CONVERGENCE PROPERTIES OF FILTERS AND NETS

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CONVERGENCE PROPERTIES OF FILTERS AND NETS

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By

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. FILTERS ON SETS</td>
<td>1</td>
</tr>
<tr>
<td>II. FILTERS AND CONVERGENCE</td>
<td>18</td>
</tr>
<tr>
<td>III. NETS AND CONVERGENCE</td>
<td>40</td>
</tr>
<tr>
<td>IV. RELATIONSHIPS BETWEEN NETS AND FILTERS</td>
<td>51</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>61</td>
</tr>
</tbody>
</table>
CHAPTER I
FILTERS ON SETS

The concept of a filter leads to an important theory of convergence in topological spaces. The purpose of the present chapter is to define a filter and to determine some of its properties.

DEFINITION 1.1 A filter $\mathcal{F}$ on a set $X$ is a collection of subsets of $X$ which has the following properties:

(1) Each subset of $X$ which contains a member of $\mathcal{F}$ belongs to $\mathcal{F}$;

(2) Each finite intersection of members of $\mathcal{F}$ belongs to $\mathcal{F}$.

(3) The empty set does not belong to $\mathcal{F}$.

EXAMPLE 1.1 If $X$ is an infinite set, then the family of complements of the finite subsets of $X$ is a filter on $X$. If $X$ is the set of positive integers, then this filter is called the Frechet filter.

DEFINITION 1.2 Let $\mathcal{F}$ and $\mathcal{G}$ be filters on a set $X$. Then $\mathcal{F}$ is said to be finer than $\mathcal{G}$ if $\mathcal{G} \subseteq \mathcal{F}$.

PROPOSITION 1.1 If the intersection of all the sets of a filter $\mathcal{F}$ on an infinite set $X$ is empty, then $\mathcal{F}$ is finer than the filter $\mathcal{G}$ of complements of finite subsets of $X$. 

1
PROOF. Suppose that there is a member \( G \in \mathcal{F} \) which does not belong to \( \mathcal{F} \). If \( F \in \mathcal{F} \), it follows that \( F \cap (X-G) \neq \emptyset \). The set \( X-G \) may be represented as \( \{x_1, \ldots, x_n\} \). If \( x_1 \in X-G \), there is a set \( F_1 \in \mathcal{F} \) such that \( x_1 \in F_1 \). Now \( F_1 \subseteq F \), but \( F_1 \cap (X-G) = \emptyset \). Thus \( G \in \mathcal{F} \).

PROPOSITION 1.2 If \( \{\mathcal{F}_a \mid a \in I\} \) is a collection of filters on a set \( X \), then \( \bigcap \{\mathcal{F}_a \mid a \in I\} \) is a filter on \( X \).

PROOF. If \( F \) and \( G \) belong to \( \bigcap \{\mathcal{F}_a \mid a \in I\} \), then \( F \cap G \) belongs to each filter \( \mathcal{F}_a \). It follows that \( F \cap G \in \mathcal{F}_a \).

If \( F \in \mathcal{F}_a \) and \( F \subseteq X \), then \( G \) belongs to each \( \mathcal{F}_a \).

Thus \( G \in \mathcal{F}_a \).

The empty set does not belong to \( \mathcal{F}_a \) because it does not belong to any \( \mathcal{F}_a \).

Hence \( \bigcap \{\mathcal{F}_a \mid a \in I\} \) is a filter on \( X \).

PROPOSITION 1.3 Let \( \{\mathcal{F}_a \mid a \in I\} \) be a collection of filters on a set \( X \). Then \( \bigcap \{\mathcal{F}_a \mid a \in I\} = \{U \mid A \subseteq U \text{ and } a \in I\} \).

PROOF. If \( A \in \bigcap \{\mathcal{F}_a \mid a \in I\} \), then \( A = \bigcup \{A_a \mid a \in I\} \) where \( A_a \subseteq A_a \) for each \( a \in I \).

Let \( B \) be a set of the form \( \bigcup A_a \) where \( A_a \subseteq A_a \). If \( a \in I \), then \( A_a \subseteq B \). Hence \( A_a \subseteq B \), and the proposition follows.

PROPOSITION 1.4 Let \( \mathcal{F} \) and \( \mathcal{G} \) be filters on a set \( X \). If \( \mathcal{F} \) and \( \mathcal{G} \) have a least upper bound in the set of all filters on \( X \), then this upper bound is \( \{M \mid M \subseteq \mathcal{F} \text{ and } M \subseteq \mathcal{G}\} \).

PROOF. Let \( \mathcal{H} = \{M \mid M \subseteq \mathcal{F} \text{ and } M \subseteq \mathcal{G}\} \). If \( M_1 \cap N_1 \) and \( M_2 \cap N_2 \) belong to \( \mathcal{H} \), then \( (M_1 \cap N_1) \cap (M_2 \cap N_2) = (M_1 \cap N_2) \cap (N_1 \cap N_2) \).

Thus \( (M_1 \cap N_1) \cap (M_2 \cap N_2) \in \mathcal{H} \).
If $M \cap N \neq H$ and $M \cap N \subseteq C = X$, then $A = (M \cup A) \cap (N \cup A)$.

Hence $A \neq H$.

If $M \not\subseteq H$ and $N \not\subseteq H$, then $M \cap N \not\subseteq \emptyset$ because $M \cap N$ belongs to the least upper bound of $H$ and $G$. Thus $H$ is a filter.

The filter $H$ is an upper bound for $H$ and $G$ because $H \subseteq H$ and $G \subseteq H$.

If $A$ is any filter which contains both $H$ and $G$, then $A$ contains $H$. Thus $H$ is the least upper bound of $H$ and $G$ in the set of all filters on $X$.

**THEOREM 1.1** A necessary and sufficient condition that there is a filter on a set $X$ containing a set $\mathcal{Y}$ of subsets of $X$ is that no finite subset of $\mathcal{Y}$ has an empty intersection.

**PROOF.** If there is a filter on $X$ which contains $\mathcal{Y}$, then it follows from (2) of Definition 1.1 that no finite subset of $\mathcal{Y}$ has an empty intersection.

Conversely, if $\mathcal{Y}'$ denotes the set of all finite intersections of members of $\mathcal{Y}$, then $\mathcal{Y}' = \{ A \subseteq X | A \cap G' \text{ for some } G \in \mathcal{Y}' \}$ is a filter on $X$ which contains $\mathcal{Y}$.

**DEFINITION 1.3** If $\mathcal{Y}$ is a collection of subsets of a set $X$, then $\mathcal{Y}$ is said to be a subbase for the filter $\mathcal{Y}'$ constructed in Theorem 1.1 above whenever $\mathcal{Y}''$ exists.

**PROPOSITION 1.5** If $\mathcal{F}$ is a filter on $X$ and $A$ is a subset of $X$, then there is a filter $\mathcal{F}'$ on $X$ which is finer than $\mathcal{F}$ and such that $A \in \mathcal{F}'$ if and only if $A$ meets each member of $\mathcal{F}$.

**PROOF.** If $\mathcal{F}'$ exists, then $A$ meets each member of $\mathcal{F}$ because the empty set does not belong to $\mathcal{F}'$. 
Conversely, if A meets each member of \( \mathcal{F} \), Theorem 1.1 states that there is a filter \( \mathcal{F}' \) on \( X \) which contains \( \mathcal{F} \cup \{ A \} \). Then \( \mathcal{F}' \) is finer than \( \mathcal{F} \), and \( A \in \mathcal{F}' \).

**THEOREM 1.2** If \( \mathcal{B} \) is a collection of subsets of a set \( X \), then \( \mathcal{B} = \{ A \in X | A \supseteq B \text{ for some } B \in \mathcal{B} \} \) is a filter on \( X \) if and only if \( \mathcal{B} \) has the following two properties:

1. The intersection of two sets of \( \mathcal{B} \) contains a set of \( \mathcal{B} \);
2. \( \mathcal{B} \) is not empty, and the empty set of \( X \) is not in \( \mathcal{B} \).

**PROOF.** Suppose that \( \mathcal{F} \) is a filter. If \( B_1 \) and \( B_2 \) belong to \( \mathcal{B} \), the \( B_1 \cap B_2 \) is in \( \mathcal{F} \). Thus there is a member \( B_3 \) of \( \mathcal{B} \) so that \( B_3 \subseteq B_1 \cap B_2 \).

If \( \mathcal{B} \) is empty or if the empty set belongs to \( \mathcal{B} \), then the empty set belongs to \( \mathcal{F} \).

Thus \( \mathcal{B} \) has properties (1) and (2).

Suppose that \( \mathcal{B} \) has properties (1) and (2). If \( A_1 \) and \( A_2 \) belong to \( \mathcal{F} \), then there are elements \( B_1 \) and \( B_2 \) of \( \mathcal{B} \) such that \( B_1 \subseteq A_1 \) and \( B_2 \subseteq A_2 \). Then \( B_1 \cap B_2 \subseteq A_1 \cap A_2 \), and there is a member \( B_3 \) of \( \mathcal{B} \) so that \( B_3 \subseteq B_1 \cap B_2 \). Thus \( B_3 \subseteq A_1 \cap A_2 \), and \( A_1 \cap A_2 \subseteq \mathcal{F} \).

If \( A_1 \in \mathcal{F} \) and \( A_1 \subseteq A_2 \subseteq X \), then there is a set \( B \in \mathcal{B} \) so that \( B \subseteq A_1 \). It follows that \( B \subseteq A_2 \). Hence \( A_2 \in \mathcal{F} \).

The empty set is not in \( \mathcal{F} \) because it is not in \( \mathcal{B} \). Thus \( \mathcal{F} \) is a filter.

**DEFINITION 1.4** A set \( \mathcal{B} \) of subsets of a set \( X \) which satisfies conditions (1) and (2) of Theorem 1.2 is said to be
a base of the filter $\mathcal{F}=\{A \subseteq X \mid A \supseteq B \text{ for some } B \in \mathcal{B}\}$.

The filter $\mathcal{F}$ is said to be generated by $\mathcal{B}$. Two filter bases are equivalent if they generate the same filter.

**THEOREM 1.3** A subset $\mathcal{B}$ of a filter $\mathcal{F}$ on $X$ is a base of $\mathcal{F}$ if and only if every set of $\mathcal{F}$ contains a set of $\mathcal{B}$.

**PROOF.** If $\mathcal{B}$ is a base of $\mathcal{F}$, then Definition 1.4 states that every set of $\mathcal{F}$ contains a set of $\mathcal{B}$.

If $B_1$ and $B_2$ belong to $\mathcal{B}$, then they also belong to $\mathcal{F}$. Hence there is a set $B_3 \in \mathcal{B}$ so that $B_3 \subseteq B_1 \cap B_2$.

Condition (2) of Definition 1.4 is also satisfied because the empty set does not belong to $\mathcal{F}$. Then $\mathcal{B}$ is a base.

Let $\mathcal{G}$ denote the filter generated by $\mathcal{B}$, and let $G \in \mathcal{G}$. Then $G \in \mathcal{F}$ because $G$ contains a member of $\mathcal{B}$. If $F \in \mathcal{F}$, then $F$ contains a member $B$ of $\mathcal{B}$. Thus $F \in \mathcal{G}$. It follows that $\mathcal{F} = \mathcal{G}$.

**THEOREM 1.4** On a set $X$ a filter $\mathcal{F}$ with base $\mathcal{B}$ is finer than a filter $\mathcal{F}'$ with base $\mathcal{B}'$ if and only if every set of $\mathcal{B}$ contains a set of $\mathcal{B}'$.

**PROOF.** If $\mathcal{F}'$ is finer than $\mathcal{F}$ and if $B \in \mathcal{B}$, then $B \in \mathcal{F}'$.

There is a set $B' \in \mathcal{B}'$ so that $B' \subseteq B$.

If every set of $\mathcal{B}$ contains a set of $\mathcal{B}'$ and if $F \in \mathcal{F}$, there is a member $B$ of $\mathcal{B}$ so that $B \subseteq F$. Now there is a set $B' \in \mathcal{B}'$ such that $B' \subseteq B$. Then $B' \subseteq F$, and it follows that $F \in \mathcal{F}'$.

**COROLLARY 1.1** Two filter bases $\mathcal{B}$ and $\mathcal{B}'$ on a set $X$ are equivalent if and only if every set of $\mathcal{B}$ contains a set of $\mathcal{B}'$ and every set of $\mathcal{B}'$ contains a set of $\mathcal{B}$.
PROOF. This statement is an immediate consequence of Theorem 1.4.

DEFINITION 1.5 If $X$ is a set, the cardinal number of $X$ is denoted by $|X|$. If $\mathcal{F}$ is a filter, the potency of $\mathcal{F}$ is denoted by $Pz(\mathcal{F})$ and is defined as $Pz(\mathcal{F}) = \min \{|M| | M \in \mathcal{F} \}$.

DEFINITION 1.6 If $\mathcal{F}$ is a filter, the rank of $\mathcal{F}$ is denoted by $Rk(\mathcal{F})$ and is defined as $Rk(\mathcal{F}) = \min \{|B| | B \text{ is a base for } \mathcal{F} \}$.

THEOREM 1.5 If $\mathcal{F}$ is a filter on a set $X$, then

1. Either $Rk(\mathcal{F}) = 1$ or $Rk(\mathcal{F}) \geq \aleph_0$.
2. $Rk(\mathcal{F}) \leq 2^{Pz(\mathcal{F})}$.

PROOF. If $Rk(\mathcal{F})$ is finite, then $\mathcal{F}$ has a finite base $B = \{B_1, \ldots, B_n\}$. The set $B = \bigcap_{i=1}^n B_i$ belongs to $\mathcal{F}$. Thus there is some $B_k \in B$ with $B_k \subseteq B$. Certainly $B \subseteq B_i$ for each $B_i \in B$. Consequently $B$ and $\{B\}$ are equivalent bases, and $\{B\}$ generates $\mathcal{F}$. Hence $Rk(\mathcal{F}) = 1$. It follows that if $Rk(\mathcal{F}) \neq 1$, then $Rk(\mathcal{F}) \geq \aleph_0$.

If $M \in \mathcal{F}$ and $|M| = Pz(\mathcal{F})$, then $\mathcal{B} = \{F \cap M | F \in \mathcal{F}\}$ is a base for $\mathcal{F}$. Then $Rk(\mathcal{F}) \leq |\mathcal{B}| \leq |\mathcal{P}(M)| = 2^{Pz(\mathcal{F})}$.

Filters which are maximal on a set play an important role in the theory of filters.

DEFINITION 1.7 An ultrafilter on a set $X$ is a filter $\mathcal{F}$ such that $\mathcal{F}$ is not properly contained in any other filter on $X$.

THEOREM 1.6 If $\mathcal{F}$ is any filter on a set $X$, there is an ultrafilter finer than $\mathcal{F}$.

PROOF. Define $\mathcal{A} = \{\mathcal{F}' | \mathcal{F}' \text{ is a filter on } X \text{ and } \mathcal{F} \subseteq \mathcal{F}' \}$. Now $\mathcal{A}$ is not empty because $\mathcal{F} \in \mathcal{A}$. Let $\{\mathcal{F}_\alpha | \alpha \in I\}$ be a chain in $\mathcal{A}$,
and consider $\bigcup \{ G_a \mid a \in I \}$. If $G_1$ and $G_2$ belong to $\bigcup \{ G_a \mid a \in I \}$, then there are members $a_1$ and $a_2$ of $I$ such that $G_1 \in G_{a_1}$ and $G_2 \in G_{a_2}$. It may be assumed that $G_{a_1} \subseteq G_{a_2}$, in which case $G_1$ and $G_2$ belong to $G_{a_2}$. It follows that $G_1 \cap G_2 \in \bigcup \{ G_a \mid a \in I \}$.

Let $A \in \bigcup \{ G_a \mid a \in I \}$, and suppose that $A \subset B$. Then $A$ belongs to some $G_a$. Thus $B \in G_a$; hence $B \in \bigcup \{ G_a \mid a \in I \}$.

The empty set does not belong to $\bigcup \{ G_a \mid a \in I \}$ because it belongs to none of the filters $G_a$. It follows that $\bigcup \{ G_a \mid a \in I \}$ is a filter, and certainly $\emptyset \in \bigcup \{ G_a \mid a \in I \}$. By Zorn's Lemma $\mathcal{A}$ has a maximal element $U$. Then $U$ is an ultrafilter on $X$ finer than $\mathcal{F}$.

**THEOREM 1.7** Let $\mathcal{F}$ be an ultrafilter on a set $X$. If $A$ and $B$ are two subsets of $X$ such that $A \cup B \in \mathcal{F}$, then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

**PROOF.** Suppose $A \notin \mathcal{F}$. Then $F \cap (X-A) \neq \emptyset$ for each $F \in \mathcal{F}$. Define $\mathcal{G} = \{ C \subseteq X \mid C \supseteq F \cap (X-A) \text{ for some } F \in \mathcal{F} \}$. If $C_1$ and $C_2$ belong to $\mathcal{G}$, there are sets $F_1$ and $F_2$ in $\mathcal{F}$ so that $F_1 \cap (X-A) \subset C_1$ and $F_2 \cap (X-A) \subset C_2$. It follows that

$C_1 \cap C_2 \supset [F_1 \cap (X-A)] \cap [F_2 \cap (X-A)] = (F_1 \cap F_2) \cap (X-A)$. Since $F_1 \cap F_2 \in \mathcal{F}$, then $C_1 \cap C_2 \in \mathcal{G}$. Properties (1) and (3) of Definition 1.1 are clear; consequently $\mathcal{G}$ is a filter on $X$. Now

$(A \cup B) \cap (X-A) \in \mathcal{G}$, and $(A \cup B) \cap (X-A) = B \cap (X-A)$. Moreover, $B \cap (X-A) \subseteq B$. Thus $B \in \mathcal{G}$. It is also true that $\mathcal{F} \subseteq \mathcal{G}$, and since $\mathcal{F}$ is an ultrafilter, it follows that $\mathcal{F} = \mathcal{G}$. Then $B \in \mathcal{F}$.

A similar proof shows that if $B \notin \mathcal{F}$, then $A \in \mathcal{F}$.

**COROLLARY 1.2** Let $\mathcal{F}$ be an ultrafilter on $X$, and let $A = X$. Then either $A \in \mathcal{F}$ or $X-A \in \mathcal{F}$. 


PROOF. Since $A \cup (X-A) = X \in \mathcal{U}$, either $A \in \mathcal{U}$ or $X-A \in \mathcal{U}$ by Theorem 1.7.

THEOREM 1.8 If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on a set $X$, then $|\mathcal{U}| = |\mathcal{V}|$.

PROOF. Define $A = \{X-A \mid A \in H\}$. Then $A \cup \mathcal{U} = \mathcal{P}(X)$ by the corollary following Theorem 1.7. If $f: \mathcal{U} \rightarrow A$ is defined by $f(A) = X-A$ for each $A \in \mathcal{U}$, $f$ is clearly one-to-one and onto. Thus $|\mathcal{U}| = |A|$. It is also clear that $\mathcal{U} \cap A = \emptyset$.

The proof now divides into two cases:

Case 1: $X$ is finite. Then $|X| = n$ for some positive integer $n$.

It follows that $|\mathcal{P}(X)| = 2^n$. However, it is true that

$$|\mathcal{P}(X)| = |A \cup \mathcal{U}| = |A| + |\mathcal{U}| = |\mathcal{U}| + |\mathcal{U}| = 2|\mathcal{U}|.$$ 

Thus $2|\mathcal{U}| = 2^n$ so that $|\mathcal{U}| = 2^{n-1}$.

Case 2: $X$ is infinite. Then $|\mathcal{P}(X)| = 2^{|X|}$, and

$$|\mathcal{P}(X)| = |A \cup \mathcal{U}| = |A| + |\mathcal{U}| = |\mathcal{U}| + |\mathcal{U}| = |\mathcal{U}|.$$ 

Thus $|\mathcal{U}| = 2^{|X|}$.

Similarly $|\mathcal{U}| = 2^{n-1}$ if $X$ is finite, and $|\mathcal{U}| = 2^{|X|}$ if $X$ is infinite. It follows that $|\mathcal{U}| = |\mathcal{V}|$.

THEOREM 1.9 Let $\mathcal{Y}$ be a subbase of a filter on a set $X$.

If for each subset $Y$ of $X$ either $Y \in \mathcal{Y}$ or $X-Y \in \mathcal{Y}$, then $\mathcal{Y}$ is an ultrafilter on $X$.

PROOF. There is an ultrafilter $\mathcal{U}$ which contains $\mathcal{Y}$. Let $Y \subseteq X$. If $Y \in \mathcal{U}$, then $X-Y \notin \mathcal{U}$. Then $X-Y \notin \mathcal{Y}$, so $Y \in \mathcal{Y}$. Hence $\mathcal{Y} = \mathcal{U}$.

THEOREM 1.10 Every filter $\mathcal{F}$ on a set $X$ is the intersection of the ultrafilters finer than $\mathcal{F}$. 
PROOF. Let \( \{ \mathcal{U}_\alpha \}_{\alpha \in I} \) be the collection of all ultrafilters on \( X \) which are finer than \( \mathcal{F} \). Then \( \mathcal{F} = \cap \{ \mathcal{U}_\alpha \}_{\alpha \in I} \).

Let \( A \) be a subset of \( X \) which does not belong to \( \mathcal{F} \). If \( \mathcal{F} \) is coarser than \( \mathcal{F} \), then \( \cap \mathcal{F} X - A \neq \emptyset \). The family \( \{ F \cap (X - A) \mid F \in \mathcal{F} \} \) is a base for a filter \( \mathcal{F}' \) on \( X \) which is finer than \( \mathcal{F} \) and which contains \( X - A \). There is an ultrafilter \( \mathcal{V} \) which is finer than \( \mathcal{F}' \) by Theorem 1.6. Then \( \mathcal{V} \subseteq \{ \mathcal{U}_\alpha \}_{\alpha \in I} \) and \( A \notin \mathcal{V} \). Hence

\[ \cap \{ \mathcal{U}_\alpha \}_{\alpha \in I} = \mathcal{F} , \text{ and the proof is complete.} \]

PROPOSITION 1.6 Every ultrafilter which is finer than the intersection of a finite number of filters is finer than at least one of them.

PROOF. Suppose that \( \{ \mathcal{F}_i \}_{i = 1, \ldots, n} \) is a finite collection of filters on a set \( X \) and that \( \mathcal{U} \) is an ultrafilter on \( X \) which is finer than \( \mathcal{F}_i \). If \( \mathcal{U} \) is not finer than any \( \mathcal{F}_i \), then if \( 1 \leq i \leq n \), there is a set \( \mathcal{F}_i \subseteq \mathcal{F}_i \) such that \( \mathcal{F}_i \notin \mathcal{U} \). Since \( \mathcal{U} \) is an ultrafilter, it follows that \( X - \mathcal{F}_i \in \mathcal{U} \) for \( 1 \leq i \leq n \). Then \( \mathcal{F}_i (X - \mathcal{F}_i) \in \mathcal{U} \), and \( \mathcal{F}_i (X - \mathcal{F}_i) \) is equal to \( X - \mathcal{F}_i \). Thus \( \mathcal{F}_i \in \mathcal{U} \). However, \( \mathcal{F}_i \notin \mathcal{U} \), and \( \mathcal{F}_i \notin \mathcal{U} \). Hence it must be the case that \( \mathcal{U} \) is finer than some \( \mathcal{F}_i \).

The following example shows that the converse of Proposition 1.6 is not true.

EXAMPLE 1.2 Let \( X = \{ 2i \mid i = 1, 2, \ldots \} \), and let \( \mathcal{F}_i \) be the ultrafilter generated by the positive even integer \( 2i \). Define \( A = \{ 2^k, 2 \cdot 2^k, 3 \cdot 2^k, \ldots \} \mid k = 1, 2, 3, \ldots \} \). Then \( A \) is a base for a filter \( \mathcal{F} \) on \( X \), and there is an ultrafilter \( \mathcal{U} \) finer than \( \mathcal{F} \). For each positive integer \( i \), \( \mathcal{U} \notin \mathcal{F}_i \).
for given \( i \), there is a positive integer \( k \) so that \( 2^k > 2i \). Then \( \{ 2^k, 2 \cdot 2^k, 3 \cdot 2^k, \ldots \} \) because \( 21 \) does not belong to \( \{ 2^k, 2 \cdot 2^k, \ldots \} \). However, \( \bigcap_{i=1}^{\infty} X_i = \{ x \} \), and \( x \in U \).

**PROPOSITION 1.7** The intersection of all the sets of an ultrafilter contains at most one point; if the intersection is one point, then the ultrafilter consists of all the sets containing this point.

**PROOF.** Let \( U \) be an ultrafilter on a set \( X \). If \( \bigcap \{ U \mid x \in U \} = \emptyset \), there is nothing to prove. If \( \bigcap \{ U \mid x \in U \} \neq \emptyset \), there is a point \( x \in X \) which belongs to \( \bigcap \{ U \mid x \in U \} \). Suppose by way of contradiction that there is a point \( y \) distinct from \( x \) which also belongs to \( \bigcap \{ U \mid x \in U \} \). Then \( \{ x \} \not\in U \). It follows that \( U \) is properly contained in the filter generated by \( \{ x \} \). However, this is impossible because \( U \) is an ultrafilter. Thus it must be the case that \( \bigcap \{ U \mid x \in U \} = \{ x \} \).

If \( \bigcap \{ U \mid x \in U \} = \{ x \} \), then \( x \) belongs to each \( U \in U \). If \( A \) is any subset of \( X \) which contains \( x \), then \( A \in U \); for otherwise, \( X - A \in U \), and then \( \bigcap \{ U \mid x \in U \} \neq \{ x \} \). Thus \( U \) consists of all the sets containing \( x \).

**DEFINITION 1.8** Let \( \mathcal{F} \) be a filter on a set \( X \), and let \( A \subseteq X \). Then the trace of \( \mathcal{F} \) on \( A \) is denoted by \( \mathcal{F}_A \) and is defined by \( \mathcal{F}_A = \{ F \cap A \mid F \in \mathcal{F} \} \).

**THEOREM 1.11** If \( \mathcal{F} \) is a filter on a set \( X \) and \( A \subseteq X \), then \( \mathcal{F}_A \) is a filter on \( A \) if and only if each set of \( \mathcal{F} \) meets \( A \).

**PROOF.** If \( \mathcal{F}_A \) is a filter on \( A \), then each set of \( \mathcal{F} \) meets \( A \) because \( A \in \mathcal{F}_A \).
Conversely, suppose that each set of $\mathcal{F}$ meets $A$. Then the empty set does not belong to $\mathcal{F}_A$. If $F_1 \cap A$ and $F_2 \cap A$ belong to $\mathcal{F}_A$, then $(F_1 \cap A) \cap (F_2 \cap A) = (F_1 \cap F_2) \cap A$. If $F \cap A = P \cap A$, then $P = (F \cup P) \cap A$. Thus $P \in \mathcal{F}_A$, and $\mathcal{F}_A$ is a filter on $A$.

**DEFINITION 1.9** Let $A$ be a subset of a set $X$, and let $\mathcal{F}$ be a filter on $X$. If $\mathcal{F}_A$ is a filter on $A$, then $\mathcal{F}_A$ is said to be induced by $\mathcal{F}$ on $A$.

**THEOREM 1.12** An ultrafilter $\mathcal{U}$ on a set $X$ induces a filter on a subset $A$ of $X$ if and only if $A \in \mathcal{U}$; if this condition is satisfied, then $\mathcal{U}_A$ is an ultrafilter on $A$.

**PROOF.** If $\mathcal{U}_A$ is a filter on $A$, then $A$ meets every set of $\mathcal{U}$. Then $A \in \mathcal{U}$; for if $A \notin \mathcal{U}$, $X-A \in \mathcal{U}$, and $A$ and $X-A$ are disjoint.

If $A \in \mathcal{U}$, then $A$ meets every set of $\mathcal{U}$. Hence $\mathcal{U}_A$ is a filter on $A$.

If $\mathcal{U}_A$ is a filter on $A$, let $B \subseteq A$. Then $B \subseteq X$, so either $B \in \mathcal{U}$ or $X-B \in \mathcal{U}$. Thus either $B \cap A = B$ or $(X-B) \cap A = A-B$ belongs to $\mathcal{U}_A$. Hence $\mathcal{U}_A$ is an ultrafilter on $A$.

**PROPOSITION 1.8** If a subset $A$ of a set $X$ does not belong to an ultrafilter $\mathcal{U}$ on $X$, then the trace $\mathcal{U}_A$ of $\mathcal{U}$ on $A$ is $\mathcal{P}(A)$.

**PROOF.** Certainly $\mathcal{U}_A \subseteq \mathcal{P}(A)$. Let $B \in \mathcal{P}(A)$. Then $B = [B \cup (X-A)] \cap A$. Now $X-A \in \mathcal{U}$ because $\mathcal{U}$ is an ultrafilter and $A \notin \mathcal{U}$. Thus $B \cup (X-A) \in \mathcal{U}$, and it follows that the intersection of $B \cup (X-A)$ and $A$ belongs to $\mathcal{U}_A$. Hence $\mathcal{P}(A) \subseteq \mathcal{U}_A$.

**THEOREM 1.13** If $\mathcal{B}$ is a filter base on a set $X$ and $f$ is a function from $X$ into a set $X'$, then $f(\mathcal{B})$ is a base on $X'$. 
PROOF. If \( f(B_1) \) and \( f(B_2) \) belong to \( f(\mathcal{B}) \), it follows that \( f(B_1 \cap B_2) \subseteq f(B_1) \cap f(B_2) \). If \( B \in \mathcal{B} \), then \( f(B) \neq \emptyset \) because \( B \neq \emptyset \). Thus \( f(\mathcal{B}) \) is a filter base on \( X \).

**THEOREM 1.14** Let \( \mathcal{B} \) be an ultrafilter base on a set \( X \), and let \( f \) be a function from \( X \) to \( X' \). Then \( f(\mathcal{B}) \) is an ultrafilter base on \( X' \).

**PROOF.** Let \( A' \) be a subset of \( X' \). Either \( f^{-1}(A') \) or \( X-f^{-1}(A') \) belongs to the ultrafilter \( \mathcal{U} \) generated by \( \mathcal{B} \). If \( f^{-1}(A') \in \mathcal{U} \), then there is a set \( B \in \mathcal{B} \) so that \( B \subseteq f^{-1}(A') \). Then \( f(B) \subseteq f[f^{-1}(A')] = A' \), and it follows that \( A' \) belongs to the filter generated by \( f(\mathcal{B}) \).

Similarly, if \( X-f^{-1}(A') \in \mathcal{U} \), then \( X-A' \) belongs to the filter generated by \( f(\mathcal{B}) \). Hence \( f(\mathcal{B}) \) is an ultrafilter base.

**THEOREM 1.15** If \( f \) is a function on a set \( X \) into a set \( X' \) and \( \mathcal{B}' \) is a filter base on \( X' \), then \( f^{-1}(\mathcal{B}') \) is a filter base on \( X \) if and only if \( f^{-1}(M') \neq \emptyset \) for each \( M' \in \mathcal{B}' \).

**PROOF.** If \( f^{-1}(\mathcal{B}') \) is a filter base on \( X \), then it must be the case that \( f^{-1}(M') \neq \emptyset \) for each \( M' \in \mathcal{B}' \).

Conversely, if \( f^{-1}(M') \neq \emptyset \) for each \( M' \in \mathcal{B}' \) and \( M_1 \) and \( M_2 \) belong to \( \mathcal{B}' \), then \( f^{-1}(M_1) \cap f^{-1}(M_2) = f^{-1}(M_1 \cap M_2) \). Thus \( f^{-1}(\mathcal{B}') \) is a filter base on \( X \).

**PROPOSITION 1.9** Let \( f \) be a mapping of a set \( X \) into a set \( X' \). Then \( f \) is one-to-one if and only if, for every filter base on \( X \), \( f^{-1}[f(\mathcal{B})] \) is a filter base equivalent to \( \mathcal{B} \).
PROOF. Suppose \( f \) is one-to-one. Then if \( B \in \mathcal{B} \),
\[ B = f^{-1}[f(B)]. \]
Thus \( \emptyset \) and \( f^{-1}[f(\emptyset)] \) are equivalent.

Conversely, suppose by way of contradiction that \( f \) is
not one-to-one. Then there are points \( x \) and \( y \) in \( X \) so that
\( f(x) = f(y) \). If \( \mathcal{B} \) is the filter base \( \{\{x\}\} \), then \( f(\mathcal{B}) \) is
equal to \( \{\{f(x)\}\} \). Consequently \( \{y\} \in f^{-1}[\{f(x)\}] = f^{-1}[f(\emptyset)]. \)
However, \( \{y\} \notin \mathcal{B} \), so \( \mathcal{B} \) and \( f^{-1}[f(\emptyset)] \) are not equivalent, a
contradiction. Thus \( f \) is one-to-one.

PROPOSITION 1.10 If \( f \) is a mapping of a set \( X \) onto a
set \( X' \), then the image \( f(\mathcal{F}) \) of every filter \( \mathcal{F} \) on \( X \) is a
filter on \( X' \).

PROOF. Let \( \mathcal{F} \) be a filter on \( X \), and let \( f(F) \in f(\mathcal{F}) \).
Suppose that \( f(F) \subseteq A \subseteq X' \). Then \( f^{-1}[f(F)] \subseteq f^{-1}(A) \). Now
\( F \subseteq f^{-1}[f(F)] \), so \( F \subseteq f^{-1}(A) \). Thus \( f^{-1}(A) \in \mathcal{F} \). Since \( f \) is
onto, \( f[f^{-1}(A)] = A \). Hence \( A \in f(\mathcal{F}) \).

Let \( f(F_1) \) and \( f(F_2) \) belong to \( f(\mathcal{F}) \). Since \( f(F_1 \cap F_2) \) is
a subset of \( f(F_1) \cap f(F_2) \) and \( f(F_1 \cap F_2) \in f(\mathcal{F}) \), it follows
from the first part of the proof that \( f(F_1) \cap f(F_2) \in f(\mathcal{F}) \).
Consequently \( f(\mathcal{F}) \) is a filter on \( X' \).

DEFINITION 1.10 Let \( \{x_n\} \) be a sequence of elements of
a set \( X \). The elementary filter associated with the sequence
\( \{x_n\} \) is the filter generated by the image of the Fréchet
filter by the mapping \( n \to x_n \) of the set \( N \) of natural numbers
into \( X \).

THEOREM 1.16 If a filter \( \mathcal{F} \) has a countable base, it is
the intersection of the elementary filters finer than it.
PROOF. Let \( \{ A_n \mid n \in \mathbb{N} \} \) be a countable base for \( \mathcal{F} \).

For each \( n \in \mathbb{N} \), define \( B_n = \bigcap_{p \in A_n} A_p \). Then \( \{ B_n \mid n \in \mathbb{N} \} \) is a subset of \( \mathcal{F} \), and by Theorem 1.3 it follows that \( \{ B_n \mid n \in \mathbb{N} \} \) is also a base for \( \mathcal{F} \). For each \( n \in \mathbb{N} \), choose an element \( b_n \in B_n \), and let \( \mathcal{G} \) be the elementary filter generated by \( \{ b_n \mid n \in \mathbb{N} \} \). If \( F \in \mathcal{F} \), there is a base element \( B_{n_0} \in F \).

Since \( B_n \subseteq B_{n_0} \) if \( n \leq n_0 \), \( \{ b_n \mid n \leq n_0 \} \subseteq B_{n_0} \). Then \( \{ b_n \mid n \leq n_0 \} \) is a subset of \( F \), and \( F \in \mathcal{G} \). Thus \( \mathcal{F} \subseteq \mathcal{G} \), and there is an elementary filter finer than \( \mathcal{F} \).

Let \( \{ \mathcal{G}_a \mid a \in I \} \) be the collection of all elementary filters on \( X \) finer than \( \mathcal{F} \). Clearly \( \mathcal{F} \subseteq \bigcap \{ \mathcal{G}_a \mid a \in I \} \).

Suppose that \( M \) is a subset of \( X \) which does not belong to \( \mathcal{F} \).

If \( F \in \mathcal{F} \), \( F \cap (X-M) / \in \mathcal{F} \) because \( F \notin M \). In particular \( B_n \cap (X-M) / \in \mathcal{F} \), for each \( n \in \mathbb{N} \). Choose some \( b_n \in B_n \cap (X-M) \) for each \( n \in \mathbb{N} \), and let \( \mathcal{G} \) be the elementary filter generated by \( \{ b_n \} \). If \( F \in \mathcal{F} \), there is a \( B_{n_0} \in F \) for some \( n_0 \in \mathbb{N} \). Then \( \{ b_n \mid n \leq n_0 \} \subseteq F \), so \( F \in \mathcal{G} \). Thus \( \mathcal{F} \subseteq \mathcal{G} \), and \( \mathcal{G} = \mathcal{G}_a \) for some \( a \in I \). Now \( X-M \notin \mathcal{G} \), so \( M \notin \mathcal{G} \). Then \( \bigcap \{ \mathcal{G}_a \mid a \in I \} \subseteq \mathcal{F} \), and the proof is complete.

PROPOSITION 1.11 Let \( X \) be an infinite set. Then the filter \( \mathcal{G} \) of complements of finite subsets of \( X \) is the intersection of the elementary filters associated with infinite sequences in \( X \) all of whose terms are distinct.

PROOF. Because \( X \) is infinite, there is a sequence of members of \( X \) all of whose terms are distinct. Let \( \{ \mathcal{G}_a \mid a \in I \} \) be the collection of all elementary filters associated with
infinite sequences in \( X \) all of whose members are distinct. Choose \( a_0 \in I \), and let \( \{ x_n \} \) be the sequence associated with \( \mathcal{A}_{a_0} \). Suppose that \( G \in \mathcal{Y} \). Then \( X - G \) is finite, so only finitely many terms of \( \{ x_n \} \) belong to \( X - G \). Since \( \{ x_n \} \) has infinitely many distinct terms, there is a natural number \( n_0 \) so that if \( n \geq n_0 \), \( x_n \in G \). Thus \( G \in \mathcal{A}_{a_0} \), and it follows that \( \mathcal{Y} \subseteq \bigcap \{ \mathcal{A}_a | a \in I \} \).

Suppose that \( A \) is a subset of \( X \) which does not belong to \( \mathcal{Y} \). Then \( X - A \) is infinite, so there is an infinite sequence \( \{ y_n \} \) of distinct members of \( X - A \). There is a \( \beta \in I \) so that \( \mathcal{F}_\beta \) is the elementary filter associated with \( \{ y_n \} \). Because \( X - A \subseteq \mathcal{F}_\beta \), \( A \notin \mathcal{F}_\beta \). Then \( A \notin \bigcap \{ \mathcal{A}_a | a \in I \} \). Hence \( \bigcap \{ \mathcal{A}_a | a \in I \} \subseteq \mathcal{Y} \). Consequently \( \mathcal{Y} = \bigcap \{ \mathcal{A}_a | a \in I \} \).

**Proposition 1.12** If \( X \) is an infinite set, then an elementary filter associated with a sequence all of whose terms are distinct is not an ultrafilter.

**Proof.** Let \( \{ a_n \} \) be a sequence in an infinite set \( X \) all of whose terms are distinct. Let \( \mathcal{F} \) be the elementary filter associated with \( \{ a_n \} \). Define \( A = \{ a_{2n} | n \in \mathbb{N} \} \). Now \( A \notin \mathcal{F} \) because there are infinitely many terms of \( \{ a_n \} \) which do not belong to \( A \). For the same reason \( X - A \notin \mathcal{F} \). Hence \( \mathcal{F} \) is not an ultrafilter.

**Proposition 1.13** Let \( \mathcal{F} \) be a countable linearly ordered set of elementary filters on a set \( X \). Then there is an elementary filter which is finer than all the filters of \( \mathcal{F} \).
PROOF. Let \( \mathcal{E} = \{ \mathcal{F}_i \mid i \in \mathbb{N} \} \). With each \( \mathcal{F}_i \), there is associated a sequence \( \{ a_n^i \} \). A terminating section of \( \{ a_n^i \} \) is defined to be a set \( A_k^i = \{ a_n^i \mid n \geq k \} \). The set \( \{ A_k^i \mid k \in \mathbb{N} \} \) of terminating sections of \( \{ a_n^i \} \) is a base for \( \mathcal{F}_i \) and is a countable set. Also the family \( \{ \{ a_n^i \} \mid i \in \mathbb{N} \} \) is countable because \( \mathcal{E} \) is countable. Thus the collection of all terminating sections of the sequences \( \{ a_n^i \} \) is countable because it is a countable union of countable sets. This collection will be denoted by \( \{ T_j \mid j \in \mathbb{N} \} \). Now \( \{ T_j \mid j \in \mathbb{N} \} \) is a filter base.

Clearly \( \{ T_j \mid j \in \mathbb{N} \} \) is not empty, and the empty set is not a terminating section of any sequence. If \( T_1 \) and \( T_2 \) belong to \( \{ T_j \mid j \in \mathbb{N} \} \), then \( T_1 \) and \( T_2 \) are terminating sections of filters \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) respectively. Because \( \mathcal{E} \) is linearly ordered, it may be assumed that \( \mathcal{F}_1 \subset \mathcal{F}_2 \). Then \( T_1 \) contains a terminating section \( T_2' \) of the sequence \( \{ a_n^2 \} \). Now \( T_2 \cap T_2' \) is a subset of \( T_1 \) and \( T_2 \), and \( T_2 \cap T_2' \) is a terminating section of the sequence \( \{ a_n^2 \} \).

Let \( \mathcal{G} \) denote the filter generated by the base \( \{ T_j \mid j \in \mathbb{N} \} \). Then \( \mathcal{G} \) is finer than each \( \mathcal{F}_i \) in \( \mathcal{E} \). Because \( \mathcal{G} \) has a countable base, there is an elementary filter \( \mathcal{G} \) finer than \( \mathcal{G} \) by Theorem 1.16. Then \( \mathcal{G} \) is finer than each of the filters in \( \mathcal{E} \).

PROPOSITION 1.14 Let \( n: \mathbb{N} \to \mathbb{N} \) be a map of the positive integers onto the positive integers so that \( f^{-1}(\{ m \}) \) is finite for each \( m \in \mathbb{N} \). If \( \{ x_n \} \) is any sequence in a set \( X \),
define \( y_n = x_{f(n)} \). Then the elementary filters associated with the sequences \( \{x_n\} \) and \( \{y_n\} \) are the same.

**PROOF.** Let \( \mathcal{F} \) and \( \mathcal{G} \) denote the elementary filters associated with the sequences \( \{x_n\} \) and \( \{y_n\} \) respectively. Let \( F \in \mathcal{F} \). There is a positive integer \( n_0 \) so that if \( n \geq n_0 \), then \( \{x_n \ | \ n \geq n_0\} \subseteq F \). For each positive integer \( i \) with \( 1 \leq i \leq n_0 \), there is a largest positive integer \( n_i \) whose image under \( f \) is \( i \). If \( n' \) is chosen to be larger than the maximum \( \{ n_i \ | \ 1 \leq i \leq n_0 \} \) and \( n \geq n' \), then \( f(n) > n_0 \). Thus \( \{x_{f(n)} \ | \ n \geq n'\} \) is a subset of \( \{x_n \ | \ n \geq n_0\} \), so \( \{y_n \ | \ n \geq n'\} \subseteq F \). Then \( F \in \mathcal{G} \).

Let \( G \in \mathcal{G} \). There is some \( n_0 \in \mathbb{N} \) so that \( \{y_n \ | \ n \geq n_0\} \subseteq G \). Now \( \{y_n \ | \ n \geq n_0\} = \{x_{f(n)} \ | \ n \geq n_0\} \). Let \( n' = \text{maximum} \ \{f(i) \ | \ 1 \leq i \leq n_0\} \), and consider \( \{x_n \ | \ n \geq n'+1\} \). If \( n \geq n'+1 \), there is an \( m \in \mathbb{N} \) so that \( f(m) = n \) because \( f \) is onto. It must be the case that \( m \geq n_0 \) because \( n \geq n' = \text{maximum} \ \{f(i) \ | \ 1 \leq i \leq n_0\} \). Thus \( \{x_n \ | \ n \geq n'+1\} \subseteq \{x_{f(n)} \ | \ n \geq n_0\} \), so \( G \in \mathcal{F} \). Hence \( \mathcal{F} = \mathcal{G} \).
CHAPTER II
FILTERS AND CONVERGENCE

The development of the concept of a filter leads to a theory of convergence in topological spaces. It is the purpose of this chapter to investigate this theory.

DEFINITION 2.1 Let $X$ be a set. A topology on $X$ is a collection $\mathcal{J}$ of subsets of $X$ so that the following conditions are satisfied:

1. Each union of members of $\mathcal{J}$ belongs to $\mathcal{J}$;
2. Each finite intersection of members of $\mathcal{J}$ belongs to $\mathcal{J}$;
3. $X$ and $\emptyset$ belong to $\mathcal{J}$.

The pair $(X, \mathcal{J})$ is called a topological space. The members of $\mathcal{J}$ are called open sets. When no confusion is likely, explicit mention of $\mathcal{J}$ is often omitted, and $X$ is referred to as a topological space. The elements of $X$ are called points.

Basic properties of topological spaces will be assumed.

DEFINITION 2.2 Let $X$ be a topological space, and let $A$ be a subset of $X$. A subset $N$ of $X$ is said to be a neighborhood of $A$ if there is an open subset $T$ of $X$ so that $A \subseteq T$ and $T \subseteq N$. A set $N$ is said to be a neighborhood of a point $x \in X$ if $N$ is a neighborhood of $\{x\}$.
THEOREM 2.1 Let \( X \) be a topological space, and let \( A \) be a non-empty subset of \( X \). Then the collection \( \mathcal{N}(A) \) of all neighborhoods of \( A \) is a filter on \( X \).

PROOF. The intersection of two neighborhoods of \( A \) is a neighborhood of \( A \) because open sets are closed under finite intersections. If a set \( B \) contains a neighborhood of \( A \), then \( B \) is also a neighborhood of \( A \). Since \( A \neq \emptyset \), each neighborhood of \( A \) is also non-empty. Thus \( \mathcal{N}(A) \) is a filter on \( X \).

DEFINITION 2.3 Let \( X \) be a topological space and \( \mathcal{F} \) a filter on \( X \). A subset of \( A \) of \( X \) is said to be a limit of \( \mathcal{F} \) if \( \mathcal{F} \) is finer than the neighborhood filter of \( A \). The filter \( \mathcal{F} \) is also said to converge to \( A \). A set \( A \) is said to be a limit of a filter base \( \mathcal{B} \) in \( X \) if the filter whose base is converges to \( A \); also \( \mathcal{B} \) is said to converge to \( A \). A filter is said to converge to a point \( x \in X \) if it converges to \( \{ x \} \).

It is an immediate consequence of Definition 2.3 that if a filter \( \mathcal{F} \) converges to \( A \), then every filter finer than \( \mathcal{F} \) converges to \( A \). In fact, if \( \mathcal{E} \) is a collection of filters each of which converges to \( A \), then the intersection of the members of \( \mathcal{E} \) converges to \( A \).

Some of the properties of convergent filters will now be established.

THEOREM 2.2 If \( \mathcal{F} \) is a filter on a topological space \( X \), then \( \mathcal{F} \) converges to \( X \).

PROOF. Since \( \mathcal{F} \) is a filter, then \( X \in \mathcal{F} \), and the only neighborhood of \( X \) is \( X \) itself. Thus \( \mathcal{F} \) converges to \( X \).
THEOREM 2.3 Each filter on a topological space $X$ has a unique limit if and only if $|X| = 1$.

PROOF. Suppose that each filter on $X$ has a unique limit and that $|X| \geq 2$. Let $x \in X$, and define $F = \{ A \subseteq X \mid x \in A \}$. Now $F$ is a filter which converges to $x$. However, $F$ also converges to $X$, and $X \not\in \{x\}$. Hence $|X| = 1$.

If $|X| = 1$, then it follows immediately that each filter has the unique limit $X$.

THEOREM 2.4 Let $\mathcal{F}$ be a filter on a topological space $X$. If $\mathcal{F}$ converges to $A$ and $A \subseteq B$, then $\mathcal{F}$ converges to $B$.

PROOF. Let $N$ be a neighborhood of $B$. Then $A \subseteq B \subseteq N$, and it follows that $N$ is a neighborhood of $A$. Because $\mathcal{F}$ converges to $A$, $N \in \mathcal{F}$. Thus $\mathcal{F}$ converges to $B$.

THEOREM 2.5 A subset $A$ of a topological space $X$ is open if and only if $A$ belongs to every filter which converges to $A$.

PROOF. If $A$ is open, then $A$ is a neighborhood of itself. Hence $A$ belongs to every filter which converges to $A$.

Conversely, if $A$ belongs to every filter which converges to $A$, then $A$ belongs to its neighborhood filter. It follows that $A$ is a neighborhood of itself. Thus $A$ is open.

DEFINITION 2.4 Let $X$ be a topological space, and let $A \subseteq X$. A point $x \in X$ is said to be an accumulation point of $A$ if every neighborhood of $x$ intersects $A - \{x\}$.

THEOREM 2.6 A point $x$ is an accumulation point of a subset $A$ of a topological space $X$ if and only if there is a filter $\mathcal{F}$ on $X$ so that $A - \{x\} \in \mathcal{F}$ and $\mathcal{F}$ converges to $x$. 
PROOF. Suppose that $x$ is an accumulation point of $A$, and define $\mathcal{B} = \{ Y \cap U \mid Y \text{ contains } A \setminus \{x\} \text{ and } U \in \mathcal{K}(x) \}$. Because $x$ is an accumulation point of $A$, $\mathcal{B}$ is a filter base. Now $\mathcal{B}$ converges to $x$, and $A \setminus \{x\}$ belongs to the filter $\mathcal{F}$ generated by $\mathcal{B}$. Thus $\mathcal{F}$ is the required filter.

If there is a filter $\mathcal{F}$ on $X$ so that $A \setminus \{x\} \in \mathcal{F}$ and $\mathcal{F}$ converges to $x$, then every neighborhood of $x$ intersects $A \setminus \{x\}$. It follows that $x$ is an accumulation point of $A$.

DEFINITION 2.5 A subset $A$ of a topological space $X$ is said to be closed if $X \setminus A$ is open.

THEOREM 2.7 A subset $A$ of a topological space $X$ is closed if and only if no filter on $X$ which contains $A$ converges to $X \setminus A$.

PROOF. Suppose that $A$ is closed, and let $\mathcal{F}$ be any filter on $X$ to which $A$ belongs. If $\mathcal{F}$ converges to $X \setminus A$, then $X \setminus A \in \mathcal{F}$ because $X \setminus A$ is open. However, since $A \in \mathcal{F}$, $X \setminus A \notin \mathcal{F}$. Thus $\mathcal{F}$ does not converge to $X \setminus A$.

Suppose that no filter which contains $A$ converges to $X \setminus A$. If $A$ is not closed, then there is an accumulation point $x$ of $A$ such that $x \notin A$. By Theorem 2.6 there is a filter $\mathcal{G}$ on $X$ so that $A \setminus \{x\} \in \mathcal{G}$ and $\mathcal{G}$ converges to $x$. Since $x \in X \setminus A$, Theorem 2.4 shows that $\mathcal{G}$ converges to $X \setminus A$. However, because $A \setminus \{x\} \in \mathcal{G}$, $A \notin \mathcal{G}$. Thus it must be the case that $A$ is closed.

THEOREM 2.8 A filter $\mathcal{F}$ on a topological space $X$ converges to a set $A$ if and only if every ultrafilter which is finer than $\mathcal{F}$ converges to $A$. 

PROOF. If \( F \) converges to \( A \), then any filter finer than \( F \) also converges to \( A \).

If every ultrafilter finer than \( F \) converges to \( A \), then the intersection of those ultrafilters converges to \( A \). Since \( F \) is the intersection of all ultrafilters finer than it by Theorem 1.10, then \( F \) converges to \( A \).

**THEOREM 2.9** Let \( F \) be a filter on a topological space \( X \), and let \( \mathcal{B} = \{ A \mid A \in F \text{ and } A \text{ is open} \} \). Then \( \mathcal{B} \) is a filter base, and \( F \) converges to a set \( Y \) if and only if \( \mathcal{B} \) converges to \( Y \).

PROOF. If \( A_1 \) and \( A_2 \) belong to \( \mathcal{B} \), then \( A_1 \cap A_2 \in \mathcal{B} \).

Since \( A_1 \cap A_2 \) is open, it follows that \( A_1 \cap A_2 \in \mathcal{B} \). The empty set is not in \( \mathcal{B} \) because it is not in \( F \), and \( \mathcal{B} \) is not empty because \( X \in \mathcal{B} \). Hence \( \mathcal{B} \) is a filter base.

Suppose that \( F \) converges to \( Y \), and let \( N \) be a neighborhood of \( Y \). There is an open set \( A \) so that \( Y \subseteq A \subseteq N \). Now \( A \) is a neighborhood of \( Y \), so \( A \in F \). Since \( A \) is open, \( A \in \mathcal{B} \). Then \( N \) belongs to the filter generated by \( \mathcal{B} \), so \( \mathcal{B} \) converges to \( Y \).

If \( \mathcal{B} \) converges to \( Y \) and \( M \) is a neighborhood of \( Y \), there is a member \( B \) of \( \mathcal{B} \) so that \( B \subseteq M \). Since \( B \in F \), then \( M \in F \). Hence \( F \) converges to \( Y \).

**THEOREM 2.10** Let \( X \) be a topological space, and let \( F \) be a filter on \( X \) which converges to a point \( x \). If \( F \in F \) and \( F \) is closed, then \( x \in F \).
PROOF. If there is a closed set $F \in \mathcal{F}$ so that $x \notin F$, then $x$ belongs to the open set $X - F$. Since $\mathcal{F}$ converges to $x$, $X - F \in \mathcal{F}$. However, $F$ and $X - F$ are disjoint. Thus $x \in F$.

**Theorem 2.11** Let $\mathcal{F}$ be a filter on a set $X$, and let $L$ denote the set of limit points of $\mathcal{F}$. Then $L$ is closed.

**Proof.** If $L$ is empty, the theorem follows. Assume that $L \neq \emptyset$, and suppose by way of contradiction that $L$ is not closed. Then there is an accumulation point $x$ of $L$ which does not belong to $L$. Let $N$ be a neighborhood of $x$. There is an open set $O \subseteq N$ so that $x \notin O$, and there is a $y \in L$ which belongs to $O$. Then $N$ is a neighborhood of $y$, so $N \in \mathcal{F}$. Then $\mathcal{F}$ converges to $x$, a contradiction to the assumption that $x \notin L$. Thus it must be the case that $x \in L$, and $L$ is closed.

**Theorem 2.12** Let $\mathcal{U}$ be an ultrafilter on a topological space $X$, and let $L$ be the set of limit points of $\mathcal{U}$. Then $L = \bigcap \{ A \mid A \in \mathcal{U} \text{ and } A \text{ is closed} \}$.

**Proof.** By Theorem 2.4, $L = A$ for each closed element $A$ of $\mathcal{U}$. Let $x \in \bigcap \{ A \mid A \in \mathcal{U} \text{ and } A \text{ is closed} \}$. If $\mathcal{U}$ does not converge to $x$, then there is an open neighborhood $N$ of $x$ so that $N \notin \mathcal{U}$. Because $\mathcal{U}$ is an ultrafilter, it follows that $X - N \notin \mathcal{U}$. However, $X - N$ is closed, and $x \notin X - N$, a contradiction. Thus $x \in L$, and $L = \bigcap \{ A \mid A \in \mathcal{U} \text{ and } A \text{ is closed} \}$.

**Corollary 2.1.** Let $\mathcal{U}$ be an ultrafilter on a set $X$. If the set $L$ of limit points of $\mathcal{U}$ is empty, then the intersection of the members of $\mathcal{U}$ is empty.
PROOF. Certainly \( \bigcap \{ U \mid U \in \mathcal{U} \} \subset \bigcap \{ A \mid A \in \mathcal{U} \text{ and } A \text{ is closed} \} \).

By Theorem 2.6, \( L = \bigcap \{ A \mid A \in \mathcal{U} \text{ and } A \text{ is closed} \} \). Since \( L = \emptyset \),
\( \bigcap \{ U \mid U \in \mathcal{U} \} = \emptyset \).

The following example shows that the converse of Corollary 2.1 is not true.

EXAMPLE 2.1. Let \( \mathbb{R} \) be the set of real numbers with the usual topology, and let \( x \in \mathbb{R} \). Let \( \{ N_\alpha \mid \alpha \in I \} \) be the collection of all neighborhoods of \( x \), and for each \( \alpha \in I \) choose \( y_\alpha \) to be a member of \( N_\alpha \) distinct from \( x \). Let \( S = \{ x_\alpha \mid \alpha \in I \} \).

The set \( \{ N_\alpha \mid \alpha \in I \} \cup \{ S \} \) is a subbase for a filter \( \mathcal{F} \) on \( \mathbb{R} \).

There is an ultrafilter \( \mathcal{U} \) on \( X \) which contains \( \mathcal{F} \). Now \( \mathcal{U} \) converges to \( x \) because \( \{ N_\alpha \mid \alpha \in I \} \subset \mathcal{U} \). However, \( x \notin S \), so \( x \notin \bigcap \{ U \mid U \in \mathcal{U} \} \). If \( y \) is any real number other than \( x \), there is a neighborhood \( N_\rho \) of \( x \) so that \( y \notin N_\rho \). Then \( y \notin \bigcap \{ U \mid U \in \mathcal{U} \} \).

Consequently \( \bigcap \{ U \mid U \in \mathcal{U} \} = \emptyset \). Thus \( \mathcal{U} \) is an ultrafilter which converges to a point, but \( \bigcap \{ U \mid U \in \mathcal{U} \} = \emptyset \).

DEFINITION 2.6. A point \( x \) is said to be a cluster point of a filter base \( \mathcal{B} \) in a topological space \( X \) if every neighborhood of \( x \) intersects every member of \( \mathcal{B} \).

It is an immediate consequence of Definition 2.6 that if \( x \) is a cluster point of a filter base \( \mathcal{B} \), then \( x \) is a cluster point of the filter whose base is \( \mathcal{B} \).

THEOREM 2.13. A point \( x \) is a cluster point of a filter \( \mathcal{F} \) if and only if there is a filter finer than \( \mathcal{F} \) which converges to \( x \).
PROOF. Suppose \( x \) is a cluster point of a filter \( \mathcal{F} \), and define \( \mathcal{B} = \{ U \cap F \mid F \in \mathcal{F} \} \) and \( U \) is a neighborhood of \( x \).

Because \( x \) is a cluster point of \( \mathcal{F} \), no member of \( \mathcal{B} \) is empty. If \( U_1 \cap F_1 \) and \( U_2 \cap F_2 \) belong to \( \mathcal{B} \), then \((U_1 \cap F_1) \cap (U_2 \cap F_2) = (U_1 \cap U_2) \cap (F_1 \cap F_2)\). Since \( U_1 \cap U_2 \) is a neighborhood of \( x \) and \( F_1 \cap F_2 \in \mathcal{F} \), \( \mathcal{B} \) is closed under finite intersections. The filter generated by \( \mathcal{B} \) is finer than \( \mathcal{F} \), and it converges to \( x \).

Conversely, suppose that there is a filter \( \mathcal{G} \) finer than \( \mathcal{F} \) such that \( \mathcal{G} \) converges to \( x \). If \( U \) is a neighborhood of \( x \) and \( F \in \mathcal{F} \), then both \( U \) and \( F \) belong to \( \mathcal{G} \). Then \( U \cap F \neq \emptyset \), and \( x \) is a cluster point of \( \mathcal{F} \).

COROLLARY 2.2 An ultrafilter \( \mathcal{U} \) converges to a point \( x \) if and only if \( x \) is a cluster point of \( \mathcal{U} \).

PROOF. If \( \mathcal{U} \) converges to \( x \), then clearly \( x \) is a cluster point of \( \mathcal{U} \).

If \( x \) is a cluster point of \( \mathcal{U} \), there is a filter \( \mathcal{G} \) which is finer than \( \mathcal{U} \) and which converges to \( x \). Because \( \mathcal{U} \) is an ultrafilter, it must be the case that \( \mathcal{U} = \mathcal{G} \). Thus \( \mathcal{U} \) converges to \( x \).

THEOREM 2.14 The set of cluster points of a filter base on a topological space \( X \) is closed in \( X \).

PROOF. Let \( \mathcal{B} \) be a filter base on \( X \), and let \( A \) be the set of cluster points of \( \mathcal{B} \). Let \( x \) be an accumulation point of \( A \), and let \( N \) be a neighborhood of \( x \). There is an open
neighborhood 0 of $x$ such that $0 \in N$. There is a point $y$

distinct from $x$ such that $y \in 0 \cap A$. Then $y \in 0$, so $N$ is a

neighborhood of $y$. Since $y$ is a cluster point of $B$, $N \cap B \neq \emptyset$ for each $B \in \mathcal{B}$. Hence $x$ is a cluster point of $B$.

Then $x \in A$, and $A$ is closed.

**DEFINITION 2.7** If $A$ is a subset of a topological

space $X$, then the closure of $A$ is denoted by $\bar{A}$ and is defined
to be the set of all $x \in X$ such that every neighborhood of $x$

intersects $A$.

It is worthwhile to note that if $A$ is a subset of a
topological space $X$ and $x \in \bar{A}$, then the trace of the neighbor-

hood filter of $x$ on $A$ is a filter on $A$. Also, if $\mathcal{F}$ is a

filter on $A$, then $\mathcal{F}$ is a base for a filter on $X$.

**DEFINITION 2.8** Let $\mathcal{F}$ be a filter on a subset $A$ of a
topological space $X$, and let $x \in \bar{A}$. If $\mathcal{F}$ is finer than the

trace of the neighborhood filter of $x$ on $A$, then $\mathcal{F}$ is said
to converge to $x$.

**THEOREM 2.15** Let $\mathcal{F}$ be a filter on a subset $A$ of a
topological space $X$, and let $x \in A$. Then $\mathcal{F}$ converges to $x$
if and only if the filter on $X$ for which $\mathcal{F}$ is a base also

converges to $x$.

**PROOF.** Suppose that $\mathcal{F}$ converges to $x$, and let $\mathcal{G}$ denote
the filter on $X$ generated by $\mathcal{F}$. If $N$ is a neighborhood of
$x$, then $N \cap A \in \mathcal{F}$. Since $N \cap A \cap N$, it follows that $N \in \mathcal{G}$. Thus
$\mathcal{G}$ is finer than the neighborhood filter of $x$. Consequently
$\mathcal{G}$ converges to $x$. 
Suppose that $\mathcal{F}$ converges to $x$. Let $U$ be a neighborhood of $x$. Then $U \in \mathcal{F}$, so there is $F \in \mathcal{F}$ with $F \subseteq U$. Thus $F \cap A \subseteq U \cap A$. Also $F \cap A \subseteq F$ because $F \subseteq A$. Hence $F \subseteq U \cap A$, and $U \cap A \in \mathcal{F}$. Then $\mathcal{F}$ is finer than the trace of the neighborhood filter of $x$ on $A$. Thus $\mathcal{F}$ converges to $x$.

**THEOREM 2.16** Let $\mathcal{B}$ be a filter base on a subset $A$ of a topological space $X$. Then every cluster point of $\mathcal{B}$ in $X$ belongs to $\overline{A}$, and every point of $\overline{A}$ is a limit point of a filter on $A$.

**PROOF.** If $x$ is a cluster point of $\mathcal{B}$, then every neighborhood of $x$ intersects $A$. Thus $x \in \overline{A}$.

If $x \in \overline{A}$, the trace of the neighborhood filter of $x$ on $A$ is a filter on $A$ which converges to $x$.

**PROPOSITION 2.1** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two topologies on the same set $X$. Show that $\mathcal{F}_2 \subseteq \mathcal{F}_1$ if and only if every filter which is convergent in the topology $\mathcal{F}_1$ converges to the same point in the topology $\mathcal{F}_2$.

**PROOF.** Suppose that $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Let $\mathcal{F}$ converge to $x$ in $\mathcal{F}_1$, and let $N$ be a neighborhood of $x$ with respect to $\mathcal{F}_2$. Then $N$ is a neighborhood of $x$ with respect to $\mathcal{F}_1$, so $N \in \mathcal{F}_1$. Thus $\mathcal{F}$ converges to $x$ in $\mathcal{F}_2$.

Conversely, let $T \in \mathcal{F}_2$. It may be assumed that $T \neq \emptyset$ since $\emptyset \in \mathcal{F}_1$. Let $x \in T$. The neighborhood filter $\mathcal{U}$ of $x$ taken with respect to $\mathcal{F}_1$ converges to $x$ in $\mathcal{F}_1$. Then $\mathcal{U}$ converges to $x$ in $\mathcal{F}_2$, so $T$ is a $\mathcal{F}_1$-neighborhood of $x$. 
Since $x$ is any member of $T$, it follows that $T$ is a $Q_i$-neighborhood of each of its points. Then $T$ is open in $Q_i$.

Hence $Q_i \subseteq Q_i$.

The convergent properties of the image of a filter with respect to a function from one topological space to another are important.

**DEFINITION 2.9** Let $f$ be a mapping of a set $X$ into a topological space $Y$, and let $\mathcal{F}$ be a filter on $X$. A point $y \in Y$ is said to be a limit point (or simply a limit) of $f$ with respect to $\mathcal{F}$ if $y$ is a limit point of the filter base $f(\mathcal{F})$.

A cluster point of $f$ is defined similarly.

**THEOREM 2.17** A point $y \in Y$ is a limit of $f$ with respect to a filter $\mathcal{F}$ if and only if for each neighborhood $V$ of $y$ in $Y$, there is a set $M \in \mathcal{F}$ with $f(M) \subseteq V$.

**PROOF.** Suppose that $y$ is a limit of $f$ with respect to $\mathcal{F}$. If $V$ is a neighborhood of $y$, $V$ belongs to the filter generated by $f(\mathcal{F})$. Then there is a member $M \in \mathcal{F}$ so that $f(M) \subseteq V$.

Conversely, let $V$ be a neighborhood of $y$. There is a set $M \in \mathcal{F}$ such that $f(M) \subseteq V$. Then $V$ belongs to the filter generated by $f(\mathcal{F})$, so $y$ is a limit of $f(\mathcal{F})$.

**THEOREM 2.18** A point $y \in Y$ is a cluster point of $f$ with respect to $\mathcal{F}$ if and only if for each neighborhood $V$ of $y$ and each $M \in \mathcal{F}$, there is a point $x \in M$ such that $f(x) \in V$.

**PROOF.** Let $y$ be a cluster point of $f$ with respect to $\mathcal{F}$. If $M \in \mathcal{F}$ and $V$ is a neighborhood of $y$, then $f(M) \cap V \neq \emptyset$. Thus
there is a point \( z \in Y \) such that \( z \in f(M) \cap V \). Since \( z \in f(M) \), there is an \( x \in M \) so that \( z = f(x) \). Then \( f(x) \in V \).

Conversely, if \( V \) is a neighborhood of \( y \) and \( M \in \mathcal{F} \), then there is a point \( x \in M \) so that \( f(x) \in V \). Now \( f(x) \in f(M) \), so \( f(M) \cap V \neq \emptyset \). Thus every neighborhood of \( y \) intersects every member of \( f(\mathcal{F}) \). It follows that \( y \) is a cluster point of \( f(\mathcal{F}) \).

If \( f \) is a function from the set of natural numbers \( \mathbb{N} \) to a topological space \( X \) defined by \( f(n) = x^n \), it is possible to define a limit of the sequence \( \{x_n\} \) to be a limit of the image of the Fréchet filter with respect to \( f \). The following example shows that this definition is equivalent to the usual definition of the limit of a sequence of real numbers if \( X \) is the set of real numbers with the usual topology.

**EXAMPLE 2.2** Suppose that \( \{x_n\} \) is a sequence of real numbers and that the limit of \( \{x_n\} \) exists in the usual sense. Let \( y = \lim x_n \), and suppose that \( U \) is a neighborhood of \( y \). There is a positive number \( \varepsilon \) so that \((y - \varepsilon, y + \varepsilon) \subseteq U \). Since \( \lim x_n = y \), there is a natural number \( n_0 \) so that if \( n \geq n_0 \), then \( x_n \in (y - \varepsilon, y + \varepsilon) \). Then \( U \) contains all except a finite number of the terms of \( \{x_n\} \). Thus \( U \) belongs to the elementary filter associated with \( \{x_n\} \).

Suppose \( y \) is a limit of the image of the Fréchet filter on \( \mathbb{N} \) with respect to the function \( f: \mathbb{N} \to X \) defined by \( f(n) = x^n \). Let \( \varepsilon > 0 \). Since \((y - \varepsilon, y + \varepsilon) \) is a neighborhood of \( y \), it contains all except finitely many terms of \( \{x_n\} \). Then there
is a natural number \( n_0 \) so that \( x_n \in (y - \epsilon, y + \epsilon) \) if \( n \geq n_0 \). Thus \( \lim x_n = y \).

**THEOREM 2.19** Let \( f \) be a mapping of a set \( X \) into a topological space \( Y \). Then \( y \in Y \) is a cluster point of \( f \) with respect to \( \mathcal{F} \) if and only if there is a filter \( \mathcal{G} \) on \( X \) which is finer than \( \mathcal{F} \) and such that \( y \) is a limit of \( f \) with respect to \( \mathcal{G} \).

**PROOF.** Suppose \( y \) is a cluster point of \( f \) with respect to \( \mathcal{F} \). Define \( \mathcal{G} \) to be the collection of all subsets \( A \) of \( X \) so that \( A \) contains \( f^{-1}(U) \cap F \) for some neighborhood \( U \) of \( y \) and some \( F \in \mathcal{F} \). By Theorem 2.12 each set \( f^{-1}(U) \cap F \neq \emptyset \). It is routine to verify that \( \mathcal{G} \) satisfies the other conditions required of filters. Clearly \( \mathcal{F} \subseteq \mathcal{G} \). If \( U \) is any neighborhood of \( y \), \( f^{-1}(U) \in \mathcal{G} \). Then \( f \left[ f^{-1}(U) \right] \in f(\mathcal{G}) \), and \( f \left[ f^{-1}(U) \right] \subseteq U \). Thus \( U \) belongs to the filter generated by \( f(\mathcal{G}) \), and \( f(\mathcal{G}) \) converges to \( y \).

Suppose there is a filter \( \mathcal{G} \) on \( X \) which is finer than \( \mathcal{F} \) and such that \( y \) is a limit of \( f \) with respect to \( \mathcal{G} \). Since \( \mathcal{F} \subseteq \mathcal{G} \), then \( f(\mathcal{F}) \subseteq f(\mathcal{G}) \). If \( U \) is a neighborhood of \( y \), then \( U \) belongs to the filter generated by \( f(\mathcal{F}) \). Thus \( U \cap f(F) \neq \emptyset \) for each \( F \in \mathcal{F} \) because \( f(\mathcal{F}) \subseteq f(\mathcal{G}) \). Hence \( y \) is a cluster point of \( f(\mathcal{G}) \).

**DEFINITION 2.10** Let \( f \) be a function from a topological space \( X \) to a space \( Y \), and let \( x \in X \). Then \( f \) is said to be continuous at \( x \) if for every neighborhood \( N \) of \( f(x) \), \( f^{-1}(N) \)
is a neighborhood of $x$. If $f$ is continuous at each $x \in X$, then $f$ is said to be continuous on $X$.

**THEOREM 2.20** Let $f$ be a function from a topological space $X$ to a topological space $Y$. Let $x \in X$, and let $\mathcal{H}(x)$ be the neighborhood filter of $x$. Then $f$ is continuous at $x$ if and only if $f[\mathcal{H}(x)]$ converges to $f(x)$.

**PROOF.** Suppose that $f$ is continuous at $x$, and let $U$ be a neighborhood of $f(x)$. Then $f^{-1}(U)$ is a neighborhood of $x$, so $f^{-1}(U) \in \mathcal{H}(x)$. It follows that $f[f^{-1}(U)] \in f[\mathcal{H}(x)]$, and since $f[f^{-1}(U)] \subseteq U$, $U$ belongs to the filter generated by $f[\mathcal{H}(x)]$. Thus $f[\mathcal{H}(x)]$ converges to $f(x)$.

Suppose that $f[\mathcal{H}(x)]$ converges to $f(x)$. Let $U$ be a neighborhood of $f(x)$. Then there is a neighborhood $N$ of $x$ so that $f(N) \subseteq U$. Thus $f^{-1}[f(N)] \subseteq f^{-1}(U)$, and since $N \subseteq f^{-1}[f(N)]$, it follows that $N \subseteq f^{-1}(U)$. Then $f^{-1}(U) \in \mathcal{H}(x)$, and $f$ is continuous at $x$.

**COROLLARY 2.3** Let $f : X \to Y$ be a function from a topological space $X$ to a topological space $Y$ which is continuous at $x \in X$. Then for every filter base $\mathcal{B}$ on $X$ which converges to $x$, the filter base $f(\mathcal{B})$ converges to $f(x)$.

**PROOF.** Suppose that $\mathcal{B}$ is a filter base on $X$ which converges to $x$, and let $\mathcal{F}$ be the filter whose base is $\mathcal{B}$. Then $\mathcal{F}$ is finer than the neighborhood filter $\mathcal{H}(x)$ of $x$. Consequently $f[\mathcal{H}(x)] \subseteq f(\mathcal{B})$. Because $f$ is continuous, $f[\mathcal{H}(x)]$ converges to $f(x)$. Thus $f(\mathcal{B})$ converges to $f(x)$. 
THEOREM 2.21 Let $f: X \rightarrow Y$ be a function from a topological space $X$ to a topological space $Y$, and let $x \in X$. If for every ultrafilter $\mathcal{U}$ on $X$ which converges to $x$, the ultrafilter base $f(\mathcal{U})$ converges to $f(x)$, then $f$ is continuous at $x$.

PROOF. Suppose that $f$ is not continuous at $x$. Then there is a neighborhood $N$ of $f(x)$ so that $f^{-1}(N)$ is not a neighborhood of $x$. It follows that there is an ultrafilter $\mathcal{U}$ finer than the neighborhood filter $\mathcal{H}(x)$ of $x$ such that $f^{-1}(N) \notin \mathcal{U}$ by Theorem 1.10. Thus $X-f^{-1}(N) = f^{-1}(X-N) \notin \mathcal{U}$. Then $f[f^{-1}(X-N)] \subseteq f(\mathcal{U})$, and since $f[f^{-1}(X-N)] \subseteq X-N$, $X-N \notin f(\mathcal{U})$. Now $\mathcal{U}$ converges to $x$ because $\mathcal{H}(x) \subseteq \mathcal{U}$. Then $f(\mathcal{U})$ converges to $f(x)$. However, $N$ is a neighborhood of $f(x)$ which does not belong to the filter generated by $f(\mathcal{U})$. Hence it must be the case that $f^{-1}(N)$ is a neighborhood of $x$, and $f$ is continuous at $x$.

THEOREM 2.22 Let $g$ be a mapping of a set $Z$ into a topological space $X$, and suppose that $g$ has a limit $a$ with respect to a filter $\mathcal{F}$ on $Z$; then if the map $f: X \rightarrow Y$ is continuous at $a$, the composition $f \circ g$ has $f(a)$ as a limit point with respect to $\mathcal{F}$.

PROOF. Let $U$ be a neighborhood of $f(a)$. Then $f^{-1}(U)$ is a neighborhood of $a$. Since $f(\mathcal{F})$ converges to $a$, there is a set $F \in \mathcal{F}$ so that $g(F) \subseteq f^{-1}(U)$. Thus $f \circ g(F) \subseteq f[f^{-1}(U)]$, and since $f[f^{-1}(U)] \subseteq U$, then $f \circ g(F) \subseteq U$. Now $f \circ g(F) \subseteq f \circ g(\mathcal{F})$. 
so $\mathcal{U} \in f \circ g(\mathcal{B})$. Hence $f(a)$ is a limit with respect to $\mathcal{F}$ of $f \circ g$.

**PROPOSITION 2.2** Let $X$ and $X'$ be topological spaces, and let $f : X \rightarrow X'$ be a function which is continuous at $x_0 \in X$. If $\mathcal{B}$ is any filter base on $X$ which has $x_0$ as a cluster point, then $f(x_0)$ is a cluster point of the filter base $f(\mathcal{B})$ on $X'$.

**PROOF.** Let $U$ be a neighborhood of $f(x_0)$, and let $\mathcal{B} \in \mathcal{F}$. Then $f^{-1}(U) \cap B \neq \emptyset$. Now $f \left[ f^{-1}(U) \cap B \right] \subseteq f \left[ f^{-1}(U) \right] \cap f(B)$, and $f \left[ f^{-1}(U) \right] \cap f(B) \subseteq f(U) \cap f(B)$. Then $U \cap f(B) \neq \emptyset$, and $f(x_0)$ is a cluster point of $f(\mathcal{B})$.

**PROPOSITION 2.3** A subset $A$ of a topological space $X$ is said to be primitive if it is the set of limit points of an ultrafilter on $X$. Let $\mathcal{F}_A$ denote the set of ultrafilters on $X$ whose set of limit points is $A$.

(a) If $A$ is a primitive subset of $X$, then every open subset of $X$ which meets $A$ belongs to all the ultrafilters in $\mathcal{F}_A$.

**PROOF.** Let $O$ be an open subset of $X$ which meets $A$, and let $\mathcal{U} \in \mathcal{F}_A$. Since $A \cap O \neq \emptyset$, there is a point $x \in X$ so that $x \in A \cap O$. Then $\mathcal{U}$ converges to $x$, and $O$ is a neighborhood of $x$. Thus $O \in \mathcal{U}$.

(b) Let $A$ and $B$ be two distinct primitive subsets of $X$. Then there is an ultrafilter $\mathcal{U} \in \mathcal{F}_A$, an ultrafilter $\mathcal{B} \in \mathcal{F}_B$, and a subset $M$ of $X$ such that $M \in \mathcal{U}$ and $X \setminus M \in \mathcal{B}$.

**PROOF.** Since $A \not\subseteq B$, either $A \not\subseteq B$ or $B \not\subseteq A$. Without loss of generality it may be assumed that $A \not\subseteq B$. Then there is a
point \( x \in X \) so that \( x \in A \) and \( x \notin B \). If \( U \in \mathcal{U}_A \) and \( B \in \mathcal{Y}_B \), then \( U \) converges to \( x \), and \( B \) does not converge to \( x \). Thus each neighborhood of \( x \) belongs to \( \mathcal{U} \), but there is a neighborhood \( N \) of \( x \) such that \( N \notin B \). Since \( B \) is an ultrafilter, \( X - N \notin B \). Hence \( N \in \mathcal{U} \), and \( X - N \in B \).

(c) Let \( f : X \rightarrow Y \) be a continuous mapping. Then the image under \( f \) of any primitive subset of \( X \) is contained in a primitive subset of \( Y \).

PROOF. Let \( A \) be a primitive subset of \( X \), and let \( U \in \mathcal{U}_A \). Then \( f(U) \) is an ultrafilter on \( Y \). Let \( B \) denote the set of limit points of \( f(U) \). If \( a \in A \), \( f(U) \) converges to \( f(a) \) because \( f \) is continuous. Thus \( f(A) \subseteq B \), and \( B \) is a primitive subset of \( Y \).

(d) If \( x \in X \) and \( \{ x \} \) is closed, then \( \{ x \} \) is a primitive subset of \( X \).

PROOF. Let \( U \) be the filter generated by \( \{ x \} \). Now \( U \) is an ultrafilter on \( X \) which converges to \( x \). Suppose that \( y \in X \) and \( y \notin x \). Then \( y \in X - \{ x \} \), and \( X - \{ x \} \) is open. Since \( \{ x \} \in U \), \( X - \{ x \} \notin U \). Thus \( U \) does not converge to \( y \). Then \( \{ x \} \) is a primitive subset of \( X \) because \( x \) is the only limit of \( U \).

It is possible to characterize certain separation properties of topological spaces in terms of filters.

DEFINITION 2.11 A topological space \( X \) is said to be \( T_1 \) if \( \{ x \} \) is closed for each \( x \in X \).

THEOREM 2.23 A topological space is \( T_1 \) if and only if the neighborhood filter of each point has a unique limit point.
PROOF. Suppose that \( X \) is \( T_1 \). Let \( x \in X \), and let \( \mathcal{N}(x) \) be the neighborhood filter of \( x \). Certainly \( \mathcal{N}(x) \) converges to \( x \). If \( y \in X \) and \( y \neq x \), then \( X - \{ x \} \) is a neighborhood of \( y \) which does not belong to \( \mathcal{N}(x) \). Thus \( \mathcal{N}(x) \) does not converge to \( y \). It follows that \( \mathcal{N}(x) \) has a unique limit.

Suppose that the neighborhood filter of each point has a unique limit point. Suppose by way of contradiction that there is a point \( x \in X \) so that \( \{ x \} \) is not closed. Then there is an accumulation point \( y \) of \( \{ x \} \). If \( N \) is a neighborhood of \( y \), there is an open set \( O \subseteq N \) so that \( y \in O \). Then \( x \in O \), and \( N \) is a neighborhood of \( x \). Thus the neighborhood filter of \( x \) converges to \( y \) because every neighborhood of \( y \) is a neighborhood of \( x \). However, the neighborhood filter of \( x \) has a unique limit. Hence it must be the case that \( \{ x \} \) is closed, and \( X \) is \( T_1 \).

THEOREM 2.24 A topological space \( X \) is \( T_1 \) if and only if for each \( x \in X \) the ultrafilter generated by \( \{ x \} \) has a unique limit point.

PROOF. Suppose that \( X \) is \( T_1 \). Let \( x \in X \), and let \( \mathcal{U} \) be the filter generated by \( \{ x \} \). Certainly \( \mathcal{U} \) converges to \( x \). If \( y \in X \) and \( y \neq x \), then \( X - \{ x \} \) is an open neighborhood of \( x \). Now \( X - \{ x \} \notin \mathcal{U} \) because \( x \notin X - \{ x \} \). Hence \( \mathcal{U} \) does not converge to \( y \).

Conversely, if \( x \in X \), then the neighborhood filter of \( x \) has a unique limit point because it is coarser than the filter generated by \( \{ x \} \). Thus \( X \) is \( T_1 \) by Theorem 2.17.
DEFINITION 2.12 A topological space $X$ is $T_2$ if and only if whenever $x$ and $y$ are distinct points of $X$, there are disjoint neighborhoods of $x$ and $y$.

THEOREM 2.25 A topological space $X$ is $T_2$ if and only if each filter on $X$ has at most one limit point.

PROOF. Suppose that $X$ is $T_2$, and let $\mathcal{F}$ be a filter on $X$. Suppose by way of contradiction that $x$ and $y$ are distinct limit points of $\mathcal{F}$. Since $X$ is $T_2$, there is a neighborhood $U$ of $x$ and a neighborhood $V$ of $y$ so that $U \cap V = \emptyset$. However, $U$ and $V$ belong to $\mathcal{F}$, so $U \cap V \neq \emptyset$. Thus it must be the case that $x = y$.

Conversely, suppose that $X$ is not $T_2$. Then there are distinct points $x$ and $y$ of $X$ so that every neighborhood of $x$ intersects every neighborhood of $y$. Define $\mathcal{B}$ to be

$$\{ U \cap V \mid U \text{ is a neighborhood of } x \text{ and } V \text{ is a neighborhood of } y \}.$$ 

It is routine to verify that $\mathcal{B}$ is a filter base on $X$ and that $\mathcal{B}$ converges to both $x$ and $y$. Then there is a filter on $X$ which does not have a unique limit. Thus $X$ is $T_2$.

COROLLARY 2.4 A topological space $X$ is $T_2$ if and only if every ultrafilter on $X$ has at most one limit point.

PROOF. If $X$ is $T_2$, then Theorem 2.19 shows that every ultrafilter on $X$ has at most one limit point.

Conversely, if $X$ is not $T_2$, then there is a filter on $X$ which has two limit points. There is an ultrafilter $\mathcal{U}$ on $X$ which is finer than it. Then $\mathcal{U}$ also has two limit points. Thus $X$ is $T_2$. 
DEFINITION 2.13 A topological space $X$ is said to be regular if for each $x \in X$ and each neighborhood $N$ of $x$, there is a closed neighborhood $U$ of $x$ so that $U \subseteq N$.

THEOREM 2.26 Let $X$ be a regular topological space, and let $\mathcal{F}$ be a filter on $X$. Let $\mathcal{C} = \{ C \in \mathcal{F} \mid C \text{ is closed} \}$. Then $\mathcal{F}$ converges to $x$ if and only if $\mathcal{C}$ converges to $x$.

PROOF. Suppose $\mathcal{F}$ converges to $x \in X$, and let $N$ be a neighborhood of $x$. There is a closed neighborhood $C$ of $x$ so that $C \subseteq N$. Because $\mathcal{F}$ converges to $x$, $C \in \mathcal{C}$. Then $N$ belongs to the filter generated by $\mathcal{C}$, so $\mathcal{C}$ converges to $x$.

Suppose $\mathcal{C}$ converges to $x$, and let $N$ be a neighborhood of $x$. Then $N$ belongs to the filter generated by $\mathcal{C}$, and there is a closed set $C \in \mathcal{C}$ such that $C \subseteq N$. Since $C \in \mathcal{F}$, then $N \in \mathcal{F}$. Thus $\mathcal{F}$ converges to $x$.

THEOREM 2.27 Let $X$ be a topological space, and for each $x \in X$, let $\mathcal{B}(x)$ be the neighborhood filter of $x$. Let $\mathcal{A}(x)$ be equal to $\{ A \in \mathcal{B}(x) \mid A \text{ is closed} \}$. Then $X$ is regular if and only if $\mathcal{A}(x)$ converges to $x$ for each $x \in X$.

PROOF. It is to be noted that $\mathcal{A}(x)$ is a filter base on $X$. Suppose that $X$ is regular, and let $x \in X$. If $N \in \mathcal{A}(x)$, there is a set $A \in \mathcal{B}(x)$ so that $A \subseteq N$. Then $N$ belongs to the filter generated by $\mathcal{B}(x)$, and $\mathcal{B}(x)$ converges to $x$.

Conversely, suppose that $\mathcal{B}(x)$ converges to $x$ for each $x \in X$. Let $x \in X$, and let $N \in \mathcal{A}(x)$. Then there is a member $A$ of $\mathcal{B}(x)$ so that $A \subseteq N$. Since $A$ is a closed neighborhood of $x$, $X$ is regular.
DEFINITION 2.14 A topological space $X$ is said to be $T_3$ if it is $T_1$ and regular.

THEOREM 2.28 Let $X$ be a topological space, and let $\mathcal{V}(x)$ be the neighborhood filter of $x$ for each $x \in X$. Let $\mathcal{B}(x)$ be equal to $\{A \in \mathcal{V}(x) \mid A$ is closed $\}$. Then $X$ is $T_3$ if and only if $\mathcal{B}(x)$ has the unique limit point $x$ for each $x \in X$.

PROOF. Suppose that $X$ is $T_3$. By Theorem 2.27, $\mathcal{B}(x)$ converges to $x$ for each $x \in X$. Suppose that $\mathcal{B}(x)$ converges to a point $y$ distinct from $x$ for some $x \in X$. If $N$ is a neighborhood of $y$, there is a set $B \in \mathcal{B}(x)$ so that $B \subseteq N$. Then $N \in \mathcal{V}(x)$, and $\mathcal{V}(x)$ converges to $y$. It follows from Theorem 2.23 that $X$ is not $T_1$. Because $X$ is $T_3$, it must be the case that $\mathcal{B}(x)$ converges only to $x$.

Suppose that $\mathcal{B}(x)$ has the unique limit point $x$. Then $X$ is regular by Theorem 2.27. If $X$ is not $T_1$, there is a point $x$ in $X$ so that $\mathcal{V}(x)$ converges to a point $y$ distinct from $x$. If $N$ is a neighborhood of $y$, there is a set $B \in \mathcal{B}(x)$ so that $B \subseteq N$. It follows that $\mathcal{B}(x)$ converges to $y$. Hence $X$ is $T_1$.

DEFINITION 2.15 A topological space is said to be normal for each closed subset $A$ of $X$ and each neighborhood $N$ of $A$, there is a closed neighborhood $U$ of $A$ so that $U \subseteq N$.

THEOREM 2.29 Let $X$ be a topological space, and let $A$ be a closed subset of $X$. Let $\mathcal{V}(A)$ denote the neighborhood filter of $A$, and define $\mathcal{B}(A) = \{Y \in \mathcal{V}(A) \mid Y$ is closed $\}$. 
Then X is normal if and only if $\mathcal{B}(A)$ converges to A for each closed subset A of X.

**PROOF.** Suppose that X is normal, and let N be a neighborhood of A. There is a closed neighborhood U of A so that $U \subseteq N$. Then $U \in \mathcal{B}(A)$, and N belongs to the filter generated by $\mathcal{B}(A)$. Thus $\mathcal{B}(A)$ converges to A.

Suppose that $\mathcal{B}(A)$ converges to A, and let N be a neighborhood of A. There is an element $U \in \mathcal{B}(A)$ so that $U \subseteq N$. Since U is a closed neighborhood of A, X is normal.

**DEFINITION 2.16** A topological space is said to be $T_4$ if it is $T_1$ and normal.

**THEOREM 2.30** A topological space X is $T_4$ if and only if the following two conditions are satisfied:

1. The neighborhood filter of each point $x$ in X has a unique limit point;

2. For each closed subset A of X, the set $\mathcal{B}(A)$ defined in Theorem 2.29 converges to A.

**PROOF.** The proof follows form Theorem 2.23 and Theorem 2.29.
CHAPTER III

NETS AND CONVERGENCE

In the previous chapter a theory of convergence in topological spaces based upon the concept of a filter was discussed. Such convergence can also be studied if the idea of a sequence is generalized.

DEFINITION 3.1 A binary relation \( \succcurlyeq \) is said to direct a non-empty set \( A \) if \( \succcurlyeq \) has the following properties:

1. If \( x \in A \), then \( x \succcurlyeq x \);
2. If \( x, y, \) and \( z \) are members of \( A \) such that \( x \succcurlyeq y \) and \( y \succcurlyeq z \), then \( x \succcurlyeq z \);
3. If \( x \) and \( y \) belong to \( A \), then there is an element \( z \) in \( A \) so that \( z \succcurlyeq x \) and \( z \succcurlyeq y \).

If \( x \) and \( y \) belong to \( A \) and \( x \succcurlyeq y \), then \( x \) is said to follow \( y \), and \( y \) is said to precede \( x \).

DEFINITION 3.2 A function \( f \) from a directed set \( A \) to a set \( X \) is called a net. The notation \( \{ x_a \mid a \in A \} \) is often used to denote a net on a directed set \( A \).

DEFINITION 3.3 A net \( \{ x_a \mid a \in A \} \) is said to be eventually in a set \( B \) if there is a member \( p \) of \( A \) so that \( x_q \in B \) for all \( q \succcurlyeq p \). The net is frequently in \( B \) if for each \( m \) in \( A \) there is \( n \) in \( A \) so that \( n \succcurlyeq m \) and \( x_n \in B \).
DEFINITION 3.4 A subset B of a directed set A is said to be cofinal if for each \( m \in A \) there is \( p \in B \) so that \( p \geq m \).

It is an immediate consequence of Definition 3.4 that if \( A \) is a set directed by \( \geq \) and \( B \) is a cofinal subset of \( A \), then \( B \) is directed by \( \geq \).

DEFINITION 3.5 A net \( \{ x_a \mid a \in A \} \) in a topological space is said to converge to a point \( x \) if it is eventually in every neighborhood of \( x \).

In Chapter II characterizations of various topological properties in terms of filters were given. The following theorem uses nets to describe some basic topological concepts.

THEOREM 3.1 Let \( X \) be a topological space. Then:

(a) A point \( x \) is an accumulation point of a subset \( A \) of \( X \) if and only if there is a net in \( A - \{ x \} \) which converges to \( x \);

(b) A point \( x \) belongs to the closure of a subset \( A \) of \( X \) if and only if there is a net in \( A \) which converges to \( x \);

(c) A subset \( A \) of \( X \) is closed if and only if no net in \( A \) converges to a point in \( X - A \);

(d) A subset \( A \) of \( X \) is open if and only if every net which converges to a point of \( A \) is eventually in \( A \).

PROOF.

(a) Suppose that \( x \) is an accumulation point of \( A \), and let \( \mathcal{N} \) denote the collection of neighborhoods of \( x \). Because the intersection of two neighborhoods is a neighborhood, \( \mathcal{N} \) is directed by \( \subseteq \). If \( N \) is a neighborhood of \( x \), there is
a point \( x_N \in N \cap A - \{ x \} \). Then \( \{ x_N \mid N \in \mathcal{H} \} \) is a net in \( A - \{ x \} \). If \( M \) and \( N \) belong to \( \mathcal{H} \) and \( M \subseteq N \), then \( x_M \in N \). Thus \( \{ x_N \mid N \in \mathcal{H} \} \) converges to \( x \).

Suppose that there is a net \( \{ x_b \mid b \in B \} \) in \( A - \{ x \} \) which converges to \( x \). Then the net is eventually in every neighborhood of \( x \). Hence \( x \) is an accumulation point of \( A \).

(b) Suppose that \( x \in A \), and let \( \mathcal{H} \) denote the collection of neighborhoods of \( x \). If \( x \in A \), then the net \( \{ x_N \mid N \in \mathcal{H} \} \) where \( x_N = x \) for each \( N \in \mathcal{H} \) converges to \( x \). If \( x \notin A \), then \( x \) is an accumulation point of \( A \). By (a) of this theorem, there is a net in \( A \) which converges to \( x \).

If there is a net in \( A \) which converges to \( x \), then every neighborhood of \( x \) contains a member of \( A \). Thus \( x \in A \).

(c) Suppose that \( A \) is closed, and let \( \{ x_b \mid b \in B \} \) be a net in \( A \). If the net converges to \( x \in X - A \), then \( x \) is an accumulation point of \( A \) which does not belong to \( A \). However, \( A \) is closed. Thus it must be the case that \( x \in A \).

Suppose that no net in \( A \) converges to a point of \( X - A \). If \( A \) is not closed, then there is an accumulation point \( x \) of \( A \) which does not belong to \( A \). It follows that there is a net in \( A \) which converges to \( x \). Thus \( A \) is closed.

(d) Suppose that \( A \) is open, and let \( \{ x_b \mid b \in B \} \) be a net which converges to a point \( x \) in \( A \). Then \( \{ x_b \mid b \in B \} \) is eventually in \( A \) because \( A \) is a neighborhood of \( x \).

Suppose that every net which converges to a point of \( A \) is eventually in \( A \). If \( A \) is not open, there is a point \( x \in A \)
so that every neighborhood of $x$ intersects $X-A$. Then $x \in X-A$, and it follows that there is a net in $X-A$ which converges to $x$. However, such a net is not eventually in $A$. Thus $A$ is open.

**DEFINITION 3.6** A net $\{y_b \mid b \in B\}$ is a subnet of a net $\{x_a \mid a \in A\}$ if there is a function $N$ on $B$ with values in $A$ so that the following conditions are satisfied:

1. $y_i = x_{N_i}$ for each $i$ in $B$;
2. For each $a \in A$ there is some $b \in B$ such that $N_p \geq a$ if $p \geq b$.

An immediate consequence of Definition 3.6 is that if a net is eventually in a set $A$, any subnet of the net is eventually in $A$.

**THEOREM 3.2** Let $\{x_d \mid d \in D\}$ be a net, and let $\mathcal{A}$ be a family of sets such that $\{x_d \mid d \in D\}$ is frequently in each member of $\mathcal{A}$ and such that the intersection of two members of $\mathcal{A}$ contains a member of $\mathcal{A}$. Then there is a subnet of $\{x_d \mid d \in D\}$ which is eventually in each member of $\mathcal{A}$.

**PROOF.** Because the intersection of two members of $\mathcal{A}$ contains a member of $\mathcal{A}$, it follows that $\mathcal{A}$ is directed by $\subseteq$. Define $E = \{(m,A) \mid m \in D, A \in \mathcal{A}, \text{ and } x_m \in A\}$, and let $(m,A)$ follow $(n,B)$ provided that $m \geq n$ and $A \subseteq B$. Then $E$ is a directed set because $\{x_d \mid d \in D\}$ is frequently in each member of $\mathcal{A}$. If $(m,A) \in E$, let $N(m,A) = m$. Then $\{x_{N(m,A)} \mid (m,A) \in E\}$ is a subnet of $\{x_d \mid d \in D\}$. If $A \in \mathcal{A}$, let $m \in D$ so that $x_m \in A$. If $(n,B)$ follows $(m,A)$, then $x_{N(n,B)} = x_n$. Since $x_n \in A$, the subnet $\{x_{N(m,A)} \mid (m,A) \in E\}$ is eventually in $A$. 

DEFINITION 3.7 Let $X$ be a topological space, and let $\{x_d \mid d \in D\}$ be a net in $X$. A point $x$ in $X$ is said to be a cluster point of the net $\{x_d \mid d \in D\}$ if the net is frequently in each neighborhood of $x$.

The following theorem states an important relationship between a cluster point and a subnet of a net.

THEOREM 3.3 A point $x$ in a topological space $X$ is a cluster point of a net $\{x_a \mid a \in A\}$ in $X$ if and only if some subnet of $\{x_a \mid a \in A\}$ converges to $x$.

PROOF. Suppose that $x$ is a cluster point of $\{x_a \mid a \in A\}$, and let $\mathcal{H}$ denote the collection of neighborhoods of $x$. The intersection of two members of $\mathcal{H}$ contains a member of $\mathcal{H}$, and $\{x_a \mid a \in A\}$ is frequently in each member of $\mathcal{H}$. By Theorem 3.2 there is a subnet of $\{x_a \mid a \in A\}$ which is eventually in each member of $\mathcal{H}$. Then this subnet converges to $x$.

Suppose that some subnet of $\{x_a \mid a \in A\}$ converges to $x$. If $x$ is not a cluster point of $\{x_a \mid a \in A\}$, then there is a neighborhood $N$ of $x$ so that $\{x_a \mid a \in A\}$ is not frequently in $N$. It follows that the net is eventually in $X-N$. However, if $\{x_a \mid a \in A\}$ is eventually in $X-N$, every subnet is also eventually in $X-N$. Then no subnet converges to $x$. Thus it must be the case that $x$ is a cluster point of $\{x_a \mid a \in A\}$.

It is also possible to characterize cluster points in terms of closure.

THEOREM 3.4 Let $\{x_n \mid n \in D\}$ be a net in a topological space $X$, and for each $n \in D$, let $A_n$ be the set of all points
Then $x$ is a cluster point of $\{x_n \mid n \in D\}$ if and only if $x$ belongs to the closure of $A_n$ for each $n \in D$.

**Proof.** Suppose that $x$ is a cluster point of $\{x_n \mid n \in D\}$, and let $n \in D$. If $N$ is a neighborhood of $x$, there is $m \in D$ so that $m \geq n$ and $x_m \in N$. Since $x_m \in A_n$, it follows that $x \in A_n$.

Suppose that $x \in A_n$ for each $n \in D$. Let $N$ be a neighborhood of $x$, and let $m \in D$. Since $x \in A_m$, there is $x_n \in A_m$ so that $x_n \in N$. Then $\{x_n \mid n \in D\}$ is frequently in $N$ because $n \geq m$.

Hence $x$ is a cluster point of $\{x_n \mid n \in D\}$.

It is possible to construct a simpler type of subnet than the one described in Theorem 3.2. If $\{x_n \mid n \in D\}$ is a net and $N$ is the identity function on a cofinal subnet $E$ of $D$, then $\{x_{N(m)} \mid m \in E\} = \{x_m \mid m \in E\}$. If $n \in D$, there is $m \in E$ so that $m \geq n$. Then $N_p \geq n$ whenever $p \geq m$. Thus $\{x_{N(m)} \mid m \in E\}$ is a subnet of $\{x_n \mid n \in D\}$. However, the following example shows that this type of subnet is not adequate for all purposes.

**Proposition 3.1** Let $X$ be the set of all pairs of non-negative integers with the topology described as follows: For each point $(m, n)$ other than $(0, 0)$, the set $\{(m, n)\}$ is open. A set $U$ containing $(0, 0)$ is open if for all except a finite number of integers $m$, the set $\{n \mid (m, n) \notin U\}$ is finite. Then:

(a) No sequence in $X-\{(0, 0)\}$ converges to $(0, 0)$;

(b) There is a sequence $S$ in $X-\{(0, 0)\}$ with $(0, 0)$ as a cluster point, and $S$ restricted to any cofinal subset of the natural numbers fails to converge to $(0, 0)$. 

PROOF. The set \( \{(m,n) \mid n = 0,1,2, \ldots \} \) will be called the mth column of X.

(a) Suppose by way of contradiction that there is a sequence S in X - \( \{(0,0)\} \) which converges to (0,0). Thus S is eventually in the complement of each column; for if S is frequently in some column C, it follows that S is not eventually in the neighborhood of (0,0) consisting of all columns except C. Then S has only a finite number of values in each column, and \( \{n \mid (m,n) \in S\} \) is finite for each m. Consequently, X - \( \{(m,n) \mid (m,n) \in S\} \) is a neighborhood of (0,0) which S is not eventually in. Hence no sequence in X - \( \{(0,0)\} \) converges to (0,0).

(b) The set \( S = \{(m,n) \mid m \text{ and } n \text{ are non-negative integers}\} \) is a countable set. If the members of S are counted according to Cantor's diagonal process, the result is a sequence in X. This sequence has infinitely many values in each column, so (0,0) is a cluster point of it. However, S restricted to any cofinal subset of the integers fails to converge to (0,0).

The following definition deals with a type of net which deserves additional study.

DEFINITION 3.8 A net in a set X is said to be universal if for each subset A of X, the net is eventually in A or eventually in X - A.

PROPOSITION 3.2
(a) If a universal net is frequently in a set, then it is eventually in the set. Hence a universal net in a topological space converges to each of its cluster points.

(b) If a net is universal, then each subnet is also universal. If \( \{ x_n | n \in D \} \) is a universal net in a set \( X \) and \( f \) is a function from \( X \) to \( Y \), then the image of \( \{ x_n | n \in D \} \) is a universal net in \( Y \).

**PROOF.**

(a) Suppose that \( \{ x_n | n \in D \} \) is a universal net in a set \( X \), and let \( A \subseteq X \). If \( \{ x_n | n \in D \} \) is frequently in \( A \), then \( \{ x_n | n \in D \} \) is not eventually in \( X - A \). Thus it is eventually in \( A \).

If \( x \) is a cluster point of \( \{ x_n | n \in D \} \) in a topological space, then the net is frequently in each neighborhood of \( x \). Thus it is eventually in each neighborhood of \( x \), and it converges to \( x \).

(b) Let \( \{ x_n | n \in D \} \) be a universal net in a set \( X \), and let \( \{ y_m | m \in E \} \) be a subnet of \( \{ x_n | n \in D \} \). Since \( \{ y_m | m \in E \} \) is eventually in each set that \( \{ x_n | n \in D \} \) is eventually in, it follows that \( \{ y_m | m \in E \} \) is also a universal net.

Let \( f \) be a function from \( X \) to a set \( Y \), and let \( A \) be a subset of \( Y \). Then \( f^{-1}(A) \subseteq X \), so \( \{ x_n | n \in D \} \) is either eventually in \( f^{-1}(A) \) or eventually in \( X - f^{-1}(A) \). If \( \{ x_n | n \in D \} \) is eventually in \( f^{-1}(A) \), then the image of \( \{ x_n | n \in D \} \) is eventually in \( f \left[ f^{-1}(A) \right] \). Since \( f \left[ f^{-1}(A) \right] \subseteq A \), then the image of \( \{ x_n | n \in D \} \) is eventually in \( A \).
If \( \{x_n \mid n \in D\} \) is eventually in \( X-f^{-1}(A) \), a similar proof shows that the image of \( \{x_n \mid n \in D\} \) is eventually in \( f[X-f^{-1}(A)] \).

**THEOREM 3.5** If \( \{x_n \mid n \in D\} = S \) is a net in \( X \), then there is a family \( \mathcal{F} \) of subsets of \( X \) such that \( S \) is frequently in each member of \( \mathcal{F} \), the intersection of two members of \( \mathcal{F} \) belongs to \( \mathcal{F} \), and for each subset \( A \) of \( X \), either \( A \) or \( X-A \) belongs to \( \mathcal{F} \).

**PROOF.** Let \( \{\mathcal{F}_\alpha \mid \alpha \in I\} \) be the collection of all families of subsets of \( X \) so that for each \( \alpha \in I \), \( \{x_n \mid n \in D\} \) is frequently in each member of \( \mathcal{F}_\alpha \) and the intersection of two members of \( \mathcal{F}_\alpha \) belongs to \( \mathcal{F}_\alpha \). Clearly \( \{\mathcal{F}_\alpha \mid \alpha \in I\} \) is not empty. Partially order \( \{\mathcal{F}_\alpha \mid \alpha \in I\} \) by \( \subset \), and let \( \{\mathcal{F}_\beta \mid \beta \in J\} \) be a chain in \( \{\mathcal{F}_\beta \mid \beta \in I\} \). If \( \alpha \in U\{\mathcal{F}_\beta \mid \beta \in J\} \), then there is \( \mathcal{F}_0 \in J \) so that \( A \in \mathcal{F}_0 \). The net \( \{x_n \mid n \in D\} \) is frequently in \( A \). Thus it is frequently in each member of \( U\{\mathcal{F}_\beta \mid \beta \in J\} \).

If \( B \) and \( C \) belong to \( U\{\mathcal{F}_\beta \mid \beta \in J\} \), then there are elements \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) in \( J \) so that \( B \in \mathcal{F}_1 \), and \( C \in \mathcal{F}_2 \). It may be assumed that \( \mathcal{F}_1 \subset \mathcal{F}_2 \) since \( \{\mathcal{F}_\beta \mid \beta \in J\} \) is a chain. The \( B \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \), and \( \mathcal{F}_2 \) is a subset of \( U\{\mathcal{F}_\beta \mid \beta \in J\} \). It follows that \( U\{\mathcal{F}_\beta \mid \beta \in J\} \subseteq \{\mathcal{F}_\alpha \mid \alpha \in I\} \). Since \( \mathcal{F}_\beta \in U\{\mathcal{F}_\beta \mid \beta \in J\} \) for each \( \beta \in J \), the conditions of Zorn's Lemma are satisfied, and \( \{\mathcal{F}_\alpha \mid \alpha \in I\} \) has a maximal element \( \mathcal{F}_M \).

Let \( A \) be a subset of \( X \). It will be shown that \( \{x_n \mid n \in D\} \) cannot be frequently in \( A \cap M \) and frequently in \( (X-A) \cap M \) for
all $M \in \mathcal{M}$; for suppose that this is true, and define
\[ \mathcal{M}' = \mathcal{M} \cup \{ A \cap M \mid M \in \mathcal{M} \}. \]
Now $\{ x_n \mid n \in D \}$ is frequently in each member of $\mathcal{M}'$, and the intersection of two members of $\mathcal{M}'$ belongs to $\mathcal{M}'$. Since $\mathcal{M} \subset \mathcal{M}'$ and $\mathcal{M}$ is maximal, it follows that $\mathcal{M} = \mathcal{M}'$. Then $A \in \mathcal{M}$.

In a similar manner it can be shown that $X-A \in \mathcal{M}$. Then $A \cap (X-A) = \emptyset \in \mathcal{M}$. However, $\{ x_n \mid n \in D \}$ is not frequently in the empty set. Hence it must be the case that for some $M_0$ in $\mathcal{M}$ the net $\{ x_n \mid n \in D \}$ is not frequently in one of $A \cap M_0$ and $(X-A) \cap M_0$.

It will now be shown that either $A$ or $X-A$ belongs to $\mathcal{M}$. Assume that $\{ x_n \mid n \in D \}$ is not frequently in $A \cap M_0$. Then it is frequently in $(X-A) \cap M_0$. If $M \in \mathcal{M}$, the net $\{ x_n \mid n \in D \}$ is frequently in $M \cap M_0$. It follows that either it is frequently in $(M \cap M_0) \cap A$ or frequently in $(M \cap M_0) \cap (X-A)$. If the net is frequently in $(M \cap M_0) \cap A$, then it is frequently in $M_0 \cap A$. Thus it must be the case that $\{ x_n \mid n \in D \}$ is frequently in $(M \cap M_0) \cap (X-A)$. Then it is frequently in $M \cap (X-A)$. If $\mathcal{N}$ is defined to be $\mathcal{M} \cup \{ X-A \} \cup \{ M \cap (X-A) \mid M \in \mathcal{M} \}$, the net $\{ x_n \mid n \in D \}$ is frequently in each member of $\mathcal{N}$, and the intersection of two members of $\mathcal{N}$ belongs to $\mathcal{N}$. Since $\mathcal{M}$ is maximal, $\mathcal{M} = \mathcal{N}$, and $X-A \in \mathcal{M}$.

Similarly, if $\{ x_n \mid n \in D \}$ is not frequently in $M_0 \cap (X-A)$, then $A \in \mathcal{M}$.

**Theorem 3.6** There is a universal subnet of each net in $X$. 
PROOF. Let \( \{ x_n \mid n \in D \} \) be a net in a set \( X \). By Theorem 3.6 there is a family \( \mathcal{F} \) of subsets of \( X \) so that \( \{ x_n \mid n \in D \} \) is frequently in each member of \( \mathcal{F} \), the intersection of two members of \( \mathcal{F} \) belongs to \( \mathcal{F} \), and for each subset \( A \) of \( X \), either \( A \) or \( X-A \) belongs to \( \mathcal{F} \). By Theorem 3.2 there is a subnet of \( \{ x_n \mid n \in D \} \) which is eventually in each member of \( \mathcal{F} \). This subnet is a universal net.
CHAPTER IV

RELATIONSHIPS BETWEEN NETS AND FILTERS

The results obtained in the two previous chapters indicate that there is a close relationship between the concept of a net and that of a filter. It is the purpose of the present chapter to investigate this connection.

THEOREM 4.1 If \( \{x_n \mid n \in D\} \) is a net in a set \( X \), and if \( E(n) = \{x_k \mid k \geq n\} \), then the collection \( \{E(n) \mid n \in D\} \) is a filter base on \( X \).

PROOF. Each set \( E(n) \) is not empty because \( x_n \in E_n \). If \( E_i \) and \( E_j \) are two such sets, there is a member \( k \in D \) so that \( k \geq i \) and \( k \geq j \). Then \( E_k \in E_i \cap E_j \). Hence \( \{E(n) \mid n \in D\} \) is a filter base on \( X \).

DEFINITION 4.1 The filter base \( \{E(n) \mid n \in D\} \) of Theorem 4.1 is said to be the filter base associated with the net \( \{x_n \mid n \in D\} \).

THEOREM 4.2 A net \( \{x_n \mid n \in D\} \) is eventually in a set \( A \) if and only if \( A \) belongs to the filter generated by the filter base \( \{E(n) \mid n \in D\} \) associated with \( \{x_n \mid n \in D\} \).

PROOF. If the net \( \{x_n \mid n \in D\} \) is eventually in \( A \), there is \( k \in D \) so that \( \{x_n \mid n \geq k\} \subseteq A \). Since \( \{x_n \mid n \geq k\} = E(k) \), it follows that \( A \) belongs to the filter generated by \( \{E(n) \mid n \in D\} \).
If $A$ belongs to the filter generated by $\{E(n) \mid n \in D\}$, then $E(k) \subseteq A$ for some $k \in D$. Consequently $x_n \in A$ if $n \geq k$. Hence the net $\{x_n \mid n \in D\}$ is eventually in $A$.

An immediate consequence of Theorem 4.2 is that a net converges to a point in a topological space if and only if each neighborhood of the point belongs to the filter associated with the net.

**Theorem 4.3** If $\{y_i \mid i \in B\}$ is a subnet of $\{x_n \mid n \in D\}$, then the filter generated by $\{F(i) \mid i \in B\}$ is finer than the filter generated by $\{E(n) \mid n \in D\}$.

**Proof.** Let $E(k) \in E(n) \mid n \in D$. There is a function $N: B \to D$ and an element $j \in B$ so that $N(i) \leq k$ whenever $i \geq j$. Since $F(j) = \{y_i \mid i \geq j\} = \{x_{N(i)} \mid i \geq j\}$, it follows that $F(j) \subseteq E(k)$. The required result follows from Theorem 1.4.

It is also possible to associate a net with any given filter base.

**Theorem 4.4** Let $\{E_\alpha \mid \alpha \in A\}$ be a filter base on a set $X$. If $d_1$ and $d_2$ belong to $A$, define $d_1 \leq d_2$ to mean that $E_{d_2} \subseteq E_{d_1}$. Then $A$ is a directed set.

**Proof.** If $d_1$ and $d_2$ belong to $A$, there is an $d_3 \in A$ so that $E_{d_3} \subseteq E_{d_1} \cap E_{d_2}$. Consequently $d_1 \leq d_3$ and $d_2 \leq d_3$. It follows that $A$ is directed.

**Definition 4.2** Let $\{B_\alpha \mid \alpha \in A\}$ be a filter base on a set $X$, and let $A$ be directed as in Theorem 4.4. If an arbitrary point $x_\alpha$ is chosen from each $B_\alpha$, then $\{x_\alpha \mid \alpha \in A\}$
is a net in $X$. Any such net is called a net associated with the filter base $\{B_\alpha | \alpha \in A\}$.

**Theorem 4.5** Let $\{B_\alpha | \alpha \in A\}$ be a filter base on a set $X$, and let $\{x_\alpha | \alpha \in A\}$ be any net associated with $\{B_\alpha | \alpha \in A\}$. If a subset $Y$ of $X$ belongs to the filter generated by $\{B_\alpha | \alpha \in A\}$, then the net $\{x_\alpha | \alpha \in A\}$ is eventually in $Y$.

**Proof.** There is an element $a_0 \in A$ so that $B_{a_0} \subseteq Y$. If $a \geq a_0$, then $x_\alpha \in B_\alpha$. Thus $\{x_\alpha | a \geq a_0\} \subseteq Y$, and the net $\{x_\alpha | \alpha \in A\}$ is eventually in $Y$.

An immediate consequence of Theorem 4.5 is that a net associated with a filter base in a topological space converges to a point whenever the filter base converges to the point. However, the following example shows that the converse of this statement is not true.

**Example 4.1** Let $\mathbb{R}$ be the set of real numbers with the usual topology, and let $A$ be a non-empty open subset of $\mathbb{R}$. Then the set $\{A\}$ is a filter base on $\mathbb{R}$, and if $x \in A$, it follows that $\{x\}$ is a net in $\mathbb{R}$. Now the net $\{x\}$ converges to $x$, but $\{A\}$ does not converge to $x$.

**Theorem 4.6** Let $\{B_\alpha | \alpha \in A\}$ be a filter base on a topological space $X$, and let $\{x_\alpha | \alpha \in A\}$ be any net associated with $\{B_\alpha | \alpha \in A\}$. If $\{x_\alpha | \alpha \in A\}$ converges to $x$, then $x$ is a cluster point of $\{B_\alpha | \alpha \in A\}$.

**Proof.** Let $N$ be a neighborhood of $x$ and $B_\rho \in \{B_\alpha | \alpha \in A\}$. There is an element $a_0 \in A$ so that $x_\alpha \in N$ whenever $a \geq a_0$. 
There is \( \delta \in A \) so that \( \delta \geq \delta_0 \) and \( \delta \geq \delta \). Then \( x \in B \cap N \).

Hence \( x \) is a cluster point of \( \{ B_\alpha \mid \alpha \in A \} \) because each neighborhood of \( x \) intersects each member of \( \{ B_\alpha \mid \alpha \in A \} \).

**THEOREM 4.7** Let \( \{ B_\alpha \mid \alpha \in A \} \) be an ultrafilter base on a topological space \( X \), and let \( \{ x_\alpha \mid \alpha \in A \} \) be any net associated with \( \{ B_\alpha \mid \alpha \in A \} \). If \( \{ x_\alpha \mid \alpha \in A \} \) converges to \( x \), then \( \{ B_\alpha \mid \alpha \in A \} \) converges to \( x \).

**PROOF.** Since \( x \) is a cluster point of \( \{ B_\alpha \mid \alpha \in A \} \), there is a filter \( \mathcal{F} \) which is finer than the filter generated by \( \{ B_\alpha \mid \alpha \in A \} \) and which converges to \( x \). Then \( \mathcal{F} \) and \( \{ B_\alpha \mid \alpha \in A \} \) are equivalent because \( \{ B_\alpha \mid \alpha \in A \} \) is an ultrafilter base. Thus \( \{ B_\alpha \mid \alpha \in A \} \) converges to \( x \).

It is not possible to obtain a result for filter bases which corresponds exactly to Theorem 4.3; nevertheless, the following theorem can be proved.

**THEOREM 4.8** Let \( \{ x_n \mid n \in A \} \) be a net in a set \( X \), and let \( \{ E(n) \mid n \in A \} \) be the filter base associated with \( \{ x_n \mid n \in A \} \). Let \( \{ F_m \mid m \in B \} \) be a filter base whose filter is finer than the filter generated by \( \{ E(n) \mid n \in A \} \). Rename the points \( x_n \) as \( y_m \) in any manner subject to the restriction that \( x_n \) cannot be renamed as \( y_m \) unless it belongs to \( F_m \). Then \( \{ y_m \mid m \in B \} \) is a subnet of \( \{ x_n \mid n \in A \} \).

**PROOF.** First it will be shown that each set \( F_m \) contains a point \( x_n \). Suppose that some \( F_m \) contains no point \( x_n \). If \( n \in A \), there is a set \( F_p \subseteq E(n) \). Since \( E(n) = \{ x_k \mid k \geq n \} \), it
follows that $F_m \cap F_p = \emptyset$, a contradiction. Thus the indicated construction can be carried out, and \( \{ y_m \mid m \in B \} \) is a net in \( X \). There is a function \( N : B \rightarrow A \) which renames the points \( x_n \).

If \( n_0 \in A \), there is \( m_0 \in B \) so that \( F_{m_0} \subseteq E_{n_0} \). Then \( F_m \subseteq E_{n_0} \) whenever \( m \geq m_0 \). Since \( x_{N(m)} = y_m \in F_m \), it follows that \( N(m) \geq n_0 \) if \( m \geq m_0 \). Thus \( \{ y_m \mid m \in B \} \) is a subnet of \( \{ x_n \mid n \in A \} \).

It is possible to associate a net with a filter base in a manner other than that described in Definition 4.2.

**Theorem 4.9** Let \( B \) be a filter base on a set \( X \), and let \( D = \{ (x, F) \mid x \in F \text{ and } F \in B \} \). Direct \( D \) by defining that \( (y, G) \geq (x, F) \) if \( G \subseteq F \), and let \( f(x, F) = x \). Then the filter \( \mathcal{F} \) generated by \( B \) is the collection of all sets \( A \) such that the net \( \{ f(x, F) \mid (x, F) \in D \} \) is eventually in \( A \).

**Proof.** It is routine to verify that the order defined on \( D \) directs \( D \). Let \( F \in \mathcal{F} \). There is \( B \in B \) so that \( B \subseteq F \).

If \( x \in B \) and \( (y, G) \geq (x, B) \), then \( f(y, G) = y \in F \). Thus the net \( \{ f(x, F) \mid (x, F) \in D \} \) is eventually in \( F \).

If \( \{ (x, F) \mid (x, F) \in D \} \) is eventually in a subset \( A \) of \( X \), then there is a member \( (y, B) \in D \) so that \( x \in A \) whenever \( (x, F) \geq (y, B) \). If \( z \in B \), then \( (z, B) \geq (y, B) \). Thus \( B \subseteq A \), and \( A \in \mathcal{F} \).

The filter associated with the net described in Theorem 4.9 is the same filter used to construct the net.

The following example shows that such a situation may not exist if a net is constructed from a filter in the manner described by Definition 4.2.
EXAMPLE 4.2 Let \( R \) be the set of real numbers with the usual topology, and let \( x \in R \). Let \( \{ N_i \mid i \in I \} \) be the neighborhood filter of \( x \), and for each \( N_i \in \{ N_i \mid i \in I \} \) choose a point \( y_i \in N_i \) so that \( y_i \neq x \). Then \( \{ y_i \mid i \in I \} \) is a net associated with \( \{ N_i \mid i \in I \} \). If \( i_o \in I \), then \( \{ y_i \mid i \geq i_o \} \) belongs to the filter base associated with \( \{ y_i \mid i \in I \} \). However, since \( x \neq y_i \) for each \( i \in I \), it follows that \( \{ y_i \mid i \geq i_o \} \) is not a neighborhood of \( x \). Thus \( \{ y_i \mid i \geq i_o \} \) does not belong to \( \{ N_i \mid i \in I \} \).

THEOREM 4.10 Let \( \{ B_a \mid a \in A \} \) be a filter base on a set \( X \). Then \( \{ B_a \mid a \in A \} \) is an ultrafilter base if and only if for every net \( \{ x_a \mid a \in A \} \) associated with \( \{ B_a \mid a \in A \} \), the filter base associated with \( \{ x_a \mid a \in A \} \) is equivalent to \( \{ B_a \mid a \in A \} \).

PROOF. Suppose that \( \{ B_a \mid a \in A \} \) is an ultrafilter base, and let \( \{ x_a \mid a \in A \} \) be any net associated with \( \{ B_a \mid a \in A \} \). Let \( \{ E(a) \mid a \in A \} \) be the filter base associated with the net \( \{ x_a \mid a \in A \} \). If \( B_{a_0} \in \{ B_a \mid a \in A \} \) and \( a \geq a_0 \), then \( x_a \in B_{a_0} \). Thus \( E(a_0) \subseteq B_{a_0} \).

If \( E(a_1) \) does not belong to the filter \( \mathcal{F} \) generated by \( \{ B_a \mid a \in A \} \) for some \( a_1 \in A \), then \( X - E(a_1) \in \mathcal{F} \). There is \( \beta \in I \) so that \( B_{\beta} \subseteq X - E(a_1) \), and there is \( \gamma \in I \) so that \( \delta \geq a_1 \) and \( \gamma \geq \beta \). Then \( x_{\gamma} \in E(a_1) \), and \( B_{\gamma} \subseteq B_{\beta} \). Since \( x_{\gamma} \in B_{\gamma} \), it follows that \( x_{\gamma} \in B_{\beta} \). Thus \( x_{\gamma} \in X - E(a_1) \). However, \( x_{\gamma} \) cannot belong to \( E(a_1) \) and \( X - E(a_1) \). Hence it must be the
case that \( E(a_i) \) belongs to \( \mathcal{F} \). Consequently \( \{ B_a | a \in I \} \) and \( \{ E(a) | a \in I \} \) are equivalent.

Conversely, suppose by way of contradiction that

\[ \{ B_a | a \in I \} \] is not an ultrafilter base. Then there is a subset \( A \) of \( X \) so that \( A \notin \mathcal{F} \) and \( X-A \notin \mathcal{F} \). Thus \( B_a \notin A \) for each \( a \in I \). It follows that \( B_a \cap (X-A) \neq \emptyset \), and a net \( \{ x_a | a \in I \} \) can be chosen so that \( x_a \in B_a \cap (X-A) \) for all \( a \in I \). Since \( \{ x_a | a \in I \} \) is in \( X-A \), the set \( X-A \) belongs to the filter associated with the net \( \{ x_a | a \in I \} \). However, \( X-A \notin \mathcal{F} \), and this contradiction completes the proof.

There is an important relationship between universal nets and ultrafilters.

**Theorem 4.11** If \( \{ x_a | a \in A \} \) is a universal net in a set \( X \), then the filter base \( \{ E(a) | a \in A \} \) associated with \( \{ B_a | a \in A \} \) is an ultrafilter base.

**Proof.** Let \( Y \subset X \). The net \( \{ x_a | a \in A \} \) is eventually in either \( Y \) or \( X-Y \). If \( \{ x_a | a \in A \} \) is eventually in \( Y \), there is an element \( a_0 \in A \) so that \( x_a \in Y \) whenever \( a \geq a_0 \). Then \( E(a_0) \subset Y \), and \( Y \) belongs to the filter generated by \( \{ E(a) | a \in A \} \).

Similarly, if \( \{ x_a | a \in A \} \) is eventually in \( X-Y \), then \( X-Y \) belongs to the filter generated by \( \{ E(a) | a \in A \} \).

**Theorem 4.12** If \( \{ E(a) | a \in A \} \) is an ultrafilter base on a set \( X \), then any net associated with \( \{ B_a | a \in A \} \) is a universal net.

**Proof.** Let \( \{ x_a | a \in A \} \) be a net associated with \( \{ B_a | a \in A \} \), and let \( Y \subset X \). Since \( \{ B_a | a \in A \} \) is an ultrafilter base, either
Y or \( X-Y \) belongs to the filter \( \mathcal{F} \) generated by \( \{ B_\alpha \mid \alpha \in \Lambda \} \).

If \( Y \in \mathcal{F} \), then there is \( \alpha_0 \in \Lambda \) so that \( B_{\alpha_0} \subseteq Y \). Thus \( \{ x_\alpha \mid \alpha \in \Lambda \} \subseteq B_{\alpha_0} \subseteq Y \), and the net \( \{ x_\alpha \mid \alpha \in \Lambda \} \) is eventually in \( Y \).

Similarly, if \( X-Y \in \mathcal{F} \), then \( \{ x_\alpha \mid \alpha \in \Lambda \} \) is eventually in \( X-Y \).

Hence \( \{ x_\alpha \mid \alpha \in \Lambda \} \) is a universal net.

It is possible to associate many nets with one filter base, and it is also possible to associate the same filter base with different nets. Nevertheless, equivalent filter bases generate equal filters. The following definition makes it possible to define a one-to-one correspondence between a filter and a certain collection of nets.

**DEFINITION 4.3** Let \( \{ x_\alpha \mid \alpha \in \Lambda \} \) and \( \{ y_\beta \mid \beta \in \Lambda \} \) be nets in a set \( X \). Then \( \{ x_\alpha \mid \alpha \in \Lambda \} \) and \( \{ y_\beta \mid \beta \in \Lambda \} \) are said to be equivalent if the filter bases associated with them are equivalent.

Clearly, the relation defined in Definition 4.3 is an equivalence relation.

**THEOREM 4.13** Let \( \mathcal{A} = \{ A_\alpha \mid \alpha \in \Lambda \} \) be the family of all collections of equivalent nets on a set \( X \), and for each \( \alpha \in \Lambda \), let \( \mathcal{F}_\alpha \) be the filter generated by the filter bases associated with the nets in \( A_\alpha \). Let \( \mathcal{B} \) be the collection of all filters on \( X \), and define \( f: \mathcal{A} \to \mathcal{B} \) by \( f(A_\alpha) = \mathcal{F}_\alpha \). Then \( f \) is one-to-one and onto.
PROOF. If \( \alpha \) and \( \beta \) are elements of \( I \) such that \( \alpha \neq \beta \), there are nets in \( A_{\alpha} \) and \( A_{\beta} \) whose associated filter bases generate distinct filters. Then \( \mathcal{F}_\alpha \neq \mathcal{F}_\beta \), and \( f \) is one-to-one.

If \( \mathcal{F} \in \mathcal{B} \), then by Theorem 4.9 there is a net in \( X \) whose associated filter base is equivalent to \( \mathcal{F} \). It follows that \( \mathcal{F} \) is the image of the equivalent class to which this net belongs. Thus \( f \) is onto.

If a topology is defined on a set \( X \), then it follows from Theorem 4.2 that equivalent nets converge to the same points. However, distinct filters may converge to the same set of points. It is possible to define an equivalence relation on the collection of all nets in a topological space in terms of convergence.

DEFINITION 4.4 Let \( \{ x_\alpha \mid \alpha \in A \} \) and \( \{ y_\beta \mid \beta \in B \} \) be nets in a topological space \( X \). Then the nets \( \{ x_\alpha \mid \alpha \in A \} \) and \( \{ y_\beta \mid \beta \in B \} \) are said to be equivalent if they converge to the same set of points.

It follows from Theorem 4.2 that two nets in a topological space are equivalent if and only if the filter bases associated with the nets converge to the same points.

THEOREM 4.14 In a topological space \( X \), let \( \mathcal{A} = \{ A_\alpha \mid \alpha \in I \} \) be the family of all collections of equivalent nets in \( X \). Let \( \mathcal{B} = \{ B_\beta \mid \beta \in J \} \) be the family of all collections of filters on \( X \) so that for each \( \beta \in J \), a filter \( \mathcal{F} \in \mathcal{B}_\beta \) if and only if the set of limit points of \( \mathcal{F} \) is the same as that of all other filters in \( \mathcal{B}_\beta \). Define \( f: \mathcal{A} \to \mathcal{B} \) so that for each \( \alpha \in I \), \( f(A_\alpha) \)
is the member of \( \mathcal{B} \) to which the filters associated with the nets in \( A_\alpha \) belong. Then \( f \) is one-to-one and onto.

**Proof.** If \( \alpha \) and \( \beta \) are members of \( I \) so that \( A_\alpha \neq A_\beta \), it may be assumed without loss of generality that \( A_\alpha \neq A_\beta \). Then there is a net \( \{ x_\delta | \delta \in \mathcal{K} \} \) in \( A_\alpha \) which does not belong to \( A_\beta \). The set of limit points of \( \{ x_\delta | \delta \in \mathcal{K} \} \) is not the same as that of any net in \( A_\beta \). Thus the set of limit points of the filter associated with \( \{ x_\delta | \delta \in \mathcal{K} \} \) is not the same as that of the filter associated with any net in \( A_\beta \). Thus \( f(A_\alpha) \neq f(A_\beta) \), and \( f \) is one-to-one.

Let \( B_\beta \in \mathcal{B} \). If \( f \in B_\beta \), then there is a net in \( X \) whose associated filter has the same limit points as \( f \). If \( A_{\alpha_0} \) denotes the member of \( A \) to which this net belongs, then \( f(A_{\alpha_0}) = B_\beta \). Thus \( f \) is onto.

It is worthwhile to note that if two nets are equivalent on a set \( X \) as defined in Definition 4.3, they remain equivalent if any topology is defined on \( X \). On the other hand, the equivalence of Definition 4.4 depends upon the topology which \( X \) carries. Nets which are equivalent for some topology on \( X \) may not be equivalent if another topology is defined on \( X \). For example, if \( X \) has the indiscrete topology, all nets on \( X \) are equivalent. If \( X \) has the discrete topology, each net on \( X \) has at most one limit point.

If a topology is defined on \( X \), then nets which are equivalent in the sense of Definition 4.4 may not be equivalent according to Definition 4.3. Distinct filters may converge to the same set of points.
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