A STUDY OF FUNCTIONS ON METRIC SPACES

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A STUDY OF FUNCTIONS ON METRIC SPACES

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TABLE OF CONTENTS

CHAPTER I

SOME FUNDAMENTAL PROPERTIES OF METRIC SPACES

A metric space is a pair of two things: A set X, whose elements are called points, and a distance, i.e., a single-valued, nonnegative real function $d(x,y)$, defined for arbitrary x and y in X and satisfying the following conditions;

(a) $d(x,y) \geq 0$

(b) $d(x,y)=0$ iff $x=y$

(c) $d(x,y)=d(y,x)$

(d) $d(x,y)+d(y,z) \geq d(x,z)$.

The following are some examples of metric spaces.

Example (1). Let $d(x,y)=\begin{Bmatrix} 0 \text{ if } x=y \\ 1 \text{ if } x \neq y \end{Bmatrix}$ for any arbitrary set.

Example (2). Consider the set D^n of ordered n-tuples of real numbers, $x=(x_1,x_2, \cdots ,x_n)$, with distance function, $d(x,y)=\left\{ \sum_{k=1}^{n} (y_k - x_k)^2 \right\}^{\frac{1}{2}}$.

Example (3). Let $\mathcal{C}$ be the set of all continuous real functions on the interval $[a,b]$, and define a distance function $d(f,g)=\sup \left| f(x)-g(x) \right|$ where $f, g \in \mathcal{C}$ and $a \leq x \leq b$.

If n is chosen from the set $\{1,2,3\}$, Example (2) becomes one of the more familiar forms of Euclidean space where the notion of distance is the usual one, and the four defining properties of the distance function are readily verified in these cases. However, if n is restricted to being a positive integer, the concept of distance becomes less intuitive.

Example (1) illustrates that a metric does not have to conform to the ordinary concept of distances. However, if the arbitrary set is chosen to be two adjacent integers, or the three vertex points of an equilateral triangle, or the four vertices of an equilateral pyramid, the distance function does seem more familiar.

Each of these examples can readily be shown to satisfy the four metric properties. However, it is worthwhile to include a proof of the third property for Example (2). This property is sometimes called the triangle inequality because of its similarity to the familiar geometric proposition which states that the sum of the lengths of two sides of a given triangle is greater than or equal to the length of the third side.

Proof. The validity of the Schwarz inequality will be assumed for this proof.

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$$

Choose three elements $x=(x_1,x_2,x_3,\cdots,x_n)$, $y=(y_1,y_2,y_3,\cdots,y_n)$, $z=(z_1,z_2,z_3,\cdots,z_n)$ from the space D^n. It will be convenient to make the following substitutions. Let $a_k=y_k-x_k$, $b_k=z_k-y_k$ and by addition we obtain $z_k-x_k=a_k+b_k$. In the equation

$$\sum_{k=1}^{n} (a_k+b_k)^2=\sum_{k=1}^{n} a_k^2+2\sum_{k=1}^{n} a_k b_k+\sum_{k=1}^{n} b_k^2$$

we can substitute

$$2\sum_{k=1}^{n} a_k b_k = 2\left\{\left(\sum_{k=1}^{n} a_k b_k\right)^{\frac{1}{2}}\right\}^2$$

and by the Schwarz inequality

$$2\left\{\left(\sum_{k=1}^{n} a_k b_k\right)^{\frac{1}{2}}\right\}^2 \leq 2\left(\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}}$$

Thus

$$\sum_{k=1}^{n}(a_k+b_k)^2 \leq \sum_{k=1}^{n} a_k^2 + 2\left(\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} + \sum_{k=1}^{n} b_k^2$$

$$= \left[\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} (b_k^2)\right)^{\frac{1}{2}}\right]^2$$

<u>i</u>. <u>e</u>.,

$$d^2(x,z) \leq [d(x,y)+d(y,z)]^2$$

or

$$d(x,z) \leq d(x,y)+d(y,z).$$

<u>Problem 1.1</u>. Let X be a metric space with metric d. Show that d_1 defined by $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is also a metric on X.

<u>Proof</u>.

(a) Let x,y be two points in the space X. Then $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{0}{1+0} = 0$ iff x=y.

(b) $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0$.

(c) $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d_1(y,x)$.

(d) Let d(x,y)=a, d(y,z)=b, and d(x,z)=c. Obviously $a+b \geq c$. Also $2ab+ac+bc+2abc \geq ac+bc+abc$. Addition of these two inequalities gives $a+b+2ab+ac+bc+2abc \geq c+ac+bc+abc$, or $(a+b+2ab)(1+c) \geq c(1+a+b+ab)$. This inequality may be written $\frac{a+b+2ab}{(1+a)(1+b)} \geq \frac{c}{1+c}$ or $\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{c}{1+c}$. Thus $\frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \geq \frac{d(x,z)}{1+d(x,z)}$ and hence $d_1(x,y)+d_1(y,z) \geq d_1(x,z)$. Since d_1 satisfies all four metric space properties and since it is obviously a non-negative function, d_1 is a metric on X.

Remark: Any arbitrary set X with this metric d_1 forms a bounded metric space.

Problem 1.2. Let X be a non-empty set and let d be a real function of ordered pairs of elements of X which satisfies the following three conditions: $d(x,y) \geq 0$, and $x=y$ implies $d(x,y)=0$; $d(x,y)=d(y,x)$; and $d(x,y)+d(y,z) \geq d(x,z)$. A function d with these properties is called a pseudo-metric. The following is an example of a pseudo-metric. Let X be the set of real numbers and let $d = \begin{cases} |x-y| \text{ for either of } x,y \text{ non-integer} \\ 0 \text{ for both } x,y \text{ integers} \end{cases}$.

This is not a metric since $d(x,y)=0$ does not imply $x=y$.

Let d be a pseudo-metric on X. Define $\sim$ on X by $x \sim y \Leftrightarrow d(x,y)=0$. It will now be shown that this is an equivalence relation whose corresponding class of equivalence sets can be made into a metric space in a natural way.

(a) reflexive property; $x \sim x \Leftrightarrow d(x,x)=0$

(b) symmetric property; if $x \sim y \Leftrightarrow d(x,y)=0$, then $y \sim x \Leftrightarrow d(y,x)=0$ since $d(y,x)=d(x,y)$

(c) transitive property; suppose $y \sim x$ and $x \sim z$.

Then $d(y,x)=0$ and $d(x,z)=0$, and $d(y,z) \leq d(x,y)+d(x,z)=0$. Also $d(y,z) \geq 0$. Thus $d(y,z)=0$ and $y \sim z$.

For the second part of the problem it must be shown that a metric can be properly defined on the following class of equivalence sets.

Let $\{x\}$ be the set of all elements of X which are equivalent to x, $\{y\}$ be the set of all elements of X equivalent to y, etc. Let $T=\{\{x\} \mid x \in X\}$.

(a) Choose two elements $\{x\}$ and $\{y\}$ from T and define d' to be as follows; $d'(\{x\},\{y\})=d(x,y)$ for $x \in \{x\}$ and $y \in \{y\}$. Obviously this is a single valued non-negative function for if $x \in \{x\}$ and $y_1, y_2 \in \{y\}$, then $d(x,y_1)+d(y_1,y_2) \geq d(x,y_2)$ or $d(x,y_1) \geq d(x,y_2)$. Similarly $d(x,y_2)+d(y_1,y_2) \geq d(x,y_1)$ or $d(x,y_2) \geq d(x,y_1)$; thus $d(x,y_1)=d(x,y_2) \geq 0$.

Suppose $d'(\{x\},\{y\})=0$. Then $d(x,y)=0$ and $x \sim y$. Thus $\{x\}=\{y\}$. Now suppose $\{x\}=\{y\}$. Then $x \sim y$ and $d(x,y)=0$; thus $d'(\{x\},\{y\})=0$.

(b) $d'(\{x\},\{y\})=d(x,y)=d(y,x)=d'(\{y\},\{x\})$.

(c) $d'(\{x\},\{y\})+d'(\{y\},\{z\})=d(x,y)+d(y,z) \geq d(x,z)$ $=d'(\{x\},\{z\})$.

Thus d' is a metric on T.

Throughout this paper, the symbol X will refer to an arbitrary metric space and d will be a metric defined on X. The symbol R will refer to the metric space composed of the set of real numbers with the usual metric, i.e., $d(x,y)=|x-y|$.

Before beginning a discussion of the basic properties of metric spaces it is necessary to define the following fundamental concepts.

Definition 1.1. A closed sphere denoted by $S[x_0,r]$ is the set of all points $x \in X$ which satisfy the condition $d(x_0,x) \leq r$.

Definition 1.2. An open sphere denoted by $S(x_0,r)$ is the set of all points $x \in X$ which satisfy the condition $d(x_0,x) < r$.

In each case x_0 will be called the center and r the radius of the sphere.

Some examples of open (closed) spheres are

Example (1). In the first example on page one, any open sphere $S(x_0,r)$ will consist of either the single point x_0 when $r \leq 1$ or the entire space when $r > 1$.

Example (2). Consider the second example on page one and let n=3. In this case, the closed sphere $S[x_0,r]$ fits the usual notion of a sphere and a point y will belong to $S[x_0,r]$ if it is contained within or lies on the surface of the geometric sphere with center x_0 and radius r.

It will sometimes be convenient to refer to an ϵ-neighborhood of a point x_0 which will mean an open sphere with center x_0 and radius $\epsilon > 0$.

Definition 1.3. A contact point of a set M in X will mean any point x such that every ϵ-neighborhood of x contains at least one point of M. The set of all contact points of

M will be called the closure of M and will be denoted by $[M]$.

Theorem 1.1. If M and M_1 are sets such that $M_1 \subseteq M$, then $[M_1] \subseteq [M]$.

Proof. Suppose $x \in [M_1]$. Then each open sphere $S(x,r)$ contains some point $y \in M_1$. But $M_1 \subseteq M$ thus $y \in M$. Therefore x is also a contact point of M, and hence $x \in [M]$. Thus $[M_1] \subseteq [M]$.

Theorem 1.2. The closure of the closure of a set $M \subset X$ is equal to the closure of M, i.e., $[[M]] = [M]$.

Proof. Let $x \in [[M]]$. Then an arbitrary sphere $S(x,r)$ contains a point $x_1 \in [M]$. Let $r_1 = r - d(x,x_1)$, and consider the sphere $S(x_1,r_1)$. Obviously $S(x_1,r_1) \subseteq S(x,r)$ since if $y \in S(x_1,r_1)$ we know by the triangle inequality that $d(x,y) \leq d(x,y_1) + d(x_1,y) < d(x,x_1) + r_1 = r - r_1 + r_1 = r$. Since $x_1 \in [M]$ it follows that the sphere $S(x_1,r_1)$ contains some point $x_2 \in M$. Thus an arbitrary sphere $S(x,r)$ about x contains a point of M and thus $x \in [M]$.

Now suppose $x \in [M]$. Obviously $x \in [[M]]$ since any sphere $S(x,r)$ about x contains x itself. Thus x is a contact point of $[M]$ and $x \in [[M]]$. Therefore $[[M]] = [M]$.

Theorem 1.3. The closure of a sum is equal to the sum of the closures. $[M_1 \cup M_2] = [M_1] \cup [M_2]$.

Proof. Suppose $x \in [M_1 \cup M_2]$. Then either $x \in (M_1 \cup M_2)$ or an arbitrary sphere $S(x,r)$ about x contains some point $y \in (M_1 \cup M_2)$. In the former case x belongs to at least one of M_1 and M_2 and hence to at least one of $[M_1]$ and $[M_2]$. Thus $x \in [M_1] \bigcup [M_2]$. In the latter case each sphere $S(x,r)$ contains some point $y \in (M_1 \bigcup M_2)$. If there is a sphere $S(x,r')$ such that $S(x,r') \bigcap M_2 = \emptyset$ then each sphere $S(x,r) \bigcap M_1 \neq \emptyset$; and hence $x \in [M_1]$ and thus belongs to $[M_1] \bigcup [M_2]$.

Now suppose $x \in [M_1] \bigcup [M_2]$. We can assume without loss of generality that $x \in [M_1]$. Thus each sphere $S(x,r)$ is such that $S(x,r) \bigcap M_1 \neq \emptyset$. It follows then that $S(x,r) \bigcap (M_1 \bigcup M_2) \neq \emptyset$. Thus $x \in [M_1 \bigcup M_2]$. Therefore $[M_1 \bigcup M_2] = [M_1] \bigcup [M_2]$.

It is necessary to define the concepts of limit point and isolated point before proceeding to the next theorem.

Definition 1.4. The point x is called a limit point of a set M if an arbitrary ϵ-neighborhood of x contains an infinite set of points of M.

Obviously the first example on page one will have no limit points, while in the third example, each point would be a limit point.

Definition 1.5. A point x belonging to a set $M \subset X$ is said to be an isolated point of this set if x has a neighborhood $S(x,r)$ which does not contain any points of M different from x. For example, in the set of positive integers with the usual metric, each element is an isolated point.

Theorem 1.4. Every contact point of the set $M \subset X$ is either a limit point or an isolated point of the set M.

Proof. Suppose x is a contact point of M and suppose there is some neighborhood $S(x,r)$ which contains a finite set of points of M different from x. Let $Y=\{y_1,y_2, \cdots, y_n\}$ be the points of M such that $y_i \in S(x,r)$ and $y_i \neq x$. Consider the set $T=\{r_i \mid r_i=d(x,y_i)\}$. Since this is a finite set of positive real numbers, there exists a positive real number k such that $0<k<r_i$. Thus the sphere $S(x,k)$ is such that $S(x,k)\cap Y=\emptyset$. But $S(x,k)\cap M\neq\emptyset$ by the definition of a contact point. Thus $x \in M$ and x is isolated.

If there exists no neighborhood $S(x,r)$ such that $S(x,r)\cap M$ is finite, then by definition x is a limit point of M. Therefore x is either an isolated point of M or a limit point of M.

The concept of a sequence is a very useful tool in this study of metric spaces. Of special importance are the notions of convergent sequence, subsequence, Cauchy sequence, and limit of a sequence which will be defined in the following manner.

Definition 1.6. A mapping of the set N of positive integers onto a set B will be called a sequence if for each $n \in N$ there is an image $b \in B$. The image of n will be called b_n.

Definition 1.7. Let $x_1,x_2,x_3, \cdots$ be a sequence of points in the metric space X. We say that this sequence converges to the point x if every neighborhood $S(x,r)$ contains all points x_n starting with some one of them, i.e., if for every $\epsilon>0$ we can find a positive integer N such that $S(x,\epsilon)$ contains all points x_n with $n>N$.

Definition 1.8. The statement that $\{b_n\}$ is a subsequence of a sequence $\{a_m\}$ means that if b_i and $b_j \in \{b_n\}$, then there exist a_k, $a_t \in \{a_m\}$, where $a_k = b_i$, $a_t = b_j$ and if $t > k$ then $j > i$.

Definition 1.9. The statement that $\{a_n\}$ is a Cauchy sequence means that for each $\epsilon > 0$ there is a positive integer M such that $d(a_i, a_j) < \epsilon$ if $i > M$ and $j > M$.

Theorem 1.5a. A necessary and sufficient condition that the point x be a contact point of the set M is that there exist a sequence of points $\{a_n\}$ of the set M which converges to x.

Proof. Necessity.

If x is a contact point of M then every open sphere $S(x, 1/n)$ where $n \in \{1, 2, 3, \cdots\}$ must contain some point x_n of M. These points obviously form a sequence which converges to x. In the case of an isolated point, the sequence would clearly be composed of the single point x itself.

Sufficiency.

If the sequence of points $\{a_n\}$ of M converges to x, then every neighborhood $S(x, r)$ of x must contain a point y of the sequence. Thus since $\{a_n\} \subseteq M$ it follows that x is a contact point of M.

Theorem 1.5b. A necessary and sufficient condition that the point x be a limit point of M is that there exist a sequence $\{a_n\}$ of distinct points of M which converges to x.

Proof. Necessity.

If x is a limit point of M then every sphere $S(x,r)$ about x contains an infinite set of points of M. Choose $k_1=1$ and from the sphere $S(x,k_1/1)$ we can choose some y_1 such that $y_1 \in M$, $y_1 \neq x$. Let $k_2=d(x,y_1)$ and from the sphere $S(x,k_2/2)$ we can choose a y_2 such that $y_2 \in M$ and $y_2 \neq x$. Let $k_3=d(x,y_2)$. Clearly in this manner we can form a sequence of spheres $S_n(x,k_n/n)$ and a sequence $\{y_n\}$ of distinct elements of M where $k_n=d(x,y_{(n-1)})$, such that $\{k_n/n\}$ will converge to 0 and thus $\{y_n\}$ will converge to x.

Sufficiency.

By the definition of convergence of a sequence of distinct points $\{a_n\}$ to a point x, it is clear that every ϵ-neighborhood of x will contain an infinite set of points of the sequence $\{a_n\}$. Since $\{a_n\} \subset M$, it follows that x is a limit point of M.

Theorem 1.6. The limit of a sequence, if it exists, is unique.

Proof. Suppose the sequence $\{a_n\}$ converges to a, and $\{a_n\}$ converges to b. If $a \neq b$ then $d(a,b)=\epsilon > 0$. Now let $\epsilon' = \epsilon/2$. The sphere $S(a,\epsilon')$ contains all but a finite set of elements of $\{a_n\}$. Obviously the sphere $S(b,\epsilon')$ contains at most a finite set of the elements of $\{a_n\}$. Thus the assumption $a \neq b$ must be false. Therefore $a=b$ and the limit of a sequence is unique.

Theorem 1.7. If $\lim a_n = a$ and $\{b_n\}$ is a subsequence of $\{a_n\}$, then $\lim b_n = a$.

Proof. Suppose $\{a_n\} \to a$ and $\{b_n\} \to b$ where $\{b_n\}$ is a subsequence of $\{a_n\}$. Further suppose $b \neq a$. Then $d(b,a) = \epsilon > 0$. Consider the spheres $S(a, \epsilon/2)$ and $S(b, \epsilon/2)$. Obviously their intersection is empty; yet each sphere must contain all but a finite set of its respective sequence. Now since $\{b_n\}$ is a subset of $\{a_n\}$, and since all but a finite subset of $\{a_n\}$ is contained in $S(a, \epsilon/2)$, it follows that the sphere $S(b, \epsilon/2)$ must contain only a finite set of points of $\{b_n\}$. Thus the assumption that $b \neq a$ is not valid. Thus $\lim b_n = \lim a_n = a$.

Definition 1.10. The derived set A' of a set A will mean the set of all points x such that x is a limit point of A.

The concepts of open and closed spheres were introduced earlier. It will now be convenient to introduce the closely related concepts of open and closed sets.

Definition 1.11. A set M in a metric space is closed if $M = [M]$.

Examples. Every closed sphere is closed. An open sphere $S(x,r)$ is closed if there are no elements $y \in M$ such that $d(x,y) = r$.

Definition 1.12. A point x is said to be an interior point of a set M if there exists an ϵ-neighborhood of x which is contained entirely in the set M.

Definition 1.13. A set all of whose points are interior points is said to be an open set.

Theorem 1.8. The intersection of an arbitrary number and the union of an arbitrary finite number of closed sets are closed.

Proof. Let $F=\bigcap F_\alpha$ where each F_α is a closed set and suppose x is a limit point of F. An arbitrary neighborhood S(x,r) of x will contain an infinite set of points of F. This implies that S(x,r) contains an infinite set of points from each F_α, and thus x is a limit point of each F_α. Since each F_α is closed, this means that $x \in F_\alpha$ for each $F_\alpha \in \bigcap F_\alpha$ and hence $x \in F$. Thus F is closed.

Let $F=\bigcup_{i=1}^{n} F_i$ where F_i is closed, and suppose x to be a limit point of F. Thus an arbitrary neighborhood S(x,r) is such that $S(x,r)\bigcap F=J$, and J is infinite. This implies the existence of some $F_i \subset F$ such that $S(x,r)\bigcap F_i$ is also infinite for otherwise would imply that S(x,r) would contain at most some finite number of elements, say k, from each $F_i \subset F$. Thus J would contain at most nk elements of F which is contrary to the fact that J is infinite. Therefore x is a limit of at least one F_i and hence $x \in F_i \subset F$. Thus F is closed.

In Theorem 1.8 it was necessary to restrict the union to a finite class of closed set since it is possible to suggest infinite unions which are not closed. In Theorem 1.9 it is necessary to restrict the arbitrary intersection to a finite class of open sets for a similar reason.

Theorem 1.9. The union of an arbitrary number and the intersection of a finite number of open sets are open.

Proof. Let $G=\bigcup G_\alpha$ where G_α are open sets and suppose $x \in G$. If $x \in G$ then obviously there exists some $G_\alpha \subset G$ such that $x \in G_\alpha$. Now G_α is open so there is some neighborhood $S(x,r)$ of x such that $S(x,r) \subset G_\alpha \subset G$. Thus x is an interior point of G, and G is open.

Now let $G=\bigcap_{i=1}^{n} G_i$ where G_i is open and suppose $x \in G$. This implies that for $i \in \{1,2, \cdots n\}$ $x \in G_i$. Thus in each G_i there exists a neighborhood $S(x,r_i)$ such that $S(x,r_i) \subset G_i$. Since i is finite it is possible to speak of the least r_i or radius of the spheres $S(x,r_i)$. If $r_t \leq r_i$ for $i \in \{1,2, \cdots n\}$, and if $y \in S(x,r_t)$ then obviously $y \in S(x,r_i)$. Thus $S(x,r_t) \subset \bigcap_{i=1}^{n} G_i = G$. Therefore there is a neighborhood of x which is contained entirely in G and thus G is open.

Theorem 1.10. A necessary and sufficient condition that the set M be open is that $\overline{M}$ (the complement of M) with respect to the whole space be closed.

Proof. Necessity.

Suppose M is open. Then if $x \in M$ there is a neighborhood of x $S(x,r)$ which belongs entirely to M. That is, $S(x,r)$ contains no point y such that $y \in \overline{M}$. Thus M contains no contact points of $\overline{M}$. Therefore if y is a contact point of $\overline{M}$, then $y \in \overline{M}$ which implies $\overline{M}$ is closed. If $\overline{M}$ has no contact points then $\overline{M}=\emptyset$ and $\emptyset$ is closed.

Sufficiency.

Suppose $\overline{M}$ is closed and suppose $x \in M$. If every neighborhood $S(x,r)$ of x contains a point of $\overline{M}$, then x is a contact point of $\overline{M}$. Since $\overline{M}$ is closed, this implies that $x \in \overline{M}$ which contradicts the assumption that $x \in M$. Thus there must be some neighborhood $S(x,r)$ of x which contains no points of $\overline{M}$. This implies that x is an interior point of M and hence M is open.

<u>Problem 1.3</u>. Show that $[A]$ is a closed superset of A which is contained in every closed superset of A.

<u>Proof</u>. Suppose B is a closed superset of A and $x \in [A]$. Then x is either a limit point or an isolated point of A. If x is an isolated point of A, then $x \in A$ and $x \in B$ since $A \subset B$. If x is a limit point of A, then x is a limit point of B and $x \in B$ since B is closed. Thus $[A] \subset B$.

<u>Problem 1.4</u>. A set $M \subset X$ is bounded $\Leftrightarrow$ it is non-empty and is contained in a closed sphere.

<u>Proof</u>. Suppose M is a bounded subset of X. By definition there is a real number $K > 0$ such that $0 \leq c(x,y) \leq K$ for $x,y \in M$. But $d(\emptyset) = -\infty$. Thus $M \neq \emptyset$, and $M \subset S[x,K]$ for $x \in M$. Clearly if M is non-empty and $M \subset S[x,K]$, then M is bounded.

<u>Problem 1.5</u>. Show $[A]$ equals the intersection of all closed supersets of A.

<u>Proof</u>. Suppose $x \in [A]$ and B is a closed superset of A. By Problem 1.3, $x \in B$, and hence $x \in \bigcap_{B \in T} B$ where T is the class of all closed supersets of A. Thus $[A] \subset \bigcap_{B \in T} B$.

Now suppose $x \in \bigcap_{B \in T} B$. Then x belongs to each $B \in T$. From Problem 1.3, $[A] \in T$; thus $\bigcap_{B \in T} B \subset [A]$. Hence $A = \bigcap_{B \in T} B$.

<u>Definition 1.14</u>. A family $\{G_\alpha\}$ of open sets in X is called a basis in X if every open set in X can be represented

as the sum of a (finite or infinite) collection of sets belonging to this family.

Definition 1.15. A system of sets M_α is a covering of a space X if $\bigcup M_\alpha = X$. A covering consisting of open (closed) sets will be called an open (closed) covering.

Example (1). In every metric space the family of all open spheres forms a basis.

Example (2). In Euclidean 2-space with the usual metric the set of open (closed) spheres S(x,r), where r=1 and x is a point with integer coordinates, will form an open (closed) covering.

Example (3). In Example (1) on page one the set of all unit spheres will form both a basis and a covering.

Theorem 1.11. A necessary and sufficient condition that a system of open sets $\{G_\alpha\}$ be a basis in X is that for every open set $G \subset X$ and for every point $x \in G$, a set G_α can be found in this system such that $x \in G_\alpha \subset G$.

Proof. Necessity.

If $\{G_\alpha\}$ is a basis then every open set G is the union of some subset of $\{G_\alpha\}$. That is $G = \bigcup G_k$ where $G_k \in \{G_\alpha\}$. Therefore if $x \in G$ then there is some G_k in G such that $x \in G_k$. But $G_k \in \{G_\alpha\}$ and thus there exists some G_α such that $x \in G_\alpha \subset G$.

Sufficiency.

Suppose G is any open set in X and for each $x \in G$ we can find some open set $G_{\alpha_x} \in \{G_\alpha\}$ such that $x \in G_{\alpha_x} \subset G$. Then obviously $\bigcup_{x\in G} G_{\alpha_x} = G$ since each $x \in G$ is in some G_α and each G_α is a subset of G. Thus $\{G_\alpha\}$ is a basis in X.

Theorem 1.12 is concerned with the concept of one set being dense in another so it is necessary to introduce the following definition at this point.

Definition 1.16. Let A and B be two sets in a metric space X. The set A is said to be dense in B if $B \subseteq [A]$. A is said to be everywhere dense in X if $[A]=X$.

Example (1). Let X be Euclidean three-space with the usual metric. Let $A=\{(x,y,z) \mid -1 < x,y,z < 1\}$ and let $B=\{(x,y,z) \mid x,y,z \in (1,-1)\}$. Then $A \cap B = \emptyset$ but A is dense in B.

Example (2). In R the set of rationals is everywhere dense.

Example (3). In Example (2) on page one, the set of all ordered n-tuples with rational coordinates is everywhere dense.

Theorem 1.12. A necessary and sufficient condition that X be a space with countable basis is that there exist in X an everywhere dense countable set.

Proof. Necessity.

Let X be a metric space with a countable basis, say $G=\{G_i \mid G_i \text{ is open and } i \in (1,2,3, \cdots)\}$. Construct a set A by choosing from each non-empty $G_i \in G$ an arbitrary $x_i \in G_i$. Suppose x is an arbitrary point of X, and let S(x,r) be a neighborhood of x. Then by the previous theorem there is a set $G_n \in G$ such that $x \in G_n \subset S(x,r)$. By definition of A there is some point $y \in A$ such that $y \in G_n \subset S(x,r)$. Therefore any neighborhood of x contains some point y of A. This implies that x is a contact point of A. Thus each $x \in X$ is a contact point of A and hence $X \subseteq [A]$. But X is the whole space; hence $[A] \subseteq X$. Thus $X=[A]$.

Sufficiency.

Suppose A is countable and A is everywhere dense in X. Then $[A]=X$. Let $S=\{S(a,1/j) \mid a \in A,\ j \in (1,2,3,\cdots)\}$. Obviously S is countable. Suppose G is an open set in X and $x \in G$. Then there exists some $S(x,r) \subset G$. By Theorem 1.4, x is either a limit point of G or an isolated point of G. Since $G \subseteq [A]$ it follows that x is an isolated point or a limit point of $[A]$. By Theorem 1.2 $[[A]]=[A]$. Thus x is either a point in A or else a limit point of A. If $x \in A$ then there is some positive integer i such that $1/i < r$. Thus $S(x,1/i) \subset S(x,r) \subset G$ and $S(x,1/i) \in \mathcal{S}$. If x is a limit point of A then there is some element $a \in A$ such that $d(a,x) < \frac{1}{2i}$. Thus $x \in S(a,\frac{1}{2i}) \subset S(x,r) \subset G$, and $S(a,\frac{1}{2i}) \in \mathcal{S}$. Thus by Theorem 1.11, S is a basis in X and S is countable.

In the proof of the next theorem the countability of the rational numbers will be assumed.

<u>Theorem 1.13</u>. Every open set on the real line is the union of a countable number of disjoint intervals. Sets of the form $(-\infty, \infty)$, (α, ∞), $(-\infty, \beta)$ will be included as intervals.

<u>Proof</u>. Let G be a non-empty open set in R. Define I to be the set of all real open intervals, (a,b), which satisfy the following conditions:

(1) $a,b \in \overline{G}$ with respect to the extended space R'.

(2) if $(a,b) \in I$, then $(a,b) \cap G = (a,b)$. First I must be shown to be non-empty. Suppose $x \in G$. Let $Y = \{y \mid y \in \overline{G},\ y > x\}$, and $Z = \{z \mid z \in \overline{G}, \text{ and } z < x\}$. If $Y \neq \emptyset$ then there exists some greatest lower bound t such that $t \leq y$ for each $y \in Y$. Obviously $t \notin G$ since $t \in G$ would imply $S(t,r) \cap \overline{G} = \emptyset$ for some $r > 0$, and the condition that t is the greatest lower bound of Y implies that $S(t,r) \cap \overline{G} \neq \emptyset$ for all $r > 0$, since $Y \subset \overline{G}$. If $Y = \emptyset$ then $(x,\infty) \subset G$ and $t = \infty$. Similarly it can be shown that there is some least upper bound u of Z and that $u \notin G$. If $Z = \emptyset$ then $(-\infty,x)$ is obviously a subset of G and $u = -\infty$. Thus $(u,t) \in I$ and hence $I \neq \emptyset$, and for each $x \in G$ there is some interval $(a,b) \in I$, such that $a < x < b$. Suppose $(a,b) \in I$ and $(c,d) \in I$, and $x \in \{(a,b) \cap (c,d)\}$. Then $\sup\{a,c\} < x < \inf\{b,d\}$.

If a=c and b=d the intervals are identical. Thus without loss of generality it can be assumed that $a < c$. Now $a < c < b$; hence $c \in (a,b)$. By Postulate 2, $(a,b) \cap G = (a,b)$, but by Postulate 1, $c \notin G$. Thus the assumption that $a < c$ is invalid and hence a=c. Similarly, b=d. Thus it has been shown that I is non-empty, each $x \in G$ belongs to some interval $(a,b) \in I$, and the elements of I are disjoint. All that remains is to show that I is countable. A subset P of the rational numbers can be formed by selecting from each $(a,b) \in I$ a rational.

Since the elements of I are disjoint, each rational so selected will be unique. Thus we can form a 1-1 correspondence from I onto P. The set M of all rationals is countable and since $P \subset M$ it follows by Cantor's theorem that P is countable and thus I is a countable collection of open intervals I_α such that $\bigcup I_\alpha = G$.

This section is primarily concerned with some of the properties of connected spaces. The concept will be encountered again in the second chapter where consideration will be given to properties of functions defined on connected spaces. First it is necessary to define what is meant by mutually separated subsets of a metric space.

Definition 1.17. Let X be a metric space. Two non-empty subsets M and N of X are said to be mutually sepatated if neither contains a point or limit point of the other; that is $M \cap [N] = \emptyset$ and $N \cap [M] = \emptyset$.

Definition 1.18. A subset M of a metric space X is called connected if M is not the union of two mutually separated sets.

A second definition of connectivity is frequently given as follows.

Definition 1.19. A space X containing no sets which are simultaneously open and closed other than the void set and the entire space X is said to be connected.

These two definitions can easily be shown to be equivalent.

In the proof of Theorem 1.14 an open interval of real numbers will be assumed to constitute an uncountable set.

<u>Theorem 1.14</u>. Let A be a connected subset of a metric space X which contains more than one point. Then A is uncountable.

<u>Proof</u>. Suppose A is a connected subset of a metric space with metric d and suppose A contains more than one point. Let x_0 and y be two points of A such that $x_0 \neq y$. Then $d(x_0,y)=k>0$. Now the set of real values r, such that $0<r<k$ is uncountable. Suppose that A is countable. Then there exists some real number $t>0$ such that $0<t<k$ and such that $d(x_0,z)\neq t$ for any $z \in A$. Now consider the two sets $C=\{y/y\in A, d(x_0,y)<t\}$, and $D=\{y|y\in A, d(x_0,y)>t\}$. Obviously neither of the two sets is empty since $x_0 \in C$ and $y \in D$. Suppose $z \in A$. Then either $d(x_0,z)<t$ or $d(x_0,z)>t$. In either case $z\in D\cup C$. Hence $A\subset(D\cup C)$. Now suppose $z\in D\cup C$. Then either $z\in D$ or $z\in C$, and in either case $z\in A$. Thus $(D\cup C)\subset A$, and therefore $D\cup C=A$. The sets C and D are obviously mutually separated and since $A=D\cup C$ it follows that A is not connected which is an invalid conclusion. Therefore the assumption that A is countable must be invalid. Hence A is uncountable.

<u>Theorem 1.15</u>. If each of M and N is a connected subset of X, and $M\cap N\neq\emptyset$, then $M\cup N$ is connected.

<u>Proof</u>. Suppose each of M and N is a connected subset of X and $M\cap N\neq\emptyset$. Further suppose that there exist sets T and S

such that $T \neq \emptyset$, $S \neq \emptyset$, T and S are mutually separated and $T \cup S = M \cup N$. Thus it can be assumed without loss of generality that there is some point $x \in M \cap N$ such that $x \in T$. The set M cannot be entirely devoid of points of S for this would imply $S \subset N$, and therefore N could be written as the union of two mutually separated sets $S \cup K$ where $K \subset T$.

Now let $T_1 = \{x \mid x \in T,\ x \in M\}$ and $S_1 = \{x \mid x \in S,\ x \in M\}$. Obviously $T_1 \cup S_1 = M$, $T_1 \neq \emptyset$, $S_1 \neq \emptyset$. Since the sets T and S are mutually separated it follows that T_1 and S_1 are mutually separated. Thus a connected subset M of X has been written as the union of two mutually separated sets, which is an invalid result. Therefore there exist no such T and S, and $M \cup N$ is connected.

Theorem 1.16. If $M \subset X$ is connected, then $[M]$ is connected.

Proof. Suppose M is connected, and suppose L to be the set of limit points of M which are not contained in M. Further suppose that $[M] = A \cup B$ where $A \neq \emptyset$, $B \neq \emptyset$. If L=A or L=B then the theorem follows. If $L \neq A$ and $L \neq B$ then $(A-L) \neq \emptyset$ and $(B-L) \neq \emptyset$. But $(A-L) \cup (B-L) = M$. Thus (A-L) and (B-L) are not mutually separated and hence A and B are not mutually separated. Therefore $[M]$ is connected.

Theorem 1.17. If M and N are mutually separated and A is a connected set such that $A \subset (M \cup N)$, then $A \subseteq M$, or $A \subseteq N$.

Proof. Suppose M and N are mutually separated and A is a connected set such that $A \subset (M \cup N)$. Further suppose $A \cap M \neq \emptyset$ and $A \cap N \neq \emptyset$. Now $A \cap M$ is a subset of M and $A \cap N$ is a subset of N.

Thus $(A \cap M)$ and $(A \cap N)$ are mutually separated. But $A=(A \cap M) \cup (A \cap N)$, and thus A is not connected. This last conclusion is invalid and therefore either $A \cap M$ $\emptyset$ or $A \cap N=\emptyset$. Since $A \subset (M \cup N)$, clearly if $A \cap M=\emptyset$ then $A \subseteq N$, and if $A \cap N=\emptyset$, $A \subseteq M$.

<u>Theorem 1.18</u>. If X is a connected metric space and M is a connected subset of X and $\overline{M}$ (complement of M)$=A \cup B$ where A and B are separated, then $A \cup M$ is connected.

<u>Proof</u>. Suppose M is a connected subset of a connected metric space X and $\overline{M}=A \cup B$ where A and B are separated. Further suppose that C and D are separated subsets of $A \cup M$ such that $C \cup D=A \cup M$. Consider the sets $C \cap M$ and $D \cap M$ and suppose $C \cap M=\emptyset$. Then $M \subseteq D$ and $C \subseteq A$. Thus C and M are mutually separated as are C and B. Hence C and $(B \cup D)$ are mutually separated. But $C \cup (B \cup D)=(C \cup D) \cup (B)=(A \cup M) \cup (B)=M \cup (A \cup B)=M \cup \overline{M}=X$. Thus X can be written as the sum of two mutually separated sets which is a contradiction of the assumption that X is connected. Therefore $C \cap M \neq \emptyset$. Similarly it can be shown that $D \cap M \neq \emptyset$. Thus $M=(C \cap M) \cup (D \cap M)$. But $(C \cap M) \subset C$and $(D \cap M) \subset D$ which implies that $(C \cap M)$ and $(D \cap M)$ are mutually separated. Thus M can be written as the sum of two mutually separated sets which contradicts the assumption that M is connected. Hence C and D are not separated and this implies that $A \cup M$ is connected.

The concept of mutually separated sets calls attention to the interesting situation of two sets which are disjoint yet not mutually separated. In order to inquire into this area it is necessary to define what is to be meant by the boundary or frontier of a set.

Definition 1.20. The statement that a point x belongs to the boundary of a set M in X means that for each real number $r>0$, the sphere $S(x,r)$ intersects both M and $\overline{M}$. Denote the set of all such x by $Fr(M)$.

Theorem 1.19. If X is a metric space and $S\subset X$, then $Fr(S)=[S]\cap[\overline{S}]$.

Proof. Suppose $x\in Fr(S)$. Then each open sphere $S(x,r)$ intersects S and $\overline{S}$. Hence $x\in[S]$ and $x\in[\overline{S}]$. Thus $Fr(S)\subset[S]\cap[\overline{S}]$. Suppose $x\in([S]\cap[\overline{S}])$. Then $x\in[S]$ and $x\in[\overline{S}]$. Therefore each open sphere $S(x,r)$ intersects S and $\overline{S}$. Thus $x\in Fr(S)$ and hence $[S]\cap[\overline{S}]\subset Fr(S)$. Therefore $Fr(S)=[S]\cap[\overline{S}]$.

Theorem 1.20. $Fr(A\cup B)\subset(Fr(A)\cup Fr(B))$.

Proof. If A or $B=\emptyset$ then the proof is trivial. Assume neither A nor B is empty, and suppose $x\in Fr(A\cup B)$. Then for $r>0$, $S(x,r)\cap(A\cup B)\neq\emptyset$. Suppose there is some $r>0$ such that $S(x,\epsilon)\cap B=\emptyset$, for $\epsilon<r$. Then $S(x,\epsilon)\cap A\neq\emptyset$, and hence $x\in Fr(A)$. Similarly if there is some $r>0$ such that $S(x,\epsilon)\cap A=\emptyset$, for $\epsilon<r$, then $S(x,\epsilon)\cap B\neq\emptyset$, in which case $x\in Fr(B)$. Thus if $x\in Fr(A\cup B)$ then $x\in Fr(A)\cup Fr(B)$ and $Fr(A\cup B)\subset\{Fr(A)\cup Fr(B)\}$.

Theorem 1.21. $Fr\left([A]\right)\subset Fr(A)$.

Proof. If $[A]=\emptyset$ then $A=\emptyset$ and $Fr([A])=Fr(A)=\emptyset$. Suppose $x\in Fr([A])$. Then for each $r>0$, the sphere $S(x,r)$ intersects

[A] and [$\bar{A}$]. Thus there is some point $y \in S(x,r)$ such that $y \in [A]$ and some point $z \in S(x,r)$ such that $z \in [\bar{A}]$. If $z \in [\bar{A}]$ then $z \notin [A]$ and hence $z \notin A$. Thus $z \in \bar{A}$, so an arbitrary sphere about x intersects $\bar{A}$. Now the sphere $S(x,r)$ must be shown to intersect A. The point $y \in [A]$ is either an element of A or a limit point of A by Theorem 1.4. If $y \in A$ then the theorem follows. If y is a limit point of A, then for every $\epsilon > 0$, the sphere $S(y, \epsilon)$ intersects A. Now $d(x,y) < r$ since $y \in S(x,r)$. Choose ϵ' such that $0 < \epsilon' < r - d(x,y)$. Obviously the sphere $S(y, \epsilon') \subset S(x,r)$ and $S(y, \epsilon')$ intersects A. Thus $S(x,r)$ intersects A. Hence $x \in Fr(A)$ since an arbitrary neighborhood of x intersects both of A and $\bar{A}$.

Definition 1.21. The set of all interior points of a set M is called the interior of M and will be denoted by M^{o}.

Theorem 1.22. The boundary of the interior of a set A is contained in the boundary of A. That is, $Fr(A^{o}) \subset Fr(A)$.

Proof. Suppose $A^{o} \neq \emptyset$, and $x \in Fr(A^{o})$. Then obviously for $r > 0$, $S(x,r) \cap A^{o} \neq \emptyset$, and $S(x,r) \cap \bar{A}^{o} \neq \emptyset$. Choose some y such that $y \in S(x,r) \cap \bar{A}^{o}$. Now $\bar{A}^{o} = (A - A^{o}) \cup \bar{A}$. Thus either $y \in (A - A^{o})$ or else $y \in \bar{A}$. If $y \in \bar{A}$, then the theorem follows since $S(x,r)$ has been shown to intersect both A and $\bar{A}$. If $y \in (A - A^{o})$ then each sphere $S(y,r')$ must intersect $\bar{A}$ since otherwise would imply $y \in A^{o}$. Choose an r' such that $0 < r' < r - d(x,r)$. Obviously the sphere $S(y,r') \subset S(x,r)$, and since $S(y,r')$ intersects $\bar{A}$ it follows that $S(x,r)$ intersects $\bar{A}$. Thus $S(x,r) \cap \bar{A} \neq \emptyset$ and $S(x,r) \cap A \neq \emptyset$ hence $x \in Fr(A)$. Consequently $Fr(A^{o}) \subset Fr(A)$.

Problem 1.6. If A is a subset of a metric space X, then the complement of the closure of A equals the interior of the complement of A. That is, $[\bar{A}] = \bar{A}^{o}$.

Proof. Suppose $x \in [\bar{A}]$ and suppose every open sphere $S(x,r)$ intersects A. Then x is a contact point and belongs to $[A]$. This is a contradiction of the assumption that $x \in [\bar{A}]$. Thus there is some sphere $S(x,r)$ which is contained in $\bar{A}$. Thus $x \in \bar{A}^{o}$, and $[\bar{A}] \subset \bar{A}^{o}$.

Now suppose $x \in \bar{A}^{o}$. Then there is a sphere $S(x,r)$ which is contained in the set $\bar{A}^{o}$. Thus $S(x,r) \cap A = \emptyset$. Hence $x \in [\bar{A}]$, and $\bar{A}^{o} \subset [\bar{A}]$. Thus $[\bar{A}] = \bar{A}^{o}$.

The concept of a compact set is quite important to this study of metric spaces since it is this property which guarantees the Bolzano-Weierstrass property of Theorem 1.24. The definition of compactness chosen here is a form of the conclusion of the Heine-Borel theorem.[1]

Definition 1.23. A set S is compact if every open covering of S has a finite subcovering.

Theorem 1.23. A compact set is closed and bounded.

Lemma. If $S(a,r)$ is an open sphere and $x,y \in S(a,r)$ then $d(x,y) < 2r$.

Proof. Suppose $S(a,r)$ is an open sphere and $x,y \in S(a,r)$. Obviously $d(a,x) < r$ and $d(a,y) < r$ and $d(x,y) \leq d(a,x) + d(a,y) < 2r$.

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[1]George F. Simmons, Introduction to Topology and Modern Analysis (New York, 1963), pp. 111-112.

Proof of Theorem 1.23. Suppose M is a compact set. Consider the open covering $G_\alpha = \{S(x,1) \mid x \in M\}$. Since M is compact there exists a finite subcovering S_i, where $S_i \subset G_\alpha$. Choose some point $a \in M$ and consider the set $C = \{x \mid x$ is the center of a sphere in $S_i\}$. Since $\{S_i\}$ is finite, it follows that C is finite and that there is some $x \in C$ such that $d(a,x) \geq d(a,y)$ for all $y \in C$. Thus $[d(a,x)+1] \geq d(a,y)$ for all $y \in M$, and hence M is bounded.

Now suppose x is a limit point of M and $x \notin M$. Consider the sequence $\{b_n\}$ of points of M which converges to x and consider the covering $G_\alpha = \{S(a, \epsilon) \mid a \in M, \epsilon = \frac{1}{2}d(a,x)\}$. Since M is compact there is a finite subcovering $\{S_i\}$ of $\{G_\alpha\}$. Clearly no sphere in S_i contains x, and no sphere in S_i contains more than a finite set of points of $\{b_n\}$. Now every point of $\{b_n\}$ belongs to some sphere in S_i; thus the sequence $\{b_n\}$ has only a finite set of points. This is an invalid conclusion; hence our assumption that $x \notin M$ must be invalid. Thus M is closed.

Theorem 1.24. Every compact infinite subset of a metric space has at least one limit point.

Proof. Suppose M is an infinite compact subset of a metric space X, and suppose that M has no limit point. Then each point of M is an isolated point. Thus if $y \in M$ there is an $\epsilon > 0$ such that $S(y, \epsilon)$ contains no point of M other than y, and we can form an open covering $\{G_\alpha\}$ by choosing for each $y \in M$, an $\epsilon > 0$ such that $S(y, \epsilon) \cap M = \{y\}$. From the definition of compactness, there is a finite subcovering $\{S_i\}$ of $\{G_\alpha\}$. But since each sphere in $\{S_i\}$ contains only a single point of M, this implies that M is finite which is invalid. Thus M must have at least one limit point.

Definition 1.24. The statement that a metric space X is a sequentially complete space means that every infinite sequence in X has a convergent subsequence in X.

CHAPTER II

PROPERTIES OF FUNCTIONS ON METRIC SPACES

The aim of this chapter is to establish some of the properties of functions defined on metric spaces. No attempt is made in this paper to examine a particular type of function in detail. Instead, some of the properties of several kinds of functions will be observed as the functions are defined on the various forms of metric spaces described in Chapter I, i.e., connected spaces, compact spaces, complete spaces, etc.

Basic to this discussion is the notion of a continuous function. The words mapping and function will be used interchangeably.

Definition 2.1. Let X and X' be metric spaces with metrics d and d'. The mapping f of the space X into the space X' is said to be continuous at the point $x_0 \in X$ if for each real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that if $y \in X$ and $d(x_0, y) < \delta$, then $d'(f(x_0), f(y)) < \epsilon$. If f is continuous at each point of X, then f is said to be continuous on X.

Example (1). Any function defined on the metric space of Example (1), page one in the previous chapter will be continuous.

Example (2). If A is a subset of a metric space X and

x_0 is a fixed element of A, then the function f defined by $f(x)=d(x_0,x)$ is continuous on A.

The first of these two examples is easily verified. The second example will be examined in more detail after some groundwork has been laid.

<u>Theorem 2.1</u>. A necessary and sufficient condition that the mapping f of the metric space X into the metric space X' be continuous is that for every sequence $\{x_n\}$ which converges to x, the sequence $\{f(x_n)\}$ converges to f(x).

<u>Proof</u>. Necessity.

Suppose f is a continuous mapping of the metric space X into the metric space X' and $\{x_n\}$ is a sequence which converges to $x_0 \in X$. Choose $\epsilon > 0$. The continuity of f at x_0 guarantees the existence of a $\delta > 0$ such that if $d(x_0,y) < \delta$ and $y \in X$, then $d'(f(x_0),f(y)) < \epsilon$. Now $\{x_n\}$ converges to x_0 so the sphere $S(x_0, \delta)$ contains all but a finite number of points of $\{x_n\}$. Thus the sphere $S(f(x_0), \epsilon)$ contains all but a finite set of points of the sequence $\{f(x_n)\}$. Therefore $\{f(x_n)\}$ converges to $f(x_0)$.

Sufficiency.

Suppose f is a function of X into X' and for every sequence $\{x_n\}$ which converges to x_0 the sequence $\{f(x_n)\}$ converges to $f(x_0)$. Further suppose that f is not continuous at some $x_0 \in X$. Then there is some $\epsilon > 0$ such that for every $\delta > 0$ there is an $x \in X$ for which $d(x_0,x) < \delta$ and $d'(f(x_0),f(x)) \geq \epsilon$.

Consider the set $N=\{S(x_0,1/n) \mid n=1,2,3 \cdots\}$. Each of these neighborhoods of x_0 must contain some $y \in X$ such that $d'(f(x_0),f(y)) \geq \epsilon$. Choose one such y_n from each of the neighborhoods $S(x_0,1/n)$. Now the sequence $\{y_n\}$ converges to x_0 but $d'(f(x_0),f(y_n)) \geq \epsilon$ for each element of $\{y_n\}$. Thus $\{f(y_n)\}$ does not converge to $f(x_0)$. This conclusion is invalid thus the assumption that f is not continuous at x_0 is invalid. Therefore f is continuous at each $x_0 \in X$ and hence f is continuous on X.

The notion of a bounded set was introduced in the first chapter. It was this property along with that of closure which were guaranteed by the Heine-Borel covering property of compact spaces. Since a function is defined in terms of sets it is natural to extend the concept of a bound to functions.

Definition 2.2. Let f be a function from a metric space X into a metric space X'. The function f is said to be bounded on a subset A of X if there exists a real number M such that $d'(f(x),f(y)) \leq M$ for all $x,y \in A$.

Theorem 2.2. If A is a compact subset of the space X and f is a continuous function on X to X', then f(A) is bounded.

Proof. If f(A) is finite it is obviously bounded. Assume f(A) to both infinite and unbounded. Thus if a is some fixed point of A, and n is a positive integer, there is some $x_n \in A$ such that $d'(f(x_n),f(a)) > n$. Obviously the sequence $\{f(x_n)\}$ is not convergent nor does it contain a convergent subsequence. Now A is bounded, thus the sequence $\{x_n\}$ is bounded. And the Bolzano-Wierstrass property of compact spaces which was established in Chapter I along with the uniqueness of limit points of a sequence guarantee that the sequence $\{x_n\}$ has exactly one limit point. Therefore there is some subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ which converges to a point x_o. Compactness guarantees closure and thus $x_o \in A$, and $f(x_o) \in f(A)$. Theorem 2.1 insures that $\{f(x_{n_k})\}$ converges to $f(x_o)$. However this is not in agreement with the conclusion reached earlier from the assumption that f(A) is unbounded. Therefore this assumption must be invalid, and hence f(A) is bounded.

Problem 2.1. Let X be a non-empty metric space and f be a real function defined on X. Show that f is bounded $\Longleftrightarrow$ there exists a real number k such that $|f(x)| \leq k$ for every $x \in X \Leftrightarrow \sup |f(x)| < +\infty$.

Proof. Suppose f is bounded in X. Then by definition there exists some real number k such that $d(f(x),f(y)) \leq k$ for all $x,y \in X$. The set X is non-empty so choose $z \in X$. Then $|f|$ is obviously bounded by $|f(z)|+k$, since otherwise would imply $k+|f(z)| < |f(x)|$, or $k < |f(x)| - |f(z)| \leq |f(x)-f(z)| = d(f(x),f(z))$ which is a contradiction. Hence $\sup |f(x)| \leq |f(z)|+k < \infty$.

Now suppose $\sup |f(x)| < \infty$. This implies the existence of a real number k such that $|f(x)| \leq k$ for all $x \in X$. Thus $-k \leq |f(x)| \leq k$, and f is bounded.

Now consider the set of all bounded real functions defined on X and define the norm of a function f in this set by $\|f\| = \sup |f(x)|$. It is obvious that $\|f\|$ is a nonnegative real number such that $\|f\| = 0 \Leftrightarrow f = 0$, and that $\|-f\| = \|f\|$. Also for such functions f and g, $\|f+g\| \leq \|f\| + \|g\|$ since $\|f+g\| = \sup |(f+g)(x)| = \sup |f(x)+g(x)| \leq \sup |f(x)| + \sup |g(x)| = \|f\| + \|g\|$.

Theorem 2.3. If A is a compact subset of a metric space X and f is a continuous function on X to R (the real numbers with usual metric), then there is a point $p \in A$ such that $f(p) \geq f(x)$ for all $x \in A$.

Proof. If A is finite, the theorem follows. Assume that A is compact and infinite. The previous theorem guarantees that f(A) is bounded and hence has a least upper bound M, such that given any real number $\epsilon > 0$ there is some $x \in A$ such that $d'(f(x),M) < \epsilon$. Thus for each real number of the form 1/n, $n \in \{1,2,3, \cdots\}$, there is some $x_n \in A$ such that $d'(f(x_n),M) < 1/n$. From Theorems 1.7 and 1.24 it is clear that the sequence $\{x_n\}$ contains some subsequence $\{x_{nk}\}$ which converges to a point x_0; and $x_0 \in A$ since A is closed. Clearly $\{f(x_n)\}$ converges to M and thus by Theorem 1.7 $\{f(x_{n_k})\}$ converges to M, and $M = f(x_0)$. Thus $x_0 \in A$ and $f(x_0) \geq f(x)$ for all $x \in A$.

Theorem 2.4. If f is a continuous function on X to X' and g is a continuous function on X' to X", then gf is a continuous function on X to X" where $gf(x) = g(f(x))$ for $x \in X$.

Proof. Suppose the above conditions and choose $\epsilon > 0$. Further suppose $x_0 \in X$. Then $f(x_0) \in X'$. Since g is continuous on X' to X" there is a $\delta' > 0$ such that if $f(y) \in X'$ and $f(y) \in S(f(x_0), \delta')$, then $d''[g(f(x_0)), g(f(y))] < \epsilon$. Also since f is continuous at x_0 there is some δ such that if $z \in X$ and $z \in S(x_0, \delta)$, then $f(z) \in S(f(x_0), \delta')$. Thus if $z \in X$ and $z \in S(x_0, \delta)$ then $f(z) \in S(f(x_0), \delta')$ which implies that $g(f(z)) \in S(g(f(x_0)), \epsilon)$. Therefore fg is continuous on X to X".

An important and familiar property of a continuous function defined on an interval of real numbers is that of assuming all values between any two of its values. A similar property holds for continuous functions defined on certain metric spaces. While the notion of a sphere in a metric space bears some similarity to that of an interval on the real line, there are some properties of intervals which are not posessed by spheres in general, and it is not sufficient to define a continuous function on an open sphere if it is desired that the function have the above mentioned property. It is necessary to restrict the metric space on which the continuous function is to be defined to be a connected space.

Before proving this theorem it will be convenient to establish the preliminary result of Theorem 2.5.

Theorem 2.5. Let f be a real valued function defined on a metric space X and assume f is continuous at a point a and that $f(a) > 0$. Then there is a sphere S with center at a so that for every $x \in S$, $f(x) > 0$.

Lemma.. If a and b are real numbers such that $a > 0$ and $b \leq 0$ then $|a-b| \geq a$.

Proof. Since $a > 0$, it follows that $|a| = a > 0$ and since $b \leq 0$ it follows that $|b| = -b \geq 0$. Now $|a-b| = |a+(-b)|$, and $-b = |b|$. Hence $|a-b| = \big||a| + |b|\big| = |a| + |b| \geq |a| = a$.

Proof of Theorem 2.5. Let f be a real valued function on X, be continuous at a, and $f(a) > 0$. Choose $\epsilon = f(a) > 0$. The continuity of f at a insures the existence of a $\delta > 0$ such that if $y \in X$ and $y \in S(a, \delta)$, then $f(y) \in S(f(a), \epsilon)$. Now suppose that $x \in X$, $x \in S(a, \delta)$, and $f(x) \leq 0$. Then $d'(f(a), f(x)) = |f(a) - f(x)| \geq f(a) = \epsilon$. This is an invalid conclusion and thus the assumption that $f(x) \leq 0$ for some $x \in S(a, \delta)$ must be invalid. Therefore $f(x) > 0$ for all $x \in S(a, \delta)$.

Theorem 2.6. Let f be a continuous function on a connected subset A of a metric space X and be real valued. Assume there is an element $a \in A$ such that $f(a) < 0$ and an element $b \in A$ such that $f(b) > 0$. Then there is an element $c \in A$ such that $f(c) = 0$.

Proof. Suppose there is no element $c \in A$ such that $f(c) = 0$. Then there are non-empty sets M and N such that $M \cup N = A$ and $M \cap N = \emptyset$ where $M = \{x \in A \mid f(x) > 0\}$ and $N = \{x \in A \mid f(x) < 0\}$. A is a connected set so we can assume without loss of generality that $M \cap [N] \neq \emptyset$. If $y \in M \cap [N]$ then $y \in M$ and y is a limit point of N. Thus there is some sequence $\{y_n\} \subset N$ such that $\{y_n\}$ converges to y. By Theorem 2.1, $\{f(y_n)\}$ converges to $f(y)$. Now $f(y) > 0$ since $y \in M$ and Theorem 2.5 asserts that there is some sphere $S(y, \epsilon)$ such that for every $x \in S(y, \epsilon)$, $f(x) > 0$ so $S(y, \epsilon)$ contains no points of $\{y_n\}$. Thus $\{y_n\}$ does not converge to y.

Hence our assumption that there is no $c \in A$ such that $f(c)=0$ is invalid and the proof is complete.

Although the details of the proofs will not be submitted in this paper, it is worthwhile to mention that the familiar property of the continuity of sums and products of continuous functions holds true for functions defined on metric spaces. That is, if each of f and g is a continuous function on a metric space X to the real numbers, then each of $f \pm g$, $f \cdot g$, $|f|$, αf (for real numbers α), and f/g (where $g(x) \neq 0$ for all $x \in X$) is continuous on X.

Definition 2.3. Let X and X' be metric spaces. The mapping f of the space X into the space X' is said to be uniformly continuous on X if for each $\epsilon > 0$ there is a $\delta > 0$ such that if $x,y \in X$ and $d(x,y) < \delta$, then $d'(f(x),f(y)) < \epsilon$.

Obviously the set of all uniformly continuous functions defined on a metric space X is a subset of all continuous functions defined on the space X. Thus it follows that the remarks made about continuous functions also are valid for uniformly continuous functions.

Theorem 2.7. Let X be a metric space and X' be a complete metric space and let A be a dense subspace of X. If f is a uniformly continuous function of A into X', then f can be extended uniquely to a uniformly continuous function g of X into X'; that is, g is uniformly continuous on X to X', $g(x) = f(x)$ for each $x \in A$, and g is unique.

<u>Proof</u>. If $A=X$ the theorem follows. Assume $A \neq X$ and define g to be the following;

(a) If $x \in A$, then $g(x)=f(x)$

(b) If $x \in (X-A)$ then x must be a limit point of A and there exists a sequence $\{x_n\}$ of elements of A such that $\{x_n\}$ converges to x. Thus the sequence $\{g(x_n)\}=\{f(x_n)\}$ converges to some y and since X' is complete, $y \in X'$. In this case let $g(x)=y$.

Now choose $\epsilon > 0$. Since f is uniformly continuous there is some $\delta > 0$ so that if $x,y \in A$ and $d(x,y)<\delta$, then $d'(g(x),g(y)) < \epsilon/2$. Suppose $x,y \in X$ and $d(x,y)<\delta$. Consider two sequences $\{x_n\}$ and $\{y_n\}$ of points of A which converge to x and y. By the triangle inequality for $x_i \in \{x_n\}$, $y_i \in \{y_n\}$,

$$d(x_i,y_i) \leq d(x_i,y)+d(y,y_i) \leq d(x_i,x)+d(x,y)+d(y_i,y).$$

The property of convergence guarantees that for an $\epsilon > 0$, there is some positive integer k such that if i is an integer and $i>k$ then $d(x,x_i)+d(y,y_i)<\epsilon$. Since $d(x,y)<\delta$ it follows that $\delta-d(x,y)>0$ and hence there is some positive integer k such that $d(x,x_i)+d(y,y_i)<\delta-d(x,y)$ for $i>k$, or $d(x,x_i)+d(y,y_i)+d(x,y)<\delta$ for $i>k$. Therefore $d(x_i,y_i) \leq d(x,x_i)+d(y,y_i)+d(x,y)<\delta$ for $i>k$. Since $x_i,y_i \in A$, it follows that $d'(g(x_i),g(y_i))<\epsilon/2$ for $i>k$. By the triangle inequality $d'(g(x),g(y)) \leq d'(g(x_i),g(y_i))+M$ where $M=d'(g(x_i),g(x))+d'(g(y_i),g(y))$. Suppose $d'(g(x),g(y))>\epsilon/2$. Then there is some $c>0$ such that $d'(g(x),g(y))=\epsilon/2+c$. From this it follows that $\epsilon/2+c \leq d'(g(x_i),g(y_i))+M < \epsilon/2+M$ for $i>k$. Thus $\epsilon/2+c<\epsilon/2+M$.

or $c < M$. But M goes to zero as $\{g(x_n)\}$ and $\{g(y_n)\}$ converges to x and y, and c is a constant, thus $c \nless M$. Therefore the assumption that $d'(g(x),g(y)) > \epsilon/2$ is invalid and hence $d'(g(x),g(y)) \leq \epsilon/2 < \epsilon$, and g is uniformly continuous on X to X'. The function g must now be shown to be unique.

Suppose h is a uniformly continuous extension of f on X to X' and there is some $x \in X$ such that $h(x) \neq g(x)$. Then by definition of an extension of a function $x \notin A$. Thus x is a limit point of A. Now $d'(h(x),g(x))=p>0$. Thus by virtue of the triangle inequality the neighborhoods $S(g(x),p/4)$ and $S(h(x),p/4)$ have an empty intersection. Since g and h are uniformly continuous, there is some $\delta > 0$ so that if $y \in X$ and $y \in S(x,\delta)$, then $d'(g(x),g(y)) < p/4$. Also $d'(h(x),h(y)) < p/4$. Now x is a limit point of A; hence there is some $y \in A$ such that $y \neq x$ and $y \in S(x,\delta)$ and $g(y)=h(y)$ by definition. But $g(y) \in S(g(x),p/4)$ and $h(y) \in S(h(x),p/4)$. This contradicts the conclusion that $S(g(x),p/4) \cap S(h(x),p/4)=\emptyset$ which was obtained through the assumption that $g(x) \neq h(x)$. Thus the assumption must be faulty. Hence g is unique.

Theorem 2.8. If f is continuous on a metric space X, then f is uniformly continuous on a conpact set $A \subset X$.

Proof. Suppose A is a compact subset of X and f is continuous on A. Let $\epsilon > 0$ and consider $\{S(f(x), \epsilon/8) \mid x \in A\}$. Clearly $f^{-1}(S(f(x), \epsilon/8))$ is an open set containing x and $\{f^{-1}(S(f(x), \epsilon/8)) \mid x \in A\}$ is an open covering of A. Now for every $f^{-1}(S(f(x), \epsilon/8))$ where $x \in A$ there is some $S(x, \delta_x) \subset f^{-1}(S(f(x), \epsilon/8))$. Then $\{S(x, \delta_x) | x \in A\}$ is an open covering of A and there is a finite subcovering of A, call it $\{S_1(x_1, \delta_1), S_2(x_2, \delta_2), \cdots, S_n(x_n, \delta_n)\}$. Consider $p = \min\{d(\{S_i \cap A\}, \{S_j \cap A\}) | d(\{S_i \cap A\}, \{S_i \cap A\}) \neq 0\}$. Let $\delta = \frac{1}{2}\min\{p, \delta_1, \delta_2, \cdots, \delta_n\}$. Suppose $x, y \in A$ and $d(x,y) < \delta$. Then $x \in S_i$ and $y \in S_j$.

Case (1). If $i=j$ then by definition of S_i, $d'(f(x),f(y)) < \epsilon$.

Case (2). If $i \neq j$ and $(S_i \cap A) \cap (S_j \cap A) \neq \emptyset$ then choose some $z \in (S_i \cap A) \cap (S_j \cap A)$. Now $d'(f(x),f(z)) < \epsilon/4$ and $d'(f(z),f(y)) < \epsilon/4$ and by the triangle inequality, $d(f(x),f(y)) < \epsilon/2 < \epsilon$.

Case (3). If $i \neq j$ and $(S_i \cap A) \cap (S_j \cap A) = \emptyset$ then for $r > 0$ there exist $a \in (S_i \cap A)$ and $b \in (S_j \cap A)$ such that $d(a,b) < r$ for otherwise would imply $d(\{S_i \cap A\}, \{S_j \cap A\}) \geq r$ and hence $d(x,y) > \delta$. For each of the real numbers $1/n$, $n \in \{1,2,3,\cdots\}$, choose $a_n \in (S_i \cap A)$ and $b_n \in (S_j \cap A)$ such that $d(a_n,b_n) < 1/n$. Clearly the sequences $\{a_n\}$ and $\{b_n\}$ contain subsequences which converge to the same limit x_0. But $x_0 \in A$ hence there is some sphere S_t such that $x_0 \in S_t$. Obviously S_t intersects $(S_i \cap A)$ and $(S_j \cap A)$. Choose $z \in S_t \cap (S_i \cap A)$ and $z' \in S_t \cap (S_j \cap A)$. Now $d'(f(x),f(z)) < \epsilon/4$ and $d'(f(z),f(z')) < \epsilon/4$ and $d'(f(z'),f(y)) < \epsilon/4$. Thus applying the triangle inequality twice gives $d(f(x),f(y)) < \epsilon$. Therefore f is uniformly continuous on A.

In Example (2), page 29, it was suggested that a real function f is continuous if defined on a metric space such that $f(x) = d(x,a)$ where a is a fixed point of X. A proof of this assertion is now offered in Problem 2.3. It is shown in Problem 2.3 that a function f so defined is not only continuous but also uniformly continuous.

Problem 2.3. Let M be a subset of a metric space X and $a \in M$. Then the function f defined by $f(x)=d(x,a)$ (a is fixed), is a uniformly continuous function on M to the real numbers.

Proof. Suppose the above conditions and choose $\epsilon > 0$. Let $\delta = \epsilon$. Now suppose $x,y \in M$ and $d(x,y) < \delta$. By the triangle inequality $d(a,x) \leq d(a,y)+d(x,y)$ or $d(a,x)-d(a,y) \leq d(x,y) < \delta = \epsilon$. Thus $f(x)-f(y) < \epsilon$. Also $d(a,y)-d(a,x) \leq d(x,y) < \delta = \epsilon$. Hence $f(y)-f(x) < \epsilon$. Thus $|f(x)-f(y)| = d'(f(x),f(y)) < \epsilon$, and f is uniformly continuous on M to R.

The set of all such functions on M, that is, the set $A=\{f_a \mid a \in M,\ f_a(x)=d(a,x)\}$ has several interesting properties. An attempt is made in Problems 2.4 and 2.5 to examine some of the more interesting ones.

Problem 2.4. Suppose X is a compact metric space with metric d, $a \in X$, and f is defined on X to R such that $f_a(x)=d(a,x)$ (a is fixed). Then f is continuous by Problem 2.3. Thus the set $A=\{f_a \mid a \in X\}$ is a subset of $\mathcal{C} = \{$all continuous real functions on X$\}$. Define a distance function ρ on $\mathcal{C}$ such that if $f,g \in \mathcal{C}$, then $\rho(f,g)=\sup_{x \in X}|f(x)-g(x)|$. Obviously ρ is a non-negative, single valued, real function and $\rho(f,g)=0 \Leftrightarrow f=g$. It is also clear that $\rho(f,g)=\sup_{x\in X}|f(x)-g(x)| = \sup_{x \in X}|g(x)-f(x)| = \rho(g,f)$. The triangle inequality is easily verified as follows.

$\rho(f,g)+\rho(g,h)=\sup_{x\in X}|f(x)-g(x)|+\sup_{x\in X}|g(x)-h(x)| \geq \sup_{x\in X}|f(x)-g(x)+g(x)-h(x)|=\sup_{x\in X}|f(x)-h(x)|=\rho(f,h)$. Thus ρ is a metric on $\mathcal{C}$.

Theorem 2.9. If a sequence $\{a_n\}\to a\in X$, then $\{f_{a_n}\}\to f_a\in \mathcal{C}$; i.e., for each $x\in M$, $\lim_{n\to\infty} f_{a_n}(x)=f_a(x)$.

Proof. Suppose $a\in X$, $\{a_n\}\to a$. Choose $\epsilon>0$ and let $\delta=\epsilon$. Now the sphere $S(a,\delta)$ contains all but a finite set of points of $\{a_n\}$. Suppose $a_i\in S(a,\delta)$. Then $d(a,a_i)<\delta=\epsilon$. Now suppose $x\in X$. By the triangle inequality $d(x,a)+d(a,a_i)\geq d(x,a_i)$ and also $d(x,a_i)+d(a,a_i)\geq d(x,a)$. Define ϵ' to be $d(a,a_i)$. Then these inequalities can be written as $\epsilon'\geq d(x,a_i)-d(x,a) =f_{a_i}(x)-f_a(x)$ and $\epsilon'\geq d(x,a)-d(x,a_i)=f_a(x)-f_{a_i}(x)$. Thus $|f_a(x)-f_{a_i}(x)|\leq \epsilon'<\delta=\epsilon$. Since x was arbitrarily chosen it follows that $\sup_{x\in X}|f_a(x)-f_{a_i}(x)|=d(f_a,f_{a_i})<\epsilon$. Hence $\{f_{a_n}\}\to f_a$.

Problem 2.5. Let M be a metric space and define a mapping $\emptyset$ of M into A by $\emptyset(a)=f_a$. Then $\emptyset$ is 1-1 onto continuous, since if $\{a_n\}\to a$ then $\{f_{a_n}\}\to f_a$.

Problem 2.5.1. Suppose $a,x,b\in M$. Then $d(a,b)+d(a,x)\geq d(b,x)$ and also $d(a,b)+d(b,x)\geq d(a,x)$. These two inequalities may be written as $d(a,b)\geq d(a,x)-d(b,x)$ and $d(a,b)\geq d(b,x)-d(a,x)$. Therefore $d(a,b)\geq |d(a,x)-d(b,x)|$ for all values of x. Thus it follows that $d(a,b)\geq \sup_{x\in X}|d(a,x)-d(b,x)|=\sup_{x\in X}|f_a(x)-f_b(x)|=d(f_a,f_b)$. But if x=a then $d(f_a,f_b)=|d(a,a)-d(b,a)|=d(a,b)$. Thus the supremum is attained for x=a or x=b and thus $d(a,b)=d(f_a,f_b)$.

Before proceeding to Problem 2.5.2 it is necessary to introduce the following definition..

Definition 2.4. A metric space X is said to be complete if every Cauchy sequence in X has a limit in X.

Problem 2.5.2. If U is a subset of a complete metric space M and $A(U)=\{f_a | a \in U\}$, then U is closed if and only if A(U) is closed. Suppose U is closed, g is a real function on M, and g is a limit point of A(U). Then some sequence $\{f_{a_n}\} \to g$. Clearly $\{f_{a_n}\}$ is a Cauchy sequence since for $\epsilon > 0$ there is some integer N such that $f_{a_i} \in S(g, \epsilon/2)$ for $i > N$ and if f_{a_i}, $f_{a_j} \in S(g, \epsilon/2)$ then $d(f_{a_i}, f_{a_j}) < \epsilon$. Now for $a,b \in U$ $d(a,b) = d(f_a, f_b)$; thus for $a_i, a_j \in \{a_n\}$, $d(a_i, a_j) < \epsilon$. Therefore $\{a_n\}$ is a Cauchy sequence and has a limit $x \in U$ since U is closed. Clearly $f_x \in A(U)$ and the sequence $\{f_{a_n}\}$ converges to f_x. The uniqueness of the limit of a sequence guarantees that $f_x = g$. Hence A(U) is closed.

Now suppose A(U) is closed and further suppose a sequence $\{a_n\} \subseteq U$ converges to some a. Then the sequence $\{f_{a_n}\} \to f_a$ and since A(U) is closed, f_a must belong to A(U). Thus by definition of A(U), $a \in U$, and U is closed.

Problem 2.5.3. A subset U of a metric space M is open if and only if A(U) is open. Suppose A(U) is open and $f_a \in A(U)$. Now $\emptyset$ is continuous on U to A(U) and thus for an arbitrary sphere $S(f_a, \epsilon)$ there is a δ so that if $y \in U$ and $y \in S(a, \delta)$, then $\emptyset(y) \in S(f_a, \epsilon)$. Obviously $a \in U$. Let $N=\{x \in S(a, \delta) | x \notin U\}$. If N contains no sequence $\{x_n\}$ which converges to the point a then a δ' can be found such that $S(a, \delta') \subseteq S(a, \delta)$ and $S(a, \delta') \subseteq U$. Clearly N contains no such sequence since this would imply the convergence of the sequence $\{\emptyset(x_n)\}$ to $\emptyset(a)=f_a$. But $S(f_a,\epsilon)\subseteq A(U)$. Thus U is open.

Now suppose U is open and $a\in U$. Then $S(a,r)\subset U$ for some $r>0$. Consider the sphere $S(f_a,r)$, and suppose $f_b\in S(f_a,r)$ and $b\notin S(a,r)$. Then $d(a,b)\geq r$; thus from Problem 2.5.1 $d(f_a,f_b)\geq r$ which contradicts the assumption that $f_b\in S(f_a,r)$. Thus $S(f_a,r)\subset A(U)$ and A(U) is open.

Problem 2.5.4. Suppose U is compact. Then every open covering of U has a finite subcovering. Suppose $\{G_\alpha\}$ is an open covering of A(U). Clearly we can form a class $\{H_\alpha\}$ of sets H such that for each $G\in\{G_\alpha\}$ there is an $H\in\{H_\alpha\}$ and $a\in H$ if and only if $\emptyset(a)=f_a\in G$. Problem 2.5.3 guarantees that each H is open and by definition $\{H_\alpha\}$ is a covering. Thus $\{H_\alpha\}$ is an open covering of U. Now U is compact; hence there must be some finite subcovering $\{H_i\}$ of U. Obviously $\{G_i\}$ is a covering for A(U) and $\{G_i\}$ is finite; thus A(U) is compact. Since no property of U was used in this proof that is not also a property of A(U) it follows immediately that A(U) compact implies U compact.

Problem 2.5.5. Let U be a subset of a metric space M. Then U is dense in M if and only if A(U) is dense in A. Suppose A(U) is dense in A, and $a\in M$. Further suppose there exists some $r>0$ such that $S(a,r)\cap U=\emptyset$. Now $f_a\in A$ and hence either $f_a\in A(U)$ or f_a is a limit point of A(U). Clearly $f_a\notin A(U)$ since $S(a,r)\cap U=\emptyset$; thus f_a is a limit point of A(U). Thus for $r>0$ there is some f_b such that $f_b\in S(f_a,r)$ and

$f_b \in A(U)$. Then $b \in U$ and $d(a,b)=d(f_a,f_b)<r$ which is a contradiction. Thus U is dense in M.

Now suppose U is dense in M and $f_a \in A$. Then $a \in M$ and $a \in U$ or an arbitrary sphere S(a,r) intersects U. In the former case the theorem follows. In the latter case there is some sequence $\{a_n\} \rightarrow a$; hence $\{f_{a_n}\} \rightarrow f_a$ and f_a is a limit point of A(U). Thus A(U) is dense in A.

An interesting class of functions is that of lower-semicontinuous functions. This class contains the continuous functions, step functions, and others.

<u>Definition 2.5</u>. A function f on a metric space M into the real numbers R is called lower-semicontinuous at $c \in M$ if for each $\epsilon > 0$ there is a $\delta > 0$ such that whenever $d(x,c) < \delta$, then $f(c) < f(x) + \epsilon$. If f is lower-semicontinuous at each point of a set $S \subset M$, then f is lower-semicontinuous on S.

<u>Theorem 2.10</u>. A function f on a metric space M into R is lower-semicontinuous if and only if for each real number k, the set $E_k = \{x \in M \mid f(x) \leq k\}$ is closed.

<u>Proof</u>. Suppose f is a lower-semicontinuous function on a metric space M to the real numbers R. Choose some real number k and suppose x_0 is a limit point of $E_k = \{x \in M \mid f(x) \leq k\}$. Now if $x_0 \in E_k$, the theorem follows. Therefore assume $x_0 \notin E_k$ and hence $f(x_0) > k$, or $f(x_0) - k = \epsilon > 0$. But f is lower-semicontinuous and hence there exists a δ such that if $d(x_0,y) < \delta$ then $f(x_0) < f(y) + \epsilon$. Since x_0 is a limit point of E_k

then $S(x_0, \delta)$ contains some point y of E_k other than x_0. Now $f(y) \leq k$ and $f(x_0)=k+\epsilon$ or $k=f(x_0)-\epsilon$. Hence $f(y) \leq k=f(x_0)-\epsilon$ or $f(x_0) \geq f(y)+\epsilon$; but since f is lower-semicontinuous, then $f(x_0) < f(y)+\epsilon$. Thus we have an invalid conclusion and hence our assumption that $x_0 \notin E_k$ is invalid. Therefore we can conclude that $x_0 \in E_k$ and hence E_k is closed.

Now suppose f is a function on the metric space M into the real numbers R, and for each real number k, the set $E_k = \{x \in M \mid f(x) \leq k\}$ is closed. Choose $\epsilon > 0$ and suppose $x_0 \in M$. By a previous theorem we know that x_0 is either an isolated point or a limit point of M. If x_0 is an isolated point of M, there exists a sphere $S(x_0, \delta)$ such that $S(x_0, \delta) \cap M = \{x_0\}$. Therefore we can say if $y \in M$ and $d(x_0,y) < \delta$, $f(x_0) < f(y)+\epsilon$. Suppose x_0 is a limit point of M and suppose that for some $\epsilon > 0$, every sphere about x_0, $S(x_0, \delta)$ where $\delta > 0$, contains some $y \in M$ such that $f(x_0) \geq f(y)+\epsilon$. Consider the set of spheres $T=\{S(x_0,1/n) \mid n=1,2,3,\cdots\}$. Each of these spheres $S(x_0,1/n)$ contains some element y_n such that $f(x_0) \geq f(y_n)+\epsilon$. Consider a sequence $\{y_n\}$ of such elements and choose $k'=f(x_0)-\epsilon$. Now obviously $f(x_0) > k'$ implies $x_0 \notin E_{k'} = \{x \in M \mid f(x) \leq k'\}$. But $\{y_n\} \to x_0$, and each $y_n \in E_{k'}$ since for y_n, $f(x_0) \geq f(y_n)+\epsilon$. Therefore x_0 is a limit point of $E_{k'}$. But this is contradictory to our supposition that $E_{k'}$ is closed. Thus there must exist some $\delta > 0$ such that if $y \in M$ and $d(x_0,y) < \delta$, then $f(x_0) < f(y)+\epsilon$.

Corollary 2.10.1. A function f on a metric space M into R is lower-semicontinuous iff for each $k \in R$, the set $G_k = \{x \in M | f(x) > k\}$ is open.

By a previous theorem we know that G_k open implies $\overline{G}_k$ closed and $\overline{G}_k$ closed implies G_k open, where $\overline{G}$ denotes the complement of G. Now $\overline{G}_k = \{x \in M | f(x) \leq k\}$. By the previous theorem, if f is lower-semicontinuous, then $\overline{G}_k$ is closed and hence G_k is open. Also G_k open implies $\overline{G}_k$ is closed, which by the previous theorem implies f is lower-semicontinuous.

Theorem 2.11. If f is lower-semicontinuous on a metric space M into R and if f is bounded below on a compact non-empty set $S \subset M$, then there is an element $c \in S$ so that $f(x) \geq f(c)$ for all $x \in S$.

Proof. If f is finite the theorem follows. Assume the above conditions. Then there is some real number k such that $f(x) \geq k$ for all $x \in S$. Now choose $k' = k + \epsilon$ where $\epsilon = d(k, f(S))$. Clearly any neighborhood of k' will intersect f(S), and $f(x) \geq k'$ for all $x \in S$. Thus from each of the neighborhoods $S(k', 1/n)$ where $n \in \{1, 2, 3, \cdots\}$, choose some element y_n such that $y_n \in f(S)$ and $f^{-1}(y_n) \in S$. Obviously $\{y_n\}$ converges to k'. The compact set S has the Bolzano-Weierstrass property and hence $\{f^{-1}(y_n)\}$ has some limit point $x_0 \in S$. Suppose $f(x_0) \neq k'$. Then since $f(x_0) \geq k'$ there exists some $\epsilon > 0$ such that $f(x_0) - \epsilon = k'$. Now f is lower-semicontinuous at x_0; thus for $\epsilon/2$ there is some $\delta > 0$ such that if $y \in S$ and $d(x_0, y) < \delta$, then $f(x_0) < f(y) + \epsilon/2$.

Since all but a finite number of the points of the sequence $\{f^{-1}(y_n)\}$ are contained in the sphere $S(x_0, \delta)$, it follows that all but a finite number of points of $\{f(y_n)\}$ are such that $f(x_0) < f(y_i) + \epsilon/2$ or $f(y_i) > f(x_0) - \epsilon/2$. Now suppose $z \in S(k', \epsilon/2)$. Then $z < k' + \epsilon/2 = f(x_0) - \epsilon + \epsilon/2 = f(x_0) - \epsilon/2$. Therefore the sphere $S(k', \epsilon/2)$ contains only a finite number of points of the sequence $\{f(y_n)\}$ and hence $\{f(y_n)\}$ does not converge to k'. Thus the assumption that $f(x_0) \neq k'$ is invalid and hence $f(x_0) = k' \leq f(x)$ for all $x \in S$.

Problem 2.6.1. Suppose each of f and g is a lower semi-continuous function on a subset S of a metric space X. Show that f+g is also lower-semicontinuous on S.

Proof. Choose $\epsilon > 0$ and let $\epsilon' = \epsilon/2$. Since f and g are lower-semicontinuous at a point $x_0 \in S$, there are real numbers r and r' such that if $y \in S$ and $y \in S(x_0, r)$ then $f(x_0) < f(y) + \epsilon'$; and if $y \in S(x_0, r')$ then $g(x_0) < g(y) + \epsilon'$. Choose the smaller of r and r'. Now addition of these two inequalities gives $f(x_0) + g(x_0) < f(y) + g(y) + 2\epsilon'$ or $(f+g)(x_0) < (f+g)(y) + \epsilon$. Thus f+g is lower-semicontinuous on S.

Problem 2.6.2. If f is lower-semicontinuous on S, then αf is lower-semicontinuous for $\alpha \geq 0$.

Proof. Suppose $x_0 \in S$. Clearly any constant function is lower-semicontinuous since for $y \in S$, $f(x_0) = f(y)$ and thus $f(x_0) < f(y) + \epsilon$ for $\epsilon > 0$. Thus if $\alpha = 0$, αf is lower-semicontinuous. Suppose $\alpha > 0$, and $\epsilon > 0$. Let $\epsilon' = \epsilon/\alpha$. Then for ϵ' there is some positive real number r such that for $y \in S(x_0, r)$,

$f(x_0) < f(y) + \epsilon'$. Multiplication by α gives $\alpha f(x_0) < \alpha f(y) + \alpha\epsilon'$, or $\alpha f(x_0) < \alpha f(y) + \epsilon$. Thus αf is lower-semicontinuous for $\alpha \geq 0$.

Problem 2.6.3. If each of f and g is a lower-semicontinuous function on S, then $f \cdot g$ is lower-semicontinuous on S.

Proof. Suppose each of f and g are lower-semicontinuous functions on S such that $f \geq 0$ and $g \geq 0$. Further suppose $x \in S$, and $\epsilon > 0$. If either of f(x) and g(x) are 0 then clearly $f(x) \cdot g(x) = 0 < f(y) \cdot g(y) + \epsilon$ for all $y \in S$. If each of f(x) and g(x) is non-zero, then choose $\epsilon' = \frac{1}{2}\min\{1, \epsilon, f(x), g(x), \epsilon/(f(x)+g(x))\}$. Clearly $\epsilon' > 0$, $1 > \epsilon'$ and hence $\epsilon' > (\epsilon')^2$. Also $(\epsilon') \cdot (f(x)+g(x)) < (\epsilon/(f(x)+g(x))) \cdot (f(x)+g(x)) = \epsilon$ and $(\epsilon')^2 < (\epsilon') \cdot (f(x)+g(x))$ or $(\epsilon') \cdot (f(x)+g(x)) - (\epsilon')^2 > 0$. Since f and g are lower-semicontinuous at x, clearly there is some δ such that for $y \in S$ and $y \in S(x, \delta)$, $f(x) < f(y) + \epsilon'$ and $g(x) < g(y) + \epsilon'$. Thus $f(x) - \epsilon' < f(y)$ and $g(x) - \epsilon' < g(y)$. Multiplication gives $f(x) \cdot g(x) - \epsilon' \cdot (f(x)+g(x)) + (\epsilon')^2 < g(y) \cdot f(y)$. This may be written as $f(x) \cdot g(x) < g(y) \cdot f(y) + \epsilon' \cdot (f(x)+g(x)) - (\epsilon')^2 < f(y) \cdot g(y) + \epsilon$. Thus $f \cdot g(x) < f \cdot g(y) + \epsilon$ and $f \cdot g$ is lower-semicontinuous at x.

From the class of step functions it is easy to offer examples showing that $f-g$, αf (for $\alpha > 0$), $f \cdot g$ (for $f < 0$ or $g < 0$), and f/g are not lower-semicontinuous.

The class of upper-semicontinuous functions was not considered in this paper since the properties which were established for the lower-semicontinuous functions can be extended in a natural way to the upper-semicontinuous function.

The last topic to be considered in this paper is that of convex functions.

Definition 2.6. A function f on a metric space X to R is called convex if for each $x,y \in M$ and each $z \in M$ such that $d(x,z)+d(z,y)=d(x,y)$, then $f(z) \leq \frac{d(z,y)}{d(x,y)} \cdot f(x) + \frac{d(x,z)}{d(x,y)} \cdot f(y)$.

Problem 2.7.1. If each of f and g is a convex function in a set S, then f+g is convex.

Proof. Suppose f and g are convex on S to R, $x,y \in S$, and $z \in S$ such that $d(x,z)+d(z,y)=d(x,y)$. Then $f(z) \leq \frac{d(z,y)}{d(x,y)} \cdot f(x) + \frac{d(x,z)}{d(x,y)} \cdot f(y)$ and also $g(z) \leq \frac{d(z,y)}{d(x,y)} \cdot g(x) + \frac{d(x,z)}{d(x,y)} \cdot g(y)$. Addition of these two inequalities gives $(f+g)(z) \leq \frac{d(z,y)}{d(x,y)} \cdot (f+g)(x) + \frac{d(x,z)}{d(x,y)} \cdot (f+g)(y)$. Thus f+g is convex on S.

Problem 2.7.2. Show αf is convex for $\alpha \geq 0$.

Proof. The inequality $f(z) \leq \frac{d(z,y)}{d(x,y)} \cdot f(x) + \frac{d(x,z)}{d(x,y)} \cdot f(y)$ follows from the convexity of f. Since $\alpha \geq 0$ the multiplication by α does not change the sense of the inequality. Thus $\alpha f(z) \leq \frac{d(z,y)}{d(x,y)} \alpha f(x) + \frac{d(x,z)}{d(x,y)} \alpha f(y)$. Therefore αf is convex.

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