CONTINUOUS SOLUTIONS OF LAPLACE'S EQUATION IN TWO VARIABLES

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CONTINUOUS SOLUTIONS OF LAPLACE'S EQUATION IN TWO VARIABLES

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CHAPTER I

INTRODUCTION

In mathematical physics, Laplace's equation, a linear homogeneous partial differential equation in two real variables, $x$ and $y,$
\[ \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \] (1)
plays an especially significant role. It is fundamental to the solution of problems in electrostatics, thermodynamics, potential theory and other branches of mathematical physics. It is for this reason that this investigation concerns the development of some general properties of continuous solutions of this equation.

Essential to a clear formulation of the development to follow are the following definitions.

Definition 1. The set $\mathbb{E}_1$ is the set of real numbers.

Definition 2. The set of all points in the rectangular coordinate plane, $\mathbb{E}_2,$ is defined by $\mathbb{E}_2 = \{(a, b): a, b \in \mathbb{E}_1\}.$

Definition 3. If $a, b \in \mathbb{E}_1$ and $a < b,$ then $\{x: x \in \mathbb{E}_1 \text{ and } a < x < b\}$ is called the "open interval from $a$ to $b,"$ or more briefly, the "open interval $I:(a,b)."$

Definition 4. If $a, b \in \mathbb{E}_1$ and $a < b,$ then $\{x: x \in \mathbb{E}_1 \text{ and } a \leq x \leq b\}$ is called the "closed interval from $a$ to $b,"$ or more briefly, the "closed interval $I:[a,b]."$
Definition 5. A point set is the set of all points satisfying a given condition or set of conditions.

Definition 6. A curve \( c \) in \( E_2 \) is a set of points \((x,y)\) given by \( c(t) = (f(t),g(t)) \), where \( f \) and \( g \) are continuous functions of a parameter \( t \) whose domain is an interval.

Definition 7. A circular neighborhood of a point \( P \) in \( E_2 \), \( P(x',y') \), is a set of all points \( Q(x,y) \) in \( E_2 \) such that, for a real number \( d > 0 \),
\[
0 \leq [(x - x')^2 + (y - y')^2]^{\frac{1}{2}} = m(PQ) < d,
\]
where \( m(PQ) \) is the measure of the line segment determined by the points \( P \) and \( Q \).

Definition 8. In \( E_2 \), a point set \( S \) is an open point set if each point \( P \) in \( S \) has some circular neighborhood which is a subset of \( S \).

Definition 9. In \( E_2 \), a point \( P \) is called a "boundary point" of a point set \( S \) if every circular neighborhood of \( P \) contains at least one point of \( S \) and one point not in \( S \).

Definition 10. The boundary of a point set \( S \) is the collection of all boundary points of \( S \).

Definition 11. An interior point \( P \) of a point set \( S \) in \( E_2 \) has some circular neighborhood which is a subset of \( S \).

Definition 12. The interior of a point set \( S \) is the set consisting of all interior points of \( S \).

Definition 13. In \( E_2 \), a point \( P \) is called an "exterior point" of a point set \( S \) if there exists a circular neighborhood of \( P \) which contains no points of \( S \).
**Definition 14.** A region \( R \) is a nonempty open point set.

**Definition 15.** A closed region \( \overline{R} \) is the union of the region \( R \) and its boundary \( B \) and will be denoted by \( \overline{R} = R + B \).

**Definition 16.** A point set \( S \) in the plane is said to be bounded if \( S \) is a subset of the interior of some sufficiently large circle.

**Definition 17.** If a plane curve \( c(t) \) has as its domain \( I: [a, b] \) and \( c(a) = c(b) \), then \( c \) is called a "closed curve." A closed curve \( c(t) \) with domain \( I: [a, b] \) is called a "simple closed curve" if \( c(t_1) \neq c(t_2) \) whenever \( t_1 \neq t_2 \), where \( t_1 \) is an element of \( I: (a, b) \) and \( t_2 \) is an element of \( I: [a, b] \).

**Definition 18.** A plane curve given by \( c(t) = (f(t), g(t)) \) in Definition 6 is called a "smooth curve" if (1) \( c(t_1) \) is not equal to \( c(t_2) \) whenever \( t_1 \) and \( t_2 \) are in \( I: (a, b) \) unless \( c(t) \) is a simple closed curve as defined in Definition 17 and (2) \( df/dt \) and \( dg/dt \) exist, are continuous, and are not both zero for any value of \( t \). A plane curve \( c(t) \) is called a "sectionally smooth curve" if the parameter interval can be partitioned so that \( c(t) \) is smooth in each subinterval (that is, \( dc/dt \) is continuous and not zero in each subinterval and both one-sided derivatives, \( dc(t_k - 0)/dt \) and \( dc(t_k + 0)/dt \) exist at each partition point \( t_k \)).

**Definition 19.** A regular region in \( \mathbb{R}^2 \) is a closed and bounded region whose boundary is a sectionally smooth, simple closed curve or a finite number of sectionally smooth, simple closed curves which do not intersect each other.
In the general theory of partial differential equations the complete solution \( u(x,y) \) of an equation of the form

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = V, \tag{2}
\]

where \( u \) is the dependent variable, \( x \) and \( y \) are the independent variables, and \( R, S, T, \) and \( V \) are functions of \( x, y, u, \frac{\partial u}{\partial x}, \) and \( \frac{\partial u}{\partial y} \), can be determined by the use of a method credited to Gaspard Monge (1746-1818).\(^1\) However, the complete solutions of certain equations belonging to the class of equations indicated by equation (2) can be obtained more easily in many cases by suitable transformations involving the independent variables. Laplace's equation, (1), belongs to this class of equations, because (1) is obtained by taking \( R = T = 1 \) and \( S = V = 0 \) in (2). If this latter approach is used, equation (1) is transformed into an easily integrated equation in the following manner. If it is assumed that

\[
u(x,y) = z(v,w), \tag{3}\]

where \( v = x + my \), \( w = x - my \), and \( m \) is a constant, then by a "chain rule" of differentiation,

\[
\frac{\delta u}{\delta x} = \frac{\delta z}{\delta v} \frac{\delta v}{\delta x} + \frac{\delta z}{\delta w} \frac{\delta w}{\delta x} = \frac{\delta z}{\delta v} + \frac{\delta z}{\delta w},
\]

since \( \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 1; \)

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial^2 z}{\partial v^2} \frac{\delta v}{\delta x} + \frac{\partial^2 z}{\partial w^2} \frac{\delta w}{\delta x} + \frac{\partial^2 z}{\partial v \partial w} \frac{\delta v}{\delta x} \frac{\delta w}{\delta x} + \frac{\partial^2 z}{\partial w \partial v} \frac{\delta w}{\delta x} \frac{\delta v}{\delta x} + 2\frac{\partial^2 z}{\partial v \partial w} \frac{\delta v}{\delta x} \frac{\delta w}{\delta x}
\]

since $\delta v/\delta x = \delta w/\delta x = 1$ and $\delta^2 z/\delta w^2 v = \delta^2 z/\delta v^2 w$ if $z$, $\delta z/\delta v$, $\delta z/\delta w$, $\delta^2 z/\delta w^2 v$, and $\delta^2 z/\delta v^2 w$ are defined and $\delta^2 z/\delta w^2 v$ and $\delta^2 z/\delta v^2 w$ are continuous.\(^2\) Similarly,

$$\frac{\delta^2 u}{\delta y^2} = \frac{\delta}{\delta y}(\delta u/\delta y) = \frac{m}{\delta v^2} \frac{\delta^2 z}{\delta y} \frac{\delta v}{\delta w} + \frac{m}{\delta v^2} \frac{\delta^2 z}{\delta w^2} \frac{\delta w}{\delta y} - \frac{m}{\delta v^2} \frac{\delta z}{\delta w} \frac{\delta w}{\delta y} - \frac{m}{\delta v^2} \frac{\delta^2 z}{\delta v^2 \delta w} \frac{\delta v}{\delta y}$$

Hence, $\delta^2 u/\delta x^2 + \delta^2 u/\delta y^2 = 0$ becomes

$$\left(\frac{\delta^2 z}{\delta v^2} + \frac{\delta^2 z}{\delta w^2} + 2\frac{\delta^2 z}{\delta v^2 \delta w}\right) + \left(m^2 \frac{\delta^2 z}{\delta v^2} + m^2 \frac{\delta^2 z}{\delta w^2} - 2m \frac{\delta^2 z}{\delta v^2 \delta w}\right) = 0,$$

or

$$\left(m^2 + 1\right)\frac{\delta^2 z}{\delta v^2} + \frac{\delta^2 z}{\delta w^2} + 2\left(1 - m^2\right)\frac{\delta^2 z}{\delta v^2 \delta w} = 0.$$

An examination of this last equation reveals that if $m^2 = -1$ the last equation is equivalent to $0(\delta^2 z/\delta v^2 + \delta^2 z/\delta w^2) + 4\delta^2 z/\delta v^2 w = 0$, or $\delta^2 z/\delta v^2 w = 0$. Then,

$$\int \frac{\delta^2 z}{\delta v^2} dw = \int 0 dw + P(v),$$

where $P(v)$ is an arbitrary function of $v$ only. Integration gives $\delta z/\delta v = P(v)$, from which

$$\int \frac{\delta z}{\delta v} dv = \int P(v) dv + g(w),$$

where $g(w)$ is an arbitrary function of $w$ only, yields

$$z(v,w) = f(v) + g(w).$$

Since the choice of $m^2 = -1$ was made, $m = \pm i$; and, hence, $v = x + iy$ and $w = x - iy$, or $v = x - iy$ and $w = x + iy$.

Hence, an arbitrary choice of $m = i$ does not affect the

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generality and correctness of \( z(v,w) = f(v) + g(w) \), since \( f \) and \( g \) are arbitrary. Therefore, if this choice is made, the assumption that \( u(x,y) = z(v,w) \) gives the complete solution of \( \partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0 \) as

\[
u(x,y) = f(x + iy) + g(x - iy) . \tag{4} \]

For \( u \) in equation (4) to be real, \( g \) must be the conjugate of \( f \). Equation (4) indicates, therefore, that Laplace's equation in two real variables has a special significance in the theory of functions of a complex variable.
CHAPTER II

DIRICHLET PROBLEM IN THE PLANE

The solution of Laplace's equation in two variables derived in Chapter I is too general for most practical purposes because of the great difficulty of determining the arbitrary functions to satisfy given conditions. In applied problems involving differential equations, to determine a particular solution of a differential equation satisfying a given set of conditions is required. These conditions are called the "boundary conditions." A differential equation together with a set of boundary conditions is called a "boundary-value problem."

The first boundary-value problem for Laplace's equation is known as the "Dirichlet problem." For general regions in \( E_2 \), it may be stated as follows:

If \( G \) is a bounded region with boundary \( B \) and \( f(s) \) is a continuous function of arc length \( s \) defined on \( B \), determine \( u(x,y) \), a function which is continuous and has continuous first and second partial derivatives in \( \bar{G} = G + B \) such that \( u(x,y) \) satisfies \( \delta^2 u / \delta x^2 + \delta^2 u / \delta y^2 = 0 \) in \( G \) and such that \( \lim_{s \to B} u(x,y) = f(s) \) on \( B \).

If \( G \) is a regular region, then it can be proved that the Dirichlet problem always has a solution.\(^1\) In this instance,

a simple, but very important, case is the one for which \( G \) is the interior of a circle.

The Dirichlet problem for the circle may be stated as follows:

If \( K \) is the interior of a circle \( C \) and \( f(s) \) is a continuous function of arc length \( s \) defined on \( C \), determine a function \( u(x,y) \) which is continuous and possesses continuous first and second partial derivatives in \( \mathbb{R} = K + C \) such that \( u(x,y) \) satisfies the equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

in \( K \) and such that the limit \( u(x,y) = f(s) \) on \( C \).

In order to derive a solution of this problem, it is convenient to place the origin of the rectangular system of coordinates \((x,y)\) at the center \( O \) of \( C \). This can be done without loss of generality, since if \( O = P(h,k) \), Laplace's equation remains invariant under the transformation \( x = x_1 + h \) and \( y = y_1 + k \); that is, \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) and also \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \). If the pole of a system of polar coordinates \((r,\theta)\) is made to coincide with the origin and the polar axis with the positive \( x \)-axis, then \( x = r \cos \theta \) and \( y = r \sin \theta \) in Figure 1.

![Figure 1](image)

Fig. 1—Diagram for the Dirichlet problem for the circle
For $x = r \cos \theta$, $y = r \sin \theta$, and $u(r, \theta)$,

$$\frac{\partial u}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial u}{\partial y} = (\cos \theta) \frac{\partial u}{\partial x} + (\sin \theta) \frac{\partial u}{\partial y}$$  \hspace{1cm} (1)

and,

$$\frac{\partial u}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial u}{\partial y} = (-r \sin \theta) \frac{\partial u}{\partial x} + (r \cos \theta) \frac{\partial u}{\partial y}.$$  \hspace{1cm} (2)

From equations (1) and (2),

$$\frac{\partial u}{\partial x} = \begin{vmatrix} \cos \theta & \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} & \sin \theta \end{vmatrix}, \quad \frac{\partial u}{\partial \theta} = \begin{vmatrix} \cos \theta & \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} & \sin \theta \end{vmatrix}$$

$$= \frac{r \cos \theta \frac{\partial u}{\partial r} - \sin \theta \frac{\partial u}{\partial \theta}}{\cos \theta \sin \theta} = \frac{\cos \theta (\frac{\partial u}{\partial r}) - (\sin \theta) \frac{\partial u}{\partial \theta}}{r \cos \theta + r \sin \theta}$$  \hspace{1cm} (3)

and,

$$\frac{\partial u}{\partial y} = \begin{vmatrix} \cos \theta & \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} & \sin \theta \end{vmatrix}, \quad \frac{\partial u}{\partial \theta} = \begin{vmatrix} \cos \theta & \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} & \sin \theta \end{vmatrix}$$

$$= \frac{r \sin \theta \frac{\partial u}{\partial r} + \cos \theta \frac{\partial u}{\partial \theta}}{\cos \theta \sin \theta} = \frac{(\cos \theta) \frac{\partial u}{\partial r} + (r \sin \theta) \frac{\partial u}{\partial \theta}}{r}$$  \hspace{1cm} (4)

In order to obtain $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, $u$ is replaced by $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in equations (3) and (4), respectively.

Hence,

$$\frac{\partial^2 u}{\partial x^2} = (\cos \theta) \frac{\partial}{\partial r} (\frac{\partial u}{\partial x}) - \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left[ (\cos \theta) \frac{\partial u}{\partial x} + (r \sin \theta) \frac{\partial u}{\partial \theta} \right]$$

$$= \cos \theta [(\cos \theta) \frac{\partial^2 u}{\partial r^2} + (1/r^2 \sin \theta) \frac{\partial u}{\partial \theta} - (1/r \sin \theta) \frac{\partial^2 u}{\partial r \partial \theta}]

- \frac{1}{r \sin \theta} \left[ (-\sin \theta) \frac{\partial u}{\partial r} + (\cos \theta) \frac{\partial^2 u}{\partial r \partial \theta} \right]

- (1/r \sin \theta) (\delta^2 u/\delta \theta^2) - (1/r \cos \theta) (\delta u/\delta \theta)$$  \hspace{1cm} (5)
And, the second partial derivative of $u$ with respect to $y$ is
\[
\frac{\partial^2 u}{\partial y^2} = \frac{1}{r} \cos \theta \left[ \frac{1}{r} \sin \theta \frac{\partial u}{\partial \theta} + \frac{1}{r} \cos \theta \frac{\partial^2 u}{\partial \theta^2} \right] + \left( \frac{\sin \theta}{r^2} \right) \frac{\partial^2 u}{\partial \theta^2} + \left( \frac{\cos \theta}{r^2} \sin \theta \right) \frac{\partial^2 u}{\partial \theta \partial r} + \left( \frac{\sin \theta}{r^2} \cos \theta \right) \frac{\partial^2 u}{\partial r^2}.
\]

Therefore, from equations (5), (6) and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$,
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left( \cos^2 \theta + \sin^2 \theta \right) \frac{\partial^2 u}{\partial x^2} + \left( \frac{2 \cos \theta \sin \theta - 2 \cos \theta \sin \theta \frac{\partial u}{\partial r}}{r^2} \right) = 0.
\]

From this last equation,
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{r} \frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial u}{\partial y} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
\]

The method of separation of variables effects a solution of equation (7). Under the assumption that $u(r, \theta) = H(r)F(\theta)$, where $H$ is a function of $r$ alone and $F$ is a function of $\theta$ alone,

\[
\frac{\partial u}{\partial r} = F \frac{dH}{dr}, \quad \text{and} \quad \frac{\partial^2 u}{\partial r^2} = F \frac{d^2 H}{dr^2}, \quad \text{and},
\]

\[
\frac{\partial u}{\partial \theta} = H \frac{dF}{d\theta}, \quad \text{and} \quad \frac{\partial^2 u}{\partial \theta^2} = H \frac{d^2 F}{d\theta^2}.
\]

Substitution of (8) and (9) in (7) gives
\[
F \frac{d^2 H}{dr^2} + \frac{1}{r} \frac{F}{dr} \frac{dH}{dr} + \frac{1}{r} H \frac{d^2 F}{d\theta^2} = 0.
\]
From the last equation
\[ r^2 \frac{d^2 H}{dr^2} + r \frac{dH}{dr} + \frac{d^2 F}{d\theta^2} = 0. \]

In this equation the expression on the left is independent of \( \theta \), and the expression on the right is independent of \( r \). Hence, each must be equal to a constant, say \( m \). Then,
\[ r^2 \frac{d^2 H}{dr^2} + r \frac{dH}{dr} - mH = 0, \]and\[ \frac{d^2 F}{d\theta^2} + mF = 0. \] (10)
The solutions of the linear ordinary differential equations (10) can be derived by using Euler's method.\(^2\) If, in the first of equations (10), it is assumed that \( H(r) = r^p \), then \[ \frac{dH}{dr} = pr^{p-1}; \quad \frac{d^2 H}{dr^2} = p(p-1)r^{p-2}; \quad r^2 \frac{d^2 H}{dr^2} = p(p-1)r^p; \quad r \frac{dH}{dr} = pr^p; \quad \text{and,} \quad mH = mr^p. \]The equation, then, becomes
\[ p(p-1)r^p + pr^p + (-m)r^p = 0, \]or
\[ (p^2 - m)r^p = 0. \]Hence, \( p^2 - m = 0 \) yields \( p = \pm \sqrt{m} \). Therefore, solutions of \( r^2 \frac{d^2 H}{dr^2} + r \frac{dH}{dr} - mH = 0 \) are \( H_1 = Ar^{\sqrt{m}} \) and \( H_2 = Br^{-\sqrt{m}} \), where \( A \) and \( B \) are arbitrary constants. Then, the complete solution of this equation is
\[ H(r) = Ar^{\sqrt{m}} + Br^{-\sqrt{m}}. \] (11)
If, in the second equation of (10), it is assumed that \( F(\theta) = e^{p\theta} \), \( \frac{dF}{d\theta} = pe^{p\theta}, \frac{d^2 F}{d\theta^2} = p^2 e^{p\theta} \), and \( mF = me^{p\theta} \), then the equation becomes
\[ p^2 e^{p\theta} + me^{p\theta} = 0, \]or \( (p^2 + m)e^{p\theta} = 0. \)

Hence \( p^2 + m = 0 \) yields \( p = \pm \sqrt{m} \). Therefore, solutions of \( d^2 F/d\theta^2 + mF = 0 \) are \( F_1(\theta) = A_m \cos \sqrt{m} \theta \) and \( F_2 = B_m \sin \sqrt{m} \theta \), and the complete solution is

\[
F(\theta) = A_m \cos \sqrt{m} \theta + B_m \sin \sqrt{m} \theta. \tag{12}
\]

Since, from the problem, it is assumed that \( f(\theta) = f(\theta+2\pi) \), it follows that \( F(\theta) \) must be periodic with period \( 2\pi \). An examination of equation (12) reveals that this can happen only if \( m \) is an integer squared, say \( n^2 \). Since \( r^2 = x^2 + y^2 \), 
\[
Br^{-\sqrt{m}} = Br^{-n}, \tag{11}
\]
from (11), is undefined at the origin. Consequently, \( H(r) \) given by (11) is discontinuous at the origin. Hence, a continuous particular solution of the first equation of equations (10) is obtained by taking \( B = 0 \) in (11). This gives

\[
H(r) = Ar^n. \tag{13}
\]

From equations (12) and (13),

\[
u_n(r, \theta) = H(r)F(\theta) = r^n(A_n \cos n\theta + B_n \sin n\theta), \tag{14}
\]

where \( A_n \) and \( B_n \) are arbitrary, is, therefore, a set of particular continuous solutions of Laplace's equation in \( k \), since the sum of the continuous functions \( A_k \cos k\theta \) and \( B_k \sin k\theta, k = 1, 2, 3, \ldots, n \), is a continuous function and the product of the continuous functions \( r^k \) and \( A_k \cos k\theta + B_k \sin k\theta), k = 1, 2, 3, \ldots, n \), is a continuous function. \(^3\)

Also, since Laplace's equation is a linear and homogeneous equation, any finite linear combination of the solutions (14)
is also a solution. For \( n = 0 \) in (14), \( u_0 \) = a constant, say \( A_0 = a_0/2 \). Then, equation (14) becomes

\[
u_k(r, \theta) = a_0/2 + \sum_{k=1}^{\infty} r^k (A_k \cos k\theta + B_k \sin k\theta). \tag{15}\]

Also, an infinite series of the solutions given by (15) is a continuous solution of (7) in \( K \) provided the series and its first and second partial derivatives are suitably convergent. Then, if the coefficients \( A_n \) and \( B_n \) in

\[
u(r, \theta) = a_0/2 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \tag{16}\]

can be determined so that

\[
u(\lambda, \theta) = f(\theta) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta), \tag{17}\]

where \( a_n = 1/\pi \int_0^{\pi} f(\sigma) \cos(n\sigma) d\sigma \) and \( b_n = 1/\pi \int_0^{\pi} f(\sigma) \sin(n\sigma) d\sigma \), and the questions of continuity and convergence can be answered satisfactorily, the problem will be solved.

The formal determination of the coefficients \( A_n \) and \( B_n \) in \( u(\lambda, \sigma) = f(\sigma) = \sum_{0}^{\infty} (A_n \cos(n\sigma) + B_n \sin(n\sigma)) \) can be made in the following manner. Multiplication of \( f(\sigma) \), given by this equation, by \( \cos m\sigma \) and integration give

\[
\int_0^{\pi} f(\sigma) \cos(\sigma) d\sigma = \int_0^{\pi} \sum_{n=0}^{\infty} A_n \cos(n\sigma) \cos(n\sigma) d\sigma + \int_0^{\pi} \sum_{n=0}^{\infty} B_n \cos(n\sigma) \sin(n\sigma) d\sigma
= \sum_{n=0}^{\infty} \int_0^{\pi} A_n \cos(n\sigma) \cos(n\sigma) d\sigma + \sum_{n=0}^{\infty} \int_0^{\pi} B_n \cos(n\sigma) \sin(n\sigma) d\sigma \tag{18}\]


5Ibid., pp. 5-6.
Because of the orthogonality of the sine and cosine functions, each of the integrals in the right-hand member of (18) involving the product \( \cos(m\sigma)\cos(n\sigma) \) is zero for \( m \neq n \), as is each of them involving the product \( \cos(m\sigma)\sin(n\sigma) \) whether \( m \) is equal or unequal to \( n \).\(^6\) For \( m = n = 0 \), equation (18) becomes

\[
\int_{-\pi}^{\pi} f(\sigma)d\sigma = \int_{-\pi}^{\pi} A_0 d\sigma = A_0 \left[ \sigma \right]_{-\pi}^{\pi} = 2\pi A_0,
\]

from which

\[
A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\sigma)d\sigma.
\]

Hence, since \( A_0 = a_0/2 \),

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\sigma)d\sigma. \quad (19)
\]

For \( m = n \neq 0 \), equation (18) becomes

\[
\int_{-\pi}^{\pi} f(\sigma)\cos(n\sigma)d\sigma = \int_{-\pi}^{\pi} \lambda_n A_n \cos^2(n\sigma)d\sigma
\]

\[
= \lambda_n A_n \int_{-\pi}^{\pi} \cos^2(n\sigma)d\sigma
\]

\[
= \frac{\pi \lambda_n A_n}{2},
\]

from which

\[
A_n = \frac{1}{\pi \lambda_n} \int_{-\pi}^{\pi} f(\sigma)\cos(n\sigma)d\sigma. \quad (20)
\]

If \( A_n = a_n/\lambda_n \), then

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\sigma)\cos(n\sigma)d\sigma. \quad (21)
\]

Similarly, multiplication of \( f(\sigma) \) given in the last paragraph by \( \sin(m\sigma) \) and integration give the coefficients

\[
B_n = \frac{1}{\pi \lambda_n} \int_{-\pi}^{\pi} f(\sigma)\sin(n\sigma)d\sigma. \quad (22)
\]

And, if \( B_n = b_n/\lambda_n \), then the coefficients \( b_n \) are given by the

\[\footnote{Ruel V. Churchill, Fourier Series and Boundary-Value Problems (New York, 1941), pp. 53-54.}\]
equation
\[ b_n = \frac{1}{\pi} \int_0^\pi f(\sigma) \sin(n\sigma) d\sigma. \quad (23) \]

From equation (16), in view of equations (20), (21), (22), and (23),
\[ u(r,\theta) = a_0/2 + \sum_{n=1}^\infty \left( a_n \cos n\theta + b_n \sin n\theta \right), \quad (24) \]
where \( a_n = \frac{1}{\pi} \int_0^\pi f(\sigma) \cos(n\sigma) d\sigma \) and \( b_n = \frac{1}{\pi} \int_0^\pi f(\sigma) \sin(n\sigma) d\sigma \), is the solution of the problem, provided the series is suitably convergent.

If the series (24) converges uniformly with respect to \((r,\theta)\) for \(0 \leq r < \lambda\) and \(0 \leq \theta \leq 2\pi\), then its limit function \( u(r,\theta) \) is continuous.\(^7\) In order to determine the uniform convergence of series (24), consideration is given to the following inequality.

\[ |a_n \cos n\theta + b_n \sin n\theta| \leq |a_n \cos n\theta| + |b_n \sin n\theta| \]
\[ \leq |a_n| + |b_n| \]
\[ \leq 2M_3, \]
where \( M_3 = \max\{M_1, M_2\} \) and
\[ |a_n| = \left| \frac{1}{\pi} \int_0^\pi f(\sigma) \cos(n\sigma) d\sigma \right| \leq \frac{1}{\pi} \int_0^\pi |f(\sigma)| |\cos n\sigma| d\sigma \leq M_1, \]
\[ |b_n| = \left| \frac{1}{\pi} \int_0^\pi f(\sigma) \sin(n\sigma) d\sigma \right| \leq \frac{1}{\pi} \int_0^\pi |f(\sigma)| |\sin n\sigma| d\sigma \leq M_2, \]

\( M_1 > 0 \) and independent of \( n \), and \( M_2 > 0 \) and independent of \( n \), since \( f(\sigma) \) is continuous by hypothesis and, therefore, is

---

bounded. Also, $|\cos n\theta| \leq 1$, and $|\sin n\theta| \leq 1$. Hence,

$$|a_n \cos n\theta + b_n \sin n\theta| [r/\lambda]^n \leq 2M_2[r'/\lambda]^n,$$

with $0 \leq r \leq r' < \lambda$ and $0 \leq \theta \leq 2\pi$. The series of terms on the right-hand side of this last inequality converges by the ratio test; for, with $0 \leq r \leq r' < \lambda$ and $0 \leq \theta \leq 2\pi$,

$$\lim_{n\to\infty} \frac{2M_2(r'/\lambda)^{n+1}}{2M_2(r'/\lambda)^n} = r'/\lambda < 1.$$ 

Therefore, the series (24) converges uniformly with respect to $(r, \theta)$ for $0 \leq r \leq r' < \lambda$ and $0 \leq \theta \leq 2\pi$ by the Weierstrass M-Test. In a similar manner it can be shown that the assertions

$$\frac{\partial u}{\partial \theta} = \sum \frac{n(r/\lambda)^n}{1} (-a_n \sin n\theta + b_n \cos n\theta);$$

$$\frac{\partial^2 u}{\partial \theta^2} = \sum \frac{n^2(r/\lambda)^n}{1} (-a_n \cos n\theta - b_n \sin n\theta);$$

$$\frac{\partial u}{\partial r} = \sum \frac{n(r^{n-1}/\lambda^n)}{1} (a_n \cos n\theta + b_n \sin n\theta);$$

$$\frac{\partial^2 u}{\partial r^2} = \sum \frac{n(n-1)(r^{n-2}/\lambda^n)}{2} (a_n \cos n\theta + b_n \sin n\theta)$$

are true; that is, each of the series (25) is a uniformly convergent series with respect to $(r, \theta)$ for $0 \leq r \leq r' < \lambda$ and $0 \leq \theta \leq 2\pi$; and, since each of these series is a power series in $r/\lambda < 1$, it converges to the continuous derivative of $u(r, \theta)$ as indicated in equations (25). 

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9Ibid., p.565. 10Ibid., pp. 596-597. 11Ibid., p. 612.
Multiplication of equation (7) by \( r^2 \) gives
\[
 r^2 \frac{\partial^2 \psi}{\partial r^2} + r \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} = 0. \tag{26}
\]

Hence, substitution of equations (25) in (26) gives
\[
r^2 \frac{\partial^2 \psi}{\partial r^2} + r \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} = \frac{\partial}{\partial r} n(n-1)(r/\lambda)^n (a_n \cos n\theta + b_n \sin n\theta)
+ n^2 r(r/\lambda)^n (a_n \cos n\theta + b_n \sin n\theta)
+ n^2 n^2 r(r/\lambda)^n (a_n \cos n\theta + b_n \sin n\theta)
= \frac{n^2}{\lambda^2} (n^2 - n(n-1)) \left[ \frac{\partial}{\partial r} (a_n \cos n\theta + b_n \sin n\theta) \right] = 0.
\]

Thus, for \( 0 < r < \lambda \) and \( 0 < \theta < 2\pi \), \( u(r, \theta) \) defined by the series (24) is continuous and possesses continuous first and second partial derivatives in \( \mathbb{R} \) and satisfies Laplace's equation in \( \mathbb{K} \).

The conditions that \( f(\theta) \) is continuous on \( C \) and \( f(\theta) \) is equal to \( f(\theta + 2\pi) \) are not sufficient to guarantee the condition that
\[
u(\lambda, \theta) = f(\theta) = a_0/2 + \sum_{n=1}^{\infty} \left[ a_n \cos n\theta + b_n \sin n\theta \right],
\]
where \( a_n = \frac{1}{2\pi} f(\sigma) \cos(n\sigma) d\sigma \) and \( b_n = \frac{1}{2\pi} f(\sigma) \sin(n\sigma) d\sigma \). However, if the integrals for the coefficients \( a_n \) and \( b_n \) are substituted into equation (24),
\[
u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\sigma) d\sigma + \sum_{n=1}^{\infty} \left[ \frac{1}{\lambda^n} \int_{-\pi}^{\pi} f(\sigma) \cos(n(\sigma - \theta)) d\sigma \right]
= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\sigma) \left[ \frac{1}{\lambda^n} \cos(n(\sigma - \theta)) d\sigma \right]
= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\sigma) \left[ \frac{1}{\lambda^n} \cos(n(\sigma - \theta)) d\sigma \right].
\]

And, if \( \alpha = \sigma - \theta \) and \( z = (r/\lambda)(\cos \alpha + i \sin \alpha) \), then
\[
z^{\lambda} = (r/\lambda)^{\lambda} [\cos(n\alpha) + i \sin(n\alpha)].
\]
and \( (r/\lambda)^n \cos(n\alpha) \) is the real part of

\[
\frac{\infty}{\delta} z^n = \frac{1}{1 - z} = \frac{1}{1 - (r/\lambda)\cos \alpha - (r/\lambda)i \sin \alpha}
\]

\[
= \frac{\lambda}{\lambda - r \cos \alpha - ir \sin \alpha}, \quad |z| < 1.
\]

From this last equation,

\[
\frac{\infty}{\delta} z^n = \frac{\lambda}{\lambda - r \cos \alpha - ir \sin \alpha} \frac{\lambda - r \cos \alpha + ir \sin \alpha}{\lambda - r \cos \alpha - ir \sin \alpha}
\]

\[
= \frac{\lambda^2 - r \lambda \cos \alpha + ir \lambda \sin \alpha}{\lambda^2 - 2r \lambda \cos \alpha + r^2}, \quad |z| < 1.
\]

Hence,

\[
\frac{\infty}{\delta} (r/\lambda)^n \cos n\alpha = \frac{\lambda^2 - r \lambda \cos \alpha}{\lambda^2 - 2r \lambda \cos \alpha + r^2}, \quad r/\lambda < 1;
\]

and,

\[
\frac{1}{2} + \frac{\infty}{\delta} (r/\lambda)^n \cos n\alpha = \frac{\lambda^2 - r \lambda \cos \alpha}{\lambda^2 - 2r \lambda \cos \alpha + r^2} - \frac{1}{2}
\]

\[
= \frac{1}{2} \frac{\lambda^2 - r^2}{\lambda^2 - 2r \lambda \cos (\alpha - \theta) + r^2}. \quad (28)
\]

Therefore, from equations (27) and (28),

\[
u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\sigma) \frac{(\lambda^2 - r^2) d\sigma}{\lambda^2 - 2r \lambda \cos (\sigma - \theta) + r^2}, \quad (29)
\]

which, from the foregoing discussion, satisfies the conditions on \( u(r, \theta) \) for \( 0 \leq r < \lambda \) and \( 0 \leq \theta < 2\pi \). It remains to be shown that limit \( u(r, \theta) = f(\theta') \), as \( r \) approaches \( \lambda^- \) and as \( \theta \) approaches \( \theta' \), with \( u(r, \theta) \) given by equation (29). To prove this, consideration is given to the special case \( f(\sigma) = 1 \).

Then, from equation (24),

\[
\alpha_0/2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\sigma) d\sigma = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma = 1, \quad \text{for } n = 0;
\]

\[
\alpha_n = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(\sigma) \cos(n\sigma) d\sigma = \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos(n\sigma) d\sigma = 0, \quad \text{for } n = 1, 2, 3, \ldots ;
\]
and, the coefficients $b_n$ are given by

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\sigma) \sin(n\sigma) d\sigma = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(n\sigma) d\sigma = 0, \text{ for } n = 0, 1, 2, \ldots$$

Hence, $u(r, \theta) = 1 + 0 = 1$. Consequently, from equation (29),

$$1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\lambda^2 - r^2) d\sigma}{\lambda^2 - 2r\lambda \cos(\sigma - \theta) + r^2}. \tag{30}$$

If $\theta'$ is a point of continuity of $f(\sigma)$, then, from equation (30),

$$f(\theta') = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta')(\lambda^2 - r^2) d\sigma}{\lambda^2 - 2r\lambda \cos(\sigma - \theta) + r^2}. \tag{31}$$

Subtraction of equation (31) from equation (29) gives

$$u(r, \theta) - f(\theta') = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{[f(\sigma) - f(\theta')](\lambda^2 - r^2) d\sigma}{\lambda^2 - 2r\lambda \cos(\sigma - \theta) + r^2}. \tag{32}$$

Since $f(\sigma)$ is continuous for $\sigma = \theta'$, for an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(\sigma) - f(\theta')| < \varepsilon$, whenever $0 \leq |\sigma - \theta'| < \delta$. Then, from equation (32),

$$|u(r, \theta) - f(\theta')| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( f(\sigma) - f(\theta') \right) (\lambda^2 - r^2) d\sigma \right| < \varepsilon \left| \int_{-\pi}^{\pi} \frac{d\sigma}{\lambda^2 - 2r\lambda \cos(\sigma - \theta) + r^2} \right|. \tag{33}$$

Or, for $L = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda^2 - r^2}{\lambda^2 - 2r\lambda \cos(\sigma - \theta) + r^2}$,

$$|u(r, \theta) - f(\theta')| = \left| \left( f(\sigma) - f(\theta') \right) L \right| \left| \int_{-\pi}^{\pi} \frac{d\sigma}{\lambda^2 - 2r\lambda \cos(\sigma - \theta) + r^2} \right| < \varepsilon \left| \int_{-\pi}^{\pi} \frac{d\sigma}{\lambda^2 - 2r\lambda \cos(\sigma - \theta) + r^2} \right|. \tag{34}$$

For $|f(\sigma) - f(\theta')| < \varepsilon$, whenever $0 \leq |\sigma - \theta'| < \delta$, we have

$$\left| f(\sigma) - f(\theta') \right| \leq \delta \left| \int_{\theta' - \delta}^{\theta' + \delta} f(\sigma) d\sigma \right| \leq \delta \int_{\theta' - \delta}^{\theta' + \delta} |f(\sigma)| d\sigma \leq \epsilon f(\theta'). \tag{35}$$

by equation (30). Also, if $M = \max \{|f(\sigma) - f(\theta')|\}$ on $(-\pi, \pi)$

$$\left| \int_{-\pi}^{\theta' - \delta} f(\sigma) - f(\theta') \right| L d\sigma \leq M \left| \int_{-\pi}^{\theta' - \delta} d\sigma \right| L d\sigma \leq M \left| \int_{-\pi}^{\theta' - \delta} d\sigma \right|. \tag{35}$$

and

$$\left| \int_{\theta'}^{\theta' + \delta} f(\sigma) - f(\theta') \right| L d\sigma \leq M \left| \int_{\theta'}^{\theta' + \delta} d\sigma \right|. \tag{35}$$
Hence, from equations (33), (34), and (35),

\[ |u(r, \theta) - f(\theta^*)| \leq \varepsilon + M_{\theta^*}^{\theta^*+\delta} I d\sigma + M_{\theta^*}^{\theta^*+\delta} I d\sigma. \tag{36} \]

For \(-\pi < \sigma < \theta^*-\delta\) and \(\theta\) approaching \(\theta^*\) so that eventually \(|\theta - \theta^*| < \delta/2\), \(\cos(\sigma-\theta)\neq 1\). Hence, for \(-\pi < \sigma < \theta^*-\delta\) and \(\theta^*-\delta/2 < \theta < \theta^*+\delta/2\),

\[ \lim_{{r \to \lambda^-}} \int_0^{\theta^*-\delta} |f(\sigma) - f(\theta^*)| I d\sigma = \lim_{{r \to \lambda^-}} M_{\theta^*}^{\theta^*+\delta} I d\sigma = 0. \tag{37} \]

Similarly, for \(\theta^*+\delta < \sigma < \pi\) and \(\theta^*+\delta/2 < \theta < \theta^*+\delta/2\),

\[ \lim_{{r \to \lambda^-}} \int_{\theta^*+\delta}^{\pi} |f(\sigma) - f(\theta^*)| I d\sigma = \lim_{{r \to \lambda^-}} M_{\theta^*}^{\theta^*+\delta} I d\sigma = 0. \tag{38} \]

Finally, since \(\varepsilon\) approaches 0 as \(\theta\) approaches \(\theta^*\), from equations (36), (37), and (38),

\[ \lim_{{r \to \lambda^-}} |u(r, \theta) - f(\theta^*)| = 0, \]

which was to be proved. Therefore, equation (29) is the solution of the Dirichlet problem for the circle.
CHAPTER III

HARMONIC FUNCTIONS

It is the purpose of this chapter to develop some general properties of continuous solutions of Laplace's equation in two variables. These solutions constitute a class of functions known as "harmonic functions."

**Definition 20.** A function $u(x,y)$ is called "harmonic at a point $P(x,y)$" if its second partial derivatives exist, are continuous, and satisfy
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]
throughout some neighborhood of $P$.

**Definition 21.** A function $u(x,y)$ is "harmonic in a region $R$" if it is harmonic at all points of $R$.

**Definition 22.** A function $u(x,y)$ is "harmonic in a closed region $\overline{R}$" if it is continuous in $\overline{R}$ and harmonic in $R$.

Very powerful tools in the development of the properties of harmonic functions are Green's Theorem in the plane and a set of identities derived from it. It will be noted in what follows that Green's Theorem in the plane involves a line integral. For this reason some additional definitions pertaining to the boundaries of the regions involved will be needed. The first definition to follow specifies what is meant by the outer normal derivative of a function $f$ at a point $P$ on the boundary of a region, a concept which is important in many physical applications. By using this
definition, the direction of integration along the boundary of a region can be specified.

**Definition 23.** If $\bar{R} = R + B$ is a regular region and $P$ is a point belonging to a smooth portion of the boundary $B$, the unit vector $n$ perpendicular to $B$ at $P$ and consisting of only exterior points of $\bar{R}$, except $P$, is called the "outer normal" at $P$. Additionally, if $\alpha$ is the measure of the counterclockwise angle formed by the positive $x$-direction and the direction of $n$ and a function $f(x,y)$ is defined and has continuous first partial derivatives in $\bar{R}$, the "outer normal derivative of $f$ at $P$" is defined to be

$$\frac{\delta f}{\delta n} = \frac{\delta f}{\delta x} \cos \alpha + \frac{\delta f}{\delta y} \sin \alpha,$$

where $\delta f/\delta x$ and $\delta f/\delta y$ are evaluated at $P$.

**Definition 24.** If $\alpha$ is the measure of the counterclockwise angle formed by the positive $x$-direction and the direction of the outer normal at a point $P$ on a smooth portion of the boundary of a regular region, then the positive direction at $P$ on the boundary of the region is the direction which forms a counterclockwise angle having a measure of $\alpha + 90^\circ$ with the positive $x$-direction. Under this specification of direction on $B$, $B$ is said to be positively oriented, and integration along $B$ is taken in the positive direction.

As a preliminary step in the proof of Green's Theorem for the general regular region, the proof of the special case in which the region is rectangular follows.
Theorem 1. If the functions $F(x,y)$ and $G(x,y)$ are continuous and possess continuous first partial derivatives on a closed rectangular region $\bar{R} = R + \beta$, where $\beta$ is the positively oriented boundary of $\bar{R}$ and is the union of the segments $AB$, $BC$, $CD$, and $DA$, $A = (a,c)$, $B = (b,c)$, $C = (b,d)$, and $D = (a,d)$ with $a < b$ and $c < d$, then

$$\iint_{\bar{R}} \left[ \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right] dx dy = \int_\beta (F dx + G dy). \quad (1)$$

Proof. From the conclusion of the theorem,

$$\iint_{\bar{R}} \left[ \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right] dx dy = \iint_{\bar{R}} \frac{\partial G}{\partial x} dx dy - \iint_{\bar{R}} \frac{\partial F}{\partial y} dx dy.$$

Each of the double integrals in the last sentence is equal to an iterated integral.\(^1\) Hence,

$$\iint_{\bar{R}} \frac{\partial G}{\partial x} dx dy = \int_c^d \int_a^b \frac{\partial G}{\partial x} dx dy$$

$$= \int_c^d [G(b,y) - G(a,y)] dy$$

$$= \int_c^d G(b,y) dy - \int_c^d G(a,y) dy$$

$$= \int_c^d G(b,y) dy + \int_a^c G(a,y) dy$$

$$= \int_\beta G dy, \quad (2)$$

since $\int_{AB} G(x,c) dy = \int_{CD} G(x,d) dy = 0$. Similarly,

$$\iint_{\bar{R}} \frac{\partial F}{\partial y} dy dx = \int_a^b \int_c^d \frac{\partial F}{\partial y} dy dx$$

$$= \int_a^b [F(x,d) - F(x,c)] dx$$

$$= \int_a^b F(x,c) dx + \int_a^b F(x,d) dx$$

$$= -\int_a^b F(x,c) dx - \int_a^b F(x,d) dx = -\int_\beta F dx, \quad (3)$$

\(^1\)Angus E. Taylor, Advanced Calculus (Boston, 1955), pp. 529-531.
since \( \int_{BC} F(b,y) \, dx = \int_{DA} F(a,y) \, dx = 0 \). Subtraction of (3) from (2) yields

\[
\int_{P} \left[ \frac{\partial G}{\partial y} - \frac{\partial F}{\partial x} \right] \, dx \, dy = \int_{\partial P} Gdy + \int_{\partial P} Fdx
\]

\[
= \int_{\partial P} (Fdx + Gdy),
\]

which was to be proved.

Additionally, the following two definitions and Theorems 2, 3, and 4 are basic to the proof of Green's Theorem for the general regular region.

**Definition 25.** If \( S \) is a subset of a closed rectangular region \( \mathbb{R} \) in \( \mathbb{E}^2 \), for every partition \( P \) of \( \mathbb{R} \), \( J(P,S) \) is defined to be the sum of the measures of those subregions of \( P \) which contain only interior points of \( S \), and \( J(P,S) \), the sum of the measures of those subregions of \( P \) which contain points of the union of \( S \) and its boundary \( \partial S \). The numbers

\[
\underline{c}(S) = \text{l.u.b.}\{J(P,S): P \in \mathcal{P}(\mathbb{R})\}
\]

and

\[
\overline{c}(S) = \text{g.l.b.}\{J(P,S): P \in \mathcal{P}(\mathbb{R})\},
\]

where \( \mathcal{P}(\mathbb{R}) \) is the set of all possible partitions of \( \mathbb{R} \), are called, respectively, the (two-dimensional) "inner and outer Jordan content of \( S \)." If \( \underline{c}(S) = \overline{c}(S) \), the set \( S \) is said to be Jordan measurable; in this case, the common value \( c(S) \) is called the "Jordan content of \( S \)."\(^2\)

**Definition 26.** If \( c(t) = (f(t),g(t)), t \in I: [a,b] \), is a curve as defined in Definition 18 and \( \{t_0, t_1, t_2, \ldots, t_n\} \)

is a partition of $I: [a, b]$, where $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ so that $P_i = c(t_i)$, $i \in \{0, 1, 2, \ldots, n\}$, then the length $L(C)$ of the inscribed polygon $C$ determined by $P_0$, $P_1$, $P_2$, $P_3$, $P_n$ is the sum of the measures of the line segments $P_0P_1$, $P_1P_2$, $P_2P_3$, $\ldots$, $P_{n-1}P_n$: that is,

$$L(C) = \frac{n-1}{6} \sum_{i=1}^{n-1} |P_{i+1} - P_i| .$$

The length of the curve $c$, $L(c)$, is defined to be

$$L(c) = \inf \{ L(C) \}$$

for all possible partitions of $I: [a, b]$. For smooth and sectionally smooth curves it can be shown that

$$L(c) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt. \tag{3}$$

**Theorem 2.** If $F(x, y)$ is defined and continuous on a sectionally smooth, simple closed curve $B$ in $E_2$ and if $L(B)$ is the length of $B$, then

$$\left| \int_B F \, dx \right| \leq \left(\frac{1}{2}\right)(M - m)L(B),$$

where $M$ and $m$ denote, respectively, the maximum and minimum of $F$ on $B$. A similar result holds for the integral $\int_B F \, dy. \tag{4}$

**Theorem 3.** If $B$ is a sectionally smooth, simple closed curve in $E_2$ of length $L(B)$ and if, for each $h > 0$, $B(h)$ denotes the set of points in the plane at a distance not exceeding $h$ from $B$, then $B(h)$ has outer Jordan content not exceeding $\pi h^2 + 2hL(B). \tag{5}$

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4Apostol, p. 285.

Theorem 4. If (1) \( d > 0 \) is given, (2) \( S(d) \) denotes the collection of closed square regions in the \( xy \)-plane determined by the lines \( x = md \) and \( y = md \), where \( m = 0, \pm 1, \pm 2, \ldots \), and (3) \( \overline{R} = R + B \), where \( B \) is the positively oriented boundary of \( \overline{R} \), is a regular region, then \( \overline{R} \) can be expressed as the union of a finite collection of nonoverlapping regions \( \overline{R}_1, \overline{R}_2, \overline{R}_3, \ldots, \overline{R}_n \) having the following properties:

(a) Those subregions contained in the interior of \( B \) are closed square regions of the collection \( S(d) \);

(b) The remaining subregions, called "border regions," have boundaries composed of curves of \( B \) and segments of the lines \( x = md, y = md \), where \( m \) is an integer;

(c) Each border region can be enclosed in a square of edge-length \( 2d \);

(d) If \( B(\overline{R}_1), \ldots, B(\overline{R}_n) \) are the positively oriented boundaries of \( \overline{R}_1, \overline{R}_2, \ldots, \overline{R}_n \), \( B(\overline{R}) = B(\overline{R}_1) + \ldots + B(\overline{R}_n) \); and,

(e) The number of closed square regions in \( S(d) \) having points in common with \( B \) does not exceed \( 4 + 4L(B)/d \), where \( L(B) \) is the length of \( B \).

Theorem 5. Green's Theorem for a Regular Region in \( E_2 \).

If the functions \( F(x,y) \) and \( G(x,y) \) are continuous and possess

\[ \text{Apostol, pp. 287-289.} \]
continuous first partial derivatives in a regular region \( \mathbb{R} \) with a positively oriented boundary \( B \) in \( \mathbb{R}^2 \), then

\[
\int_B (F dx + G dy) = \int_{\mathbb{R}} \left( \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy. \tag{4}
\]

**Proof.** Since \( F \) and \( G \) are independent of each other, equation (4) is equivalent to two equations,

\[
\int_B \frac{\partial F}{\partial x} dx dy \quad \text{and} \quad \int_B \frac{\partial G}{\partial y} dx dy. \tag{5}
\]

The proof of (5) will be given. Equation (6) is proved similarly.

If \( d > 0 \) is given, \( S(d) \) is the collection of closed square regions in the \( xy \)-plane determined by the lines \( x = md \) and \( y = nd \), where \( m = 0, \pm 1, \pm 2, \ldots \). Then, \( \mathbb{R} \) is decomposed into a finite collection of nonoverlapping subregions as described in Theorem 4. If \( \mathbb{R}_1, \ldots, \mathbb{R}_k \) denote the square subregions which are in \( S(d) \) and are contained in \( \mathbb{R} \) and if \( \mathbb{R}_{k+1}, \mathbb{R}_{k+2}, \ldots, \mathbb{R}_n \) denote the border regions, then \( T_d \) will denote the union of the border regions.

Since no point of a border region is more than a distance \( d/2 \) from \( B \), \( T_d \) is a subset of \( B(d/2) \), in the notation of Theorem 3. Hence, if \( c(T_d) \) denotes the Jordan content of \( T_d \) and \( L(B) \) denotes the length of \( B \),

\[
c(T_d) \leq c[B(d/2)] \leq \pi (d/2)^2 + (2d/2) L(B).
\]

\(|\delta F/\delta y| \leq N \), a constant, for all \((x,y) \) in \( \mathbb{R} \). For an arbitrary \( \varepsilon > 0 \), \( d > 0 \), depending on \( \varepsilon \), can be chosen so that \( d < 1 \) and

\[\text{Apostol, pp. 289-292.}\]
so that
\[ n(d/2)^2 + (2d/2)L(B) < \varepsilon/2N. \]

If \( M_p \) and \( m_p \) denote the maximum and minimum of \( F \) on the sub-region \( \overline{B_p} \), then, by continuity, it can be assumed that \( d \) is chosen so that
\[ M_p - m_p < \frac{\varepsilon}{2[16 + 17L(B)]} \text{ for } p = 1, 2, \ldots, n. \]

For this \( d \),
\[ \left| \sum_{p=1}^{k+1} \int_{B_p} AF \, dxdy - \frac{k}{L(B)} \int_{\overline{B}} AF \, dxdy \right| = \left| \int_{T_d} AF \, dxdy \right| \leq N\sigma(T_d) < \varepsilon/2. \quad (7) \]

Also, by Theorem 2,
\[ \left| \int_{k+1}^{n} \int_{B_p} F \, dF \right| \leq \frac{1}{k+1} \sum_{p=1}^{k+1} (M_p - m_p)L(B_p) < \frac{\varepsilon}{2[16 + 17L(B)]} \sum_{p=1}^{n} L(B_p), \]

where \( B_p \) is the boundary of \( \overline{B_p} \). Each curve \( B_p \) is composed of portions of \( B \) and portions of the edges of squares of the system \( S(d) \). By Theorem 4, at most \( 4 + 4L(B)/d \) of these squares have points in common with \( B \); and, hence,
\[ \sum_{p=1}^{n} L(B_p) \leq L(B) + 4d(4 + \frac{4L(B)}{d}) = 16d + 17L(B) < 16 + 17L(B). \quad (8) \]

Therefore, (8) implies
\[ \left| \int_{k+1}^{n} \int_{B_p} F \, dF \right| < \varepsilon/2. \]

Since
\[ \int_{B} F \, dF = \int_{1}^{k} \int_{B_p} F \, dF + \int_{k+1}^{n} \int_{B_p} F \, dF, \]
the last inequality implies
\[ \left| \int_{B} F \, dF - \int_{1}^{k} \int_{B_p} F \, dF \right| < \varepsilon/2. \quad (9) \]

If Green's theorem is applied to each square region \( \overline{B_p} \), where \( p = 1, 2, \ldots, k \),
\[ \int_{1}^{k} \int_{B_p} F \, dF = -\int_{1}^{k} \int_{B_p} \frac{\partial F}{\partial y} \, dxdy. \]
Combining this last equation with (7) and (9) gives

\[ \left| \int f \frac{\partial F}{\partial y} \, dx \, dy + \int_B F \, dx \right| < \varepsilon. \]

Since \( \varepsilon \) is arbitrary, this proves that

\[ \int_B F \, dx = -\int f \frac{\partial F}{\partial y} \, dx \, dy. \]

**Theorem 6.** If, in addition to the hypothesis of Theorem 5, the function \( u(x,y) \) and its first and second partial derivatives are continuous in \( \Omega \), then

\[ \int f \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \, dx \, dy = \int_B \frac{\partial u}{\partial n} \, ds. \]  

where \( n \) and \( s \) refer to the outer normal on \( B \) and arc length along \( B \), respectively.

**Proof.** If \( G = \partial u/\partial x \) and \( F = -\partial u/\partial y \) in equation (4), then

\[ \int_B [-\partial u/\partial y + \partial u/\partial x] \, dx \, dy \]

It must now be shown that on \( B \),

\[ \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y} \frac{dx}{ds} + \frac{\partial u}{\partial x} \frac{dy}{ds}. \]  

**Figure 2.** Diagram for the Green's identity of Theorem 6

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In Figure 2,
\[ \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \sin \alpha \quad (13) \]
\[ \frac{dx}{ds} = \cos \beta, \quad \frac{dy}{ds} = \sin \beta, \]
where \( \beta \) is the measure of the angle made by the positive tangent to \( B \) with the positive \( x \)-axis. Since \( \alpha = \beta - \pi/2 \),
\[ \cos \alpha = \frac{dy}{ds} \text{ and } \sin \alpha = -\frac{dx}{ds}. \quad (14) \]
Substitution of (14) into (13) gives (12). Since (12) is valid, its substitution into (11) gives (10); and, the proof is complete.

**Theorem 7.** If, in addition to the hypothesis of Theorem 5, the function \( u(x,y) \) and its first partial derivatives and the function \( v(x,y) \) and its first and second partial derivatives are continuous in \( \mathcal{R} \), then
\[ \int_{\mathcal{R}} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] dx \, dy + \int_{\mathcal{R}} \left[ \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right] dx \, dy = \int_B u \frac{\partial v}{\partial n} \, ds. \quad (15) \]

**Proof.** If, in equation (4),
\[ F = -u \frac{\partial v}{\partial y} \text{ and } \]
\[ G = u \frac{\partial v}{\partial x}, \quad (16) \]
then
\[ \frac{\partial F}{\partial y} = -\frac{\partial u}{\partial y} \frac{\partial v}{\partial y} - u \frac{\partial^2 v}{\partial y^2} \text{ and } \]
\[ \frac{\partial G}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x^2}. \quad (17) \]
Substitution of (16), (17), (18), and (19) into (4) gives
\[ \int_B (-u \frac{\partial v}{\partial y} dx + u \frac{\partial^2 v}{\partial y^2} dy) = \int_{\mathcal{R}} \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx \, dy. \quad (20) \]
By equation (12), with \( v \) replacing \( u \), the outer normal
derivative of \(v\) is given by

\[
\frac{\delta v}{\delta n} = - \frac{\delta v}{\delta y} \frac{\delta s}{\delta s} + \frac{\delta v}{\delta x} \frac{\delta s}{\delta s}.
\] (21)

Then, multiplying (21) by \(u\),

\[
u \frac{\delta v}{\delta n} = -u \frac{\delta v}{\delta y} \frac{\delta s}{\delta s} + u \frac{\delta v}{\delta x} \frac{\delta s}{\delta s}.
\] (22)

Substitution of (22) into equation (20) yields

\[
\int_B u \frac{\delta v}{\delta n} ds = \int_R u \left( \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} \right) dx dy + \int_R \left( \frac{\delta u}{\delta x} \frac{\delta v}{\delta y} + \frac{\delta u}{\delta y} \frac{\delta v}{\delta y} \right) dx dy,
\]

by repeated application of the associative and commutative properties of addition, distributive property of multiplication with respect to addition, and the additive property of the double integral. This last equation becomes equation (15) by the application of the symmetric property of equality.

This completes the proof of Theorem 7.

**Theorem 8.** If, in addition to the hypothesis of Theorem 5, the functions \(u(x,y)\) and \(v(x,y)\) and their first and second partial derivatives are continuous in \(\mathbb{R}\), then

\[
\int_R u [\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}] dx dy = \int_R v [\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}] dx dy = \int_B u \frac{\delta v}{\delta n} ds.
\] (23)

**Proof.** Since, by hypothesis, both \(u\) and \(v\) and their first and second partial derivatives are continuous in \(\mathbb{R}\),

\[
\int_R v [\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}] dx dy + \int_R \left[ \frac{\delta u}{\delta x} \frac{\delta v}{\delta y} + \frac{\delta u}{\delta y} \frac{\delta v}{\delta y} \right] dx dy = \int_B v \frac{\delta u}{\delta n} ds,
\] (24)

by Theorem 7. Subtraction of (24) from (15) yields

\[
\int_R u \left( \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} \right) dx dy + \int_R \left( \frac{\delta u}{\delta x} \frac{\delta v}{\delta y} + \frac{\delta u}{\delta y} \frac{\delta v}{\delta y} \right) dx dy - \int_R v \left( \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} \right) dx dy
\]

\[
- \int_R \left( \frac{\delta u}{\delta x} \frac{\delta v}{\delta y} + \frac{\delta u}{\delta y} \frac{\delta v}{\delta y} \right) dx dy = \int_B u \frac{\delta v}{\delta n} ds - \int_B v \frac{\delta u}{\delta n} ds.
\]
which becomes by repeated application of the commutative and associative properties of addition and the additive property of zero
\[
\iint_{R} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx\,dy = \iint_{R} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) dx\,dy = \int_{B} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds,
\]
which is equation (23); and, the proof of Theorem 8 is complete.

**Theorem 9.** The integral of the normal derivative of a function vanishes when extended over the positively oriented boundary of any regular region in which the function is harmonic and its first and second partial derivatives exist and are continuous.

**Proof.** Since, by hypothesis, a function \( u(x,y) \) is harmonic in a regular region \( \overline{R} \) with a positively oriented boundary \( B \), then, by Definitions 19, 20, 21, and 22,
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{25}
\]
Since the hypothesis of Theorem 6 is satisfied, equation (10) becomes, by the symmetric property of equality and equation (25),
\[
\int_{B} \frac{\partial u}{\partial n} ds = \iint_{\overline{R}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx\,dy = 0,
\]
where \( B \) is the boundary of \( \overline{R} \); and, the proof of Theorem 9 is complete.

**Theorem 10.** If \( u \) is a harmonic function and possesses continuous first and second partial derivatives in a regular region \( \overline{R} \) and vanishes at all points of the positively oriented
boundary \( B \) of \( \bar{R} \), it vanishes at all points of \( R \).

**Proof.** By hypothesis \( u \) is harmonic in \( \bar{R} \), and \( u = 0 \) on \( B \). Then, by Definitions 19, 20, 21, and 22, \( \delta^2u/\delta x^2 + \delta^2u/\delta y^2 = 0 \), and the hypothesis of Theorem 7 is satisfied with \( u = v \). Hence, since \( \delta^2v/\delta x^2 + \delta^2v/\delta y^2 = 0 \) in \( R \), and \( v = 0 \) on \( B \), the first double integral and the line integral in equation (15) vanish. This latter fact yields, from equation (15),

\[
\iint_{\bar{R}} \left[ \left( \frac{\delta u}{\delta x} \right)^2 + \left( \frac{\delta u}{\delta y} \right)^2 \right] \, dx \, dy = 0.
\]

Because \( (\delta u/\delta x)^2 + (\delta u/\delta y)^2 \) is continuous and non-negative in \( \bar{R} \), \( (\delta u/\delta x)^2 + (\delta u/\delta y)^2 = 0 \) in \( \bar{R} \). Hence, \( \delta u/\delta x = \delta u/\delta y = 0 \) in \( \bar{R} \). Therefore, \( u \) is constant in \( \bar{R} \). But, \( u = 0 \) on \( B \), and since \( u \) is continuous in \( \bar{R} \), \( u = 0 \) in \( R \), which was to be proved.

**Theorem 11.** A function \( u \) which is harmonic and possesses continuous first and second partial derivatives in a regular region \( \bar{R} \) is uniquely determined by its values on the boundary \( B \) of \( \bar{R} \).

**Proof.** If both of the functions \( u \) and \( v \) satisfy the hypothesis of the theorem and have the same values on \( B \), then \( u - v = 0 \) on \( B \) and is harmonic in \( \bar{R} \), since Laplace's equation is a linear and homogeneous equation. By Theorem 10, \( u - v = 0 \) in \( R \). Hence, \( u = v \) in \( \bar{R} \); and, the proof is complete.

**Theorem 12.** If \( u \) is a single-valued function, possesses continuous first and second partial derivatives, and is harmonic in a regular region \( \bar{R} \), and if its normal derivative
vanishes at all points of the positively oriented boundary \( B \) of \( \overline{\Omega} \), then \( u \) is constant in \( \overline{\Omega} \).

Proof. Since, by hypothesis, \( \partial u / \partial n = 0 \) on \( B \) and \( u \) is harmonic in \( \overline{\Omega} \), \( \partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0 \) in \( \Omega \); and,

\[
\iint_{\overline{\Omega}} [(\partial u / \partial x)^2 + (\partial u / \partial y)^2] \, dx \, dy = 0,
\]

with \( u = v \) in equation (15). Because \( (\partial u / \partial x)^2 + (\partial u / \partial y)^2 \) is continuous and non-negative in \( \overline{\Omega} \), \( (\partial u / \partial x)^2 + (\partial u / \partial y)^2 = 0 \) in \( \overline{\Omega} \). Hence, \( \partial u / \partial x = \partial u / \partial y = 0 \) in \( \overline{\Omega} \). Therefore, \( u \) is constant in \( \overline{\Omega} \), which was to be proved.

Corollary. A function \( u \) which is single-valued and harmonic in a regular region \( \overline{\Omega} \) with a positively oriented boundary \( B \) is determined by the values of its normal derivative on \( B \), except for an additive constant.

Proof. If both of the functions \( u \) and \( v \) satisfy the hypothesis and \( \partial u / \partial n = \partial v / \partial n \) on \( B \), then \( \partial (u - v) / \partial n = 0 \) on \( B \). Hence, by Theorem 12, the function \( (u - v) \) is constant in \( \overline{\Omega} \). Therefore, \( u = v + c \), where \( c \) is a constant, in \( \overline{\Omega} \), which was to be proved.

Theorem 13. If \( u \) is harmonic, possesses continuous first and second partial derivatives in a regular region \( \overline{\Omega} \), and satisfies the condition that \( \partial u / \partial n + hu = g \) on the positively oriented boundary \( B \), where \( h \) and \( g \) are continuous functions of position on \( B \) and \( h \) is never negative, then there exists no different function satisfying the same conditions.

Proof. If \( u \) and \( v \) are two functions which satisfy the hypothesis, then \( \partial (u - v) / \partial n + h(u - v) = 0 \); that is,
\[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \text{ in } \Omega, \]
and with \( u = v \), this last equation becomes
\[ \int_{\Omega} \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \, dx dy = \int_{\partial \Omega} \frac{\partial u}{\partial n} \, ds. \]

Since the function \((u - v)\) is harmonic in \( \Omega \), applying this last equation to this function results in
\[ \int_{\Omega} \left[ \left( \frac{\partial(u - v)}{\partial x} \right)^2 + \left( \frac{\partial(u - v)}{\partial y} \right)^2 \right] \, dx dy = \int_{\partial \Omega} (u - v) \, ds, \]
and
\[ \int_{\Omega} \left[ \left( \frac{\partial(u - v)}{\partial x} \right)^2 + \left( \frac{\partial(u - v)}{\partial y} \right)^2 \right] \, dx dy + \int_{\partial \Omega} h(u - v)^2 \, ds = 0. \]

Since \([\frac{\partial(u - v)}{\partial x}]^2 + [\frac{\partial(u - v)}{\partial y}]^2 \geq 0\) and \(h(u - v)^2 \geq 0\) and the integrands are continuous in \( \Omega \), the integrals can vanish only if the integrands vanish. Hence, \( u - v = 0 \) in \( \Omega \), that is, \( u = v \) in \( \Omega \), which was to be proved.

**Theorem 14.** If the functions \( u \) and \( v \) are harmonic and possess continuous first and second partial derivatives in a regular region \( \Omega \) with a positively oriented boundary \( \partial \Omega \), then
\[ \int_{\partial \Omega} \left( \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} \right) ds = 0. \]

**Proof.** Since the hypothesis of Theorem 8 is satisfied and \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \) in \( \Omega \), the first two integrals in equation (23) are each zero, and (23) becomes
\[ 0 = \int_{\partial \Omega} (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}) ds, \]
which is equivalent to the conclusion to be proved.
Theorem 15. If the function $u$ is harmonic and has continuous first and second partial derivatives in a circular region $K$, where $K = K + C = \{(x, y): x^2 + y^2 \leq r^2, 0 \leq r \leq a\}$, the value of $u$ at the center $0$ of the positively oriented circle $C$ is the arithmetic mean of its values at the points $P(a, b)$ on $C$.

Proof. By Theorem 9, $\int_C \frac{\partial u}{\partial n} ds = 0$. Since $ds = r d\theta$ and $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}$, this integral becomes

$$\int_0^{2\pi} \frac{\partial u}{\partial r} (rd\theta) = 0, \quad \text{or} \quad \int_0^{2\pi} \frac{\partial u}{\partial r} d\theta = 0,$$

since $r$ is constant with respect to the integration. Then, multiplication of this last equation by $dr$ and integration from $r = 0$ to $r = a$ yields

$$\int_0^{2\pi} d\theta \int_0^a \frac{\partial u}{\partial r} dr = 0, \quad \text{or} \quad \int_0^{2\pi} [u(a, b) - u(0, 0)] d\theta = 0.$$

But, since $u(a, b)$ depends, in general, upon $\theta$ and $u(0, 0)$ is independent of $\theta$, this last equation is equivalent to

$$\int_0^{2\pi} u(a, b) d\theta - u(0, 0) \int_0^{2\pi} d\theta = 0 \quad \text{or} \quad \int_0^{2\pi} u(a, b) d\theta - 2\pi u(0, 0) = 0,$$

from which

$$u(0, 0) = (1/2\pi) \int_0^{2\pi} u(a, b) d\theta,$$

the conclusion to be proved.

The final theorem deals with the very important topic of the maxima and minima of a non-constant harmonic function in a regular region of a plane.

Theorem 16. If $\bar{R}$ is a regular region of a plane and $u$ is harmonic, but not constant, in $\bar{R}$, then $u$ attains its
maximum and minimum values only on the boundary $B$ of $\overline{R}$.

**Proof.** Let $m = \max\{u\}$ on $B$. Suppose that the point $P$ belongs to $R$ such that $u(P) = M > m$, and move the origin to $P$; $u$ remains harmonic under this transformation.

Let

$$v(x,y) = u(x,y) + \frac{M - m}{2d^2}(x^2 + y^2),$$

where $d = \text{l.u.b. of distances between pairs of points in } R$.

Then, $x^2 + y^2 < d^2$ for $(x,y)$ in $R$.

$$v(0,0) = u(0,0) = M.$$

On the other hand, if $(x,y)$ belongs to $B$, then

$$v(x,y) \leq m + \frac{1}{2}(M - m) = \frac{1}{2}(M + m) < M.$$

Consequently, $v(x,y)$ must attain its maximum at a point in $R$.

This, however, contradicts the fact that at all points of $R$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{M - m}{d^2} = 2\frac{M - m}{d^2} > 0,$$

for at a maximum none of the second derivatives of a function can be positive. Therefore, $u$ attains its maximum only on the boundary $B$ of $\overline{R}$. To prove the minimum part of the theorem, this result is applied to the function $-u$.

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Books


