\mathcal{A}$, so $x$ is an element of the intersection of $\mathcal{A}$. Thus the intersection of $\mathcal{B}$ is a subset of the intersection of $\mathcal{A}$.

Theorem 5-16. Suppose $\mathcal{A}$ is a class of subsets of $X$ such that each set of $\mathcal{A}$ is closed in $E$. Then the intersection $P$ of $\mathcal{A}$ is closed in $E$.

Proof. If $\mathcal{A}$ is empty, $P$ is $X$ and $P' \cap E$ is a subset of $P$. Suppose $\mathcal{A}$ is not empty. Let $\mathcal{B}$ denote the class of all sets $S$ such that $S = A' \cap E$ for some $A$ in $\mathcal{A}$. Now for each $A$ in $\mathcal{A}$, $P$ is a subset of $A$, so that $P'$ is contained in $A'$.

Hence

1. $P' \cap E \subseteq A' \cap E$ for every $A$ in $\mathcal{A}$.

But $A$ in $\mathcal{A}$ implies that $A$ is closed in $E$, so

2. $A' \cap E \subseteq A$ for every $A$ in $\mathcal{A}$.

Now (1) implies that $P' \cap E$ is contained in each set of $\mathcal{B}$, that is,
(3) $P' \cap E \subseteq \cap \mathcal{A}$.

Also, from (2) and Lemma 5-15, we can write

(4) $\cap \mathcal{A} \subseteq \cap \mathcal{A}$.

But the intersection of $\mathcal{A}$ is $P$. Consequently (3) and (4) imply

(5) $P' \cap E \subseteq P$.

Thus $P$ is closed in $E$.

Remark. If $\mathcal{A}$ is a class of subsets of $X$, we will use $\mathcal{A}'$ to denote the class of all sets $K$ such that $K = A'$ for some $A$ in $\mathcal{A}$ and $A'(E)$ to denote the class of all sets $J$ such that $J = A' \cap E$ for some $A$ in $\mathcal{A}$.

Lemma 5-17. If $\mathcal{A}$ is a finite class of subsets of $X$, then

$$(\bigcup \mathcal{A})' = \bigcup \mathcal{A}'$$

Proof. If $\mathcal{A}$ is the empty class of subsets of $X$, then $\mathcal{A}'$ is also empty; and the union of each of $\mathcal{A}$ and $\mathcal{A}'$ is $\emptyset$. But $\emptyset = \emptyset'$. Hence the lemma holds if $\mathcal{A}$ is the class of subsets of $X$ containing no elements.

Suppose $t$ is a nonnegative integer and $\mathcal{A}$ a class of subsets of $X$ containing exactly $t$ sets implies $(\bigcup \mathcal{A})' = \bigcup \mathcal{A}'$.

Suppose also that $\mathcal{A}$ is a class of subsets of $X$ containing exactly $t+1$ sets. Choose one set $G$ from $\mathcal{A}$ and let $\mathcal{A}' = \mathcal{A} - \{G\}$.

Let $U = \bigcup \mathcal{A}$, $U = \bigcup \mathcal{A}'$, $V = \bigcup \mathcal{A}'$, and $V = \bigcup \mathcal{A}'$. Then by hypothesis, since $\mathcal{A}$ contains exactly $t$ sets, $U = V$. Now $U = U \cup G$ and $V = V \cup G$. Hence using Theorem 1-19, we have

$U' = (U \cup G)' = U' \cup G' = V \cup G' = V$.

Therefore the lemma follows by the principle of mathematical induction.
Lemma 5-18. If \( \mathcal{A} \) is a finite class of subsets of \( X \), then \( (\bigcup \mathcal{A}) \cap E = \bigcup \mathcal{A}'(E) \).

Proof. If \( \mathcal{A} \) is the empty class of subsets of \( X \), then each of \( \mathcal{A}' \) and \( \mathcal{A}'(E) \) is empty, and we have
\[
(\bigcup \mathcal{A}) \cap E = \emptyset \cap E = \emptyset = \bigcup \mathcal{A}'(E).
\]
Hence the lemma is true if \( \mathcal{A} \) is the class of subsets of \( X \) which contains no elements.

Suppose \( t \) is a nonnegative integer and if \( \mathcal{A} \) is a class of subsets of \( X \) containing exactly \( t \) sets, then \( (\bigcup \mathcal{A}) \cap E \) is \( \bigcup \mathcal{A}'(E) \). Suppose also that \( \mathcal{B} \) is a class of subsets of \( X \) which contains exactly \( t+1 \) sets. Choose one set \( G \) from \( \mathcal{B} \) and let \( \mathcal{A}_1 = \mathcal{A} - \{G\} \). Then \( (\bigcup \mathcal{A}_1) \cap E = \bigcup \mathcal{A}_1'(E) \) by hypothesis. Hence we have
\[
(\bigcup \mathcal{A}) \cap E = ((\bigcup \mathcal{A}_1) \cup G') \cap E \\
= ((\bigcup \mathcal{A}_1) \cap E) \cup (G' \cap E) \\
= (\bigcup \mathcal{A}_1'(E)) \cup (G' \cap E) \\
= \bigcup \mathcal{A}'(E).
\]
Therefore by the principle of mathematical induction, it follows that

\[
(\bigcup \mathcal{A}) \cap E = \bigcup \mathcal{A}'(E)
\]
for any finite set \( \mathcal{A} \).

Theorem 5-19. Suppose \( \mathcal{A} \) is a finite class of subsets of \( X \) each of which is closed in \( E \). Then \( \bigcup \mathcal{A} \) is closed in \( E \).

Proof. By Lemma 5-17 and Lemma 5-18,
\[
(\bigcup \mathcal{A})' \cap E = (\bigcup \mathcal{A}) \cap E = \bigcup \mathcal{A}'(E);
\]
and by hypothesis, each set of \( \mathcal{A}'(E) \) is a subset of some set
Therefore

\[ U^C \cup E \subseteq U^C. \]

that is, \( U^C \) is closed in \( E \).

**Theorem 5-20.** The nucleus \( N \) of \( E \) is closed in \( E \).

**Proof.** \( N \) is a subset of \( E \), and since \( N \) is dense-in-itself, \( N \) is contained in \( N' \). Consequently \( N \) is a subset of \( N' \cap E \), and we have

\[ N \subseteq N' \cap E \subseteq N'. \]

Hence, by Theorem 1-37, \( N' \cap E \) is dense-in-itself. Thus \( N' \cap E \) is a subset of \( E \) which is dense-in-itself and therefore contained in \( N \), that is,

\[ N' \cap E \subseteq N. \]

Hence \( N \) is closed in \( E \).

**Corollary 5-21.** Suppose \( E \) is closed. Then the nucleus \( N \) of \( E \) is perfect.

**Proof.** \( N \) is a subset of the closed set \( E \). Also, by Theorem 5-20, \( N \) is closed in \( E \). Hence Theorem 5-5 implies that \( N \) is closed. In addition, the nucleus of any set is dense-in-itself. Therefore \( N \) is perfect.
CHAPTER VI

COMPACTNESS

Covering. The statement that $\mathcal{A}$ is a covering of a set $S$ (or $\mathcal{A}$ covers $S$) means that $\mathcal{A}$ is a class of sets such that $S$ is a subset of the union of $\mathcal{A}$.

Open covering. Suppose $(X, \mathcal{T})$ is a topological space. Then the statement that $\mathcal{A}$ is an open covering of a set $S$ with respect to $\mathcal{T}$ (or $(X, \mathcal{T})$) means that $\mathcal{A}$ is a covering of $S$ and $\mathcal{A}$ is a subset of $\mathcal{T}$.

Compact set. The statement that $S$ is a compact set in a topological space $(X, \mathcal{T})$ means that $S$ is a subset of $X$ such that if $\mathcal{A}$ is an open covering of $S$ with respect to $\mathcal{T}$, then there is a finite subclass of $\mathcal{A}$ which also covers $S$ (1, p. 96).

In all further discussion we will say that $\mathcal{A}$ is an open covering of $S$ or $S$ is compact when only one topological space is being considered, the dependencies being understood.

Theorem 6-1. Suppose $S$ is a finite set in a topological space $(X, \mathcal{T})$. Then $S$ is compact.

Proof. Suppose $\mathcal{A}$ is an open covering of $S$. If $S = \emptyset$ the empty subclass of $\mathcal{A}$ is a finite class whose union is $S$.

Suppose $S \neq \emptyset$. Since $\mathcal{A}$ covers $S$, each point of $S$ is in some set of $\mathcal{A}$. If $x$ is in $S$ choose one set $A(x)$ from $\mathcal{A}$ which contains $x$ as an element, and let $\mathcal{G}$ denote the class of all
A(x)'s chosen. Since S is finite, \( \mathcal{T} \) is finite. Also S is a subset of the union of \( \mathcal{T} \). Hence S is compact.

**Theorem 6-2.** Suppose \((X, \mathcal{T})\) is a topological space, \(H\) is a subspace of \(X\), and \(\mathcal{U}\) is the relative topology for \(H\) induced by \(\mathcal{T}\). Then \(H\) is compact in \((X, \mathcal{T})\) if and only if \(H\) is compact in \((H, \mathcal{U})\).

**Proof.** Suppose \(H\) is compact in \((X, \mathcal{T})\) and \(\mathcal{A}\) is an open covering of \(H\) with respect to \(\mathcal{U}\). Let \(\mathcal{B}\) denote the class of all sets \(B\) in \(\mathcal{T}\) such that \(A = B \cap H\) for some \(A\) in \(\mathcal{A}\). Then \(\mathcal{B}\) is an open covering of \(H\) with respect to \(\mathcal{T}\).

Now since \(H\) is compact in \((X, \mathcal{T})\), there is a finite subclass \(\mathcal{B}'\) of \(\mathcal{B}\) which covers \(H\). Let \(\mathcal{A}'\) denote the class of all sets \(A'\) such that \(A' = B' \cap H\) for some \(B'\) of \(\mathcal{B}'\). Then \(\mathcal{A}'\) is a finite subclass of \(\mathcal{A}\) which covers \(H\). Therefore \(H\) is compact in \((H, \mathcal{U})\).

Suppose \(H\) is compact in \((H, \mathcal{U})\) and \(\mathcal{E}\) is an open covering of \(H\) with respect to \((X, \mathcal{T})\). Let \(\mathcal{G}\) be the class of all sets \(G\) such that \(G = E \cap H\) for some \(E\) in \(\mathcal{E}\). Then \(\mathcal{G}\) is a subclass of \(\mathcal{U}\) which covers \(H\), that is, \(\mathcal{G}\) is an open covering of \(H\) with respect to \((H, \mathcal{U})\). Hence there is a finite subclass \(\mathcal{G}'\) of \(\mathcal{G}\) which also covers \(H\). Let \(\mathcal{E}'\) designate the class of all sets \(E'\) such that \(G' = E' \cap H\) for some \(G'\) in \(\mathcal{G}'\). Then \(\mathcal{E}'\) is a finite subclass of \(\mathcal{E}\) which covers \(H\). Therefore \(H\) is compact in \((X, \mathcal{T})\), and the proof is complete.

**Theorem 6-3.** Suppose that each of \((X, \mathcal{T})\) and \((Y, \mathcal{K})\) is a topological space and \(f\) is a continuous mapping of \(X\) onto \(Y\).
Proof. Suppose \( X \) is compact in \((X, T)\) and \( \mathcal{A} \) is an open covering of \( Y \) with respect to \( T \). Let \( \mathcal{B} \) denote the class of all sets \( B \) such that \( B = f^{-1}(A) \) for some \( A \) in \( \mathcal{A} \). Then \( \mathcal{B} \) covers \( X \) and, by Theorem 2-1, each set of \( \mathcal{B} \) is an open set of \((X, T)\). Thus \( \mathcal{B} \) is an open covering of \( X \) with respect to \( T \); and since \( X \) is compact in \((X, T)\), there is a finite subclass \( \mathcal{B}_1 \) of \( \mathcal{B} \) which also covers \( X \). Let \( \mathcal{A} \) be the class of all sets \( A_1 \) such that \( A_1 = f(B_1) \) for some \( B_1 \) in \( \mathcal{B}_1 \). Then \( \mathcal{A} \) is a finite subclass of \( \mathcal{A} \) which covers \( Y \), since \( f \) is a function. Hence \( Y \) is compact in \((Y, T')\).

In the remainder of this chapter we will consider a single topological space \((X, T)\).

Theorem 6-4. If \( X \) is compact and \( S \) is a closed subset of \( X \), then \( S \) is compact.

Proof. Suppose \( X \) is compact and \( S \) is a closed subset of \( X \). If \( S \) is empty, it follows by Theorem 6-1 that \( S \) is compact. Assume that \( S \) is not empty and \( \mathcal{S} \) is an open covering of \( S \). Let \( \mathcal{A}_1 \) denote \( \mathcal{S} \cup \{S^c\} \). Since \( S \) is closed, \( S^c \) is open. Hence \( \mathcal{A}_1 \) is an open covering of \( X \); and since \( X \) is compact, there is a finite subclass \( \mathcal{A}_2 \) of \( \mathcal{A}_1 \) which covers \( X \). Consequently \( \mathcal{A}_2 - \{S^c\} \) is a finite subclass of \( \mathcal{A} \) which covers \( S \). Therefore \( S \) is compact, and the theorem is established.

Theorem 6-5. Suppose \( T \) is a compact subset of \( X \). Then every infinite subset of \( T \) has an accumulation point in \( T \).

Proof. Suppose \( S \) is a subset of \( T \) such that \( S \cap T \) is empty. Then \( t \) an element of \( T \) implies there is a neighborhood
V(t) of t such that $V(t) \cap S$ contains no point other than t. For each point t of T choose one such neighborhood $V(t)$ and let $\mathcal{A}$ be the class of neighborhoods chosen. Then each point of T is in some set of $\mathcal{A}$, so $\mathcal{A}$ is an open covering of T. But T is compact. Hence there is a finite subclass $\mathcal{B}$ of $\mathcal{A}$ which covers T. Now S is a subset of T, so $\mathcal{B}$ also covers S. By definition of $\mathcal{A}$ we see that no set of $\mathcal{B}$ can contain more than one point of S. Therefore, since $\mathcal{B}$ is finite, S is finite. Consequently if $Q$ is a subset of T and $Q$ is infinite, then $Q$ has at least one accumulation point in the set T.

**Property V.** The statement that a class $\mathcal{A}$ of sets has property V means that the intersection of $\mathcal{A}$ is empty.

**Property F.** The statement that a class $\mathcal{A}$ of sets has property F means that if $\mathcal{A}_i$ is a finite subclass of $\mathcal{A}$, then the intersection of $\mathcal{A}_i$ is not empty.

**Remark.** If $\mathcal{A}$ is a class of subsets of X, we will use $\overline{\mathcal{A}}$ to denote the class of all sets $S$ such that $S = \overline{A}$ for some $A$ in $\mathcal{A}$.

**Theorem 6-6.** X is compact if and only if each class of closed subsets of X having property V does not have property F.

**Proof.** Suppose X is compact and $\mathcal{A}$ is a class of closed subsets of X having property V. Since $\mathcal{A}$ has property V, the intersection I of $\mathcal{A}$ is empty. Let $\mathcal{B}$ denote the class of all sets B such that $B = A^c$ for some $A$ in $\mathcal{A}$. Then each set of $\mathcal{B}$ is open; and by Theorem 1-7, the union $U$ of $\mathcal{B}$ is the complement of I, that is, $U = X$. Hence $\mathcal{B}$ is an open covering of X;
and since $X$ is compact, there is a finite subclass $\mathcal{B}_i$ of $\mathcal{B}$ such that $\mathcal{B}_i$ covers $X$. Let $\mathcal{A}_i$ be the class of all sets $A_1$ such that $A_1 = B_1^c$ for some $B_1$ in $\mathcal{B}_i$. Then $\mathcal{A}_i$ is a finite subclass of $\mathcal{A}$, and the intersection of $\mathcal{A}_i$ is the complement of the union of $\mathcal{B}_i$; that is, the intersection of $\mathcal{A}_i$ is $\emptyset$.

Thus $\mathcal{A}$ does not have property $F$.

Suppose that each class of closed subsets of $X$ which has property $V$ fails to have property $F$ and $\mathcal{E}$ is an open covering of $X$. Let $\mathcal{G}$ denote the class of all sets $G$ such that $G = E^c$ for some $E$ in $\mathcal{E}$. Since $\mathcal{E}$ is a class of open sets, $\mathcal{G}$ is a class of closed sets. Now $\mathcal{E}$ covers $X$, so the union of $\mathcal{E}$ is $X$. Hence, by Theorem 1-7 and the definition of $\mathcal{G}$, the intersection of $\mathcal{G}$ is empty. Thus $\mathcal{G}$ is a class of closed sets having property $V$, so $\mathcal{G}$ does not have property $F$. Thus there is a finite subclass $\mathcal{G}_i$ of $\mathcal{G}$ such that the intersection of $\mathcal{G}_i$ is empty. Let $\mathcal{E}_i$ be the class of all sets $E_1$ such that $E_1 = G_1^c$ for some $G_1$ in $\mathcal{G}_i$. Then $\mathcal{E}_i$ is a finite subclass of $\mathcal{E}$; and by Theorem 1-7, the union of $\mathcal{E}_i$ is $X$. Thus $X$ is compact, and the theorem follows.

Remark. Theorem 6-6 can be restated to say that $X$ is compact if and only if each class of closed subsets of $X$ which has property $F$ fails to have property $V$.

Corollary 6-7. $X$ is compact if and only if each class $\mathcal{A}$ of subsets of $X$ which has property $F$ is such that $\mathcal{A}$ does not have property $V$. 
Proof. Suppose $X$ is compact and $\mathcal{A}$ is a class of subsets of $X$ which has property $F$. Consider $\overline{\mathcal{A}}$. Clearly $\overline{\mathcal{A}}$ is a class of closed sets which has property $F$. Thus from Theorem 6-6, we see that $\overline{\mathcal{A}}$ does not have property $V$.

Suppose that $\mathcal{E}$ a class of subsets of $X$ having property $F$ implies that $\overline{\mathcal{E}}$ does not have property $V$, and assume that $\mathcal{B}$ is a class of subsets of $X$ having property $F$. Then $\overline{\mathcal{B}}$ is a class of closed subsets of $X$ having property $F$. Consequently, by hypothesis, $\overline{\mathcal{B}}$ does not have property $V$. Therefore we see from Theorem 6-6 (and the remark preceding this corollary) that $X$ is compact.
CHAPTER BIBLIOGRAPHY

CHAPTER VII

AXIOMS OF COUNTABILITY, SEPARABILITY,
REGULARITY, AND TOPOLOGICAL PROPERTIES

Basis of neighborhoods at a point. Suppose $(X, \mathcal{T})$ is a
topological space, $y$ is a point of $X$, and $\mathcal{B}(y)$ is a class of
neighborhoods of $y$. Then the statement that $\mathcal{B}(y)$ is a basis
of neighborhoods at $y$ means that $V$ a neighborhood of $y$ implies
there is a set $B$ of $\mathcal{B}(y)$ such that $B$ is a subset of $V$.

Basis of neighborhoods for a topological space. The
statement that $\mathcal{B}$ is a basis of neighborhoods for a topological
space $(X, \mathcal{T})$ means that $\mathcal{B}$ is a class of open sets such that if
$y$ is a point of $X$, then there is a subclass $\mathcal{B}(y)$ of $\mathcal{B}$ such
that $\mathcal{B}(y)$ is a basis of neighborhoods at $y$.

Theorem 7-1. Suppose $\mathcal{B}$ is a basis of neighborhoods for
a topological space $(X, \mathcal{T})$. Then the union of $\mathcal{B}$ is $X$.

Proof. Since each set of $\mathcal{B}$ is a subset of $X$, the union
of $\mathcal{B}$ is a subset of $X$. Now suppose that $y$ is a point of $X$.
Then there is a subclass $\mathcal{B}(y)$ of $\mathcal{B}$ such that $\mathcal{B}(y)$ is a basis
of neighborhoods at $y$. Choose an open set $V$ which contains $y$.
Then there is a set $B$ in $\mathcal{B}(y)$ such that $B$ is a subset of $V$.
Hence $y$ a point of $X$ implies $y$ is an element of some set of $\mathcal{B}$.
Thus $X$ is a subset of the union of $\mathcal{B}$, and it follows that $X$
is the union of $\mathcal{B}$.
The first axiom of countability. A topological space 
\((X, \mathcal{F})\) is said to satisfy the first axiom of countability if 
and only if \(y\) in \(X\) implies there is a countable basis of 
neighborhoods at \(y\).

**Theorem 7-2.** Suppose \((X, \mathcal{F})\) is a topological space which 
satisfies the first axiom of countability, \(S\) is a subset of \(X\), 
and \(y\) is an accumulation point of \(S\). Then there is a sequence 
\(\{x_n\}\) of elements of \(S\) with the property that \(x_n \neq y\) for any \(n\) 
and such that if \(V\) is a neighborhood of \(y\), then there is a 
positive integer \(N(V)\) such that \(n \geq N(V)\) implies that \(x_n\) is an 
element of \(V\).

**Proof.** Choose a countable basis of neighborhoods \(\mathcal{B}(y)\) 
at \(y\). Let a sequence \(\{B_n\}\) be formed from all of the sets in 
\(\mathcal{B}(y)\). (If \(\mathcal{B}(y)\) is finite with \(M\) sets, let \(B_n = B_M\) for each 
\(n\) such that \(n > M\).) Also let \(\{\mathcal{E}_n\}\) denote the sequence of classes 
such that if \(t\) is a positive integer, then \(\mathcal{E}_t\) is composed of the 
first \(t\) sets of \(\{B_n\}\). Consider the sequence \(\{T_n\}\), where \(t\) a 
positive integer implies that \(T_t\) is the intersection of \(\mathcal{E}_t\). 
Since each set of each \(\mathcal{E}_n\) is a neighborhood of \(y\) and each \(\mathcal{E}_n\) is 
composed of a finite number of sets, it follows that each \(T_n\) is 
also a neighborhood of \(y\); and since \(y\) is an accumulation point 
of \(S\), it follows that \(t\) a positive integer implies there is a 
point in \(T_t \cap S\) which is not \(y\). Finally if \(t\) is a positive 
integer, choose one point \(x_t\) such that \(x_t\) is in \(T_t \cap S\) and \(t \neq y\). 
Let \(\{x_n\}\) denote the sequence formed. Now suppose that \(V\) is a 
neighborhood of \(y\). Then there is a set \(B_{N(V)}\) from the sequence
\{B_n\} such that \(B_n(V)\) is a subset of \(V\). Also for \(n \geq N(V)\), \(\mathcal{B}_n\) contains \(B_n(V)\) as an element; so \(T_n\) is a subset of \(B_n(V)\), and \(T_n\) is therefore a subset of \(V\). Thus for \(n \geq N(V)\), \(x_n\) is in \(T_n\) and consequently in \(V\), which completes the proof.

The second axiom of countability. The statement that a topological space \((X, \mathcal{T})\) satisfies the second axiom of countability means that there is a countable basis of neighborhoods for \((X, \mathcal{T})\).

Separable topological space. The statement that a topological space \((X, \mathcal{T})\) is separable means that there is a countable subset \(S\) of \(X\) such that \(\overline{S} = X\).

Perfectly separable topological space. A topological space is said to be perfectly separable if and only if it satisfies the second axiom of countability.

Theorem 7-3. Suppose \((X, \mathcal{T})\) is a perfectly separable topological space, \(S\) is a subset of \(X\), and \(\mathcal{A}\) is an open covering of \(S\). Then there is a countable subclass of \(\mathcal{A}\) which also covers \(S\).

Proof. If \(S\) is empty, the empty subclass of \(\mathcal{A}\) covers \(S\) and we are through. Suppose \(S \neq \emptyset\) and choose a countable basis \(\mathcal{B}\) of neighborhoods for \((X, \mathcal{T})\). If \(B\) is a set in \(\mathcal{B}\), let \(\mathcal{A}(B)\) denote the class of all sets \(A\) from \(\mathcal{A}\) which contain \(B\) as a subset. Let \(\mathcal{H}\) be a class formed by choosing one set from each of the nonempty \(\mathcal{A}(B)\)'s. Then \(\mathcal{H}\) is a subclass of \(\mathcal{A}\); and since \(\mathcal{B}\) is countable, so is \(\mathcal{H}\).
Suppose $x$ is an element of $S$. Then since $\mathcal{A}$ is an open covering of $S$, there is a set $A$ in $\mathcal{A}$ such that $x$ is in $A$. Also since $\mathcal{B}$ is a basis of neighborhoods for $(X,\mathcal{J})$, there is a set $B$ of $\mathcal{B}$ such that $B$ is a neighborhood of $x$ and a subset of $A$. Consequently there is a set $G$ of $\mathcal{B}$ which contains $B$ as a subset and $x$ as an element. Therefore $\mathcal{B}$ covers $S$, and the theorem is established.

**Theorem 7-4.** A perfectly separable topological space is separable.

**Proof.** Suppose $(X,\mathcal{J})$ is a perfectly separable topological space and choose a countable basis of neighborhoods $\mathcal{B}$ for $(X,\mathcal{J})$.

Let $S$ be a set formed by choosing one element from each set of $\mathcal{B}$. Since $\mathcal{B}$ is countable, so is $S$. Consider the interior $T$ of $S^C$.

Suppose, for contradiction, that there is a point $t$ in $T$. Since $T$ is an open set, it is a neighborhood of $t$. Hence there is a set $B$ in $\mathcal{B}$ such that $B$ is a neighborhood of $t$ and a subset of $T$. Thus $B$ is a set from $\mathcal{B}$ containing no point of $S$, which is contrary to the definition of $S$. Therefore $T$ is empty. Consequently $S = X$ as a result of Corollary 1-32. Thus $(X,\mathcal{J})$ is separable.

**Regularity.** A topological space $(X,\mathcal{J})$ is called regular if and only if $y$ in $X$ and $V$ a neighborhood of $y$ imply there is a neighborhood $T$ of $y$ such that $T$ is a subset of $V$ (1, p. 98).

**Property Q.** The statement that a topological space $(X,\mathcal{J})$ has property $Q$ means that if $x$ is in $X$ and $A$ is a closed set which does not contain $x$, then there are two disjoint open sets $U$ and $V$ such that $x$ is in $U$ and $A$ is a subset of $V$. 
Theorem 7-5. A topological space \((X, \mathcal{T})\) is regular if and only if it has property \(Q\).

Proof. Suppose \((X, \mathcal{T})\) has property \(Q\), \(y\) is a point of \(X\), and \(W\) is a neighborhood of \(y\). Let \(B\) denote the complement of \(W\). Then \(B\) is a closed set which does not contain \(y\); and since \((X, \mathcal{T})\) has property \(Q\), there is a pair of disjoint open sets \(S\) and \(P\) such that \(y\) is in \(S\) and \(B\) is a subset of \(P\). If \(P\) is empty \(B = W^c\) is empty, so \(W = X\) and \(\bar{S}\) is a subset of \(W\). Suppose \(P\) is not empty. Then by Theorem 3-4, \(S\) and \(P\) are separated; and by Theorem 3-13, \(\bar{S}\) is a subset of \(P^c\). Now \(B\) a subset of \(P\) implies that \(P^c\) is a subset of \(B^c = W\). Thus \(\bar{S}\) is a subset of \(W\). Hence \((X, \mathcal{T})\) is regular.

Suppose \((X, \mathcal{T})\) is regular, \(x\) is a point of \(X\), and \(A\) is a closed set which does not contain \(x\) as an element. Since \(A\) is a closed set which does not contain \(x\), \(A^c\) is a neighborhood of \(x\); and since \((X, \mathcal{T})\) is regular, there is a neighborhood \(U\) of \(x\) such that \(\bar{U}\) is a subset of \(A^c\). Now \(\bar{U}\) a subset of \(A^c\) implies \(A\) is contained in \((\bar{U})^c\). Also since \(\bar{U}\) is closed, \((\bar{U})^c\) is open; and since \(U\) is a subset of \(\bar{U}\), \(U \cap (\bar{U})^c = \emptyset\). Let \(V\) denote \((\bar{U})^c\). Then \(U\) and \(V\) are disjoint open sets such that \(x\) is in \(U\) and \(A\) is a subset of \(V\). Hence \((X, \mathcal{T})\) has property \(Q\).

Topological property. The statement that \(K\) is a topological property of a topological space \((X, \mathcal{T})\) means that if \((Y, \mathcal{A})\) is a topological space which is homeomorphic to \((X, \mathcal{T})\), then \(K\) is a property of \((Y, \mathcal{A})\).
Theorem 7-6. Suppose \((X, \mathcal{T})\) is a topological space. Then each of the following is a topological property of \((X, \mathcal{T})\):

(a) compactness of \(X\),
(b) connectedness of \(X\),
(c) \(x\) in \(X\) implies \(X - \{x\}\) is connected,
(d) perfect separability,
(e) separability,
(f) \(X = X'\),
(g) regularity.

Proof. Suppose \((Y, \mathcal{A})\) is a topological space and \(f\) is a homeomorphic mapping of \(X\) onto \(Y\). If \(X\) is compact, then \(Y\) is compact by Theorem 6-3. Also if \(X\) is connected, it follows from Theorem 3-3 that \(Y\) is connected. Thus (a) and (b) are topological properties of \((X, \mathcal{T})\).

Suppose \(z\) is an element of \(Y\) and \(x\) in \(X\) implies \(X - \{x\}\) is connected. Let \(Y_1 = Y - \{z\}\), \(X_1 = X - \{f^{-1}(z)\}\), and \(f_1 = f - \{[f^{-1}(z), z]\}\). Also let \(\mathcal{T}_1\) be the relative topology for \(X_1\) induced by \(\mathcal{T}\) and \(\mathcal{A}_1\) the relative topology for \(Y_1\) induced by \(\mathcal{A}_1\). By hypothesis \(X_1\) is connected in \((X, \mathcal{T})\); and by Theorem 4-8, \(X_1\) connected in \((X, \mathcal{T})\) implies \(X_1\) is connected in \((X_1, \mathcal{T}_1)\). Now since \(f_1\) is a one-to-one mapping of \(X_1\) onto \(Y_1\) and \(X_1\) is a connected set in \((X_1, \mathcal{T}_1)\), Theorem 3-3 implies that \(Y_1\) is a connected set in \((Y_1, \mathcal{A}_1)\). Consequently Theorem 4-8 guarantees that \(Y_1\) is connected in \((Y, \mathcal{A})\), so \((X, \mathcal{T})\) has (c) as a topological property.

Assume that \((X, \mathcal{T})\) is perfectly separable and choose a countable basis of neighborhoods \(\mathcal{B}\) for \((X, \mathcal{T})\). Let \(\mathcal{A}\) be the
class of all sets $A$ such that $A = f(B)$ for some $B$ in $\mathcal{B}$. Since $\mathcal{B}$ is countable, so is $\mathcal{A}$; and since $f$ is a one-to-one mapping of $X$ onto $Y$ and $f^{-1}$ is continuous, Theorem 2-1 guarantees that each set of $\mathcal{A}$ is an open set in $(Y, \mathcal{G})$. Now suppose that $y$ is an element of $Y$. Consider $f^{-1}(y)$. There is a subclass $\mathcal{B}[f^{-1}(y)]$ of $\mathcal{B}$ such that $\mathcal{B}[f^{-1}(y)]$ is a basis of neighborhoods at $f^{-1}(y)$. Let $\mathcal{A}(y)$ be the class of all sets $A_1$ such that $A_1 = f(B_1)$ for some $B_1$ in $\mathcal{B}[f^{-1}(y)]$. Then $\mathcal{A}(y)$ is a subclass of $\mathcal{A}$. Suppose $V$ is a neighborhood of $y$. Then $f^{-1}(V)$ is a neighborhood of $f^{-1}(y)$, so there is a set $U$ in $\mathcal{B}[f^{-1}(y)]$ such that $U$ is a subset of $f^{-1}(V)$, which implies that $f(U)$ is a subset of $V$. But $f(U)$ is in $\mathcal{A}(y)$. Hence $\mathcal{A}(y)$ is a basis of neighborhoods at $y$. Consequently $\mathcal{A}$ is a countable basis of neighborhoods for $(Y, \mathcal{G})$. Thus $(Y, \mathcal{G})$ is perfectly separable. Hence (d) is a topological property of $(X, \mathcal{I})$.

Suppose $(X, \mathcal{I})$ is separable and choose a countable subset $S$ of $X$ such that $S = X$. Since $S$ is countable, so is $f(S)$. Also $S = X$ implies $f(S) = Y$; and by Corollary 2-6, $f(S) = f(S)$. Thus $f(S) = Y$. Therefore $(Y, \mathcal{G})$ is separable and (e) is a topological property of $(X, \mathcal{I})$.

Suppose $X = X'$. Then $f(X) = f(X')$. Now $f(X) = Y$ and, by Theorem 2-3, $f(X') = [f(X)]'$. But $[f(X)]' = Y'$, so $Y = Y'$. Thus (f) is a topological property of $(X, \mathcal{I})$.

Suppose $(X, \mathcal{I})$ is regular, $y$ is a point of $Y$, and $V$ is a neighborhood of $y$. Since $f$ is a homeomorphism, $f^{-1}(V)$ is a neighborhood of $f^{-1}(y)$; and since $(X, \mathcal{I})$ is regular, there is
a neighborhood \( T \) of \( f^{-1}(y) \) such that \( T \) is a subset of \( f^{-1}(V) \).

But by Theorem 2-1, \( f^{-1} \) continuous implies \( f(T) \) is open. Thus \( f(T) \) is a neighborhood of \( y \). Also by Corollary 2-6, \( f(\overline{T}) = \overline{f(T)} \).

Hence \( T \) a subset of \( f^{-1}(V) \) implies that \( f(T) \) is a subset of \( V \).

Therefore \((Y, \mathcal{H})\) is regular, (g) is a topological property of \((X, \mathcal{T})\), and the theorem is established.
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CHAPTER VIII

SOME COMMON TOPOLOGICAL SPACES

\( T_1 \) topological space. The statement that a topological space \((X, \mathcal{T})\) is \( T_1 \) or a \( T_1 \) topological space means that if \( x \) and \( y \) are distinct points of \( X \), then there is a neighborhood \( V \) of \( x \) such that \( V \) does not contain \( y \).

\( T_2 \) topological space. The statement that a topological space \((X, \mathcal{T})\) is \( T_2 \) or a \( T_2 \) (Hausdorff) topological space means that if \( x_1 \) and \( x_2 \) are distinct points of \( X \), then there exist disjoint sets \( U_1 \) and \( U_2 \) such that \( U_1 \) is a neighborhood of \( x_1 \) and \( U_2 \) is a neighborhood of \( x_2 \).

\( T_3 \) topological space. The statement that a topological space \((X, \mathcal{T})\) is \( T_3 \) or a \( T_3 \) topological space means that \((X, \mathcal{T})\) is regular and \( T_1 \).

Normal topological space. A topological space \((X, \mathcal{T})\) is said to be normal if and only if \( A \) and \( B \) a pair of disjoint closed sets in \((X, \mathcal{T})\) implies that there is a pair of disjoint open sets \( U \) and \( V \) such that \( A \) is a subset of \( U \) and \( B \) is a subset of \( V \) (1, p. 104).

\( T_4 \) topological space. The statement that a topological space \((X, \mathcal{T})\) is \( T_4 \) or a \( T_4 \) topological space means that \((X, \mathcal{T})\) is normal and \( T_1 \).
Theorem 8-1. A topological space \((X, \mathcal{T})\) is T\(_1\) if and only if \(S\) a subset of \(X\) consisting of a single point implies that \(S\) is closed.

Proof. Suppose \((X, \mathcal{T})\) is T\(_1\), \(S\) is a subset of \(X\) consisting of a single point \(x\), \(y\) is an element of \(X\) and \(y \neq x\). Since \((X, \mathcal{T})\) is T\(_1\), there is a neighborhood \(V\) of \(y\) such that \(V\) does not contain \(x\). Hence \(y\) is not in \(S'\). Consequently no point of \(S'\) can be an accumulation point of \(S\), that is, \(S'\) is a subset of \(S\). Thus \(S\) is closed.

Suppose that \(T\) a subset of \(X\) consisting of a single point implies that \(T\) is closed. If \(X\) does not contain two distinct points, it follows vacuously that \(X\) is T\(_1\). Assume then that \(z\) and \(w\) are distinct points of \(X\). Let \(W\) denote the set consisting of the single point \(w\). Then \(W\) is closed by hypothesis, so \(W^c\) is an open set which contains \(z\) but not \(w\). Hence \((X, \mathcal{T})\) is T\(_1\).

Theorem 8-2. Suppose \((X, \mathcal{T})\) is a T\(_1\) topological space, \(V\) is an open subset of \(X\), \(t\) is a nonnegative integer, and \(T\) is a subset of \(V\) containing \(t\) elements. Then \(V - T\) is open.

Proof. Suppose \(t = 0\). Then \(T\) is empty, and \(V - T = V\) which is open. Hence the theorem is true for \(t = 0\).

Suppose \(n\) is a nonnegative integer, the theorem holds for \(t = n\), and \(T\) is a subset of \(V\) containing \(n + 1\) elements. Choose an element \(x\) of \(T\) and let \(U\) denote \(T - \{x\}\). Then \(W = V - U\) is open since \(U\) is a subset of \(V\) containing \(n\) elements. Consider \(V - T\), which can be expressed as \(W - \{x\}\). By Theorem 8-1, \(\{x\}\) is closed. Consequently \(X - \{x\}\) is open. Also

\[ W - \{x\} = W \cap [X - \{x\}] \]
Hence  $W - \{x\}$  is open, since each of $W$ and $X - \{x\}$ is open; that is, $V - T$ is an open set. Thus the theorem holds for $t = n + 1$. Therefore the theorem follows by the principle of mathematical induction.

**Theorem 8-3.** Suppose $(X, \mathcal{T})$  is a $T_1$  topological space and $S$  is a subset of $X$. Then $S'$ is closed.

**Proof.** Suppose $y$  is a point of $(S')'$  and $V$  is a neighborhood of $y$. Then there is a point $z$  in $V \cap S'$  such that $z \neq y$. Also since $V$  is a neighborhood of $z$  and $z$  is in $S'$, there is a point $w_1$  in $V \cap S$  such that $w_1 \neq z$. Consider the set $U = V - \{w_1\}$. By Theorem 8-2, $U$  is an open set. Thus $U$  is an open subset of $V$  which contains $z$  but not $w_1$. Consequently $U \cap S$  contains a point $w_2$  such that $w_2 \neq z$. Furthermore since $w_1$  is not in $U$, $w_1 \neq w_2$. Thus $V \cap S$  contains two points, $w_1$  and $w_2$, at least one of which is not $y$. Therefore $y$  is an accumulation point of $S$. Hence $(S')'$ is a subset of $S'$. It follows from Theorem 1-10 that $S'$ is closed.

**Theorem 8-4.** Suppose $(X, \mathcal{T})$  is a $T_1$  topological space and $S$ is a subset of $X$. Then a point $y$ is in $S'$ if and only if $V$  a neighborhood of $y$ implies that $V \cap S$ is infinite.

**Proof.** Suppose, for contradiction, that there is a point $y$  of $S'$, a nonnegative integer $n$, and a neighborhood $U$  of $y$ such that $U \cap S$  is a finite set containing exactly $n$  elements which are not $y$. Let $T$  denote $(U \cap S) - \{y\}$. Then by Theorem 8-2, $W = U - T$  is an open set. Hence $W$  is a neighborhood of $y$ such that $W \cap S$  contains no element other than
possibly \( y \), which is impossible since \( y \) is an accumulation point of \( S \). Therefore \( y \) in \( S' \) and \( U \) a neighborhood of \( y \) imply that \( U \cap S \) is infinite.

Suppose \( Q \) is a neighborhood of a point \( t \) and \( V \) a neighborhood of \( t \) implies that \( V \cap S \) is infinite. Then \( Q \cap S \) contains two distinct points \( x \) and \( z \). Now at most one of \( x \) and \( z \) can be \( t \). Hence \( Q \cap S \) contains a point which is not \( t \). Therefore \( t \) is in \( S' \).

**Corollary 8-5.** Suppose \( S \) is a finite set in a \( T_1 \) topological space. Then \( S' = \emptyset \).

**Proof.** Suppose \( y \) is a point and \( V \) is a neighborhood of \( y \). Then since \( S \) is finite, \( V \cap S \) is finite. Hence Theorem 8-4 guarantees that \( y \) is not an accumulation point of \( S \). Therefore \( S' = \emptyset \).

**Theorem 8-6.** Suppose \( (X, \mathcal{T}) \) is a Hausdorff topological space, \( S \) is a compact subset of \( X \), and \( x \) is a point in \( S^c \). Then there exist disjoint open sets \( U \) and \( V \) such that \( S \) is a subset of \( U \) and \( x \) is an element of \( V \).

**Proof.** If \( S \) is empty, \( U = \emptyset \) and \( V = X \) will suffice. Suppose \( S \) is not empty. If \( y \) is a point of \( S \) choose one neighborhood \( U(y) \) of \( y \) and one neighborhood \( V(y) \) of \( x \) such that \( U(y) \cap V(y) = \emptyset \). (\( U(y) \) and \( V(y) \) exist since the topological space is Hausdorff.) Let \( \mathcal{A} \) denote the class of \( U(y) \)'s chosen and \( \mathcal{B} \) the class of \( V(y) \)'s chosen. Since \( \mathcal{A} \) covers \( S \) and \( S \) is compact, there is a finite subclass \( \mathcal{A}' \) of \( \mathcal{A} \) which also covers \( S \). Let \( \mathcal{B}' \) denote the class of all sets \( V(y) \) from \( \mathcal{B} \) such that
U(y) is in $\mathcal{A}$. Since $\mathcal{A}$ is finite so is $\mathcal{B}$. Next let $U$ represent the union of $\mathcal{A}$ and $V$ the intersection of $\mathcal{B}$. Since $\mathcal{A}$ covers $S$, $S$ is a subset of $U$; and since each set of $\mathcal{A}$ is open, so is $U$. Also since $\mathcal{B}$ is a finite class of neighborhoods of $x$, $V$ is a neighborhood of $x$. Finally $U \cap V = \emptyset$. Otherwise there would be a point common to all the sets of $\mathcal{A}$ and in some set of $\mathcal{B}$, which is contrary to the definitions of $\mathcal{A}$ and $\mathcal{B}$. The theorem follows.

Theorem 8-7. Suppose $(X, \mathcal{F})$ is a Hausdorff topological space and $S$ is a compact subset of $X$. Then $S$ is closed.

Proof. If $S = X$ we are through. Suppose $S$ is not $X$ and $x$ is a point of $S^C$. By Theorem 8-6 there is a pair of disjoint open sets $U$ and $V$ such that $S$ is a subset of $U$ and $x$ is in $V$. Thus $V$ is a neighborhood of $x$ which contains no point of $S$. Consequently $x$ cannot be in $S'$. Hence $S^C$ is a subset of $(S')^C$, which implies that $S'$ is a subset of $S$. Therefore $S$ is closed.

Theorem 8-8. Suppose $S$ and $T$ are disjoint compact sets in a Hausdorff topological space. Then there exist disjoint open sets $U$ and $V$ such that $S$ is a subset of $U$ and $T$ is a subset of $V$.

Proof. If $t$ is a point of $T$ choose a pair of disjoint open sets $A(t)$ and $B(t)$ such that $S$ is a subset of $A(t)$ and $t$ is in $B(t)$. (The existence of $A(t)$ and $B(t)$ is guaranteed by Theorem 8-6.) Let $\mathcal{A}$ be the class of $A(t)$'s chosen and $\mathcal{B}$ the class of $B(t)$'s chosen. Then $\mathcal{B}$ is an open covering of $T$; and since $T$ is compact, there is a finite subclass $\mathcal{B}_1$ of $\mathcal{B}$ which
also covers \( T \). Let \( A_i \) be the set of all \( A(t)'s \) from \( A \) such that \( B(t) \) is in \( A_i \). Since \( A_i \) is finite so is \( A_i \). Consider the intersection \( U \) of \( A_i \) and the union \( V \) of \( A_i \). Since \( S \) is contained in each set of \( A_i \), \( A_i \) is finite, and each set of \( A_i \) is open, \( U \) is open and \( S \) is a subset of \( U \). Also since \( S_i \) is a class of open sets covering \( T \), \( T \) is a subset of \( V \) and \( V \) is open. Furthermore \( U \cap V = \emptyset \) since each point of \( V \) fails to be in some set of \( A_i \), and the proof is complete.

**Theorem 8-9.** Suppose \((X, \mathcal{F})\) is a \( T_3 \) topological space. Then \((X, \mathcal{F})\) is also \( T_2 \).

**Proof.** Suppose \( x \) and \( y \) are distinct points of \( X \). Since \((X, \mathcal{F})\) is \( T_3 \) it is regular and hence, by Theorem 7-5, has property \( Q \). Let \( A \) be the set consisting of the single point \( y \). Then \( x \) is not in \( A \); and since \((X, \mathcal{F})\) is \( T_1 \), Theorem 8-1 guarantees that \( A \) is closed. Also since \((X, \mathcal{F})\) has property \( Q \), there is a pair of disjoint open sets \( U \) and \( V \) such that \( x \) is in \( U \) and \( A \) is a subset of \( V \). But \( A \) a subset of \( V \) implies that \( y \) is a point of \( V \). Hence \((X, \mathcal{F})\) is \( T_2 \).

**Theorem 8-10.** Suppose \((X, \mathcal{F})\) is a \( T_4 \) topological space. Then \((X, \mathcal{F})\) is also \( T_3 \).

**Proof.** Since \((X, \mathcal{F})\) is \( T_4 \), it is normal and \( T_1 \). Suppose \( y \) is a point of \( X \) and \( A \) is a closed set which does not contain \( y \). Let \( B \) denote the set consisting of the single point \( y \). By Theorem 8-1, and since \((X, \mathcal{F})\) is \( T_1 \), \( B \) is closed. Furthermore since \( y \) is not a point of \( A \), \( A \cap B = \emptyset \). Now as a consequence of the normality of \((X, \mathcal{F})\), there exist open disjoint sets \( U \)
and $V$ such that $A$ is a subset of $U$ and $B$ is a subset of $V$.

But $B$ a subset of $V$ implies $y$ is in $V$. Hence $(X, \mathcal{T})$ has
property Q. Thus by Theorem 7-3, $(X, \mathcal{T})$ is regular. Therefore $(X, \mathcal{T})$ is $T_3$. 
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